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On Hopkins' Picard group Pic_2 at the prime 3

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In this paper we calculate the algebraic Hopkins Picard group $\mathrm{Pic}_2^{\mathrm{alg}}$ at the prime $p=3$, which is a subgroup of the group of isomorphism classes of invertible $K(2)$–local spectra, ie of Hopkins' Picard group Pic_2. We use the resolution of the $K(2)$–local sphere introduced by Goerss, Henn, Mahowald and Rezk in [3] and the methods from Henn, Karamanov and Mahowald [5] and Karamanov [7].

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1 Introduction

Let $\mathcal{C}$ be a symmetric monoidal category with product $\wedge$ and unit I. We say that an object X in $\mathcal{C}$ is invertible if there exists an object Y in $\mathcal{C}$ such that $X \wedge Y \cong I$. If the collection of equivalence classes of invertible objects is a set, then the product defines a group structure on it. We denote this group by $\mathrm{Pic}(\mathcal{C})$, the Picard group of $\mathcal{C}$.

For example, the homomorphism $\mathbb{Z} \to \mathrm{Pic}(\mathcal{S})\colon n \mapsto S^n$ defines an isomorphism between the integers and the Picard group of $\mathcal{S}$, the stable homotopy category, by Hopkins, Mahowald and Sadofsky [6].

Let $\mathcal{K}_n$ be the category of $K(n)$–local spectra, where $K(n)$ is the n–th Morava K–theory at the prime p. The unit in $\mathcal{K}_n$ is given by $L_{K(n)}S^0$ and the product of two $K(n)$–local spectra by $X \wedge Y := L_{K(n)}(X \wedge Y)$ (as the ordinary smash product of two $K(n)$–local spectra need not be $K(n)$–local). Hopkins' Picard group is the group $\mathrm{Pic}(\mathcal{K}_n)$ which we denote by Pic_n. The first account of it appears in Strickland [8] and the case $n=1$ is treated in detail in [6] where also some examples of elements of Pic_2 at the prime $p=2$ are given.

In this paper we are interested in Pic_2 at the prime $p=3$.

One way to study $\mathcal{K}_n$ is through the functor $E_{n*}X := \pi_* L_{K(n)}(E_n \wedge X)$ where E_n is the Lubin–Tate spectrum with coefficients ring $E_{n*} \cong \mathbb{W}\mathbb{F}_{p^n}[\![u_1, \dots, u_{n-1}]\!][u, u^{-1}]$, where the power series ring is over the Witt vectors of $\mathbb{F}_{p^n}$. Recall that E_n is acted on by the (big) Morava stabilizer group $\mathbb{G}_n = \mathbb{S}_n \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ by E_∞–maps; see Goerss

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and Hopkins [4]. Let $\mathcal{E}\mathcal{G}_n$ be the category of profinite $E_{n*}[\![\mathbb{G}_n]\!]$–modules, ie E_{n*}–modules with a continuous $\mathbb{G}_n$–action compatible with the action of $\mathbb{G}_n$ on E_{n*} (see Goerss, Henn, Mahowald and Rezk [3] or Hopkins, Mahowald and Sadofsky [6] for details). The tensor product (over $(E_n)_*$) gives a monoidal structure on $\mathcal{E}\mathcal{G}_n$.

Proposition 1.1 [6] *Let $X \in \mathcal{K}_n$. Then the following conditions are equivalent:*

(a) *X is invertible in $\mathcal{K}_n$.*

(b) *$E_{n*}X$ is free E_{n*}–module of rank 1.*

(c) *$E_{n*}X$ is invertible in $\mathcal{E}\mathcal{G}_n$.*

1.1. Let $\mathrm{Pic}_n^{\mathrm{alg}} := \mathrm{Pic}(\mathcal{E}\mathcal{G}_n)$. By Proposition 1.1 there is a homomorphism

$$\begin{aligned} \epsilon_n\colon \mathrm{Pic}_n &\to \mathrm{Pic}_n^{\mathrm{alg}} \\ X &\mapsto E_{n*}X\ . \end{aligned}$$

Let $\mathrm{Pic}_n^{\mathrm{alg},0}$ be the subgroup of $\mathrm{Pic}_n^{\mathrm{alg}}$ of index 2 of modules concentrated in even degrees. Let $M \in \mathrm{Pic}_n^{\mathrm{alg},0}$ and ι_M be a generator of M in degree 0, as an $(E_n)_*$–module. Then for all $g \in \mathbb{G}_n$ there exists a unique element $u_g \in (E_n)_0^\times$ such that $g_*(\iota_M) = u_g\iota_M$. The map $\theta_M\colon g \mapsto u_g$ is a crossed homomorphism and is a well defined element in $H^1(\mathbb{G}_n; (E_n)_0^\times)$ that does not depend on ι_M. Thus we have a homomorphism $\mathrm{Pic}_n^{\mathrm{alg},0} \to H^1(\mathbb{G}_n; (E_n)_0^\times)$.

Proposition 1.2 [6] $\mathrm{Pic}_n^{\mathrm{alg},0} \cong H^1(\mathbb{G}_n; (E_n)_0^\times)$.

Not much is known about the kernel κ_n of ϵ_n (cf [8]). When $n^2 \le 2(p-1)$ and $n > 1$ or when $n = 1$ and $p > 2$, it is known to be trivial. It is conjectured (by Hopkins – see Strickland [8]) that κ_n is a finite p–group.

The next theorem is an unpublished result of Goerss, Henn, Mahowald and Rezk.

Theorem 1.3 *At the prime $p = 3$, $\kappa_2 \cong \mathbb{Z}/3 \times \mathbb{Z}/3$.*

The next two theorems describe some known results for Pic_n.

Theorem 1.4 [6]

$$\begin{aligned} \mathrm{Pic}_1 &\cong \mathbb{Z}_2 \times \mathbb{Z}/2 \times \mathbb{Z}/4 \quad && \textit{for } p = 2 \\ \mathrm{Pic}_1 &\cong \mathbb{Z}_p \times \mathbb{Z}/2(p-1) \quad && \textit{for } p > 2\,. \end{aligned}$$

The spectrum S^1 is a generator of Pic_1 in the case $p > 2$. In an unpublished result and using Shimomura's calculations of $\pi_* L_2 S$ at primes $p > 3$, Hopkins shows:

Theorem 1.5 *For primes* $p > 3$

$$\mathrm{Pic}_2 \cong \mathbb{Z}_p^2 \times \mathbb{Z}/2(p^2 - 1).$$

The main result of this paper is the following theorem.

Theorem 1.6 *At the prime* 3

$$\mathrm{Pic}_2^{\mathrm{alg}} \cong \mathbb{Z}_3^2 \times \mathbb{Z}/16$$

generated by $(E_2)_* S^1$ *and* $(E_2)_* S^0[\det]$, *where* $\det$ *is a suitable character of* $\mathbb{G}_2$.

Theorem 1.3 and Theorem 1.6 imply the following theorem.

