Algebraic & Geometric Topology 10 (2010) 1781–1806 **1781**

Γ–homology of algebras over an operad

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The purpose of this paper is to study generalizations of Gamma-homology in the context of operads. Good homology theories are associated to operads under appropriate cofibrancy hypotheses, but this requirement is not satisfied by usual operads outside the characteristic zero context. In that case, the idea is to pick a cofibrant replacement Q of the given operad P. We can apply to P–algebras the homology theory associated to Q in order to define a suitable homology theory on the category of P–algebras. We make explicit a small complex to compute this homology when the operad P is binary and Koszul. In the case of the commutative operad $\mathsf{P} = \mathsf{Com}$, we retrieve the complex introduced by Robinson for the Gamma-homology of commutative algebras.

16E40; 18D50, 18G55, 18G60

The classical homology theories of commutative algebras (Harrison homology in the differential graded setting over a field of characteristic 0, see Harrison [11], and André–Quillen homology in the simplicial setting over a ring of any characteristic, see Quillen [16] and André [1]) can be considered as homology theories associated to the commutative operad Com. There is another homology theory for commutative algebras, Γ–homology (Gamma-homology in plain words, also called topological André–Quillen), which has been introduced by Robinson and Whitehouse in [18], and by Basterra in [3] (with a different point of view), to solve obstruction problems in homotopy theory. In the setting of [18], Gamma-homology is defined as the homology theory associated to an E_∞–operad (a cofibrant replacement of Com). This new homology can be defined in the context of differential graded or simplicial context or in the context of spectra, and gives the same result in each case (see Mandell [15]), in contrast with the usual André–Quillen homology.

The purpose of this paper is to study generalizations of Γ–homology in the context of operads.

Usual methods of homotopical algebra apply to the categories of algebras associated to operads which are cofibrant, or at least which fulfill sufficiently strong cofibrancy requirements. As a consequence, we have a good homology theory H_*^{Q} associated to any such operad Q. But many usual operads, like the commutative operad Com or the Lie operad Lie, do not fit this framework (unless we work with differential graded modules over a field of characteristic 0). In this situation, a natural idea is to pick a

Published: 30 August 2010 DOI: 10.2140/agt.2010.10.1781

cofibrant replacement of the given operad P, let $\mathsf{Q}\xrightarrow{\sim}\mathsf{P}$, and to apply the homology H_*^{Q} to P–algebras in order to obtain a consistent homology theory on the category of P–algebras. We use the notation $H\Gamma_*^{\mathsf{P}} = H_*^{\mathsf{Q}}$ and the name Γ–homology to refer to this homology theory after observing that different choices of Q give the same result.

This generalizes the usual notion of Γ–homology where $\mathsf{P} = \mathsf{Com}$ and Q is an E_∞–operad. The homology H_*^{Q} associated to a cofibrant replacement of the operad Lie has also been used by Chataur, Rodriguez and Scherer in [4].

The problem is that the choice of a cofibrant replacement is satisfying in theory, but making such a cofibrant replacement explicit is often very difficult (especially when the ground ring is not a field of characteristic 0). We give a direct definition of $H\Gamma_*^{\mathsf{P}}$, which agrees with the initial one, but where the choice of an operadic cofibrant replacement is avoided. The idea is to use the model category on P–bimodules, which only needs mild assumptions on P. We show how to define a complex to determine $H\Gamma_*^{\mathsf{P}}$ from a choice of a cofibrant replacement of the operad P, not in the category of operads, but in the category of P–bimodules, the operad P being viewed as a bimodule over itself. The category of P–bimodules is easier to deal with than the category of operads.

In [17], Robinson makes explicit a small complex, analogous to Harrison's complex, which computes usual Γ–homology. In the case where the operad P is Koszul, we define an explicit complex to compute the Γ–homology associated to P. Recall that an operad is Koszul if we have a quasi-isomorphism between $(\mathsf{P} \circ K\mathsf{P} \circ \mathsf{P}, \partial)$ and P, where $K\mathsf{P}$ is the Koszul construction, defined by $K(\mathsf{P})_{(s)} := H_s(B_*(\mathsf{P})_{(s)}, \partial)$. In [2], Balavoine defined a complex computing H_*^{P} when working over a field of characteristic 0, using the Koszul construction. When we work over a ring of any characteristic, finding a complex is more complicated, as we need to resolve the symmetries in $K\mathsf{P}$. This can be done by tensoring the Koszul construction by the acyclic bar construction of the symmetric group. Finally, we get a small explicit complex computing the Γ–homology of P–algebras. In the case $\mathsf{P} = \mathsf{Com}$, we obtain a variant of Robinson's complex. As an illustration, we make our complex explicit in the case $\mathsf{P} = \mathsf{Lie}$.

We also define a cohomology theory $H\Gamma_{\mathsf{P}}^*$ associated to any operad P.

In Section 1, we recall the model category structures we use in the paper: dg-modules, Σ_*–modules, bimodules, algebras over operads. Most of the model structures we consider are defined by a transfer of structure. We make the cofibrations explicit in each case. In a second part, we recall the usual notion of homology for algebras over a cofibrant operad, and show how to reduce the complex when we are given a cofibrant replacement of bimodules. Then we make a similar construction of a reduced complex when the operad is not cofibrant. This leads us to the definition of Γ–homology of algebras over an operad (without any cofibrancy hypothesis). In Section 3, we construct an explicit complex for any binary Koszul operad P to compute Γ–homology. This complex is defined using the Koszul construction $K\mathsf{P}$ and the acyclic bar construction of the symmetric group.

Convention We work in the differential graded setting. We take a category of differential graded modules (for short dg-modules) over a fixed base ring $\mathbb{K}$ as a base category (see Section 1.2 for details). We use the letter $\mathcal{C}$ to denote this category. When necessary, we assume tacitly that any dg-module, and more generally that any object defined over this base category, consists of projective modules over the ground ring.

We review the definition of the model category of Σ_*–modules underlying the category of operads in Section 1.4, the model category of bimodules in Section 1.5. All operads P will be assumed to be connected, in the sense that $\mathsf{P}(0) = 0$ and $\mathsf{P}(1) = \mathbb{K}$. All Σ_*–modules M, and more generally any object defined over the category of Σ_*–modules, will be assumed to be connected, that is $M(0) = 0$.

1 Model categories

We review here the model structures for the categories which are used in this paper. For general references on the subject, we refer the reader to the survey of Dwyer and Spalinksi [5] and the books of Hirschhorn [13] and Hovey [14]. For model structures in the operadic context, we refer to the articles of Hinich [12] and of Goerss and Hopkins [10], and the book of Fresse [7].

1.1 Transfer of structure

We use the notion of a pair of adjoint functors to transport model structures. Suppose we have an adjunction

$$F\colon \mathcal{X} \rightleftarrows \mathcal{A} :U$$

such that $\mathcal{X}$ is a cofibrantly generated model category and $\mathcal{A}$ is a category equipped with colimits and limits. We can then define classes of weak equivalences, fibrations and cofibrations in $\mathcal{A}$.

- The weak equivalences in $\mathcal{A}$ are morphisms f such that $U(f)$ is a weak equivalence in $\mathcal{X}$.
- The fibrations in $\mathcal{A}$ are morphisms f such that $U(f)$ is a fibration in $\mathcal{X}$.
- The cofibrations are the morphisms which have the left lifting property (in short, LLP) with respect to acyclic fibrations.

Under some technical hypotheses (see Hirschhorn [13, Theorem 11.3.2]), a classical result says that $\mathcal{A}$ is equipped with a model structure given by the weak equivalences, fibrations and cofibrations above. Under weaker hypotheses (see Fresse [7, Theorem 12.1.4]), the category $\mathcal{A}$ is equipped with a semi-model category, that is the lifting and factorization axioms only hold when the morphisms have a cofibrant domain. Semi-model categories will be enough for us here.

We can describe the generating (acyclic) cofibrations of the semi-model category $\mathcal{A}$ explicitly: they are the morphisms $F(i)\colon F(C) \to F(D)$ such that i ranges over the generating (acyclic) cofibrations of $\mathcal{X}$.

1.2 Model category structure for dg-modules

In this paper, the dg-modules we consider are $\mathbb{Z}$–graded modules endowed with a differential δ decreasing the degree by 1. The category of dg-modules is denoted by $\mathcal{C}$. The internal hom of this category is denoted by $\mathrm{Hom}_{\mathcal{C}}(C,D)$, for all $C,D \in \mathcal{C}$. This dg-module is spanned in degree d by the linear maps $f\colon C \to D$ which raises degrees by d. The differential of such a map in $\mathrm{Hom}_{\mathcal{C}}(C,D)$ is defined by its graded commutator with the internal differential of C and D. We adopt the terminology of homomorphisms to distinguish the elements of the dg-hom $\mathrm{Hom}_{\mathcal{C}}(C,D)$ from the actual morphisms of dg-modules, the linear maps which preserve gradings and commute with differentials.

