Infinite generation of the kernels of the Magnus and Burau representations

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Consider the kernel Mag_g of the Magnus representation of the Torelli group and the kernel Bur_n of the Burau representation of the braid group. We prove that for $g \geq 2$ and for $n \geq 6$ the groups Mag_g and Bur_n have infinite rank first homology. As a consequence we conclude that neither group has any finite generating set. The method of proof in each case consists of producing a kind of "Johnson-type" homomorphism to an infinite rank abelian group, and proving the image has infinite rank. For the case of Bur_n, we do this with the assistance of a computer calculation.

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1 Introduction

1.1 The Magnus kernel

Let $S := S_{g,1}$ be a compact, connected, oriented surface of genus $g \geq 2$ with one boundary component. Let $\mathrm{Mod}_{g,1}$ denote the *mapping class group* of S, which is the group of homotopy classes of orientation-preserving homeomorphisms of S which fix ∂S pointwise. Let $\mathcal{I}_{g,1}$ denote the *Torelli group*, which is the subgroup of $\mathrm{Mod}_{g,1}$ consisting of elements that act trivially on $H := H_1(S, \mathbb{Z})$.

The group $\mathrm{Mod}_{g,1}$ acts on the fundamental group $\pi_1(S)$, inducing an action on the solvable quotient Γ/Γ^3, where $\Gamma := \pi_1(S)$, $\Gamma^2 = [\Gamma, \Gamma]$ and $\Gamma^3 = [\Gamma^2, \Gamma^2]$ are the first three terms of the derived series of Γ. In this paper we consider the group

$$\mathrm{Mag}_g := \ker(\mathrm{Mod}(S) \to \mathrm{Aut}(\Gamma/\Gamma^3)).$$

It follows from work of Fox [4, Theorem 4.9] that Mag_g coincides with the kernel of the so-called *Magnus representation* (see Birman [2, Chapter 3])

$$r\colon \mathcal{I}_{g,1} \to \mathrm{GL}_{2g}(\mathbb{Z}H).$$

The group Mag_g is called the *Magnus kernel*. It was an open question for some time whether or not Mag_g is nontrivial. This was settled in the affirmative by Suzuki in [12]. The first main result of this paper is that Mag_g is in fact quite large.

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Theorem 1.1 *For $g \geq 2$ the group $H_1(\mathrm{Mag}_g, \mathbb{Z})$ has infinite rank.*

As the abelianization of a finitely-generated group has finite rank, we deduce the following.

Corollary 1.2 For $g \geq 2$ the group Mag_g has no finite generating set.

The idea of our proof of Theorem 1.1 is to define a kind of "Johnson-type" homomorphism (see Johnson [5]):

$$\Psi\colon \mathrm{Mag}_g \to \mathrm{Hom}\left(G^{\mathrm{ab}}, \textstyle\bigwedge^2 G^{\mathrm{ab}}\right)$$

where $G = [\Gamma, \Gamma]$ and G^{ab} denotes the abelianization of G. We then construct infinitely many linearly independent elements contained in the image.

It will follow from the definition of Ψ that Ψ extends to $\mathrm{Mag}(F_n)$, the "Magnus kernel" for $\mathrm{Aut}(F_n)$. Thus as an immediate corollary we obtain that $\mathrm{Mag}(F_n)$ is not finitely generated. Since the first posting of this paper, a different proof of this last result has been given by Satoh [11]. Satoh's approach shows that the image of $\mathrm{Mag}(F_n)$ under Ψ has abelian quotients of arbitrarily large finite rank.

1.2 The Burau kernel

Let B_n denote the braid group on n strands. B_n can be realized (see Section 4 below) as a subgroup of the automorphism group $\mathrm{Aut}(F_n)$ of the free group of rank n. The *Burau representation* is a homomorphism

$$\rho_n\colon B_n \to \mathrm{GL}_n(\mathbb{Z}[t, t^{-1}]).$$

We define the *Burau kernel*, denoted Bur_n, to be the kernel of ρ_n. Let K be the kernel of the homomorphism $F_n \to \mathbb{Z}$ taking each fixed generator of F_n to 1. It follows easily from Fox [4] that

$$\mathrm{Bur}_n = \ker(B_n \to \mathrm{Aut}(F_n/[K, K])).$$

While ρ_3 is faithful, it was a longstanding problem as to whether or not ρ_n is faithful (that is, whether Bur_n is nontrivial) for $n > 3$. This was solved by Moody [9], Long–Paton [7], and Bigelow [1] in various cases, with the result that Bur_n is nontrivial for $n \geq 5$; the case of $n = 4$ is still open. Our next main result is that Bur_n is in fact quite large for $n \geq 6$.

Theorem 1.3 *For $n \geq 6$ the group $H_1(\mathrm{Bur}_n, \mathbb{Z})$ has infinite rank; in particular, Bur_n has no finite generating set.*

To prove Theorem 1.3 we construct, similarly to the proof of Theorem 1.1 above, a homomorphism

$$\Phi\colon \mathrm{Bur}_n \to \mathrm{Hom}\left(K^{\mathrm{ab}}, \textstyle\bigwedge^2 K^{\mathrm{ab}}\right).$$

The elements which have been constructed in the kernel of the Burau representation are geometrically elegant, but algebraically very complicated; for example, the element of Bur_7 found by Long–Paton can be described by a single diagram, but as a free group automorphism sends generators of F_7 to words of length up to 475137. Thus we need the assistance of a computer in order to calculate Φ explicitly (see Section 4 below for a full discussion). For the computations in this paper we use a simpler element $\phi_B \in \mathrm{Bur}_n$ for $n \geq 6$ found by Bigelow, which takes generators to words of length no more than 9841. Once we compute the form of $\Phi(\phi_B)$, we then use an equivariance property of Φ to prove that the image of Φ has infinite rank, from which Theorem 1.3 follows.

We remark that in [10, Problem 6.24] Morita posed the problem of determining the kernel of the Magnus and Burau (among other) representations. Theorem 1.1 and Theorem 1.3 can be viewed as a partial answer to this problem.