Theorem 1.7 *At the prime* 3

$$\mathrm{Pic}_2 \cong \mathbb{Z}_3^2 \times \mathbb{Z}/3 \times \mathbb{Z}/3 \times \mathbb{Z}/16.$$

1.2. This paper is organized as follows. In Section 2 we recall the basic properties of the Morava stabilizer group $\mathbb{G}_n$ and describe some important subgroups in the case $n = 2$ and $p = 3$. We also recall the GHMR resolution of [3] and the spectral sequence of [5; 7] that we use for the most difficult part of our calculation. In Section 3 we define two elements of $\mathrm{Pic}_n^{\mathrm{alg}}$ that turn out to be generators in the case of $\mathrm{Pic}_2^{\mathrm{alg}}$. In Section 4 we present three short exact sequences that we use to simplify the calculations. In Section 5 we describe the part of the first page of the spectral sequence that is needed for the calculations. The final calculations for $\mathrm{Pic}_2^{\mathrm{alg},0}$ are done in Section 6, and $\mathrm{Pic}_2^{\mathrm{alg}}$ is treated in Section 7.

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2 On the Morava stabilizer group and the GHMR resolution

In this section we recall some basic properties of the Morava stabilizer group and some important finite subgroups in the case $n = 2$ and $p = 3$. We also describe the main tool of this work, that is the algebraic GHMR resolution of the $K(2)$–local sphere constructed in [3]. This resolution is used in [5; 7] to determine the homotopy of the mod–3 Moore spectrum localized at $K(2)$. We will use some of the calculations of [5; 7] and the spectral sequence used there. For more details the reader is referred to the corresponding papers.

2.1. Recall that $\mathbb{S}_n$ is the group of automorphisms of the Honda formal group law Γ_n with p–series $[p]_{\Gamma_n}(x) = x^{p^n}$, that is, the group of units in the endomorphism ring $\mathrm{End}(\Gamma_n)$. Let $\mathcal{O}_n$ be the noncommutative ring extension of $\mathbb{W}\mathbb{F}_{p^n}$ (the Witt vectors over $\mathbb{F}_{p^n}$, that we denote by $\mathbb{W}$ from now on) generated by an element S satisfying $S^n = p$ and $Sw = w^\sigma S$ where $w \in \mathbb{W}$ and σ is the lift of the Frobenius automorphism of $\mathbb{F}_{p^n}$. Then $\mathrm{End}(\Gamma_n)$ can be identified with $\mathcal{O}_n$. For example, in the case $n = 2$ and $p = 3$ each element g of $\mathbb{S}_2$ can be written as $g = g_1 + g_2 S$ with $g_1 \in \mathbb{W}^\times$ and $g_2 \in \mathbb{W}$.

2.2. Right multiplication of $\mathbb{S}_n$ on $\mathrm{End}(\Gamma_n)$ defines a homomorphism $\mathbb{S}_n \to \mathrm{GL}(\mathbb{W})$. Composition with the determinant can be extended to $\mathbb{G}_n$ to obtain a homomorphism $\mathbb{G}_n \to \mathbb{W}^\times \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ and it is easy to check that this lands in $\mathbb{Z}_p^\times \times \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$. The quotient of $\mathbb{Z}_p^\times$ by its torsion subgroup, isomorphic to $\mathbb{Z}/(p-1)$ when $p > 2$, can be identified with $\mathbb{Z}_p$ and we get a homomorphism called *reduced determinant* or *reduced norm*:

$$\mathbb{G}_n \to \mathbb{Z}_p .$$

The kernel of this homomorphism is denoted by $\mathbb{G}_n^1$ and in the case when p does not divide n we have $\mathbb{G}_n \cong \mathbb{G}_n^1 \times \mathbb{Z}_p$.

2.3. The element S generates a two sided maximal ideal $\mathfrak{m}$ in $\mathcal{O}_n$ with quotient $\mathcal{O}_n/\mathfrak{m} \cong \mathbb{F}_{p^n}$. The strict Morava stabilizer group S_n is the kernel of $\mathcal{O}_n^\times \to \mathbb{F}_{p^n}^\times$ induced by reduction modulo $\mathfrak{m}$. We denote by S_n^1 its intersection with $\mathbb{G}_n^1$.

2.4. Let $n = 2$ and $p = 3$ from now on. Let ω be a primitive eighth root of unity, $\phi \in \mathrm{Gal}(\mathbb{F}_9/\mathbb{F}_3)$ the generator, $t := \omega^2$, $\psi := \omega\phi$ and $a := \frac{1}{2}(1 + \omega S)$. It is easy to verify that a is an element of order 3 . These elements satisfy $\psi a = a\psi$, $t\psi = \psi t^3$, $ta = a^2 t$ and $\psi^2 = t^2$. Then a, ψ and t generate a subgroup of order 24, denoted G_{24}, ω and ϕ a subgroup isomorphic to the semidihedral group of order 16, denoted SD_{16}. The elements t and ψ generate a subgroup of SD_{16} isomorphic to the quaternion group of order 8, denoted Q_8 and we have

$$\mathrm{SD}_{16} \cong Q_8 \rtimes \mathrm{Gal}(\mathbb{F}_9/\mathbb{F}_3) . \tag{1}$$

2.5. The action of the element a on $(E_2)_*/(3) \cong \mathbb{F}_9[\![u_1]\!][u, u^{-1}]$ is described in [5, Corollary 4.7]. For our purposes we only need the following formulae:

$$\begin{aligned} a_* u &\equiv (1 + (1+\omega^2)u_1)u && \mod (u_1^3) \\ a_* u_1 &\equiv u_1 - (1+\omega^2)u_1^2 && \mod (u_1^3) . \end{aligned}$$

The (integral) action of ω is given by

$$\omega_* u_1 = \omega^2 u_1 \quad \text{and} \quad \omega_* u = \omega u \tag{2}$$

and the Frobenius ϕ acts $\mathbb{Z}_3$–linearly by extending the action of the Frobenius on $\mathbb{W}$ via

$$\phi_* u_1 = u_1 \quad \text{and} \quad \phi_* u = u\,. \tag{3}$$

2.6. The GHMR resolution In [3] a resolution of the trivial $\mathbb{G}_2^1$–module $\mathbb{Z}_3$ is constructed that has the following form

$$0 \longrightarrow C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} \mathbb{Z}_3 \longrightarrow 0$$

where $C_0 = C_3 = \mathbb{Z}_3[\![\mathbb{G}_2^1]\!] \otimes_{\mathbb{Z}_3[G_{24}]} \mathbb{Z}_3$ and $C_1 = C_2 = \mathbb{Z}_3[\![\mathbb{G}_2^1]\!] \otimes_{\mathbb{Z}_3[\mathrm{SD}_{16}]} \chi$ and χ is the nontrivial character of SD_{16} defined over $\mathbb{Z}_3$, on which ω and ϕ act by multiplication by -1. The complete ring $\mathbb{Z}_p[\![G]\!]$ is by definition $\lim_{U,n} \mathbb{Z}_p/p^n[G/U]$ where U runs through the open subgroups of G. Then we have the following lemma (cf [5, Lemma 6.1]).

