Correction to "Topological nonrealization results via the Goodwillie tower approach to iterated loopspace homology"

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Manfred Stelzer has pointed out that part of Corollary 4.5 of [1] was not sufficiently proved, and, indeed, is likely incorrect as stated. This necessitates a little more argument to finish the proof of the main theorem of [1]. The statement of this theorem, and all the examples, remain unchanged.

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In [1], the author showed that certain unstable modules over the mod 2 Steenrod algebra couldn't be realized as the reduced mod 2 cohomology of a space. The modules have the form $\Sigma^n M$, where M is an unstable module of a special sort. The method of proof was to use a 2nd quadrant spectral sequence converging to $H^*(\Omega^n X;\mathbb{Z}/2)$ to show that, were a space X to exist whose cohomology realized $\Sigma^n M$, $H^*(\Omega^n X;\mathbb{Z}/2)$ could not admit a cup product compatible with Steenrod operations.

The spectral sequence for $n > 1$ is a newish one, arising from the Goodwillie tower of the functor $X \mapsto \Sigma^\infty \Omega^n X$, and Section 4 of [1] is devoted to collecting and proving some basic facts about this spectral sequence. I thank Manfred Stelzer for pointing out that part of Corollary 4.5 is likely over optimistic, and certainly was not sufficiently proved.

We assume notation as in [1].

In Corollary 4.5, it was asserted that if $\widetilde{H}^*(X;\mathbb{Z}/2) \simeq \Sigma^n M$ with M unstable and also has no nontrivial cup products, then in the spectral sequence, one will have $E_3^{-1,*} = E_2^{-1,*} = E_1^{-1,*}$ and $E_2^{-2,*} = E_1^{-2,*}$. My mistake was in not adequately considering possible differentials on elements in $E_1^{-3,*}$ of the form $\sigma^3 L_{n-1}(x \otimes y \otimes z)$. Under the hypotheses on the cup product, the d_1 differential on such terms *will* be 0, by the same argument given explaining why d_1 is zero on terms of the form $L(x \otimes y)$: by comparison to the classical Eilenberg–Moore spectral sequence. But there is no apparent reason why d_2 need also be zero on such terms. We can only conclude that $E_2^{-1,*} = E_1^{-1,*}$, and $E_2^{-2,*} = E_1^{-2,*}$.

Corollary 4.5 is used at one critical point in the proof of the main theorem given in Section 5. Lemma 5.3 asserts that a certain element in $E_1^{-1,2d+2^{k+2}+1}$ is not a

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boundary. The argument given is that for dimension reasons, no d_r for $r > 2$ could have nonzero image in this bigrading. Implicit is that Corollary 4.5 takes care of d_1 and d_2. In light of the comments above, one needs a new argument for d_2.

It turns out that, except for one special case, a dimension argument still works: $E_3^{-3,2d+2^{k+2}+2}$ contains no elements of the form $\sigma^3 L_{n-1}(x \otimes y \otimes z)$. There are two extreme cases to consider: if x, y, and z are all chosen from the top of N_0, and if x and y are chosen from the bottom of N_0 and z is chosen from the bottom of M_1.

In the first case, $|x| = |y| = |z| = m + 2^k$, and so $\sigma^3 L_{n-1}(x \otimes y \otimes z)$ has bidegree $(-3, 3m + 3 \cdot 2^k + 2n + 1)$. In the second case, $|x| = |y| = d + 2^k$ and $|z| = l + 2^{k+1}$, and so $\sigma^3 L_{n-1}(x \otimes y \otimes z)$ has bidegree $(-3, 2d + l + 2^{k+2} + 2n + 1)$.

We are assuming inequality (5–3), which says that $2^k > 4m - 2l + 2n - 2$. One also has that $0 \leq l \leq d \leq m$ and $n \geq 1$. One can then check that, indeed,

$$3m + 3 \cdot 2^k + 2n + 1 < 2d + 2^{k+2} + 2 < 2d + l + 2^{k+2} + 2n + 1,$$

unless we are in the special case $k = 0, n = 1, l = d = m = 0$.