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2 Defining the homomorphisms

The following construction works for any group G whenever one considers automorphisms of the universal 2–step nilpotent quotient G/G_3 acting trivially on the abelianization G^{ab}. Johnson [5] considered the case $G = \Gamma = \pi_1(S)$.

With Γ equal to $\pi_1(S)$ or F_n as in the introduction, we take $G := [\Gamma, \Gamma]$ or $G := K$ respectively. In either case, let G_i be the lower central series of G, defined inductively by $G_1 = G$ and $G_{i+1} = [G, G_i]$. Consider the exact sequence

$$1 \to G_2 \to G \to G^{\mathrm{ab}} \to 1. \tag{1}$$

Centralizing (1) gives

$$1 \to G_2/G_3 \to G/G_3 \to G^{\mathrm{ab}} \to 1. \tag{2}$$

Since G is free, taking (1) as a presentation for G^{ab}, Hopf's formula gives that

$$G_2/G_3 \approx \textstyle\bigwedge^2 G^{\mathrm{ab}}.$$

$\mathrm{Aut}(\Gamma)$ acts on Γ, and thus on G, and the isomorphism $\nu\colon G_2/G_3 \approx \bigwedge^2 G^{\mathrm{ab}}$ respects the action of $\mathrm{Aut}(\Gamma)$ on both sides. In particular, conjugation by Γ descends to an action on G^{ab} by $H = \Gamma/[\Gamma,\Gamma]$ or by $\mathbb{Z} = \Gamma/K$ respectively. In the case $G = [\Gamma,\Gamma]$, the fact that Mag_g acts trivially on Γ/Γ^3 implies that Mag_g acts trivially on $G^{\mathrm{ab}} = \Gamma^2/\Gamma^3$ and on $\bigwedge^2 G^{\mathrm{ab}}$. Similarly, in the case $G = K$, we have that Bur_n acts trivially on G^{ab} and on $\bigwedge^2 G^{\mathrm{ab}}$.

Let $f \in \mathrm{Mag}_g$ (respectively, $f \in \mathrm{Bur}_n$) be given. For $x \in G^{\mathrm{ab}}$, pick any lift $\tilde{x} \in G$. Since f acts trivially on both the quotient and kernel of (2), we see that $f(\tilde{x})\tilde{x}^{-1}$ lies in the kernel G_2/G_3, which we identify with $\bigwedge^2 G^{\mathrm{ab}}$ via the isomorphism above. One checks, exactly as in Johnson [5], that

$$\delta_f\colon G^{\mathrm{ab}} \to \textstyle\bigwedge^2 G^{\mathrm{ab}}$$

defined by $\delta_f(x) := f(\tilde{x})\tilde{x}^{-1}$ is a well-defined homomorphism; in fact, the resulting map δ_f is $\mathbb{Z}H$–linear (respectively, $\mathbb{Z}[t,t^{-1}]$–linear) with respect to the conjugation action on G^{ab}. This is equivalent to the claim that

$$\delta_f(\gamma x \gamma^{-1}) \equiv \gamma \delta_f(x) \gamma^{-1} \bmod G_3,$$

which can be checked as follows. The difference between the left and right side is

$$\big(f(\gamma x \gamma^{-1})\gamma x^{-1}\gamma^{-1}\big)\big(\gamma f(x) x^{-1}\gamma^{-1}\big)^{-1} = f(\gamma) f(x) f(\gamma)^{-1} \gamma f(x)^{-1} \gamma^{-1},$$

which is conjugate to $[\gamma^{-1} f(\gamma), f(x)]$. The condition on f implies that $f(\gamma) \equiv \gamma \bmod G_2$, so $\gamma^{-1} f(\gamma) \in G_2$ and $[\gamma^{-1} f(\gamma), f(x)] \in G_3$ as desired.

One also checks, exactly as in [5], that in the case $G = [\Gamma,\Gamma]$, defining the map Ψ by $\Psi(f) := \delta_f$ gives a well-defined homomorphism;

(3) $$\Psi\colon \mathrm{Mag}_g \to \mathrm{Hom}\big(G^{\mathrm{ab}}, \textstyle\bigwedge^2 G^{\mathrm{ab}}\big).$$

and, in the case $G = K$, defining $\Phi(f) := \delta_f$ gives a well-defined homomorphism:

(4) $$\Phi\colon \mathrm{Bur}_n \to \mathrm{Hom}\big(G^{\mathrm{ab}}, \textstyle\bigwedge^2 G^{\mathrm{ab}}\big).$$

The homomorphisms Ψ and Φ are equivariant with respect to the natural actions of $\mathrm{Aut}(\Gamma)$ on the source and target.

3 Computing the image of Ψ

Let $S_{0,4}$ denote the 2–sphere with 4 open disks removed. A *lantern* in S is an embedding $S_{0,4} \hookrightarrow S$. Consider the two simple closed curves α and β and the three arcs A_1, A_2 and A_3 on $S_{0,4}$ given in Figure 1.

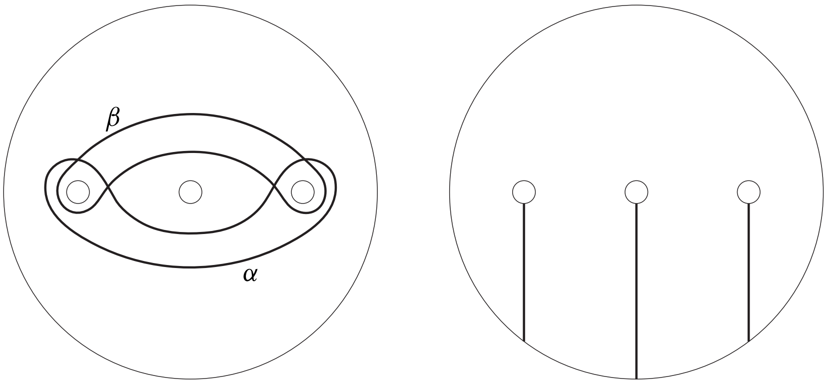

Figure 1: The simple closed curves α and β, and the arcs A_1, A_2, A_3

One directly computes the action of $f := T_\alpha T_\beta^{-1}$ on A_1, A_2 and A_3, as follows (see Figure 2). Let x, y, and z be the loops which begin with A_1, A_2 and A_3, respectively, go clockwise around the appropriate boundary component of $S_{0,4}$, then come back along the same arc A_i. Let X, Y, Z be the inverses of x, y, z in $\pi_1(S_{0,4})$. Then:

$$\begin{aligned} f(A_1) &= xyXzxYXZA_1 = [xyX, z]A_1 \\ f(A_2) &= ZXzxA_2 = [Z, X]A_2 \\ f(A_3) &= ZXzxYXZxzxyXA_3 = [ZXz, xYX]A_3 \end{aligned}$$