The category of dg-modules is equipped with its usual model structure: The weak equivalences are the quasi-isomorphisms and the fibrations are degreewise surjective maps (cofibrations are characterized by the LLP with respect to acyclic fibrations).

Let $D_n = \mathbb{K}d_n \oplus \mathbb{K}c_{n-1}$ where d_n is a homogeneous element in degree n sent by the differential to c_{n-1} in degree $n-1$. Let C_n be $\mathbb{K}c_{n-1}$, submodule of D_n. The embeddings $C_n \to D_n$, $n \in \mathbb{Z}$, define a set of generating cofibrations in $\mathcal{C}$. The maps $0 \to D_n$ define generating acyclic cofibrations.

In what follows, the underlying dg-module of any object is tacitly assumed to be cofibrant.

1.3 Twisted dg-modules

In general, we assume that a dg-module C is equipped with a differential $\delta\colon C \to C$. We sometimes twist this internal differential by a cochain $\partial \in \mathrm{Hom}_{\mathcal{C}}(C,C)$ of degree -1 in order to get a new differential $\delta+\partial$. We assume the relation $\delta\circ\partial+\partial\circ\delta+\partial^2=0$, in order to obtain that $\delta+\partial$ satisfies $(\delta+\partial)^2=0$. We usually omit the internal differential

δ in the notation: We write C for the module C with differential δ and write (C,∂) to denote the module C with differential $\delta+\partial$.

We are going to define quasi-free objects (algebras over operads, bimodules), twisted objects (C,∂) such that C is free with respect to an algebraic structure.

1.4 Model category structure for Σ_*-modules

We use the notation $\mathcal{M}$ for Σ_*–modules. We have an adjunction between the forgetful functor U (from the category $\mathcal{M}$ to the category of chain complexes) and the free Σ_*–module functor $\Sigma_*\otimes -$.

The transfer process of Section 1.1 gives us a model structure on Σ_*–modules where weak equivalences are morphisms all of whose components are weak equivalences of dg-modules and where fibrations are morphisms all of whose components are epimorphisms of dg-modules. Cofibrations are given by the LLP with respect to acyclic fibrations. Again, we can say more precisely which maps are cofibrations (see Fresse [7, Proposition 11.4.A]). The generating cofibrations are given by tensor products

$$i\otimes F_r\colon C\otimes F_r\to D\otimes F_r$$

where $i\colon C\to D$ ranges over the generating cofibrations of dg-modules and $F_r, r\in\mathbb{N}$ denote the Σ_*–modules such that

$$F_r(n)=\begin{cases}\mathbb{K}[\Sigma_r], & \text{for } n=r,\\ 0, & \text{otherwise.}\end{cases}$$

We will use the composition product $\circ$ of Σ_*–modules. Recall that for a constant Σ_*–module N (such that $N(0)=C$ and $N(r)=0$ for $r>0$), the composition $M\circ C$ represents the application of a symmetric functor with coefficients in M to C:

$$M\circ C=\bigoplus_{r=1}^{+\infty}(M(r)\otimes C^{\otimes r})_{\Sigma_r}$$

This object is denoted by $S(M,C)$ in the book by Fresse [7], but we use the notation $M\circ C$ in the paper.

In general, the composite $M\circ N$ is defined such that the associativity relation $M\circ(N\circ C)=(M\circ N)\circ C$ is satisfied for all constant Σ_*–modules C. The composition product $\circ$ is a monoidal product for the category of Σ_*–modules.

Recall that an operad is a Σ_*–module P equipped with an initial morphism $\mathsf{I}\to\mathsf{P}$ (where I is the unit Σ_*–module, with $\mathbb{K}$ in arity 1 and 0 everywhere else) and a composition product $\gamma\colon\mathsf{P}\circ\mathsf{P}\to\mathsf{P}$. As mentionned in the introduction, we assume that any operad P satisfies $\mathsf{P}(0) = 0$ and $\mathsf{P}(1) = \mathbb{K}$, so that the initial morphism of P is an isomorphism in arity 0 and in arity 1. We use the notation $\overline{\mathsf{P}}$ for the Σ_*–submodule of P formed by the components $\mathsf{P}(n)$ of arity $n > 1$ and trivial in arity 0 and in arity 1.

In what follows, we will often consider Σ_*–cofibrant operads, operads P such that the initial morphism $\mathsf{I} \to \mathsf{P}$ is a cofibration of Σ_*–modules.

1.5 Model category structure for bimodules over operads

Let P and Q be operads. Let ${}_{\mathsf{P}}\mathcal{M}^0_{\mathsf{Q}}$ be the category of connected (that is $M(0) = 0$) P–Q–bimodules in the sense of Fresse [7]. We have an adjunction

$$\mathsf{P} \circ - \circ \mathsf{Q} \colon \mathcal{M} \rightleftarrows {}_{\mathsf{P}}\mathcal{M}^0_{\mathsf{Q}} : U,$$

where U is the forgetful functor.

The transfer process gives us a semi-model structure on P–Q–bimodules, where weak equivalences are morphisms all of whose components are weak equivalences of dg-modules and where fibrations are morphisms all of whose components are epimorphisms of dg-modules. Cofibrations are given by the LLP with respect to acyclic fibrations.

We now describe a particular class of cofibrant P–Q–bimodules that we will use extensively later.

1.6 Proposition *Let P and Q be connected operads and M a cofibrant Σ_*–module.*

The quasi-free P–Q–bimodule $(\mathsf{P} \circ M \circ \mathsf{Q}, \partial)$ is cofibrant if the differential is decomposable (that is $\partial(M)$ has no component in $\mathsf{I} \circ M \circ \mathsf{I}$, or equivalently that each element in $\partial(M)$ is a sum of composites with at least a non-trivial element of P or of Q).

This result will be deduced from the following lemmas.

1.6.1 Lemma *Let Q be a connected operad and M a cofibrant Σ_*–module.*

The quasi-free Q–module $(M \circ \mathsf{Q}, \partial)$ is cofibrant if the differential is decomposable (that is $\partial(M)$ has no component in $M \circ \mathsf{I}$).

Proof The complex $(M \circ \mathsf{Q}, \partial)$ is filtered by

$$ar_\lambda(M \circ \mathsf{Q}, \partial) = (ar_\lambda M \circ \mathsf{Q}, \partial)$$

where $ar_\lambda M(n) = M(n)$ if $n \leq \lambda$ and 0 otherwise, and where the differential is just the restriction of the differential on $(M \circ \mathsf{Q}, \partial)$.

Note that $\partial(ar_\lambda M) \subset ar_{\lambda-1}(M \circ \mathsf{Q})$.

We have the following pushout of right Q–modules:

$$\begin{tikzcd}
(\partial M(n)\circ \mathsf{Q}, 0) \arrow[r] \arrow[d] & (ar_{n-1}M\circ \mathsf{Q}, \partial) \arrow[d] \\
(\partial M(n)\circ \mathsf{Q} \bigoplus M(n)\circ \mathsf{Q}, \partial) \arrow[r] & (ar_n M\circ \mathsf{Q}, \partial).
\end{tikzcd}$$

The arrow on the left is a generating cofibration. Thus the arrow on the right is a cofibration too.

Thus $(M\circ \mathsf{Q}, \partial) = \operatorname{colim}_\lambda ar_\lambda(M\circ \mathsf{Q}, \partial)$ is a cofibrant right Q–module. $\square$

1.6.2 Lemma *Let $N = (M\circ \mathsf{Q}, \partial)$ be a right Q–module with the hypothesis of the above lemma. Let P be a connected operad.*

The quasi-free P–Q–bimodule $(\mathsf{P}\circ N, \partial)$ is cofibrant if the differential is decomposable (that is $\partial(N)\subset \overline{\mathsf{P}}\circ N$, or equivalently $\partial(N)$ has no component in $\mathsf{I}\circ N$).

Proof First, note that $\partial(ar_\lambda N)\subset \overline{\mathsf{P}}\circ ar_{\lambda-1}N$. Therefore we can define a filtration by $ar_\lambda(\mathsf{P}\circ N, \partial) = (\mathsf{P}\circ ar_\lambda N, \partial)$.

Note that $\partial(ar_\lambda N)\subset ar_{\lambda-1}(\mathsf{P}\circ N)$. Using a similar argument as in the above proof, the obvious arrow $ar_{\lambda-1}(\mathsf{P}\circ N, \partial)\to ar_\lambda(\mathsf{P}\circ N, \partial)$ is a cofibration of P–Q–bimodules.