Lemma 2.1 *Let M be a left $\mathbb{G}_2^1$–module. Then there is a first quadrant cohomological spectral sequence $E_r^{*,*}$, $r \geq 1$ with*

$$E_1^{s,t} = \mathrm{Ext}^t_{\mathbb{Z}_3[\![\mathbb{G}_2^1]\!]}(C_s; M) \Longrightarrow H^{s+t}(\mathbb{G}_2^1; M) \tag{4}$$

in which $E_1^{s,t} = 0$ for $0 < s < 3$ and $t > 0$, and for $s \geq 0$ and $t > 3$, and also

$$E_1^{0,t} \cong E_1^{3,t} \cong H^t(G_{24}; M) \text{ and } E_1^{1,0} \cong E_1^{2,0} \cong \mathrm{Hom}_{\mathrm{SD}_{16}}(\chi, M)\,.$$

Note that $\mathrm{Hom}_{\mathrm{SD}_{16}}(\chi, M) \cong \{m \in M \mid \omega_* m = \phi_* m = -m\}$.

2.7. Let N_0 be the kernel of ∂_0 and $j\colon N_0 \to C_0$ the inclusion. As explained in the remark after [5, Lemma 6.1] the differentials in the spectral sequence can be evaluated if we know projective resolutions $Q_\bullet$ of N_0 and $P_\bullet$ of C_0 as well as a chain map $\varphi\colon Q_\bullet \to P_\bullet$ covering j. These data can be assembled in a double complex $T_{\bullet\bullet}$ with $T_{\bullet 0} = P_\bullet$, $T_{\bullet 1} = Q_\bullet$, vertical differentials δ_P and δ_Q and horizontal differentials $(-1)^n\varphi_n\colon Q_n \to P_n$. The filtration of the spectral sequence of this double complex agrees (up to reindexing) with that of the spectral sequence of the lemma. Hence extension problems in the spectral sequence (4) can be studied by using the double complex. As in [5] we obtain a resolution $P_\bullet := \mathbb{Z}_3[\![\mathbb{G}_2^1]\!] \otimes_{\mathbb{Z}_3[G_{24}]} P'_\bullet$ induced from an explicit resolution of the trivial G_{24}–module $\mathbb{Z}_3$.

Lemma 2.2 [5, Lemma 6.2] *Let $\overline{\chi}$ be the $\mathbb{Z}_3[Q_8]$–module whose underlying $\mathbb{Z}_3$–module is $\mathbb{Z}_3$ and on which t acts by multiplication by -1 and ψ by the identity. Then* *the trivial $\mathbb{Z}_3[G_{24}]$–module $\mathbb{Z}_3$ admits a projective resolution $P'_\bullet$ of period 4 of the following form*

$$\xrightarrow{a^2-a} 1\uparrow_{Q_8}^{G_{24}} \xrightarrow{e+a+a^2} 1\uparrow_{Q_8}^{G_{24}} \xrightarrow{a^2-a} \overline{\chi}\uparrow_{Q_8}^{G_{24}} \xrightarrow{e+a+a^2} \overline{\chi}\uparrow_{Q_8}^{G_{24}} \xrightarrow{a^2-a} 1\uparrow_{Q_8}^{G_{24}} \longrightarrow \mathbb{Z}_3 .$$

We obtain $Q_\bullet$ from splicing the exact complex $0 \to C_3 \to C_2 \to C_1 \to N_0 \to 0$ with the projective resolution $P_\bullet$ of $C_3 = C_0$ (as C_1 and C_2 are projective). If we denote by e the unit of $\mathbb{G}_2^1$, by e_i the generators $e \otimes 1$ of C_i and by $\widetilde{e}_i$ the generators $e \otimes 1$ of P_i, then by [5, Lemma 6.3] there is a chain map $\varphi_\bullet\colon Q_\bullet \to P_\bullet$ covering the homomorphism j such that $\varphi_0\colon Q_0 = C_1 \to P_0$ sends e_1 to $(e-\omega)\widetilde{e}_0$.

2.8. We denote by E the spectral sequence for $(E_2)_*/(3) \cong \mathbb{F}_9[\![u_1]\!][u, u^{-1}]$. The structure of the E_1–page is well known (cf [2] for the group $\mathbb{S}_n$, the case of $\mathbb{G}_n$ can be deduced in the same way).

Proposition 2.3 *Let $M = (E_2)_*/(3) \cong \mathbb{F}_9[\![u_1]\!][u, u^{-1}]$.*

(a) *There are elements*

$$\beta \in H^2(G_{24}, M_{12}), \quad \alpha \in H^1(G_{24}, M_4) \quad \text{and} \quad \widetilde{\alpha} \in H^1(G_{24}, M_{12}),$$

an invertible G_{24}–invariant element $\Delta \in M_{24}$ and an isomorphism of graded algebras

$$H^*(G_{24}, M) \cong \mathbb{F}_3[\![v_1^6\Delta^{-1}]\!][\Delta^{\pm 1}, v_1, \beta, \alpha, \widetilde{\alpha}]/(\alpha^2, \widetilde{\alpha}^2, v_1\alpha, v_1\widetilde{\alpha}, \alpha\widetilde{\alpha} + v_1\beta) .$$

(b) *The ring of SD_{16}–invariants of M is given by the subalgebra*

$$M^{\mathrm{SD}_{16}} = \mathbb{F}_3[\![u_1^4]\!][v_1, u^{\pm 8}]$$

and $\mathrm{Hom}_{\mathbb{Z}_3[\mathrm{SD}_{16}]}(\chi, M)$ is a free $M^{\mathrm{SD}_{16}}$–module of rank 1 with generator $\omega^2 u^4$, ie

$$\mathrm{Hom}_{\mathbb{Z}_3[\mathrm{SD}_{16}]}(\chi, M) \cong \omega^2 u^4 \mathbb{F}_3[\![u_1^4]\!][v_1, u^{\pm 8}] . \qquad \square$$

Recall that $v_1 = u_1 u^{-2}$ is invariant modulo 3 with respect to the action of $\mathbb{G}_2$ and therefore all the differentials in the spectral sequence are v_1–linear. The element $\alpha \in H^1(G_{24}; (E_2)_4/(3))$ is defined as the modulo 3 reduction of $\delta^0(v_1)$, where δ^0 is the Bockstein with respect to the short exact sequence

$$0 \longrightarrow (E_2)_* \xrightarrow{\times 3} (E_2)_* \longrightarrow (E_2)_*/(3) \longrightarrow 0 .$$

The element $\tilde{\alpha} \in H^1(G_{24};(E_2)_{12}/(3))$ is defined as $\delta^1(v_2)$, where $v_2 = u^{-8}$ and δ^1 is the Bockstein with respect to the short exact sequence

$$0 \longrightarrow (E_2)_*/(3) \xrightarrow{\times u_1} (E_2)_*/(3) \longrightarrow (E_2)_*/(3,u_1) \longrightarrow 0$$

and $\beta \in H^2(G_{24};(E_2)_{12}/(3))$ is the modulo 3 reduction of $\delta^0\delta^1(v_2)$. The definition of Δ is more complicated and we have the following formula (cf [5, Proposition 5.1]).

$$\Delta \equiv (1-\omega^2u_1^2+u_1^4)\omega^2u^{-12} \mod (u_1^6)\,. \tag{5}$$

One of the main results in [7] (see also [5, Theorem 1.2]) is the following theorem.