Let L be an embedding of a lantern in S with the property that each of the four boundary curves of L are separating in S.[1] In this case we can observe that $T_\alpha T_\beta^{-1} \in \mathrm{Mag}_g$, as follows. Note that the elements corresponding to x, y, z all lie in Γ^2. Furthermore, $\Gamma = \pi_1(S)$ has a basis where each element c is either disjoint from L, or else of the form $c = A\gamma A^{-1}$, where A is an arc intersecting L in some A_i and γ is a loop disjoint from L. In the former case the element $f = T_\alpha T_\beta^{-1}$ fixes c. In the latter case, assume for example that A intersects L in A_2; then we have

$$f(c) = f(A\gamma A^{-1}) = f(A)\gamma f(A)^{-1} = [Z, X]A\gamma A^{-1}[X, Z] = [Z, X]c[X, Z]$$

[1] To formally identify x, y, z with elements of $\Gamma = \pi_1(S)$, we choose a basepoint on ∂S, and arcs from this basepoint to L meeting L in one point. Since f is the identity off of L, any ambiguity in the choice of these paths to L does not affect the computation.

Since $x, y, z \in \Gamma^2$, we have $[Z, X] \in \Gamma^3$; thus $f(c) \equiv c \bmod \Gamma^3$. The same is true for A_1 and A_3, so we conclude that $f(c) \equiv c \bmod \Gamma^3$ for all elements of a basis for Γ, implying $T_\alpha T_\beta^{-1} \in \mathrm{Mag}_g$. Suzuki gave a more illuminating proof that elements of this form lie in Mag_g in [13].

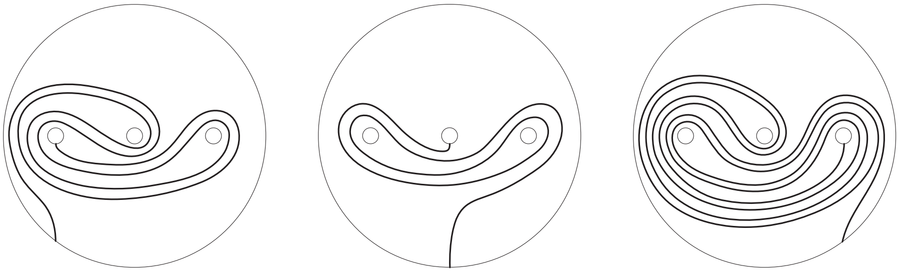

Figure 2: The arcs $f(A_1)$, $f(A_2)$ and $f(A_3)$

We are now ready to compute Ψ. For $a, b \in \Gamma$, we denote by $\{a, b\}$ the image in G^{ab} of $[a, b] \in G$ under the abelianization map.

Proposition 3.1 *Let L be a lantern embedded in S so that each of the four boundary curves of L are separating in S. Let a and b be loops intersecting L in A_1 and A_2. Then*

$$\Psi(T_\alpha T_\beta^{-1})(\{a, b\}) = (a-1)(b-1)\big[x \wedge z + y \wedge z\big]. \tag{5}$$

Note that the right hand side of (5) is an element of $\bigwedge^2 G^{\mathrm{ab}}$, considered as a $\mathbb{Z}H$–module, and the action of a and b on this module factors through H.

Proof As in the computation above, we have

$$f([a, b]) = [f(a), f(b)] = [wa, vb]$$

where

$$w = [[xyX, z], a] \quad \text{and} \quad v = [[Z, X], b].$$

From the assumption on the embedding of L we have $x, y, z \in G$, and thus $w, v \in G_2$. We will use the following commutator identities, which hold in any group; we write ${}^x y$ for xyx^{-1}.

$$[wa, b] = {}^w[a, b]\,[w, b] \qquad\qquad [a, vb] = [a, v]\,{}^v[a, b]$$

We then find that

$$[wa, vb] = {}^w[a, v]\,{}^{wv}[a, b]\,[w, v]\,{}^v[w, b].$$

Note that the second term lies in G, the first and fourth in G_2, and the third in G_3.

We want to compute $f([a,b])[a,b]^{-1}$ as an element of the quotient G_2/G_3. Note that $[w,v] \equiv 0 \bmod G_3$, and that conjugating an element of G by an element of G_2 is a trivial operation modulo G_3. Finally, since $[[a,b],[w,b]] \in G_3$, we can move $[a,b]$ to the right to cancel $[a,b]^{-1}$. We thus obtain

$$\begin{aligned} f([a,b])[a,b]^{-1} &= {}^{w}[a,v]\,{}^{wv}[a,b]\,[w,v]\,{}^{v}[w,b]\,[a,b]^{-1} \\ &\equiv [a,v][a,b][w,b][a,b]^{-1} \bmod G_3 \\ &\equiv [a,v][w,b] \bmod G_3. \end{aligned}$$

Recall that the action of Γ on Γ by conjugation descends to a $\mathbb{Z}H$ action on G^{ab}. Recall from above the isomorphism $\nu\colon G_2/G_3 \to \bigwedge^2 G^{\mathrm{ab}}$. Since the homology class of x is trivial in H, we have

$$\nu([xyX,z]) = y \wedge z \quad \text{and} \quad \nu([Z,X]) = z \wedge x.$$

It follows that

$$\nu(w) = \nu([[xyX,z],a]) = (1-a)y \wedge z$$

and

$$\nu(v) = \nu([[Z,X],b]) = (1-b)z \wedge x.$$

We therefore have that

$$\nu([a,v][w,b]) = (a-1)v - (b-1)w = (a-1)(1-b)z \wedge x - (b-1)(1-a)y \wedge z.$$