Thus $(\mathsf{P}\circ N, \partial) = \operatorname{colim}_\lambda ar_\lambda(\mathsf{P}\circ N, \partial)$ is a cofibrant P–Q–bimodule. $\square$

Proof of Proposition 1.6 The proof follows from Lemmas 1.6.1 and 1.6.2. $\square$

1.7 Model category structure for algebras over an operad

We have an adjunction between the forgetful functor U from P–algebras to dg-modules and the free P–algebra functor $\mathsf{P}\circ -$.

If P is Σ_*–cofibrant, the transfer process of Section 1.1 gives us a semi-model category on P–algebras, where weak equivalences are morphisms which are weak equivalences of dg-modules and where fibrations are morphisms which are epimorphisms of dg-modules. Cofibrations are given by the LLP with respect to acyclic fibrations.

The model category structure allows us to define the cofibrant replacement of a P–algebra A. It is a cofibrant P–algebra Q_A such that we have a weak equivalence of P–algebras $Q_A\xrightarrow{\sim} A$.

If we are given a cofibrant replacement ${}_{\mathsf{P}}\mathrm{Res}_{\mathsf{P}} \xrightarrow{\sim} \mathsf{P}$ in the category of P–bimodules, we can easily make explicit a cofibrant replacement of a P–algebra A.

First we need to recall the definition of the relative composition product of P–modules. Suppose that M is a right P–module and A a P–algebra. We denote by $M \circ_{P} A$ the quotient of $M \circ A$ coequalizing the right action of P on M and the left action of P on A. When M is a P–bimodule, the relative composite $M \circ_{P} A$ inherits a P–algebra structure.

We can now give the result:

1.7.1 Lemma *We get a cofibrant replacement* $({}_{\mathsf{P}}\mathrm{Res}_{\mathsf{P}} \circ_{\mathsf{P}} A, \partial')$ *of* A *in the category of* P*–algebras, with* $\partial' = \partial \circ_{\mathsf{P}} A$.

Proof The P–algebra $({}_{\mathsf{P}}\mathrm{Res}_{\mathsf{P}} \circ_{\mathsf{P}} A, \partial')$ is cofibrant, following the same argument of the proof of Lemma 1.6.1. The P–bimodule ${}_{\mathsf{P}}\mathrm{Res}_{\mathsf{P}}$ is cofibrant, thus it is cofibrant as a right P–module. The operad P is also cofibrant as a right P–module. As the functor $- \circ_{\mathsf{P}} A$ preserve weak equivalences between cofibrant objects (see Fresse [7, Theorem 15.1.A]), we get that $({}_{\mathsf{P}}\mathrm{Res}_{\mathsf{P}} \circ_{\mathsf{P}} A, \partial')$ is a cofibrant replacement of $\mathsf{P} \circ_{\mathsf{P}} A$. But $\mathsf{P} \circ_{\mathsf{P}} A = A$, thus $({}_{\mathsf{P}}\mathrm{Res}_{\mathsf{P}} \circ_{\mathsf{P}} A, \partial')$ is a cofibrant replacement of A in the category of P–algebras. Explicitly, the differential ∂' is given by $\partial'(m \circ (a_1, \dots, a_n)) = (\partial(m)) \circ (a_1, \dots, a_n)$ where m lies in ${}_{\mathsf{P}}\mathrm{Res}_{\mathsf{P}}$ and $\partial(m)$ in $\mathsf{P} \circ ({}_{\mathsf{P}}\mathrm{Res}_{\mathsf{P}}) \circ \mathsf{P}$. Note that we use the structure of P–algebra of A on the right hand side to get an element of ${}_{\mathsf{P}}\mathrm{Res}_{\mathsf{P}} \circ_{\mathsf{P}} A$. □

2 Gamma-homology of P-algebras

In this section, we recall the definition of the homology of Q–algebras for Q a Σ_*–cofibrant operad. In the differential graded setting over a field of characteristic 0, homology with trivial coefficients was defined by Ginzburg and Kapranov in [9] and by Getzler and Jones in [8]. Homology with coefficients was defined by Balavoine in [2]. The extension to any category of dg-modules can be found in Hinich [12]. We adopt conventions of Fresse [7] where these notions are reviewed. We define Γ–homology of P–algebras for any operad P, using bimodule resolutions. Then we prove the identity $H^{\mathsf{Q}}_* = H\Gamma^{\mathsf{P}}_*$ when Q is a Σ_*–cofibrant replacement of P.

2.1 Recollections on the homology of Q-algebras

We refer the reader to Fresse [7, Section 4] for the first definitions.

Let Q be a Σ_*–cofibrant operad, B an algebra over Q.

We denote by $U_{\mathsf{Q}}(B)$ the enveloping algebra of B and by $\Omega_{\mathsf{Q}}(B)$ the module of Kähler differentials of B.

The enveloping algebra $U_{\mathsf{Q}}(B)$ is spanned by elements $q(\diamond, b_1, \dots, b_n)$, where $q \in \mathsf{Q}(n+1)$, $b_1, \dots, b_n \in B$ and the symbol $\diamond$ denotes a free input, divided out by the relations

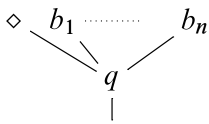
$$p(\diamond, b_1, \dots, b_{i-1}, q(b_i, \dots, b_n), b_{n+1}, \dots, b_m) = \\ p \circ_{i+1} q(\diamond, b_1, \dots, b_{i-1}, b_i, \dots, b_m).$$

The product is given by

$$p(\diamond, a_1, \dots, a_n).q(\diamond, b_1, \dots, b_m) = p \circ_1 q(\diamond, b_1, \dots, b_m, a_1, \dots, a_n).$$

We can put $\diamond$ at any place since the action of Σ_{n+1} on $\mathsf{Q}(n+1)$ allows us to permute the inputs of any operation $q \in \mathsf{Q}(n+1)$.

We represent graphically an element $q(\diamond, b_1, \dots, b_n)$ by



The module of Kähler differentials $\Omega_{\mathsf{Q}}(B)$ is a left module over $U_{\mathsf{Q}}(B)$ such that

$$\mathrm{Hom}_{U_{\mathsf{Q}}(B)}(\Omega_{\mathsf{Q}}(B), F) = \mathrm{Der}_{\mathsf{Q}}(B, F)$$

for all left modules F over $U_{\mathsf{Q}}(B)$, where $\mathrm{Der}_{\mathsf{Q}}(B, F)$ denotes the dg-module of Q–derivations $B \to F$ (not necessarily preserving the degree) and $\mathrm{Hom}_{U_{\mathsf{Q}}(B)}(\Omega_{\mathsf{Q}}(B), F)$ is the dg-module of homomorphisms of left $U_{\mathsf{Q}}(B)$–modules between $\Omega_{\mathsf{Q}}(B)$ and F.

The module of Kähler differentials $\Omega_{\mathsf{Q}}(B)$ can be seen as the dg-module spanned by elements $q(b_1, \dots, db_i, \dots, b_n)$, where $q \in \mathsf{Q}(n)$, $b_1, \dots, b_n \in B$ and d denotes a formal differentiation symbol, divided out by the relations

$$p(b_1, \dots, q(b_i, \dots, b_n), b_{n+1}, \dots, db_j, \dots, b_m) = \\ p \circ_i q(b_1, \dots, b_i, \dots, b_n, \dots, db_j, \dots, b_m) \quad \text{for } i \neq j,$$

and

$$p(b_1, \dots, dq(b_i, \dots, b_n), b_{n+1}, \dots, \dots, \dots, b_m) = \\ \sum_{j=i}^{n} p \circ_i q(b_1, \dots, b_i, \dots, db_j, \dots, b_n, \dots, b_m).$$

Let us now define the homology and the cohomology of an algebra over Q.

We choose Q_B a cofibrant replacement of B. Let E be a right $U_{\mathsf{Q}}(Q_B)$–module and F be a left $U_{\mathsf{Q}}(Q_B)$–module.

The homology of B as a Q–algebra with coefficients in E is defined by $H_*^{\mathsf{Q}}(B,E) = H_*(E \otimes_{U_{\mathsf{Q}}(Q_B)} \Omega_{\mathsf{Q}}(Q_B))$. In a similar way, the cohomology of B is defined by $H_{\mathsf{Q}}^*(B,F) = H^*(\mathrm{Hom}_{U_{\mathsf{Q}}(Q_B)}(\Omega_{\mathsf{Q}}(Q_B),F))$.

We will use the following usual lemma to reduce the complex appearing in the calculation of the homology and the cohomology.