Theorem 2.4 *There are elements*

$$\Delta_k \in E_1^{0,0,24k}, \quad b_{2k+1} \in E_1^{1,0,8(2k+1)}, \quad \overline{b}_{2k+1} \in E_1^{2,0,8(2k+1)}$$

for each $k \in \mathbb{Z}$ *satisfying*

$$\Delta_k \equiv \Delta^k, \quad b_{2k+1} \equiv \omega^2u^{-4(2k+1)}, \quad \overline{b}_{2k+1} \equiv \omega^2u^{-4(2k+1)}$$

(where the first congruence is modulo (u_1^2) *and the last two modulo* (u_1^4)*) such that*

$$d_1(\Delta_k) = \begin{cases} (-1)^{m+1}b_{2(3m+1)+1} \equiv (-1)^{m+1}\omega^2(1+u_1^4)u^{-12k} & k=2m+1,\\ v_1^{4\cdot 3^n-2}b_{2\cdot 3^n(3m-1)+1} & k=2\cdot 3^n m, 3\nmid m,\\ 0 & k=0,\end{cases}$$

$$d_1(b_{2k+1}) = \begin{cases} (-1)^n v_1^{6\cdot 3^n+2}\overline{b}_{3^{n+1}(6m+1)} & k=3^{n+1}(3m+1),\\ (-1)^n v_1^{10\cdot 3^n+2}\overline{b}_{3^n(18m+11)} & k=3^n(9m+8),\\ 0 & \textit{otherwise}.\end{cases}$$

3 Two elements of $\mathrm{Pic}_n^{\mathrm{alg},0}$

3.1. In this section n and p are arbitrary. We have two distinguished elements in $\mathrm{Pic}_n^{\mathrm{alg},0} \cong H^1(\mathbb{G}_n;(E_n)_0^\times)$. In the case $n=2$ and $p=3$ these will generate the first cohomology. The first one is given by the crossed homomorphism

$$\begin{aligned} \eta\colon \mathbb{G}_n &\to (E_n)_0^\times \\ g &\mapsto \frac{g_*u}{u}\,. \end{aligned}$$

The second one is given as the composition of the norm and the canonical inclusion

$$\det\colon \mathbb{G}_n \to \mathbb{Z}_p^\times \to (E_n)_0^\times.$$

We denote the corresponding elements of $H^1(\mathbb{G}_n;(E_n)_0^\times)$ again by η and det. Note that by the isomorphism of Proposition 1.2 the element $(E_n)_*S^2 \in \mathrm{Pic}_n^{\mathrm{alg},0}$ is sent to η (as $u \in (E_n)_{-2}$ gives rise to a generator of $(E_n)_0S^2$).

3.2. The reduction $\mathbb{W}[\![u_1]\!]^\times \to \mathbb{W}^\times$ is equivariant with respect to the inclusion $\mathbb{W}^\times \to \mathbb{G}_n$. Taking into account the Galois group $\mathrm{Gal} := \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ we obtain a homomorphism:

$$\mathrm{red}\colon H^1(\mathbb{G}_n;(E_n)_0^\times) \to H^1(\mathbb{W}^\times \rtimes \mathrm{Gal}; \mathbb{W}^\times) \to H^1(\mathbb{W}^\times;\mathbb{W}^\times)^{\mathrm{Gal}}$$

and the last homomorphism is induced by the short exact sequence $1 \to \mathbb{W}^\times \to \mathbb{W}^\times \rtimes \mathrm{Gal} \to \mathrm{Gal} \to 1$ and the corresponding spectral sequence.

Proposition 3.1 *Let $n = 2$ and $p > 2$. Then the image of the homomorphism* red *is (topologically) generated by the images of η and* det.

Proof Recall that when $p > 2$ then $\mathbb{W}^\times \cong \mathbb{W} \times \mathbb{F}_{p^n}$ with the obvious Galois action. Thus

$$H^1(\mathbb{W}^\times;\mathbb{W}^\times)^{\mathrm{Gal}} \cong \mathrm{End}(\mathbb{W}^\times)^{\mathrm{Gal}} \cong \mathbb{Z}_p^n \times \mathbb{Z}/(p^n-1).$$

The image of det is given by the composition

$$\mathbb{W}^\times \to \mathbb{G}_n \xrightarrow{\det} \mathbb{Z}_p^\times \to (E_n)_0^\times \to \mathbb{W}^\times.$$

If $g = g_0 + g_1S + \cdots + g_{n-1}S^{n-1}$ with $g_i \in \mathbb{W}$ and $w \in \mathbb{W}$ then

$$gw = (g_0 + g_1S + \cdots + g_{n-1}S^{n-1})w = g_0w + g_1w^\phi S + \cdots g_{n-1}w^{\phi^{n-1}}S^{n-1}$$

and the composition above sends w to $ww^\phi \dots w^{\phi^{n-1}}$. The image of η is given by the composition

$$\mathbb{W}^\times \to \mathbb{G}_n \xrightarrow{\eta} (E_n)_0^\times \to \mathbb{W}^\times$$

and this is easily verified to be the identity. □

4 Reductions

In this short section we present three short exact sequences that we use in our calculations. The last two were also used by Hopkins in the case $n = 2$ and $p > 3$.

4.1. The first one

$$1 \to \mathbb{G}_2^1 \to \mathbb{G}_2 \to \mathbb{Z}_3 \to 1 \tag{6}$$

was described in Section 2. We use the Lyndon–Hochschild–Serre spectral sequence associated to (6) to calculate $H^1(\mathbb{G}_2;\mathbb{W}[\![u_1]\!]^\times)$. The main difficulty is computing $H^1(\mathbb{G}_2^1;\mathbb{W}[\![u_1]\!]^\times)$. This is done in Theorem 6.4.

4.2. The reduction modulo 3 gives a short exact sequence

$$0 \to \mathbb{W}[\![u_1]\!] \xrightarrow{\exp(p-)} \mathbb{W}[\![u_1]\!]^\times \longrightarrow \mathbb{F}_9[\![u_1]\!]^\times \to 0. \tag{7}$$

We will use the long exact sequence associated to (7) to calculate $H^1(\mathbb{G}_2^1; \mathbb{W}[\![u_1]\!])$. The difficult part is $H^1(\mathbb{G}_2^1; \mathbb{F}_9[\![u_1]\!]^\times)$ (cf Corollary 6.2).

4.3. We have another short exact sequence coming from the reduction modulo u_1

$$1 \to U_1 \to \mathbb{F}_9[\![u_1]\!]^\times \to \mathbb{F}_9^\times \to 1 \tag{8}$$

where $U_1 := \{h \in \mathbb{F}_9[\![u_1]\!]^\times \mid h \equiv 1 \bmod (u_1)\}$. The hard part is to calculate $H^1(\mathbb{G}_2^1; U_1)$. This is by far the hardest part of this work (cf Theorem 5.10). Note that the group U_1 is 3–profinite.

