# Quasimorphisms and laws

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Stable commutator length vanishes in any group that obeys a law.

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If $G$ is a group and $g$ is an element of the commutator subgroup $[G,G]$, the *commutator length* of $g$, denoted $\mathrm{cl}(g)$, is the least number of commutators in $G$ whose product is $g$. The *stable commutator length*, denoted $\mathrm{scl}(g)$, is the limit $\mathrm{scl}(g) := \lim_{n\to\infty} \mathrm{cl}(g^n)/n$.

A group $G$ is said to obey a *law* if there is a free group $F$ (which may be assumed to have finite rank) and a nontrivial element $w \in F$ so that for every homomorphism $\rho\colon F \to G$, we have $\rho(w) = \mathrm{id}$. For example, abelian (or, more generally, nilpotent or solvable) groups obey laws. The free Burnside groups $B(m,n)$ with $m \geq 2$ generators and odd exponents $n \geq 665$ are perhaps the best known examples of non-amenable groups that obey laws; see for example Adyan [1].

The point of this note is to prove the following:

**Main Theorem** *Let $G$ be a group that obeys a law. Then $\mathrm{scl}(g) = 0$ for every $g \in [G,G]$.*

The proof is very short, given some basic facts about stable commutator length, which we recall for the convenience of the reader. A basic reference is Bavard's paper [2] or the author's monograph [3], especially Chapter 2.

**Definition 1** A *homogeneous quasimorphism* on a group $G$ is a function $\phi\colon G \to \mathbb{R}$ that restricts to a homomorphism on every cyclic subgroup, and for which there is a least number $D(\phi) \geq 0$ (called the *defect*) so that for any $g,h \in G$ there is an inequality $|\phi(gh) - \phi(g) - \phi(h)| \leq D(\phi)$.

The defect satisfies the following formula:

**Lemma 2** [2, Lemma 3.6] or [3, Lemma 2.24] *Let $\phi$ be a homogeneous quasimorphism. Then there is an equality*

$$\sup_{g,h\in G} |\phi([g,h])| = D(\phi).$$

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Bavard duality (see [2] or [3, Theorem 2.70]) says that for any $g \in [G,G]$, there is an equality $\mathrm{scl}(g) = \sup_\phi \phi(g)/2D(\phi)$ where the supremum is taken over all homogeneous quasimorphisms $\phi$ with nonzero defect. In particular, scl is nontrivial on $G$ if and only if $G$ admits a homogeneous quasimorphism with nonzero defect.

On the other hand, there is a topological formula for scl. Let $X$ be a space with $\pi_1(X) = G$, and let $\gamma\colon S^1 \to X$ be a free homotopy class representing the conjugacy class of $g \in G$. If $\Sigma$ is a compact, oriented surface without sphere or disk components, a map $f\colon \Sigma \to X$ is *admissible* if the map $\partial f\colon \partial\Sigma \to X$ can be factorized as $\partial\Sigma \xrightarrow{d} S^1 \xrightarrow{\gamma} X$. For an admissible map, define $n(\Sigma)$ by the equality $d_*[\partial\Sigma] = n(\Sigma)[S^1]$ in $H_1$; i.e. $n(\Sigma)$ is the degree with which $\partial\Sigma$ wraps around $\gamma$. By reversing the orientation of $\Sigma$ if necessary, we assume $n(\Sigma) \ge 0$. With this notation, one has the following formula:

**Lemma 3** [3, Proposition 2.10] *With notation as above,*

$$\mathrm{scl}(g) = \inf_\Sigma \frac{-\chi(\Sigma)}{2n(\Sigma)}$$

*where $\chi$ denotes Euler characteristic, and the infimum is taken over all compact, oriented surfaces and all admissible maps.*

Notice that both $\chi(\cdot)$ and $n(\cdot)$ are multiplicative under finite covers.

**Proof of the Main Theorem** Suppose that $G$ obeys a law. Then there is a free group $F$ and a nontrivial word $w \in F$ so that any homomorphism from $F$ to $G$ sends $w$ to id. Let $F_2$ be free on generators $x, y$. We can embed $F$ in $F_2$, and express $w$ as a word $v$ in the generators $x, y$. Hence any homomorphism from $F_2$ to $G$ sends $v$ to id.