2.1.1 Lemma *If Q_A is a quasi-free Q–algebra $Q_A = (\mathsf{Q}(C),\partial')$, then we have an isomorphism of left $U_{\mathsf{Q}}(Q_A)$–modules*

$$(U_{\mathsf{Q}}(Q_A) \otimes C, \partial'') \simeq \Omega_{\mathsf{Q}}(Q_A)$$

where the differential $\partial''\colon U_{\mathsf{Q}}(Q_A)\otimes C \to U_{\mathsf{Q}}(Q_A)\otimes C$ is a twisting homomorphism on $U_{\mathsf{Q}}(Q_A)\otimes C$ induced by the action of the twisting derivation of Q_A on $U_{\mathsf{Q}}(Q_A)\otimes C$ (see the detailed representation in Figure 2).

Proof We begin by proving the result for a free algebra $Q_A = \mathsf{Q}(C)$.
First, we have $\mathrm{Der}_{\mathsf{Q}}(\mathsf{Q}(C),F) = \mathrm{Hom}_{\mathcal{C}}(C,F)$. To prove this identification, we define $\Phi\colon \mathrm{Der}_{\mathsf{Q}}(\mathsf{Q}(C),F) \to \mathrm{Hom}_{\mathcal{C}}(C,F)$ by $\Phi(\theta) = \theta_{|C}\colon C \to F$. This map is an isomorphism. The inverse map Φ^{-1} associates to any $f\colon C \to F$ the derivation θ_f such that $\theta_f(q(c_1,\dots,c_n)) = \pm\sum_i q(c_1,\dots,f(c_i),\dots,c_n)$ where the signs are induced by the usual Koszul rule.

We have

$$\mathrm{Hom}_{\mathcal{C}}(C,F) = \mathrm{Hom}_{U_{\mathsf{Q}}(Q_A)}(U_{\mathsf{Q}}(Q_A)\otimes C, F),$$

which gives us

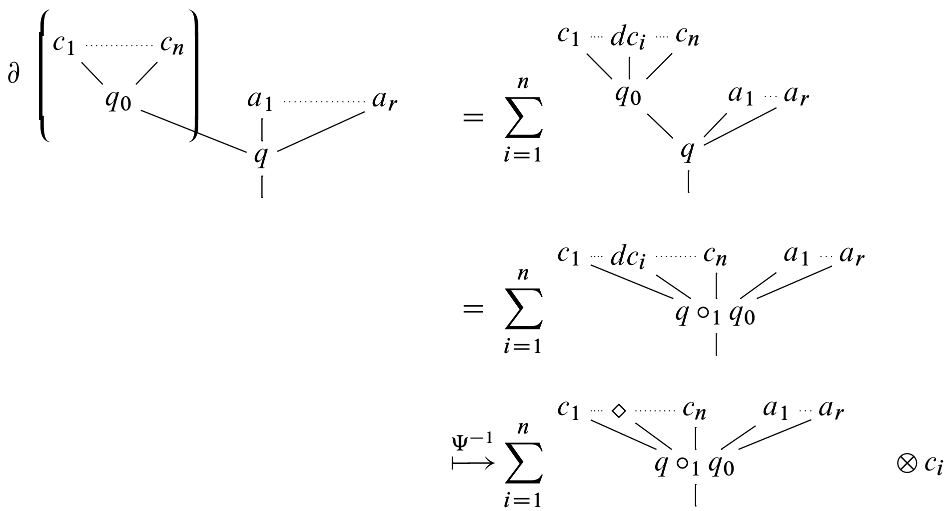
$$\mathrm{Der}_{\mathsf{Q}}(Q_A,F) = \mathrm{Hom}_{U_{\mathsf{Q}}(Q_A)}(U_{\mathsf{Q}}(Q_A)\otimes C, F).$$

But $\Omega_{\mathsf{Q}}(Q_A)$ is defined by

$$\mathrm{Hom}_{U_{\mathsf{Q}}(Q_A)}(\Omega_{\mathsf{Q}}(Q_A), F) = \mathrm{Der}_{\mathsf{Q}}(Q_A,F).$$

Thus Yoneda's lemma gives us an isomorphism Ψ of $U_{\mathsf{Q}}(Q_A)$–modules between $U_{\mathsf{Q}}(Q_A)\otimes C$ and $\Omega_{\mathsf{Q}}(Q_A)$.

The map $\Psi\colon U_{\mathsf{Q}}(Q_A)\otimes C \to \Omega_{\mathsf{Q}}(Q_A)$ associates to the element $q(\diamond,a_1,\dots,a_n)\otimes c$ the element $q(dc,a_1,\dots,a_n)$.



where the empty box is the ith input of the tree.

Figure 1: A graphical representation of the inverse isomorphism Ψ^{-1}.

Its inverse $\Psi^{-1}\colon \Omega_{\mathsf{Q}}(Q_A) \to U_{\mathsf{Q}}(Q_A)\otimes C$ sends

$$q(dq_0(\underline{c}_0), a_1, \dots, a_r) = \sum_i q \circ_1 q_0(c_1, \dots, dc_i, \dots, c_n, a_1, \dots, a_r)$$

to

$$\sum_i q \circ_1 q_0(c_1, \dots, \diamond, \dots, c_n, a_1, \dots, a_r) \otimes c_i,$$

where $\underline{c}_0 = (c_1, \dots, c_n)$.

A graphical representation of the isomorphism Ψ^{-1} is given in Figure 1.

This morphism Ψ commutes with the internal differential of C.

We now consider a quasi-free Q–algebra $Q_A = (\mathsf{Q}(C), \partial')$ with a twisting differential ∂' and explain the twisting differential ∂'' we obtain on $U_{\mathsf{Q}}(Q_A)\otimes C$. A graphical representation of the twisting part of the differential is given in Figure 2.

We consider an element $\omega = q(\diamond, a_1, \dots, a_n)\otimes c$ in $U_{\mathsf{Q}}(\mathsf{Q}(C))\otimes C$. We compute

$$\begin{aligned}\partial'(\Psi(\omega)) &= \partial'(q(dc, a_1, \dots, a_n)) \\ &= \underbrace{q(d\partial' c, a_1, \dots, a_n)}_{\partial''(\Psi(w))} + \sum_i \underbrace{q(dc, a_1, \dots, \partial' a_i, \dots, a_n)}_{\Psi(q(\diamond, a_1, \dots, \partial' a_i, \dots, a_n)\otimes c)}.\end{aligned}$$

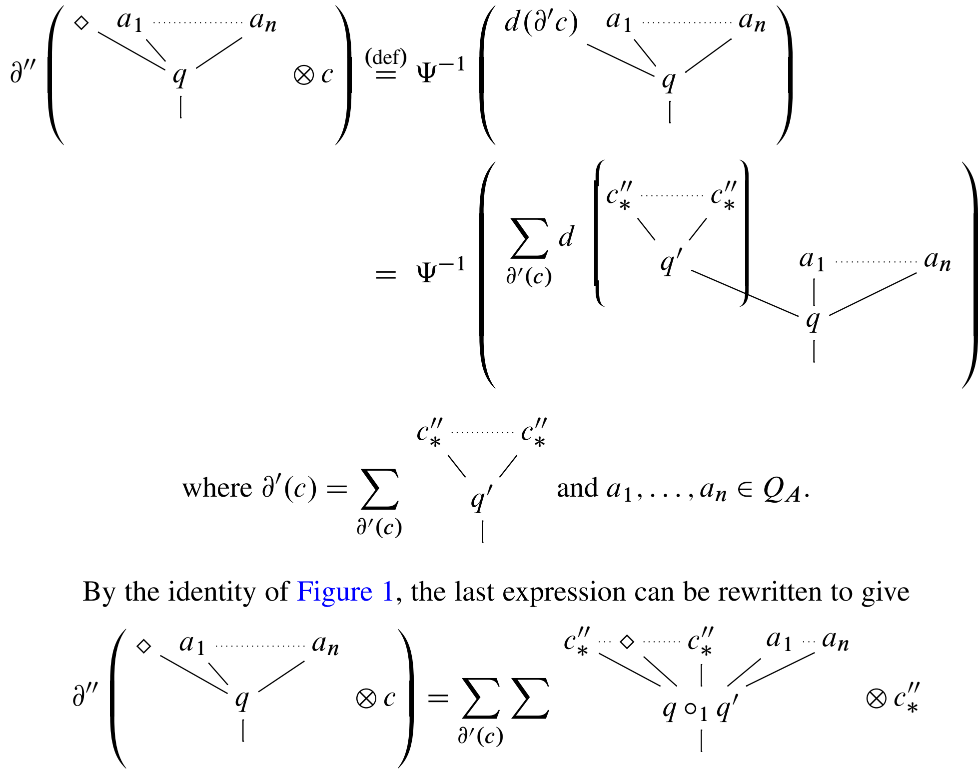


Figure 2: A graphical representation of the twisting differential in $U_{\mathsf{Q}}(Q_A) \otimes C$.