5 The spectral sequence

We use the spectral sequence (4) with $M = U_1$ and denote it by $\bar{E}$ to distinguish it from the (additive) case $M = (E_2)_*/(3)$ that we also make use of. We start with the $\bar{E}_1$–page. As we only need to calculate the first cohomology, it is sufficient to determine $\bar{E}_1^{0,0}$, $\bar{E}_1^{0,1}$ and $\bar{E}_1^{1,0} \cong \bar{E}_1^{2,0}$ and the corresponding differentials and extension problems.

5.1. The term $\bar{E}_1^{0,1}$

Proposition 5.1 $H^1(\bar{G}_{24}; \mathbb{F}_9[\![u_1]\!]^\times) \cong \mathbb{Z}/6$, *where* $\bar{G}_{24} := G_{24}/\langle t^2\rangle$.

Proof Let $\mathbb{F}_9(\!(u_1)\!)^\times$ be the multiplicative group of the field of fractions of $\mathbb{F}_9[\![u_1]\!]$. Each element of $\mathbb{F}_9(\!(u_1)\!)^\times$ is of the form $u_1^n \cdot f$ with $f \in \mathbb{F}_9[\![u_1]\!]^\times$ and $n \in \mathbb{Z}$. The map

$$\mathbb{F}_9(\!(u_1)\!)^\times \to \mathbb{Z}\colon u_1^n \cdot f \mapsto n$$

is a group homomorphism with kernel $\mathbb{F}_9[\![u_1]\!]^\times$. Thus we have a short exact sequence of $\bar{G}_{24}$–modules

$$1 \to \mathbb{F}_9[\![u_1]\!]^\times \to \mathbb{F}_9(\!(u_1)\!)^\times \to \mathbb{Z} \to 1 \tag{9}$$

where $\bar{G}_{24}$ acts trivially on $\mathbb{Z}$.

By Hilbert 90, the multiplicative version, we have $H^1(\bar{G}_{24}; \mathbb{F}_9(\!(u_1)\!)^\times) = 0$ and thus the long exact sequence induced by (9) yields

$$H^0(\bar{G}_{24}; \mathbb{F}_9[\![u_1]\!]^\times) \to H^0(\bar{G}_{24}; \mathbb{F}_9(\!(u_1)\!)^\times) \to H^0(\bar{G}_{24}; \mathbb{Z}) \twoheadrightarrow H^1(\bar{G}_{24}; \mathbb{F}_9[\![u_1]\!]^\times).$$

By Proposition 2.3 we have $H^0(\overline{G}_{24};\mathbb{F}_9[\![u_1]\!]^\times)\cong\mathbb{F}_3[\![v_1^6\Delta^{-1}]\!]^\times$ and by a similar argument we conclude $H^0(\overline{G}_{24};\mathbb{F}_9(\!(u_1)\!)^\times)\cong\mathbb{F}_3(\!(v_1^6\Delta^{-1})\!)^\times$. By (5) we know that $v_1^6\Delta^{-1}\equiv u_1^6\omega^{-2} \mod (u_1^8)$ so the image of the homomorphism

$$H^0(\overline{G}_{24};\mathbb{F}_9(\!(u_1)\!)^\times)\to H^0(\overline{G}_{24};\mathbb{Z})\cong\mathbb{Z}$$

is $6\mathbb{Z}$, and the result follows. □

Note that η is not defined on U_1 (as for example $\omega_* u/u=\omega\notin U_1$), but 8η is well defined.

Proposition 5.2 $\overline{E}_1^{0,1}\cong H^1(G_{24};U_1)\cong\mathbb{Z}/3$ *generated by the restriction of* 8η.

Proof The short exact sequence (8) induces a long exact sequence

$$\to H^1(\overline{G}_{24};U_1)\to H^1(\overline{G}_{24};\mathbb{F}_9[\![u_1]\!]^\times)\to H^1(\overline{G}_{24};\mathbb{F}_9^\times)\to .$$

The group in the middle is isomorphic to $\mathbb{Z}/6$ by Proposition 5.1 and the group on the right is 2–torsion. The group U_1 is 3–profinite, so there is no 2–torsion in $H^*(\overline{G}_{24};U_1)$. As $H^0(\overline{G}_{24};\mathbb{F}_9^\times)$ is 2–torsion, the first morphism above is injective. As U_1 is 3–profinite, $H^1(G_{24};U_1)\cong H^1(\overline{G}_{24};U_1)\cong\mathbb{Z}/3$.

For the second part of the proposition we use the resolution $P'_\bullet$ of G_{24} constructed in Lemma 2.2, to show that the cocycle of 8η can not be a coboundary. Recall that η was defined as a crossed homomorphism and thus it can be easily described using the standard (bar) resolution (cf [1, III.3]). A cocycle representing the image of 8η in the standard resolution $B_\bullet$ of G_{24} is given by $B_1\to U_1\colon [g]\mapsto g_*u^8/u^8$. By comparing these resolutions we find a representing cocycle in $P'_\bullet$. A homomorphism $\theta_\bullet\colon P'_\bullet\to B_\bullet$ over the identity of $\mathbb{Z}_3$ is given by

$$\begin{array}{rcl} \theta_0\colon P'_0 & \to & B_0 \\ e'_0 & \mapsto & \displaystyle\frac{1}{8}\sum_{g\in Q_8} g \end{array} \qquad \begin{array}{rcl} \theta_1\colon P'_1 & \to & B_1 \\ e'_1 & \mapsto & \displaystyle\frac{1}{8}(\sum_{g\in Q_8}\overline{\chi}(g^{-1})g)a[a] \end{array}$$

where e'_i are the generators $e\otimes 1$ of P'_i and $\{[g]\}_{g\in G_{24}}$ is a G_{24}–basis of B_1 (cf [1, I.5]). Thus the composition

$$P'_1\to B_1\to U_1\colon e'_1\mapsto\frac{a_*^2u^8}{a_*u^8}$$

is the desired cocycle. Using the formula from paragraph 2.5 we obtain

$$\frac{a_*^2u^8}{a_*u^8}\equiv 1-(1+\omega^2)u_1 \mod (u_1^2).$$

Now we will show that this cocycle can not be a coboundary. A morphism from $P_0' \to U_1$ sends e_0' to a Q_8–invariant element h of U_1. By Proposition 5.3(c) we know that $h \equiv 1 \bmod (u_1^2)$ and thus the composition $P_1' \to P_0' \to U_1$ sends e_1' to an element congruent to 1 modulo u_1^2 which is not the case for 8η. □

5.1 The 0–th line

In the following proposition we give the structure of the 0–th line of the first page of the spectral sequence $\bar{E}$. We end up with a nice description of the corresponding groups as products of copies of the 3–adics.