Let $X$ be a space with $\pi_1(X) = G$. Let $\Sigma$ be a once-punctured torus. We choose generators for $\pi_1(\Sigma)$, and identify this group with $F_2 = \langle x, y\rangle$. Let $\alpha$ be a loop on $\Sigma$ whose free homotopy class represents the conjugacy class of $v$. Then any continuous map $f\colon \Sigma \to X$ sends $\alpha$ to a null-homotopic loop.

Now suppose contrary to the theorem that scl does not vanish on $[G,G]$. By Bavard duality there is a homogeneous quasimorphism $\phi$ with nonzero defect. Scale $\phi$ to have $D(\phi) = 1$. Then by Lemma 2, for any $\epsilon > 0$ there are elements $g, h$ in $G$ with $\phi([g,h]) \ge 1 - \epsilon$, and consequently $\mathrm{scl}([g,h]) \ge 1/2 - \epsilon/2$ by Bavard duality.

Let $\gamma\colon S^1 \to X$ be a loop representing the conjugacy class of $[g,h]$. There is a map $f\colon \Sigma \to X$ whose boundary represents the free homotopy class of $\gamma$. As above, the loop $\alpha$ on $\Sigma$ maps to a null-homotopic loop in $X$. By Scott [4], there is a finite cover

$\widetilde{\Sigma}$ of $\Sigma$ of degree $d$ (depending on $\alpha$), so that some lift $\widetilde{\alpha}$ of $\alpha$ is homotopic to an embedded loop $\alpha'$. Composing the covering map with $f$ gives a map $\widetilde{f}\colon \widetilde{\Sigma} \to X$ for which $\widetilde{f}(\alpha')$ is null-homotopic in $X$. Since $\alpha'$ is embedded, we can compress $\widetilde{\Sigma}$ along $\alpha'$ to produce a new surface $\Sigma'$ mapping to $X$ by $f'$. The map $f'$ is admissible for $\gamma$, and satisfies $n(\Sigma') = d$. Moreover, $\chi(\widetilde{\Sigma}) = -d$, and $\chi(\Sigma') = 2-d$. Consequently, by Lemma 3, we have $\mathrm{scl}([g,h]) \le 1/2 - 1/d$.

Since $d$ is fixed (depending only on the law satisfied by $G$) but $\epsilon$ is arbitrary, we obtain a contradiction. Hence scl vanishes identically on $[G,G]$, as claimed. □

**Remark 4** The statement of the Main Theorem may be rephrased positively as saying that if scl is nonzero on $G$, then for any positive integer $n$, there are homomorphisms $F_2 \to G$ which are injective on the ball of radius $n$.

If $w$ is a word in a free group $F$, define a *$w$–word* in $G$ to be the image of $w$ under a homomorphism $F \to G$. Let $G(w)$ be the subgroup of $G$ generated by $w$–words. The *$w$–length* of $g \in G(w)$, denoted $\mathrm{l}(g|w)$, is the smallest number of $w$–words and their inverses whose product is $g$ (commutator length is the case $w = xyx^{-1}y^{-1} \in \langle x, y\rangle$), and the *stable $w$–length*, denoted $\mathrm{sl}(g|w)$ is $\mathrm{sl}(g|w) := \lim_{n\to\infty} \mathrm{l}(g^n|w)/n$.

**Question 5** Is there an example of a group that obeys a law, but for which $\mathrm{sl}(\cdot|w)$ is nontrivial for some $w$?

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## References

[1] **S I Adyan**, *Random walks on free periodic groups*, Izv. Akad. Nauk SSSR Ser. Mat. 46 (1982) 1139–1149, 1343 MR682486

[2] **C Bavard**, *Longueur stable des commutateurs*, Enseign. Math. (2) 37 (1991) 109–150 MR1115747

[3] **D Calegari**, *scl*, MSJ Memoirs 20, Mathematical Society of Japan, Tokyo (2009) MR2527432

[4] **P Scott**, *Subgroups of surface groups are almost geometric*, J. London Math. Soc. (2) 17 (1978) 555–565 MR0494062

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