The second term of this sum is induced by the action of $\partial'\colon Q_A \to Q_A$. The image by Ψ^{-1} of the first term is computed in Figure 2. We denote $\Psi^{-1}\partial''(\Psi(\omega))$ by $\partial''(\omega)$.

To conclude, the twisting differential added to δ is the sum of $\partial'\colon Q_A \to Q_A$ and of ∂'' induced by the action of ∂' on C in $U_{\mathsf{Q}}(Q_A) \otimes C$. There are two equivalent ways to see the module $U_{\mathsf{Q}}(Q_A) \otimes C$ with its twisting differential: $(U_{\mathsf{Q}}(Q_A) \otimes C, \partial'')$ or $(U_{\mathsf{Q}}(Q(C)) \otimes C, \partial' + \partial'')$. □

Note that we have not used the cofibrancy hypothesis on Q in the proof of the lemma.

2.2 From quasi-free Q-bimodules to resolutions of an algebra

We suppose here that Q is a Σ_*–cofibrant operad. Let B be a Q–algebra and E a right $U_{\mathsf{Q}}(Q_B)$–module.

Suppose we have a quasi-free Q–bimodule $(\mathsf{Q} \circ N \circ \mathsf{Q}, \partial)$ weakly equivalent to Q as Q–bimodules, with N a cofibrant Σ_*–module satisfying $N(0) = 0$.

Applying Lemma 1.7.1, we get a cofibrant replacement $(\mathsf{Q} \circ N \circ B, \partial')$ of B in the category of Q–algebras, with $\partial' = \partial \circ_{\mathsf{Q}} B$.

This particular cofibrant replacement allows us to compute the homology and the cohomology of B as a Q–algebra using a smaller complex. We first get

$$H_*^{\mathsf{Q}}(B, E) = H_*(E \otimes_{U_{\mathsf{Q}}(\mathsf{Q} \circ N \circ B)} \Omega_{\mathsf{Q}}(\mathsf{Q} \circ N \circ B)).$$

By Lemma 2.1.1 (applied to $C = N \circ B$), this homology is identified with

$$H_*^{\mathsf{Q}}(B, E) = H_*(E \otimes N \circ B, \partial'')$$

where ∂'' is induced by ∂ in two steps explained in the proofs of Lemmas 1.7.1 and 2.1.1.

2.3 An analogous smaller complex for all operads

Let P be an operad and A an algebra over P.

Suppose we have a quasi-free P–bimodule $(\mathsf{P} \circ M \circ \mathsf{P}, \partial)$ weakly equivalent to P as a P–bimodule and such that

(1) the Σ_*–module M is connected and cofibrant as a Σ_*–module;

(2) the differential ∂ is decomposable, that is $\partial(M)$ has no component in $\mathsf{I} \circ M \circ \mathsf{I}$.

Under these hypotheses, Proposition 1.6 implies that $(\mathsf{P} \circ M \circ \mathsf{P}, \partial)$ is cofibrant as a P–bimodule.

Let $Q_A = (\mathsf{P} \circ M \circ A, \partial')$ be the P–algebra defined by the construction of Section 2.2 with the operad P instead of the operad Q. Form the dg-module $(E \otimes_{U_{\mathsf{P}}(Q_A)} \Omega_{\mathsf{P}}(Q_A), \partial'')$ associated to this P–algebra. We have again a map from Q_A to A, but this map is not a weak equivalence without a cofibrancy hypothesis on P. Nevertheless, with the result of Lemma 2.1.1, we can again reduce $(E \otimes_{U_{\mathsf{P}}(Q_A)} \Omega_{\mathsf{P}}(Q_A), \partial')$ to $(E \otimes M \circ A, \partial'')$.

Moreover, we have the following lemma of homology invariance:

2.3.1 Lemma *A weak equivalence of* P*–bimodules* $(\mathsf{P} \circ M_1 \circ \mathsf{P}, \partial_1) \xrightarrow{\phi} (\mathsf{P} \circ M_2 \circ \mathsf{P}, \partial_2)$ *(both satisfying the above hypotheses (1) and (2)) induces a quasi-isomorphism* $(E \otimes M_1 \circ A, \partial_1'') \to (E \otimes M_2 \circ A, \partial_2'')$.

Proof We consider a filtration on $(E \otimes M_1 \circ A, \partial_1'')$ and then use a spectral argument. We set $F_s(E \otimes M_1 \circ A) = \mathrm{Span}_{r \leq s}\{\xi \otimes m(a_1, \dots, a_r)\}$. This complex is a subcomplex of $E \otimes M_1 \circ A$.

Let $\overline{\phi}\colon M_1 \to M_2$ denote the map $I \circ_{\mathsf{P}} \phi \circ_{\mathsf{P}} I$. The hypothesis (2) for M_1 and M_2 implies that $\overline{\phi}$ is the indecomposable part of ϕ, and is a trivial Σ_*–cofibration. We get that $\overline{\phi} \circ A\colon M_1 \circ A \to M_2 \circ A$ is a trivial cofibration of dg modules.

Abusing the notation, we let ϕ denote also $E \otimes \phi \circ A\colon E \otimes M_1 \circ A \to E \otimes M_2 \circ A$.

Let us now prove that $\phi(F_s(E \otimes M_1 \circ A)) \subseteq F_s(E \otimes M_2 \circ A)$ and that $E^0\phi = E \otimes \overline{\phi} \circ A$.

$$\begin{aligned}
&\phi(\xi \otimes m(a_1, \dots, a_r)) \\
&\overset{(1)}{=} \xi \otimes \phi(m)(a_1, \dots, a_r) \\
&\overset{(2)}{=} \xi \otimes \overline{\phi}(m)(a_1, \dots, a_r) + \sum \xi \otimes p(y_1, \dots, y_t)(q_1, \dots, q_s)(\underline{a}) \\
&\overset{(3)}{=} \xi \otimes \overline{\phi}(m)(a_1, \dots, a_r) + \sum \xi \otimes p(y_1(\underline{q}_1(\underline{a}_1), \dots, y_t(\underline{q}_t(\underline{a}_t)))) \\
&\overset{(4)}{=} \xi \otimes \overline{\phi}(m)(a_1, \dots, a_r) + \sum \sum_i \xi . u_i \otimes y_i(\underline{q}_i(\underline{a}_i)).
\end{aligned}$$

Underlined elements denote sequences of elements. Equality (2) is just using the definition of $\overline{\phi}$ as the indecomposable part of ϕ. Equality (3) comes from the composition of the subtree above each y_i. In the equality (4), we use the isomorphism of Lemma 2.1.1, and $u_i = p(y_1(\underline{q}_1(\underline{a}_1)), \dots, \diamond, \dots, y_t(\underline{q}_t(\underline{a}_t)))$ with the hole in the ith position. The important thing to notice is that the arity of each y_i is smaller than r, as the differential is decomposable. This proves $\phi(F_s(E \otimes M_1 \circ A)) \subseteq F_s(E \otimes M_2 \circ A)$.

We now consider the associated graded complex

$$E^0_s(E \otimes M_1 \circ A) = F_s(E \otimes M_1 \circ A)/F_{r<s}(E \otimes M_1 \circ A) = \mathrm{Span}\{\xi \otimes m(a_1, \dots, a_s)\}.$$

The above calculation implies that $E^0\phi = E \otimes \overline{\phi} \circ A$.

With this equality and as $\overline{\phi}$ is a trivial cofibration, we get that $E^1(\phi) = H_*(E \otimes \overline{\phi} \circ A)$ is an isomorphism. Moreover, the spectral sequence converges, as it is a homological spectral sequence with an increasing exhaustive filtration which is bounded below.

This result implies that $H_*(\phi)$ is an isomorphism. □

Thus we have the following result:

2.3.2 Lemma *The homology of $(E \otimes M \circ A, \partial'')$ does not depend on the choice of the bimodule $(\mathsf{P} \circ M \circ \mathsf{P}, \partial)$ weakly equivalent to P (as a P–bimodule) such that hypotheses (1) and (2) are satisfied.*

Proof Suppose we have the following configuration:

$$\begin{tikzcd}[column sep=small] (\mathsf{P}\circ M_1\circ \mathsf{P},\partial_1) \arrow[dr, "\sim"'] & & (\mathsf{P}\circ M_2\circ \mathsf{P},\partial_2) \arrow[dl, "\sim"] \\ & \mathsf{P} & \end{tikzcd}$$

First the semi-model structure on P–bimodules gives a weak equivalence between $(\mathsf{P}\circ M_1\circ \mathsf{P},\partial_1)$ and $(\mathsf{P}\circ M_2\circ \mathsf{P},\partial_2)$ (as $(\mathsf{P}\circ M_1\circ \mathsf{P},\partial_1)$ is cofibrant). Then Lemma 2.3.1 implies that the induced arrow $(E\otimes M_1\circ A,\partial_1'')\to(E\otimes M_2\circ A,\partial_2'')$ is a quasi-isomorphism. □

2.4 Definition of Γ–homology

Let P be an operad, A an algebra over P and E a right $U_{\mathsf{P}}(A)$–module. Suppose we have a quasi-free P–bimodule $(\mathsf{P}\circ M\circ \mathsf{P},\partial)$ weakly equivalent to P as a P–bimodule, satisfying hypotheses (1) and (2) of Section 2.3.