Recall that $v_1 = u_1 u^{-2}$ is in degree 4 and Δ in degree 24. Thus $v_1^6\Delta^{-1}$ is in degree 0.

Proposition 5.3

(a) $\bar{E}_1^{0,0} \cong \{g \in ((E_2)_0/(3))^{G_{24}} \cong \mathbb{F}_3[\![v_1^6\Delta^{-1}]\!]^\times \mid g \equiv 1 \bmod (u_1)\}$.

(b) *Let* $g_k \in \bar{E}_1^{0,0}$ *be such that* $g_k \equiv 1 + v_1^{6k}\Delta^{-k} \bmod (u_1^{6k+2})$. *Then*

$$\bar{E}_1^{0,0} \cong \prod_{\substack{k\geq 1\\ k\not\equiv 0 \bmod (3)}} \mathbb{Z}_3\{g_k\}.$$

(c) $\bar{E}_1^{1,0} = \{h \in U_1 \mid \exists k \in ((E_2)_0/(3))^{Q_8} \cong \mathbb{F}_3[\![\omega^2 u_1^2]\!], h = \omega_* k/k\}$.

(d) *Let* $h_k \in \bar{E}_1^{1,0}$ *be such that* $h_k \equiv 1 + \omega^2 u_1^{4k+2} \bmod (u_1^{4k+4})$. *Then*

$$\bar{E}_1^{1,0} \cong \bar{E}_1^{2,0} \cong \prod_{\substack{k\geq 0\\ k\not\equiv 1 \bmod (3)}} \mathbb{Z}_3\{h_k\}.$$

Proof Proposition 2.3 implies (a). By (a) each element $g \in \bar{E}_1^{0,0}$ can be written as a product $\prod_{k\geq 1} g_k^{\lambda_k}$ with $\lambda_k \in \{-1,0,1\}$. As $g_{3k} \equiv g_k^3 \bmod (u_1^{18k+2})$ we obtain the result. To get the action of Q_8 on $(E_2)_0/(3)$ we use formulae (2) and (3) and then (c) follows. As $\omega^2 = -\omega^6$ we have $h_{3k+1}^{-1} \equiv 1 + \omega^6 u_1^{4(3k+1)+2} \equiv (1+\omega^2 u_1^{4k+2})^3 \equiv h_k^3 \bmod (u_1^{12k+8})$ and we obtain (d). □

5.2. The goal of what follows is to construct families of generators $\{g_k\}$ for $k \geq 1$ and $k \not\equiv 0 \bmod (3)$ and $\{h_k\}$ for $k \geq 0$ and $k \not\equiv 1 \bmod (3)$ as in Proposition 5.3 on which the differential $\bar{d}_1$ is easy to describe.

We start with a particular element $m \in \bar{E}_1^{1,0}$ that is related to 8η and plays the same role as the element b_1 in [5].

Proposition 5.4 *There exists $m \in \bar{E}_1^{1,0}$ such that*

(a) $m \equiv 1 + \omega^2 u_1^2 \bmod (u_1^4)$

(b) $\bar{d}_1(m) = 1$

(c) $24\eta = m$ *in* $H^1(\mathbb{G}_2^1; U_1)$.

Proof We imitate the proof of [5, Proposition 5.5] and use paragraph 2.7. By definition 8η is a permanent cycle so there are cochains $c \in \mathrm{Hom}_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(P_1, U_1)$ and $d \in \mathrm{Hom}_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(Q_0, U_1)$ such that $c + d$ is a cocycle in the total complex of the double complex $\mathrm{Hom}_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(T_{\bullet\bullet}, U_1)$ and such that c represents the restriction of 8η in $H^1(G_{24}; U_1)$. From the proof of the Proposition 5.2 we have an explicit cocycle c_1 for 8η in the resolution $P_\bullet$, so there exists $h_c \in \mathrm{Hom}_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(P_0, U_1)$ such that $c = c_1 + \delta_P(h_c)$. As $24\eta = 1$ in $H^1(G_{24}; U_1)$ (Proposition 5.2) there exists $h \in \mathrm{Hom}_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(P_0, U_1)$ such that $\delta_P(h) = 3c_1$. In the double complex the cochain $3c + 3d$ is cohomologous to $3d - \varphi_0(h + 3h_c)$. One can easily check that $h = u^{24}/(u^8(a_* u^8)(a_*^2 u^8)) = 1 + \omega^2 u_1^2 \bmod (u_1^3)$. Then $3d - (e - \omega)_*(h + 3h_c) = 1 + \omega^2 u_1^2 \bmod (u_1^3)$ is a cocycle concentrated in $\mathrm{Hom}_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(T_{1,0}, U_1)$ representing 24η (cf paragraph 2.5 for the action of a and ω). □

The next lemma is elementary but crucial as it relates the differentials of $\bar{E}$ and E, and thus suggests that we could use the generators from Theorem 2.4 to construct convenient generators g_k and h_k.

Lemma 5.5 *Let $f \in \mathbb{F}_9[[u_1]]$ be such that $f \equiv 0 \bmod (u_1^k)$ for some $k > 0$. Then*

$$\frac{1}{1+f} \equiv 1 - f \quad \bmod (u_1^{2k}) .$$

Proposition 5.6 *Let $g_k := 1 + v_1^{6k} \Delta_{-k}$ for $k \geq 1, k \not\equiv 0 \bmod (3)$. Then*

$$\bar{d}_1(g_k) \equiv \begin{cases} 1 + (-1)^{m+1} \omega^2 (u_1^{12m+6} + u_1^{12m+10}) \bmod (u_1^{12m+12}) & k = 2m+1, \\ 1 + \omega^2 u_1^{12m+2} \bmod (u_1^{12m+4}) & k = 2m . \end{cases}$$

Proof By Lemma 5.5 we have $\bar{d}_1(1 + v_1^{6k} \Delta_{-k}) \equiv 1 + v_1^{6k} d_1(\Delta_{-k}) \bmod (u_1^{12k})$ where we have used the v_1–linearity of d_1. Then the result follows from the formulae from Theorem 2.4. □

5.3. As in [5] we will use the image of $\bar{d}_1$ to define h_k, for $k \not\equiv 1 \bmod (3)$. From Proposition 5.6 and using $\omega^2 = -\omega^6$ we get

$$1 + (-1)^{k+1}\omega^2 u_1^{12k+10} \equiv \bar{d}_1(g_{1+2k})(1 + (-1)^{k+1}\omega^2 u_1^{4k+2})^3 \mod (u_1^{12k+12}).$$

Thus if h_k is already defined and satisfies $h_k \equiv 1 + (-1)^{k+1}\omega^2 u_1^{4k+2} \bmod (u_1^{4k+6})$ we can define $h_{3k+2} := \bar{d}_1(g_{1+2k})h_k^3$ or recursively (as $3k+2 = 3(k+1)-1$)

$$h_{3^{n+1}(k+1)-1} := \bar{d}_1(g_{1+2(3^n(k+1)-1)})h^3_{3^n(k+1)-1}$$

The generator $h_{3(3k+1)+2} = h_{9k+5}$ needs to be defined separately and we also need to define h_{3k} for $k \geq 0$. Using Proposition 5.6 for $k \not\equiv 0 \bmod (3)$ we define

$$h_{3k} := \bar{d}_1(g_{2k})^k.$$

The reason for the power is to get the right sign.