We conclude that

$$\Psi(T_\alpha T_\beta^{-1})(\{a,b\}) = (a-1)(b-1)\big[x \wedge z + y \wedge z\big]$$

as desired. □

Theorem 3.2 *The image of Ψ has infinite rank for $g \geq 3$.*

Proof Let γ and δ_k be the curves depicted in Figure 3. The figure depicts the case $k = 3$; in general δ_k has k twists around the upper right handle. (Specifically, the curve δ_k is equal to $T_{a_3}^k(\delta_0)$, where a_3 is as in Figure 5.) The regular neighborhood of $\gamma \cup \delta_k$ is a lantern L_k, and we fix an identification of L_k with our reference lantern L by specifying that γ and δ_k should correspond to xy and yz respectively. Let $f_k \in \mathrm{Mag}_g$ be the element corresponding under this identification to the mapping class $T_\alpha T_\beta^{-1}$ on L; it is easy to check using the lantern relation that f_k is in fact $[T_\gamma^{-1}, T_{\delta_k}^{-1}]$. We will show that the images $\Psi(f_k)$ are linearly independent (over $\mathbb{Z}$).

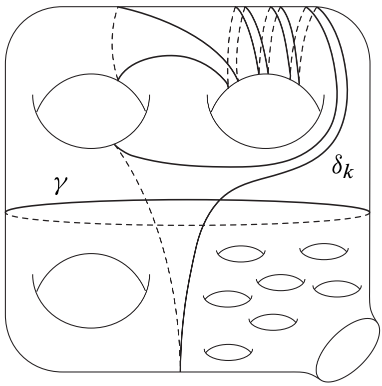

Figure 3: The curves γ and δ_k for $k = 3$

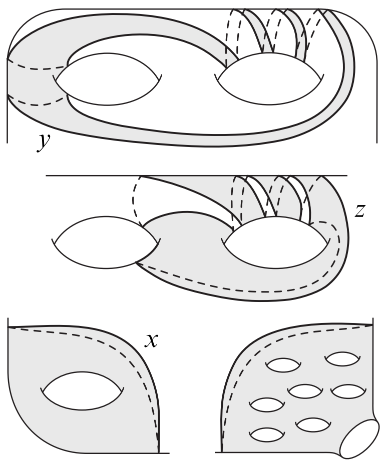

Figure 4: The boundary curves of L_k; the subsurfaces cut off by these curves are shaded

The boundary curves of L_k are depicted in Figure 4.

With the basis $a_1, b_1, \ldots, a_g, b_g$ for $\pi_1(S_{g,1})$ as illustrated in Figure 5, we see that as curves x, y and z can be represented by $[a_1, b_1]$, $[a_2, b_3 a_3^k b_2]$ and $[b_2 a_2^{-1} b_2^{-1} a_3, b_3 a_3^k]$ respectively. As based loops, we actually have the conjugate $z = {}^c[b_2 a_2^{-1} b_2^{-1} a_3, b_3 a_3^k]$, where $c = [b_3, a_3][b_2, a_2]a_2$. Note that with this representative for z, we have $xyz = [a_1, b_1][a_2, b_2][a_3, b_3]$, the fourth boundary curve in Figure 4.

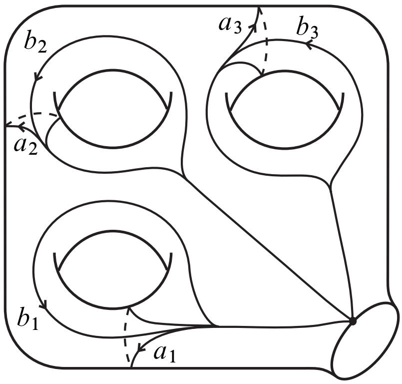

Figure 5: A basis for $\pi_1(S_{g,1})$

Note that a_1 and a_2 intersect each L_k in arcs corresponding to A_1 and A_2. Thus by Proposition 3.1, we have

$$\Psi(f_k)(\{a_1,a_2\}) = \\ (a_1-1)(a_2-1)\big[\big(\{a_1,b_1\}+\{a_2,b_3a_3^kb_2\}\big)\wedge a_2\{b_2a_2^{-1}b_2^{-1}a_3,b_3a_3^k\}\big].$$

Denote this element of $\bigwedge^2 G^{\mathrm{ab}}$ by α_k. We now check that the elements $\{\alpha_k\}$ are linearly independent as follows. There is a standard embedding $G^{\mathrm{ab}} \hookrightarrow (\mathbb{Z}H)^{2g}$ given by sending the class $[x]$ to $(\partial x/\partial z_1,\dots,\partial x/\partial z_{2g})$, where $\{z_i\}$ is our basis for $\pi_1(S)=F_{2g}$ and where $\partial/\partial z_i$ are the Fox derivatives (see, for example, Church–Pixton [3] for a detailed explanation of this embedding). The only property of this embedding that we will need is that the elements below, which together make up α_k, are mapped as follows by the embedding. Here the A_i and B_i make up a $\mathbb{Z}H$–basis for $(\mathbb{Z}H)^{2g}$.

$$\begin{aligned}
\{a_1,b_1\} &\mapsto (1-b_1)A_1-(1-a_1)B_1\\
\{a_2,b_3a_3^kb_2\} &\mapsto (1-b_3a_3^kb_2)A_2\\
&\qquad -(1-a_2)\big(B_3+b_3(1+\cdots+a_3^{k-1})A_3+b_3a_3^kB_2\big)\\
\{b_2a_2^{-1}b_2^{-1}a_3,b_3a_3^k\} &\mapsto (1-b_3a_3^k)\big((1-a_2^{-1})B_2-a_2^{-1}b_2A_2+a_2^{-1}A_3\big)\\
&\qquad -(1-a_2^{-1}a_3)\big(B_3+b_3(1+\cdots+a_3^{k-1})A_3\big)
\end{aligned}$$

By expanding out α_k, we see that α_N is the only such element which contains the term $A_1\wedge b_3a_3^NB_2$ with nonzero coefficient; it follows that the α_k are linearly independent, as desired. $\square$

As the image of Ψ is abelian, Theorem 3.2 immediately implies Theorem 1.1 for $g \geq 3$. Note that the proof of Theorem 3.2 used in an essential way that $g \geq 3$. So in order to complete the proof of Theorem 1.1, we need another argument when $g = 2$.