Define the Γ–homology of the P–algebra A with coefficients in E to be the homology of the small complex defined in Section 2.2:

$$H\Gamma_*^{\mathsf{P}}(A,E)=H_*(E\otimes M\circ A,\partial'').$$

Lemma 2.3.2 proves that the notion of Γ–homology is well defined, as it does not depend on the choice of the bimodule $(\mathsf{P}\circ M\circ \mathsf{P},\partial)$.

Moreover:

2.5 Theorem *Let Q be a Σ_*–cofibrant replacement of P.*
For A a P–algebra and E a right $U_{\mathsf{P}}(A)$–module, we have $H\Gamma_*^{\mathsf{P}}(A,E)=H_*^{\mathsf{Q}}(A,E)$.

Proof First, note that a P–algebra will also be a Q–algebra and E will also be a right $U_{\mathsf{Q}}(A)$–module. Suppose that we are given $(\mathsf{Q}\circ M\circ \mathsf{Q},\partial)\xrightarrow{\sim}\mathsf{Q}$ a cofibrant replacement as Q–bimodules with the hypotheses above. The functor $\mathsf{P}\circ_{\mathsf{Q}}-\circ_{\mathsf{Q}}\mathsf{P}$ induces a Quillen's adjunction, and therefore we get a weak equivalence $(\mathsf{P}\circ M\circ \mathsf{P},\partial)\xrightarrow{\sim}\mathsf{P}$ between quasi-free P–bimodules. Seeing A as a Q–algebra, we get $H_*^{\mathsf{Q}}(A,E)=H_*(E\otimes M\circ A,\partial'')$. But the right hand side is by definition $H\Gamma_*^{\mathsf{P}}(A,E)$, as long as the differential is the same. It is the case, as both differentials are induced by the initial differential of $\mathsf{Q}\circ M\circ \mathsf{Q}$. □

Thus the definition of homology by replacement of bimodules is equivalent to the natural definition by replacement of operads. Also, when the operad is Σ_*–cofibrant, we recover the usual notion of homology:

2.6 Corollary *Let Q be a Σ_*–cofibrant operad, B a Q–algebra and E a right $U_{\mathsf{Q}}(B)$–module. Then $H\Gamma_*^{\mathsf{Q}}(B,E)=H_*^{\mathsf{Q}}(B,E)$.*

2.7 Definition of Γ–cohomology

Let P be an operad, A an algebra over P and F a left $U_{\mathsf{P}}(A)$–module. Suppose we have a quasi-free P–bimodule $(\mathsf{P}\circ M\circ\mathsf{P},\partial)$ weakly equivalent to P as a P–bimodule, satisfying hypotheses (1) and (2) of Section 2.3.

When the operad P is Σ_*–cofibrant, we can make a similar reduction of the complex $\mathrm{Hom}_{U_{\mathsf{P}}(Q_A)}(\Omega_{\mathsf{P}}(Q_A),F)$ computing cohomology. We take for Q_A the explicit cofibrant replacement $(\mathsf{P}\circ M\circ A,\partial')$ given by Lemma 1.7.1. We apply now Lemma 2.1.1 and we get $\mathrm{Hom}_{U_{\mathsf{P}}(Q_A)}((U_{\mathsf{P}}(Q_A)\otimes M\circ A,\partial''),F)$. By adjunction, this complex is just $(\mathrm{Hom}_{\mathcal{C}}(M\circ A,F),\partial'')$.

Following the same ideas as in Section 2.3, we consider this complex even when the operad P does not satisfy any cofibrancy hypothesis.

We define the Γ–cohomology of the P–algebra A with coefficients in F

$$H^*_{\mathsf{P}}(A,F)=H^*(\mathrm{Hom}_{\mathcal{C}}(M\circ A,F),\partial'')$$

A lemma similar to Lemma 2.3.2 proves that this notion is well-defined. We recover also the usual notion of cohomology when P is Σ_*–cofibrant.

2.8 Theorem *Let Q be a Σ_*–cofibrant replacement of P.
For A a P–algebra and F a left $U_{\mathsf{P}}(A)$–module, we have $H\Gamma^*_{\mathsf{P}}(A,F)=H^*_{\mathsf{Q}}(A,F)$.*

2.9 Corollary *Let Q be a Σ_*–cofibrant operad, B a Q–algebra and F a left $U_{\mathsf{Q}}(B)$–module. Then $H\Gamma^*_{\mathsf{Q}}(B,F)=H^*_{\mathsf{Q}}(B,F)$.*

2.10 Remark If the ground ring $\mathbb{K}$ is a field of characteristic 0, then every operad P is Σ_*–cofibrant. Hence in that case Corollary 2.6 and Corollary 2.9 imply that our Γ–(co)homology agrees with the standard (co)homology of P–algebras.

3 Explicit complex à la Robinson

From now on, we assume that P is a connected binary (quadratic) Koszul operad. We define an explicit P–bimodule complex, using the Koszul construction $K\mathsf{P}$ and the bar construction of the symmetric group. Then we prove we can use this complex to compute Γ–homology of P–algebras. In the case $\mathsf{P}=\mathsf{Com}$, we retrieve the complex introduced by Robinson.

Before defining the P–bimodules involved in the complex, we construct maps which will be needed to define the differential.

3.1 Maps between bijections

Let r be a positive integer. Let $\underline{X}$ and $\underline{Y}$ be two ordered sets with r elements.

We represent an element w of $\mathrm{Bij}(\underline{X},\underline{Y})$ by a table of values:

$$w = \begin{pmatrix} x_1 & x_2 & \dots & x_r \\ w(x_1) & w(x_2) & \dots & w(x_r) \end{pmatrix}$$

The ordering amounts to a fixed bijection between $\{1,\dots,r\}$ and $\underline{X}$ (respectively $\underline{Y}$). We can use these bijections to identify elements of $\mathrm{Bij}(\underline{X},\underline{Y})$ with permutation of $\{1,\dots,r\}$.

For each pair $\{i,j\} \subset \underline{Y}$ and e a dummy variable, we form the bijection

$$c^e_{i,j}(w) = \begin{pmatrix} x_1 & x_2 & \dots & w^{-1}(i) & \dots & \widehat{w}^{-1}(j) & \dots & x_r \\ w(x_1) & w(x_2) & \dots & e & \dots & \widehat{j} & \dots & w(x_r) \end{pmatrix}$$

if $w^{-1}(i) < w^{-1}(j)$ or the bijection

$$c^e_{i,j}(w) = \begin{pmatrix} x_1 & x_2 & \dots & w^{-1}(j) & \dots & \widehat{w}^{-1}(i) & \dots & x_r \\ w(x_1) & w(x_2) & \dots & e & \dots & \widehat{i} & \dots & w(x_r) \end{pmatrix}$$

if $w^{-1}(j) < w^{-1}(i)$.

If $w^{-1}(i) < w^{-1}(j)$, we have removed the column where j is the image, and i has been replaced by e. The map $c^e_{i,j}(w)$ is a bijection from $\underline{X} \smallsetminus \{w^{-1}(j)\}$ to $\underline{Y} \smallsetminus \{i,j\} \coprod e$. In $\underline{X} \smallsetminus \{w^{-1}(j)\}$, we consider the restriction of the order of $\underline{X}$. In $\underline{Y} \smallsetminus \{i,j\} \coprod e$, we consider the restriction of the order in $\underline{Y}$ with e at the place of i. Note that the map $c^e_{i,j}(w)$ can be identified with an element of Σ_{r-1}.

In the case where $w^{-1}(j) < w^{-1}(i)$, we have removed the column where i is the image, and j has been replaced by e. The map $c^e_{i,j}(w)$ is a bijection from $\underline{X} \smallsetminus \{w^{-1}(i)\}$ to $\underline{Y} \smallsetminus \{i,j\} \coprod e$ and can be identified with an element of Σ_{r-1}.