To complete the definition of all generators h_k, we define h_0, h_{9k} and h_{9k+5} as follows (again, the power is needed to get the right sign):

$$\begin{aligned} h_0 &:= m^2 \\ h_{9k} &:= \omega_*(1 + v_1^{36k+2}b_{-1-18k})^k/(1 + v_1^{36k+2}b_{-1-18k})^k && k > 0 \\ h_{9k+5} &:= \omega_*(1 + v_1^{36k+22}b_{-11-18k})^k/(1 + v_1^{36k+22}b_{-11-18k})^k && k \geq 0. \end{aligned}$$

As $-1-18k = 1+2(9(-k-1)+8)$ and $-11-18k = 1+2\cdot 3(3(-k-1)+1)$ the elements b_{-1-18k} and $b_{-11-18k}$ belong to the two families in Theorem 2.4 that have nontrivial image under d_1. In both of these cases we can apply Lemma 5.5. This would not have been the case if we would have defined h_0 as $1 + v_1^2 b_{-1}$ as then Lemma 5.5 does not give the enough precision. The following proposition and the recursive definition of the generators describe $\bar{d}_1$: $\bar{E}_1^{1,0} \to \bar{E}_1^{2,0}$. The proof uses Theorem 2.4 and Lemma 5.5.

Proposition 5.7

$$\begin{aligned} z_{1,k} := \bar{d}_1(h_{9k}) &\equiv 1 + (-1)^{k+1}\omega^2 u_1^{36k+14} && \mod (u_1^{36k+18}) \\ z_{2,k} := \bar{d}_1(h_{9k+5}) &\equiv 1 + (-1)^{k+1}\omega^2 u_1^{36k+30} && \mod (u_1^{36k+34}). \end{aligned}$$

A more complete description of $\bar{d}_1$ is given with the following proposition.

Proposition 5.8 *The following complexes are exact:*

$$\begin{array}{ll}
\mathbb{Z}_3\{g_{2k}\}\times\prod_{n\geq 0}\mathbb{Z}_3\{g_{1+2\cdot(3^n(3k+1)-1)}\}\to\prod_{n\geq 0}\mathbb{Z}_3\{h_{3^n(3k+1)-1}\}\to 1 & \text{for } k\notin 3\mathbb{N}\\
\prod_{n\geq 0}\mathbb{Z}_3\{g_{1+2\cdot(3^n(9k+1)-1)}\}\to\prod_{n\geq 0}\mathbb{Z}_3\{h_{3^n(9k+1)-1}\}\to\mathbb{Z}_3\{z_{1,k}\} & \text{for } k>0\\
\prod_{n\geq 1}\mathbb{Z}_3\{g_{1+2\cdot(3^n(3k+2)-1)}\}\to\prod_{n\geq 1}\mathbb{Z}_3\{h_{3^n(3k+2)-1}\}\to\mathbb{Z}_3\{z_{2,k}\} & \text{for } k\geq 0\,.
\end{array}$$

The first homology of the complex

$$\prod_{n\geq 0}\mathbb{Z}_3\{g_{1+2\cdot(3^n-1)}\}\to\prod_{n\geq 0}\mathbb{Z}_3\{h_{3^n-1}\}\to 1$$

is isomorphic to $\mathbb{Z}_3\{h_0\}$. *The matrix of the first homomorphism of the first complex has the form*

$$\begin{pmatrix} -k & -3 & & & \\ & 1 & -3 & & \\ & & 1 & -3 & \\ & & & \cdots & \end{pmatrix}$$

and in the other complexes

$$\begin{pmatrix} -3 & & & \\ 1 & -3 & & \\ & 1 & -3 & \\ & & \cdots & \end{pmatrix}$$

and when nontrivial the matrix of the second morphism has the form $(1\ \ 3\ \ 9\ \ 27\ \ \cdots)$.

Proof The proof is a consequence of Proposition 5.3, Proposition 5.4, the definitions of the generators in paragraph 5.3 and Proposition 5.8. □

Corollary 5.9 $\bar{E}_2^{1,0}\cong\mathbb{Z}_3$.

Proof This is consequence of the previous proposition. Indeed, the relevant part of the 0–th line of the first page of the spectral sequence $\bar{E}$ is the product over k of the four complexes of the previous proposition. □

Theorem 5.10 $H^1(\mathbb{G}_2^1;U_1)\cong\mathbb{Z}_3$ *is generated by* 8η.

Proof We only need to resolve the extension problem

$$0\to\bar{E}_2^{1,0}\cong\mathbb{Z}_3\to H^1(\mathbb{G}_2^1;U_1)\to\mathbb{Z}/3\mathbb{Z}\cong\bar{E}_2^{0,1}\to 0\,.$$

But this is immediate due to Proposition 5.4. □

6 $\mathrm{Pic}_2^{\mathrm{alg},0}$

In this section we calculate $\mathrm{Pic}_2^{\mathrm{alg},0}$ by using the short exact sequences from Section 4. The element η again plays an important role in the proof.

Proposition 6.1 $H^1(\mathbb{G}_2^1;\mathbb{F}_9^\times)\cong\mathbb{F}_9^\times$.

Proof There is a short exact sequence

$$1\to S_2^1\to\mathbb{G}_2^1\to\mathrm{SD}_{16}\to 1$$

that gives a spectral sequence and as S_2^1 acts trivially on $\mathbb{F}_9^\times$ we have

$$H^*(\mathbb{G}_2^1;\mathbb{F}_9^\times)\cong H^*(\mathrm{SD}_{16};\mathbb{F}_9^\times)\,.$$

There is another short exact sequence

$$1\to C_8\to\mathrm{SD}_{16}\to\mathrm{Gal}\to 1$$

(where $\mathrm{Gal}:=\mathrm{Gal}(\mathbb{F}_9/\mathbb{F}_3)$ and C_8 is the cyclic subgroup of order 8 generated by ω) and thus a spectral sequence

$$H^*(\mathrm{Gal};H^*(C_8;\mathbb{F}_9^\times))\Rightarrow H^*(\mathrm{SD}_{16};\mathbb{F}_9^\times).$$

By using the standard resolution it is easily seen that the group $H^*(C_8;\mathbb{F}_9^\times)$ is isomorphic to $\mathbb{F}_9^\times$ in each degree as ω acts trivially on $\mathbb{F}_9^\times$.