Theorem 3.3 *$H_1(\mathrm{Mag}_2)$ has infinite rank; in fact, Mag_2 surjects to a free group of infinite rank.*

Proof Suzuki showed that the element $f = [T_\gamma, T_\delta]$ is in Mag_2 for γ and δ as in Figure 6; in particular Mag_2 is nontrivial. Let S_2 be a closed surface of genus 2; we denote by $\mathcal{I}_{2,*}$ the Torelli group of S_2 with respect to a marked point $*$, and by $\mathcal{I}_2$ the Torelli group of the closed surface S_2. By Johnson [6], we have the exact sequence

$$1 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{I}_{2,1} \xrightarrow{p} \mathcal{I}_{2,*} \longrightarrow 1,$$

where the kernel is generated by a twist T_ω around the boundary $\omega = \partial S_2$. It is easy to check that the action of T_ω on $\pi_1(S_{2,1})$ is conjugation by ω; since $\omega \notin \Gamma^3$, we see that $T_\omega \notin \mathrm{Mag}_2$.

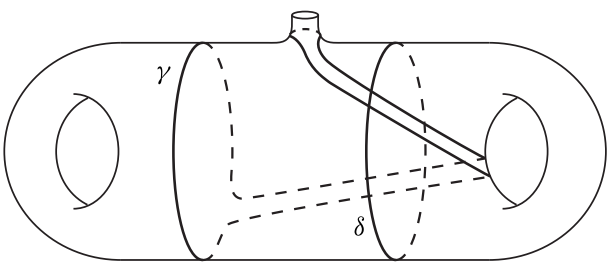

Figure 6: The commutator $[T_\gamma, T_\delta]$ lies in Mag_2

It follows that p restricts to an isomorphism between Mag_2 and a subgroup $p(\mathrm{Mag}_2) < \mathcal{I}_{2,*}$.

Again by Johnson [6], we have the exact sequence

$$1 \longrightarrow \Lambda \longrightarrow \mathcal{I}_{2,*} \xrightarrow{\pi} \mathcal{I}_2 \longrightarrow 1,$$

where $\Lambda \approx \pi_1(S_2, *)$; note that $\mathcal{I}_{2,*}$ acts on $\pi_1(S_2, *)$, and the restriction to Λ is just the action by conjugation. Mess [8] proved that $\mathcal{I}_2$ is free of infinite rank. It is easy to see from Figure 6 that f lies in $\ker \pi = \Lambda$. We use the following well-known lemma.

Lemma 3.4 *Any nontrivial infinite index normal subgroup of a surface group or free group is an infinite rank free group.*

If the image $\pi \circ p(\mathrm{Mag}_2) < \mathcal{I}_2 \approx F_\infty$ is nontrivial, it is an infinite rank free group; it either has finite index in F_∞ and thus infinite rank, or infinite index, in which case Lemma 3.4 applies. Thus Mag_2 surjects to the infinite rank free group $\pi \circ p(\mathrm{Mag}_2)$, and we are done.

Otherwise $p(\mathrm{Mag}_2) \subset \ker \pi = \Lambda$. Any $\varphi \in \mathrm{Mag}_2$ acts trivially on Γ / Γ^3; thus $p(\varphi)$ acts trivially on $\pi_1(S_2)/\pi_1(S_2)^3$. Since the action of Λ is by conjugation, this implies that $p(\varphi)$ lies in Λ^3. Thus $p(\mathrm{Mag}_2)$ has infinite index in Λ, and so by Lemma 3.4, $p(\mathrm{Mag}_2) \approx \mathrm{Mag}_2$ is an infinite rank free group. □

Theorem 1.1, and hence Corollary 1.2, follows immediately from Theorems 3.2 and 3.3.

Remark One can check by explicit computation that for Suzuki's element $f \in \mathrm{Mag}_2$ above, $\Psi(f) = 0$. It would be interesting to know whether Ψ in fact vanishes on Mag_2.

4 Computing the image of Φ

The kernel K of the map from $F_n = \langle x_1, \dots, x_n \rangle$ to $\mathbb{Z} = \langle t \rangle$ which sends each $x_i \mapsto t$ is normally generated by the elements $x_i x_j^{-1}$. If we set $x_{i,k} := x_1^k x_i x_1^{-k-1}$ for $i \neq 1$ and $k \in \mathbb{Z}$, then $\{x_{i,k}\}$ gives a basis for K as a free group. As above, the action of F_n on K by conjugation descends to a $\mathbb{Z}[t, t^{-1}]$ action on K^{ab}. With respect to this action we have $x_{i,k} = t^k x_{i,0}$, and thus K^{ab} is a free $\mathbb{Z}[t, t^{-1}]$–module with basis $\{y_i = x_{i,0}\}_{i \neq 1}$.

The braid group B_n has generators $\sigma_1, \dots, \sigma_{n-1}$; the action of σ_i on F_n sends $x_i \mapsto x_i x_{i+1} x_i^{-1}$, $x_{i+1} \mapsto x_i$, and fixes the other generators. The action of B_n on K^{ab} commutes with the $\mathbb{Z}[t, t^{-1}]$ action.