For each element i in $\underline{Y}$, we form the bijection

$$c_{\varnothing,i}(w) = \begin{pmatrix} x_1 & x_2 & \dots & \widehat{w}^{-1}(i) & \dots & x_r \\ w(x_1) & w(x_2) & \dots & \widehat{i} & \dots & w(x_r) \end{pmatrix}.$$

Here we have only removed the column where i is the image.

The map $c_{\varnothing,i}(w)$ is a bijection from $\underline{X} \smallsetminus \{w^{-1}(i)\}$ to $\underline{Y} \smallsetminus \{i\}$. Again, it can be identified with an element of Σ_{r-1} by considering the induced orders. These maps $c^e_{i,j}$ and $c_{\varnothing,i}$ play different roles, but note that $c_{\varnothing,i}(w)$ is just $c^y_{y,i}(w)$ for any y in $\underline{Y}$ such that $w^{-1}(y) < w^{-1}(i)$.

3.1.1 Lemma *Let σ be an element of* $\mathrm{Bij}(\underline{Y}) \simeq \Sigma_r$ *and w an element of* $\mathrm{Bij}(\underline{X}, \underline{Y})$.

The maps $c^e_{i,j}(w)$ and $c_{\varnothing,i}(w)$ are compatible with the action of the symmetric group on the left, that is $\overline{\sigma}.c^e_{i,j}(w) = c^e_{\sigma(i),\sigma(j)}(\sigma.w)$, where $\overline{\sigma}$ is the bijection fixing e induced by σ on $\underline{Y} \setminus \{i,j\} \coprod e$.

Proof We prove the lemma in the case where $w^{-1}(i) < w^{-1}(j)$. The proof for the other case is obtained by permuting i and j.

We already know that

$$c^e_{i,j}(w) = \begin{pmatrix} x_1 & \dots & w^{-1}(i) & \dots & \widehat{w}^{-1}(j) & \dots & x_r \\ w(x_1) & \dots & e & \dots & \widehat{j} & \dots & w(x_r) \end{pmatrix}.$$

We get

$$\overline{\sigma}.c^e_{i,j}(w) = \begin{pmatrix} x_1 & \dots & w^{-1}(i) & \dots & \widehat{w}^{-1}(j) & \dots & x_r \\ \sigma(w(x_1)) & \dots & e & \dots & \widehat{\sigma}(j) & \dots & \sigma(w(x_r)) \end{pmatrix}.$$

On the right hand side, we have

$$\sigma.w = \begin{pmatrix} x_1 & \dots & w^{-1}(i) & \dots & w^{-1}(j) & \dots & x_r \\ \sigma(w(x_1)) & \dots & \sigma(i) & \dots & \sigma(j) & \dots & \sigma(w(x_r)) \end{pmatrix}$$

and then

$$c^e_{\sigma(i),\sigma(j)}(\sigma.w) = \begin{pmatrix} x_1 & \dots & w^{-1}(i) & \dots & \widehat{w}^{-1}(j) & \dots & x_r \\ \sigma(w(x_1)) & \dots & e & \dots & \widehat{\sigma}(j) & \dots & \sigma(w(x_r)) \end{pmatrix}. \qquad \square$$

We extend the definition of $c^e_{i,j}$ to sequences of bijections $\underline{w} = (w_0, \dots w_n)$ by $c^e_{i,j}(\underline{w}) = (c^e_{i,j}(w_0), \dots, c^e_{i,j}(w_n))$.

3.2 Definition of the complex

We now define a Σ_*–module M involved in our explicit complex computing Γ–homology. We are given P a connected binary (quadratic) Koszul operad.

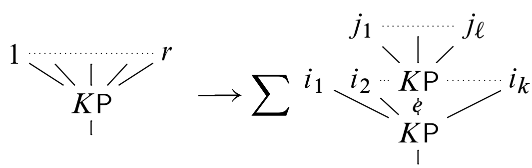
We consider $K\mathsf{P}$ the Koszul construction of P, defined by $K(\mathsf{P})_{(s)} := H_s(B_*(\mathsf{P})_{(s)})$. It is a cooperad. The Koszul construction with coefficients $\mathsf{P} \circ K\mathsf{P} \circ \mathsf{P}$ is equipped with a differential, such that $(\mathsf{P} \circ K\mathsf{P} \circ \mathsf{P}, \partial)$ is quasi-isomorphic to P. For more details, we refer the reader to the initial article of Ginzburg and Kapranov [9] or the article of Fresse [6], of which we adopt the convention.

We also consider the chain complex $C_*(E\Sigma_\bullet)$ of the total space of the universal Σ_n–bundles in simplicial spaces, $n \in \mathbb{N}$. The chain complex $C_*(E\Sigma_n)$ is the acyclic homogeneous bar construction of the symmetric group Σ_n, the module spanned in degree t by the $(t+1)$–tuples of permutations $\underline{w} = (w_0, \dots, w_t)$ together with the differential δ such that $\delta(\underline{w}) = \sum_i (-1)^i (w_0, \dots, \widehat{w}_i, \dots, w_t)$. We consider the left action of the symmetric group on this chain complex.

We define the Σ_*–module $M = K\mathsf{P} \boxtimes C_*(E\Sigma_\bullet)$ by $M(r) = K\mathsf{P}(r) \otimes C_*(E\Sigma_r)$. The action of the symmetric group is the diagonal action.

Now we construct a map $\Delta\colon M \to \mathsf{P} \circ M \circ \mathsf{P}$ which defines a twisting differential once extended by P–linearity on the right and as a P–derivation on the left.

Recall that the quadratic component of the cooperad product of $K\mathsf{P}$ is given by the dual of the operadic composition in P:

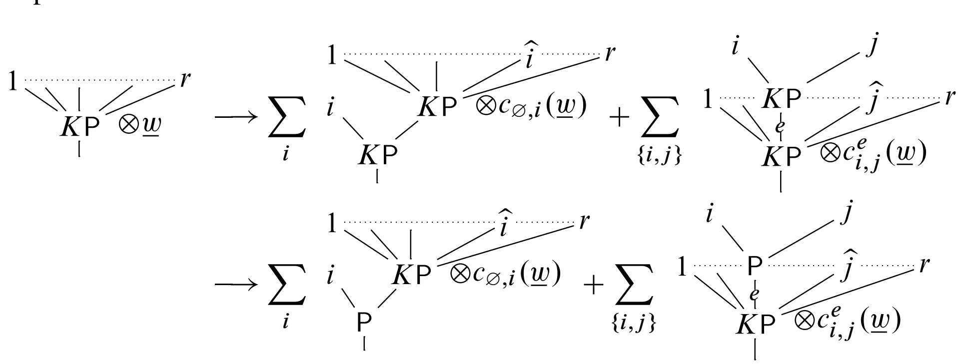


where the sum ranges over all partitions $\{i_1, \dots, i_k\} \coprod \{j_1, \dots, j_\ell\} = \{1, \dots, r\}$ and e is a dummy variable. We define two restrictions of this coproduct:

- Δ_- where we only keep the components of the differential where the set $\{i_1, \dots, i_k\}$ is reduced to one index (when the element below in the composition is binary).
- Δ_+ where we only keep the components of the differential where the set $\{j_1, \dots, j_\ell\}$ is composed of two indices (when the element above in the composition is binary).

Note that $\Delta_-(\gamma) = \Delta_+(\gamma)$ when γ is an element with three inputs.

We use this coproduct Δ on $K\mathsf{P}$ to define Δ on $K\mathsf{P} \boxtimes C_*(E\Sigma_\bullet)$ by the following composite:



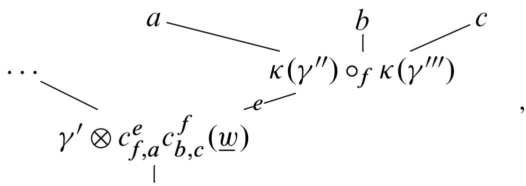
The first arrow consists in using Δ_- and Δ_+ on $K\mathsf{P}$ and the $c^e_{i,j}$ defined in the previous paragraph. The second arrow comes from the twisting cochain $\kappa\colon K\mathsf{P} \to \mathsf{P}$ (which identifies elements of arity 2 in $K\mathsf{P}$ with elements of arity 2 in P).

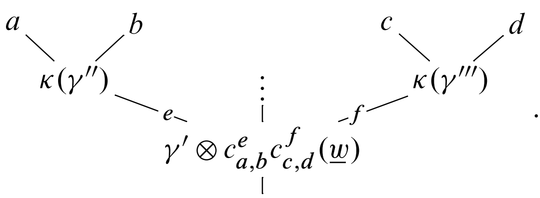
This construction defines Δ on representatives with the entries ordered from 1 to r. We apply Lemma 3.1.1 to extend this definition to $K\mathsf{P}$.