In this final special case, $n = 1$, so we are trying to use the classical Eilenberg–Moore spectral sequence to show that, if M is a $\mathbb{Z}/2$ vector space concentrated in degree 0, there cannot exist a space X with $\widetilde{H}^*(X; \mathbb{Z}/2) \simeq \Sigma M \otimes \Phi(0, 2)$, if all cup products are zero. Such a space will necessarily fit into a cofibration sequence of the form

$$\bigvee S^4 \to \bigvee \Sigma\mathbb{R}P^2 \to X.$$

We leave it to the reader to check that, by appropriately including S^4 into the first wedge, and projecting out onto a $\Sigma\mathbb{R}P^2$ in the second wedge, one sees that X will have a "subquotient" Y with $\widetilde{H}^*(Y; \mathbb{Z}/2) \simeq \Sigma\Phi(0, 2)$, and still with all cup products 0.

Similar to, but simpler than, arguments in Section 6 of [1] (which dealt with $\Sigma^2\Phi(1, 3)$), our arguments show that such a Y can't exist. Repressing some suspensions from the notation, Figure 1 shows all of $E_1^{*,*}$ in total degree less than or equal to 4, in the Eilenberg–Moore spectral sequence converging to $H^*(\Omega Y; \mathbb{Z}/2)$.

As cup products are assumed zero, $E_2^{*,*} = E_1^{*,*}$. Furthermore, $d_2(a \otimes a \otimes a) = 0$ (and thus *not* c), because $a \otimes a \otimes a = (a \otimes a) * a$ and d_2 is a derivation with respect to the shuffle product $*$. Thus through degree 4, $F^{-2}H^*(\Omega Y; \mathbb{Z}/2)$ would have a basis given by elements 1, α, β, δ, ϵ_1, ϵ_2, γ, and ω, in respective degrees 0, 1, 2, 2, 3, 3, 4, and 4, and represented by 1, a, b, $a \otimes a$, $a \otimes b$, $b \otimes a$, c, and $b \otimes b$. The structure of $\Phi(0, 2)$ ($\mathrm{Sq}^1 a = b$, $\mathrm{Sq}^2 b = c$) shows that $\gamma = \beta^2 = \alpha^4$. Furthermore,

$$
\begin{array}{ccccc|c}
a \otimes a \otimes a \otimes a & & & & & 8 \\
 & & & & & 7 \\
 & a \otimes a \otimes a & b \otimes b & & & 6 \\
 & & a \otimes b, b \otimes a & c & & 5 \\
 & & a \otimes a & & & 4 \\
 & & & b & & 3 \\
 & & & a & & 2 \\
 & & & & & 1 \\
 & & & & 1 & 0 \\
\hline
-4 & -3 & -2 & -1 & 0 & s \backslash t
\end{array}
$$

Figure 1: $E_1^{s,t}$ when $\widetilde{H}^*(Y;\mathbb{Z}/2) \simeq \Sigma\Phi(0,2)$

$\mathrm{Sq}^1\,\delta = \epsilon_1 + \epsilon_2 = \alpha \cup \beta$, as all three are represented by $a \otimes b + b \otimes a$. One then gets a contradiction, as

$$0 = \mathrm{Sq}^1\,\mathrm{Sq}^1\,\delta = \mathrm{Sq}^1(\alpha \cup \beta) = \beta^2 = \gamma \neq 0.$$

We end by observing that $\widetilde{H}^*(\mathrm{SU}(3)/\mathrm{SO}(3);\mathbb{Z}/2) \simeq \Sigma\Phi(0,2)$. Here, of course, cup products are not zero, due to Poincaré duality.

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References

[1] **N Kuhn**, *Topological nonrealization results via the Goodwillie tower approach to iterated loopspace homology*, Algebr. Geom. Topol. 8 (2008) 2109–2129 MR2460881

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