Theorem 4.1 *The image of Φ has infinite rank for $n \geq 6$.*

Proof The element of Bur_6 found by Bigelow in [1] is the commutator of the half-twists along the arcs displayed in Figure 7. In terms of the Artin generators, this is

$$\phi_B = [\psi_1 \sigma_3^{-1} \psi_1^{-1}, \psi_2 \sigma_3^{-1} \psi_2],$$

where

$$\psi_1 = \sigma_4 \sigma_5^{-1} \sigma_2^{-1} \sigma_1 \quad \text{and} \quad \psi_2 = \sigma_4^{-1} \sigma_5^2 \sigma_2 \sigma_1^{-1}.$$

In Appendix A, we give the computation of $\alpha := \Phi(\phi_B)([x_2 x_1^{-1}]) = \Phi(\phi_B)(y_2)$; it has 262 terms. The only fact about α that we will need is that its highest term of the form

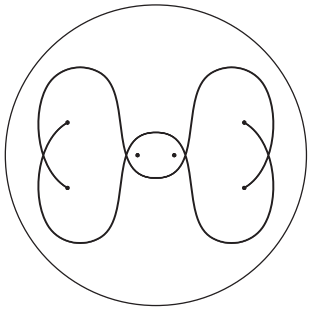

Figure 7: The two arcs defining Bigelow's element ϕ_B

$y_2 \wedge t^j y_4$ is $-2y_2 \wedge t^3 y_4$, and its highest term of the form $y_2 \wedge t^j y_5$ is $+2y_2 \wedge t^2 y_5$ (these terms are set in boxes in the appendix).

It is easy to check that

$$\begin{aligned}\sigma_4^2(x_4) &= x_4 x_5 x_4 x_5^{-1} x_4^{-1}\\ \sigma_4^2(x_5) &= x_5 x_4 x_5^{-1}\\ \sigma_4^2(x_i) &= x_i \quad \text{for } i \neq 4, 5.\end{aligned}$$

By induction, for $k \geq 1$ we have

$$\begin{aligned}\sigma_4^{2k}(x_4) &= (x_4 x_5)^k x_4 (x_4 x_5)^{-k}\\ \sigma_4^{2k}(x_5) &= (x_4 x_5)^{k-1} x_4 x_5 x_4^{-1} (x_4 x_5)^{k-1}\\ \sigma_4^{2k}(x_i) &= x_i \quad \text{for } i \neq 4, 5.\end{aligned}$$

The action of σ_4^{2k} on K^{ab} in terms of our basis is thus given by:

$$\begin{aligned}y_4 &\mapsto (1 - t + t^2 - \cdots - t^{k-1} + t^k) y_4 + (t - t^2 + \cdots + t^{k-1} - t^k) y_5\\ y_5 &\mapsto (1 - t + t^2 - \cdots - t^{k-1}) y_4 + (t - t^2 + \cdots + t^{k-1}) y_5\\ y_i &\mapsto y_i \qquad \text{for } i \neq 4, 5\end{aligned}$$

Now for $k \geq 0$ set

$$\alpha_k := \Phi(\sigma_4^{2k} \phi_B \sigma_4^{-2k})(y_2).$$

By the equivariance of Φ, and since σ_4 fixes y_2, we have $\alpha_k = \sigma_4^{2k} \cdot \alpha$. From the action of σ_4^{2k} on K^{ab}, we can see that the highest term in α_N of the form $y_2 \wedge t^j y_4$ will be $-2y_2 \wedge t^{3+N} y_4$. Thus α_N is not contained in the span of $\{\alpha_1, \ldots, \alpha_{N-1}\}$; it follows that the α_k are linearly independent over $\mathbb{Z}$, and thus the image of Φ has infinite rank. $\square$

Theorem 1.3 follows immediately.

Appendix A Appendix

The following computation was made, with the method explained in Section 4, with the help of *Mathematica*. A *Mathematica* notebook implementing these computations can be found at http://math.uchicago.edu/∼tchurch/infinitegeneration.html or from the abstract page for this article.

The output of this notebook is $\Phi(\phi_B)(y_2)$, which is:

$$
\begin{array}{rrrrr}
-t^{-3}y_2\wedge t^{-2}y_2 & +t^{-3}y_2\wedge t^{-1}y_2 & -t^{-3}y_2\wedge y_2 & -t^{-2}y_2\wedge y_2 & +t^{-1}y_2\wedge y_2 \\
+t^{-2}y_2\wedge ty_2 & +t^{-1}y_2\wedge ty_2 & -2y_2\wedge t^{2}y_2 & +ty_2\wedge t^{3}y_2 & +t^{2}y_2\wedge t^{3}y_2 \\
-t^{3}y_2\wedge t^{4}y_2 & +t^{-3}y_2\wedge t^{-4}y_3 & -t^{-2}y_2\wedge t^{-4}y_3 & -t^{-3}y_2\wedge t^{-3}y_3 & +t^{-1}y_2\wedge t^{-3}y_3 \\
+t^{-2}y_2\wedge t^{-2}y_3 & -t^{-1}y_2\wedge t^{-2}y_3 & +t^{-3}y_2\wedge t^{-1}y_3 & -y_2\wedge t^{-1}y_3 & +ty_2\wedge t^{-1}y_3 \\
-t^{2}y_2\wedge t^{-1}y_3 & -2t^{-2}y_2\wedge y_3 & +t^{3}y_2\wedge y_3 & +t^{-1}y_3\wedge y_3 & +2t^{-1}y_2\wedge ty_3 \\
-t^{-1}y_3\wedge ty_3 & -2y_2\wedge t^{2}y_3 & -t^{4}y_2\wedge t^{2}y_3 & +t^{-1}y_3\wedge t^{2}y_3 & +ty_2\wedge t^{3}y_3 \\
+t^{4}y_2\wedge t^{3}y_3 & -y_3\wedge t^{3}y_3 & +ty_3\wedge t^{3}y_3 & -t^{2}y_3\wedge t^{3}y_3 & +t^{-3}y_2\wedge t^{-3}y_4 \\
-t^{-2}y_2\wedge t^{-3}y_4 & -t^{-3}y_2\wedge t^{-2}y_4 & +t^{-1}y_2\wedge t^{-2}y_4 & +t^{-2}y_2\wedge t^{-1}y_4 & -t^{-1}y_2\wedge t^{-1}y_4 \\
+t^{-3}y_2\wedge y_4 & -y_2\wedge y_4 & +ty_2\wedge y_4 & -t^{2}y_2\wedge y_4 & -y_3\wedge y_4 \\
+ty_3\wedge y_4 & -t^{2}y_3\wedge y_4 & -2t^{-2}y_2\wedge ty_4 & +t^{3}y_2\wedge ty_4 & +t^{-1}y_3\wedge ty_4 \\
+t^{3}y_3\wedge ty_4 & +y_4\wedge ty_4 & +2t^{-1}y_2\wedge t^{2}y_4 & -t^{-1}y_3\wedge t^{2}y_4 & -t^{3}y_3\wedge t^{2}y_4 \\
-y_4\wedge t^{2}y_4 & \boxed{-2y_2\wedge t^{3}y_4} & -t^{4}y_2\wedge t^{3}y_4 & +t^{-1}y_3\wedge t^{3}y_4 & +t^{3}y_3\wedge t^{3}y_4 \\
+y_4\wedge t^{3}y_4 & +ty_2\wedge t^{4}y_4 & +t^{4}y_2\wedge t^{4}y_4 & -y_3\wedge t^{4}y_4 & +ty_3\wedge t^{4}y_4 \\
-t^{2}y_3\wedge t^{4}y_4 & -ty_4\wedge t^{4}y_4 & +t^{2}y_4\wedge t^{4}y_4 & -t^{3}y_4\wedge t^{4}y_4 & +t^{-3}y_2\wedge t^{-3}y_5 \\
-t^{-2}y_2\wedge t^{-3}y_5 & +t^{-3}y_2\wedge t^{-2}y_5 & -t^{-2}y_2\wedge t^{-2}y_5 & +y_2\wedge t^{-2}y_5 & -t^{-4}y_3\wedge t^{-2}y_5 \\
+t^{-3}y_3\wedge t^{-2}y_5 & -t^{-1}y_3\wedge t^{-2}y_5 & -t^{-3}y_4\wedge t^{-2}y_5 & +t^{-2}y_4\wedge t^{-2}y_5 & -y_4\wedge t^{-2}y_5 \\
-t^{-3}y_5\wedge t^{-2}y_5 & -2t^{-3}y_2\wedge t^{-1}y_5 & +t^{-1}y_2\wedge t^{-1}y_5 & +y_2\wedge t^{-1}y_5 & -ty_2\wedge t^{-1}y_5 \\
+t^{-4}y_3\wedge t^{-1}y_5 & -t^{-2}y_3\wedge t^{-1}y_5 & +2y_3\wedge t^{-1}y_5 & +t^{-3}y_4\wedge t^{-1}y_5 & -t^{-1}y_4\wedge t^{-1}y_5 \\
+2ty_4\wedge t^{-1}y_5 & +t^{-3}y_5\wedge t^{-1}y_5 & +t^{-3}y_2\wedge y_5 & +2t^{-2}y_2\wedge y_5 & -2t^{-1}y_2\wedge y_5 \\
-y_2\wedge y_5 & -t^{2}y_2\wedge y_5 & -t^{-3}y_3\wedge y_5 & +t^{-2}y_3\wedge y_5 & -y_3\wedge y_5 \\
-ty_3\wedge y_5 & -t^{2}y_3\wedge y_5 & -t^{-2}y_4\wedge y_5 & +t^{-1}y_4\wedge y_5 & -ty_4\wedge y_5 \\
-t^{2}y_4\wedge y_5 & -t^{3}y_4\wedge y_5 & +t^{-1}y_5\wedge y_5 & -t^{-3}y_2\wedge ty_5 & -t^{-1}y_2\wedge ty_5 \\
+y_2\wedge ty_5 & +ty_2\wedge ty_5 & +t^{3}y_2\wedge ty_5 & +t^{-1}y_3\wedge ty_5 & -y_3\wedge ty_5 \\
+2ty_3\wedge ty_5 & +t^{3}y_3\wedge ty_5 & +y_4\wedge ty_5 & -ty_4\wedge ty_5 & +2t^{2}y_4\wedge ty_5 \\
+t^{4}y_4\wedge ty_5 & -t^{-2}y_5\wedge ty_5 & -y_5\wedge ty_5 & +t^{-2}y_2\wedge t^{2}y_5 & -t^{-1}y_2\wedge t^{2}y_5 \\
\boxed{+2y_2\wedge t^{2}y_5} & -t^{2}y_2\wedge t^{2}y_5 & +t^{3}y_2\wedge t^{2}y_5 & -t^{-1}y_3\wedge t^{2}y_5 & -t^{2}y_3\wedge t^{2}y_5 \\
-y_4\wedge t^{2}y_5 & -t^{3}y_4\wedge t^{2}y_5 & +t^{-1}y_5\wedge t^{2}y_5 & -2y_5\wedge t^{2}y_5 & +ty_5\wedge t^{2}y_5 \\
-ty_2\wedge t^{3}y_5 & -t^{2}y_2\wedge t^{3}y_5 & -t^{4}y_2\wedge t^{3}y_5 & +y_3\wedge t^{3}y_5 & +ty_4\wedge t^{3}y_5 \\
+ty_5\wedge t^{3}y_5 & +t^{2}y_5\wedge t^{3}y_5 & +t^{2}y_2\wedge t^{4}y_5 & +t^{3}y_2\wedge t^{4}y_5 & -ty_3\wedge t^{4}y_5 \\
+t^{3}y_3\wedge t^{4}y_5 & -t^{2}y_4\wedge t^{4}y_5 & +t^{4}y_4\wedge t^{4}y_5 & -t^{2}y_5\wedge t^{4}y_5 & -t^{3}y_5\wedge t^{4}y_5
\end{array}
$$