The group $H^1(C_8;\mathbb{F}_9^\times)$ is generated by the identity which is Galois invariant thus

$$H^0(\mathrm{Gal};H^1(C_8;\mathbb{F}_9^\times))\cong\mathbb{F}_9^\times$$

and by Hilbert 90

$$H^1(\mathrm{Gal};H^0(C_8;\mathbb{F}_9^\times))\cong H^1(\mathrm{Gal};\mathbb{F}_9^\times)=0\,.$$

As the image of η in $H^1(\mathbb{G}_2^1;\mathbb{F}_9^\times)\cong H^1(\mathrm{SD}_{16};\mathbb{F}_9^\times)$ reduces to the identity in the group $H^1(C_8;\mathbb{F}_9^\times)$, the differential

$$d_2\colon H^0(\mathrm{Gal};H^1(C_8;\mathbb{F}_9^\times))\to H^2(\mathrm{Gal};H^0(C_8;\mathbb{F}_9^\times))$$

has to be trivial. □

Corollary 6.2 $H^1(\mathbb{G}_2^1;\mathbb{F}_9[\![u_1]\!]^\times)\cong\mathbb{Z}_3\times\mathbb{F}_9^\times$ *generated by* η.

Proof The short exact sequence (8) induces a long exact sequence

$$\to H^0(\mathbb{G}_2^1;\mathbb{F}_9^\times)\to H^1(\mathbb{G}_2^1;U_1)\to H^1(\mathbb{G}_2^1;\mathbb{F}_9[\![u_1]\!]^\times)\to H^1(\mathbb{G}_2^1;\mathbb{F}_9^\times)\to H^2(\mathbb{G}_2^1;U_1)\,.$$

By Theorem 5.10 we have $H^1(\mathbb{G}_2^1;U_1)\cong\mathbb{Z}_3$ and by Proposition 6.1 $H^1(\mathbb{G}_2^1;\mathbb{F}_9^\times)\cong\mathbb{F}_9^\times$. As U_1 is a 3–profinite group, the first and the last homomorphisms are trivial. □

Proposition 6.3

(a) $H^1(\mathbb{G}_2^1;\mathbb{F}_9[\![u_1]\!])=0$.

(b) *The group* $H^1(\mathbb{G}_2^1;\mathbb{W}[\![u_1]\!])$ *is* 3*–profinite.*

(c) $H^1(\mathbb{G}_2^1;\mathbb{W}[\![u_1]\!])=0$.

Proof (a) is direct consequence of [5, Theorem 1.6]. The group $\mathbb{W}[\![u_1]\!]$ is 3–profinite and there is a resolution of finite type (Lazard) of the trivial $\mathbb{G}_2^1$–module $\mathbb{Z}_3$ and (b) follows. Multiplication by 3 induces a short exact sequence

$$\mathbb{W}[\![u_1]\!]\xrightarrow{\times 3}\mathbb{W}[\![u_1]\!]\longrightarrow\mathbb{F}_9[\![u_1]\!]$$

which induces a long exact sequence

$$\longrightarrow H^1(\mathbb{G}_2^1;\mathbb{W}[\![u_1]\!])\longrightarrow H^1(\mathbb{G}_2^1;\mathbb{W}[\![u_1]\!])\longrightarrow H^1(\mathbb{G}_2^1;\mathbb{F}_9[\![u_1]\!])\longrightarrow$$

From (a) and the long exact sequence above it follows that the homomorphism

$$H^1(\mathbb{G}_2^1;\mathbb{W}[\![u_1]\!])\longrightarrow H^1(\mathbb{G}_2^1;\mathbb{W}[\![u_1]\!])$$

is surjective ie the group $G:=H^1(\mathbb{G}_2^1;\mathbb{W}[\![u_1]\!])$ is 3-divisible. As G is 3–profinite, it is the limit of finite 3-groups G/I_n. Thus the homomorphism $G/I_n\to G/I_n$ induced by the multiplication by 3 is surjective and therefore G is trivial. □

Theorem 6.4 $H^1(\mathbb{G}_2^1;\mathbb{W}[\![u_1]\!]^\times)\cong\mathbb{Z}_3\times\mathbb{F}_9^\times$ *generated by* η.

Proof We use the long exact sequence in $H^1(\mathbb{G}_2^1;-)$ induced from the short exact sequence (7). The homomorphism $H^1(\mathbb{G}_2^1;\mathbb{W}[\![u_1]\!]^\times)\to H^1(\mathbb{G}_2^1;\mathbb{F}_9[\![u_1]\!]^\times)$ is injective by Proposition 6.3 (c) and also surjective as the image of $\eta\in H^1(\mathbb{G}_2^1;\mathbb{W}[\![u_1]\!]^\times)$ is a generator (by Corollary 6.2). □

Finally we get to the main result of this section.

Theorem 6.5 $\mathrm{Pic}_2^{\mathrm{alg},0}\cong H^1(\mathbb{G}_2;\mathbb{W}[\![u_1]\!]^\times)\cong\mathbb{Z}_3^2\times\mathbb{F}_9^\times$ *generated by* η *and* $\det$.

Proof We use the short exact sequence (6). We have

$$H^1(\mathbb{Z}_3; H^0(\mathbb{G}_2^1; \mathbb{W}[\![u_1]\!]^\times)) \cong H^1(\mathbb{Z}_3; \mathbb{Z}_3) \cong \mathbb{Z}_3$$

generated by the image of det (cf Section 3) and

$$H^0(\mathbb{Z}_3; H^1(\mathbb{G}_2^1; \mathbb{W}[\![u_1]\!]^\times)) \cong H^0(\mathbb{Z}_3; \mathbb{Z}_3 \times \mathbb{F}_9^\times) \cong \mathbb{Z}_3 \times \mathbb{F}_9^\times .$$

generated by the image of η. The theorem follows from the short exact sequence

$$H^1(\mathbb{Z}_3; H^0(\mathbb{G}_2^1; \mathbb{W}[\![u_1]\!]^\times)) \rightarrowtail H^1(\mathbb{G}_2; \mathbb{W}[\![u_1]\!]^\times) \twoheadrightarrow H^0(\mathbb{Z}_3; H^1(\mathbb{G}_2^1; \mathbb{W}[\![u_1]\!]^\times)). \quad \square$$

7 $\mathrm{Pic}_2^{\mathrm{alg}}$

In this short section we prove Theorem 1.6 (ie we calculate $\mathrm{Pic}_2^{\mathrm{alg}}$).

We are left with the short exact sequence

$$0 \to \mathrm{Pic}_2^{\mathrm{alg},0} \to \mathrm{Pic}_2^{\mathrm{alg}} \to \mathbb{Z}/2 \to 0$$

that comes from the definition of $\mathrm{Pic}_2^{\mathrm{alg},0}$ (cf paragraph 1.1). Note that the isomorphism of Proposition 1.2 sends $(E_2)_*S^2$ to η (cf paragraph 3.1). Thus $(E_2)_*S^2$ is an element of $\mathrm{Pic}_2^{\mathrm{alg},0}$ that generates $\mathbb{Z}_3 \times \mathbb{Z}/8$ in $\mathrm{Pic}_2^{\mathrm{alg}}$. But $(E_2)_*S^1$ is not an element of $\mathrm{Pic}_2^{\mathrm{alg},0}$, therefore its image in the above sequence is a generator of $\mathbb{Z}/2$. Thus $(E_2)_*S^1$ itself must generate $\mathbb{Z}_3 \times \mathbb{Z}/16$.

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