$$
\begin{array}{rrrrr}
-t^3 y_2{\wedge}t^5 y_5 & +t^2 y_3{\wedge}t^5 y_5 & -t^3 y_3{\wedge}t^5 y_5 & +t^3 y_4{\wedge}t^5 y_5 & -t^4 y_4{\wedge}t^5 y_5 \\
+t^3 y_5{\wedge}t^5 y_5 & -t^{-3} y_2{\wedge}t^{-3} y_6 & +t^{-2} y_2{\wedge}t^{-3} y_6 & -t^{-2} y_5{\wedge}t^{-3} y_6 & +t^{-1} y_5{\wedge}t^{-3} y_6 \\
+t^{-3} y_2{\wedge}t^{-2} y_6 & -t^{-1} y_2{\wedge}t^{-2} y_6 & +t^{-2} y_5{\wedge}t^{-2} y_6 & -y_5{\wedge}t^{-2} y_6 & +t^{-3} y_2{\wedge}t^{-1} y_6 \\
-t^{-2} y_2{\wedge}t^{-1} y_6 & +y_2{\wedge}t^{-1} y_6 & -t^{-4} y_3{\wedge}t^{-1} y_6 & +t^{-3} y_3{\wedge}t^{-1} y_6 & -t^{-1} y_3{\wedge}t^{-1} y_6 \\
-t^{-3} y_4{\wedge}t^{-1} y_6 & +t^{-2} y_4{\wedge}t^{-1} y_6 & -y_4{\wedge}t^{-1} y_6 & -t^{-3} y_5{\wedge}t^{-1} y_6 & +t y_5{\wedge}t^{-1} y_6 \\
+t^{-3} y_6{\wedge}t^{-1} y_6 & -t^{-2} y_6{\wedge}t^{-1} y_6 & -t^{-3} y_2{\wedge}y_6 & -t^{-2} y_2{\wedge}y_6 & +t^{-1} y_2{\wedge}y_6 \\
+2 y_2{\wedge}y_6 & -2t y_2{\wedge}y_6 & +t^2 y_2{\wedge}y_6 & +t^{-3} y_3{\wedge}y_6 & -t^{-2} y_3{\wedge}y_6 \\
-t^{-1} y_3{\wedge}y_6 & +3 y_3{\wedge}y_6 & -t y_3{\wedge}y_6 & +t^2 y_3{\wedge}y_6 & +t^{-2} y_4{\wedge}y_6 \\
-t^{-1} y_4{\wedge}y_6 & -y_4{\wedge}y_6 & +3t y_4{\wedge}y_6 & -t^2 y_4{\wedge}y_6 & +t^3 y_4{\wedge}y_6 \\
-y_5{\wedge}y_6 & +t y_5{\wedge}y_6 & -2t^2 y_5{\wedge}y_6 & -t^{-2} y_6{\wedge}y_6 & +t^{-3} y_2{\wedge}t y_6 \\
+t^{-2} y_2{\wedge}t y_6 & -y_2{\wedge}t y_6 & -t y_2{\wedge}t y_6 & -t^3 y_2{\wedge}t y_6 & -t^{-1} y_3{\wedge}t y_6 \\
+y_3{\wedge}t y_6 & -2t y_3{\wedge}t y_6 & -t^3 y_3{\wedge}t y_6 & -y_4{\wedge}t y_6 & +t y_4{\wedge}t y_6 \\
-2t^2 y_4{\wedge}t y_6 & -t^4 y_4{\wedge}t y_6 & +t^{-2} y_5{\wedge}t y_6 & +t^{-1} y_5{\wedge}t y_6 & +t^2 y_5{\wedge}t y_6 \\
+t^3 y_5{\wedge}t y_6 & +t^{-1} y_6{\wedge}t y_6 & +2 y_6{\wedge}t y_6 & -t^{-2} y_2{\wedge}t^2 y_6 & -t^{-1} y_2{\wedge}t^2 y_6 \\
+t^2 y_2{\wedge}t^2 y_6 & +t^{-1} y_3{\wedge}t^2 y_6 & +t^2 y_3{\wedge}t^2 y_6 & +t^3 y_3{\wedge}t^2 y_6 & +y_4{\wedge}t^2 y_6 \\
+t^3 y_4{\wedge}t^2 y_6 & +t^4 y_4{\wedge}t^2 y_6 & -t^{-1} y_5{\wedge}t^2 y_6 & +t y_5{\wedge}t^2 y_6 & -t^2 y_5{\wedge}t^2 y_6 \\
-t^4 y_5{\wedge}t^2 y_6 & -2 y_6{\wedge}t^2 y_6 & -t y_6{\wedge}t^2 y_6 & +2 y_2{\wedge}t^3 y_6 & +t^4 y_2{\wedge}t^3 y_6 \\
-t^{-1} y_3{\wedge}t^3 y_6 & -t^3 y_3{\wedge}t^3 y_6 & -y_4{\wedge}t^3 y_6 & -t^4 y_4{\wedge}t^3 y_6 & -y_5{\wedge}t^3 y_6 \\
-t^2 y_5{\wedge}t^3 y_6 & +t^5 y_5{\wedge}t^3 y_6 & +y_6{\wedge}t^3 y_6 & +t^2 y_6{\wedge}t^3 y_6 & -t y_2{\wedge}t^4 y_6 \\
-t^2 y_2{\wedge}t^4 y_6 & -t^4 y_2{\wedge}t^4 y_6 & +y_3{\wedge}t^4 y_6 & +t y_4{\wedge}t^4 y_6 & +t y_5{\wedge}t^4 y_6 \\
+t^2 y_5{\wedge}t^4 y_6 & +t^4 y_5{\wedge}t^4 y_6 & -t^5 y_5{\wedge}t^4 y_6 & -t y_6{\wedge}t^4 y_6 & +t^3 y_2{\wedge}t^5 y_6 \\
-t^2 y_3{\wedge}t^5 y_6 & +t^3 y_3{\wedge}t^5 y_6 & -t^3 y_4{\wedge}t^5 y_6 & +t^4 y_4{\wedge}t^5 y_6 & -t^3 y_5{\wedge}t^5 y_6 \\
+t^3 y_6{\wedge}t^5 y_6 & -t^4 y_6{\wedge}t^5 y_6 & & &
\end{array}
$$

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