3.2.1 Lemma *The map Δ determines a differential of Σ_*–modules on $\mathsf{P} \circ M \circ \mathsf{P}$.*

Proof For an element $\gamma \in K\mathsf{P}(r)$ and $\underline{w}$ a sequence of permutations in Σ_r, we decompose $\Delta^2(\gamma \otimes \underline{w})$ in the sum of three terms: the part induced by $\Delta_+\Delta_+$, the part induced by $\Delta_-\Delta_-$ and the part induced by $\Delta_+\Delta_- + \Delta_-\Delta_+$.

The composite $\Delta_+\Delta_+$ yields terms of the form

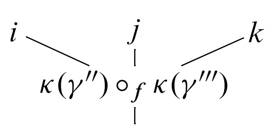
(I)  ,

(II)  .

- Let $\{i < j < k\} = \{a, b, c\}$ denote the ordered subset formed by the triple $\{a, b, c\}$ in the indexing set. We can identify the permutation occuring in terms of the form (I):

$$c^e_{f,a} c^f_{b,c}(\underline{w}) = \begin{pmatrix} \dots & w^{-1}(i) & \dots & \widehat{w}^{-1}(j) & \dots & \widehat{w}^{-1}(k) & \dots \\ \dots & e & \dots & \widehat{j} & \dots & \widehat{k} & \dots \end{pmatrix}.$$

  Thus the result of the composite $c^e_{a,b} c^f_{c,d}$ only depends on $\{i < j < k\}$. The sum of the terms associated to a given triple $\{i < j < k\}$ is 0 because the sum of the compositions



cancels in P by construction of the Koszul dual (see Fresse [6, Section 5.2]) and the sum of terms (I) is 0.

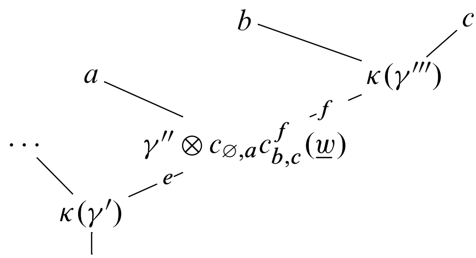
- For terms (II), we have the relation $c^e_{a,b}c^f_{c,d}(\underline{w}) = c^f_{c,d}c^e_{a,b}(\underline{w})$. By coassociativity of the coproduct in KP, the terms (II) cancel each another. Note simply that a permutation of κ with a suspension produces a sign opposition.

Thus the part of Δ^2 induced by $\Delta_+\Delta_+$ is 0.

The cancellation of the part induced by $\Delta_-\Delta_-$ is similar to the proof of the cancellation of terms (I).

We now study the part induced by $\Delta_+\Delta_- + \Delta_-\Delta_+$. The composite $\Delta_-\Delta_+$ yields terms of the form

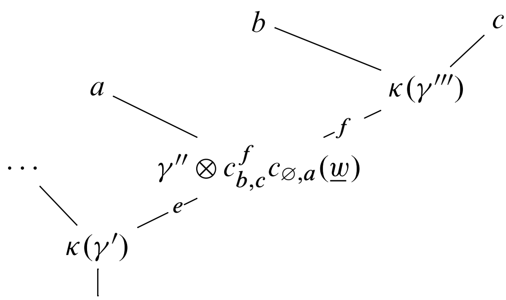
(III')



,

while the composite $\Delta_+\Delta_-$ yields terms of the form

(III")



.

But $c^f_{b,c}c_{\varnothing,a}(\underline{w}) = c_{\varnothing,a}c^f_{b,c}(\underline{w})$. So by coassociativity of the coproduct in KP, we prove the cancellation of terms (III') with terms (III"). We use again that a permutation of κ with a suspension produces a sign opposition.

Thus the part of Δ^2 induced by $\Delta_+\Delta_- + \Delta_-\Delta_+$ is 0.

Finally, we have proved that $\Delta^2 = 0$.

Moreover, the map is compatible with the symmetric action by Lemma 3.1.1. □

We now consider the differential of the bar construction of the symmetric group and use it to define another differential δ on $\mathsf{P} \circ M \circ \mathsf{P}$.

For $\underline{w} = (w_0, \dots w_n)$, recall that $\delta(\underline{w}) = \sum_i (-1)^i (w_0, \dots, \widehat{w}_i, \dots, w_n)$.

We define the map δ on $K\mathsf{P} \boxtimes C_*(\Sigma_\bullet) \to K\mathsf{P} \boxtimes C_*(\Sigma_\bullet)$ by

$$\delta(\gamma \otimes \underline{w}) = (-1)^{|\gamma|} \gamma \otimes \delta(\underline{w}).$$

3.2.2 Lemma *The map δ induces a differential on $\mathsf{P} \circ (K\mathsf{P} \boxtimes C_*(E\Sigma_\bullet)) \circ \mathsf{P}$ that anticommutes with Δ.* □

Putting all this together, we get:

3.3 Theorem *We have defined a quasi-free dg P–bimodule*

$$(\mathsf{P} \circ (K\mathsf{P} \boxtimes C_*(E\Sigma_\bullet)) \circ \mathsf{P}, \Delta + \delta),$$

where Δ and δ are both a differential.

3.4 Homology of the complex

The goal of this paragraph is to prove that we have a quasi-isomorphism

$$(\mathsf{P} \circ (K\mathsf{P} \boxtimes C_*(E\Sigma_\bullet)) \circ \mathsf{P}, \Delta + \delta) \xrightarrow{\sim} \mathsf{P}$$

of P–bimodules.

First, we consider a dg-module morphism defined by:

$$\begin{cases} K\mathsf{P}(r) \otimes C_0(\Sigma_r) \to K\mathsf{P}(r) \\ K\mathsf{P}(r) \otimes C_{\geq 1}(\Sigma_r) \to 0. \end{cases}$$

The first part of the arrow just forgets the permutation.

This morphism induces a P–bimodule morphism

$$(\mathsf{P} \circ (K\mathsf{P} \boxtimes C_*(E\Sigma_\bullet)) \circ \mathsf{P}, \Delta + \delta) \xrightarrow{\sim} (\mathsf{P} \circ K\mathsf{P} \circ \mathsf{P}, \partial),$$

by extension by linearity on the right, and as a derivation on the left. We call ϵ this P–bimodule morphism. Note that Δ is sent to the usual differential ∂ of the Koszul construction with coefficients $\mathsf{P} \circ K\mathsf{P} \circ \mathsf{P}$, while δ is sent to 0.

We will now use a spectral argument to show that $(\mathsf{P} \circ (K\mathsf{P} \boxtimes C_*(E\Sigma_\bullet)) \circ \mathsf{P}, \Delta + \delta)$ is quasi-isomorphic to $(\mathsf{P} \circ K\mathsf{P} \circ \mathsf{P}, \Delta)$.

We see $(\mathsf{P}\circ(K\mathsf{P}\boxtimes C_*(E\Sigma_\bullet))\circ\mathsf{P})$ as a bimodule, with differentials Δ and δ. The first graduation is the bar degree r in $K\mathsf{P}$ and the second graduation is the number $*$ of permutations.

$$(E^0_{r,*},d^0)=(\mathsf{P}\circ(K\mathsf{P}_r\boxtimes C_*(E\Sigma_\bullet))\circ\mathsf{P},\delta)$$

We now use that $C_*(E\Sigma_\bullet)$ is acyclic, that is $H_n(C_*(E\Sigma_\bullet))=\mathbb{K}$ if $n=0$ and 0 otherwise. We also use that the functors $\mathsf{P}\circ -$, $-\circ\mathsf{P}$ and $K\mathsf{P}_r\ \otimes -$ preserve quasi-isomorphisms (for instance, see Fresse [6, Theorem 2.1.15]).

Thus we get that $H_n(\mathsf{P}\circ(K\mathsf{P}_r\boxtimes C_*(E\Sigma_\bullet))\circ\mathsf{P},\delta)=\mathsf{P}\circ K\mathsf{P}_r\circ\mathsf{P}$.

$$(E^1_{r,0},d^1)=(\mathsf{P}\circ K\mathsf{P}_r\circ\mathsf{P},\partial)$$
$$E^2_{r,0}=H_r(\mathsf{P}\circ K\mathsf{P}\circ\mathsf{P},\partial)$$

We know that the spectral sequence of a bicomplex (both graduations being bounded below) converges to the total homology of the bicomplex.

Thus $H_*(\mathsf{P}\circ(K\mathsf{P}\boxtimes C(E\Sigma_\bullet))\circ\mathsf{P},\Delta+\delta)=H_*(\mathsf{P}\circ K\mathsf{P}\circ\mathsf{P},\partial)$.

This proves that ϵ is a quasi-isomorphism

$$(\mathsf{P}\circ(K\mathsf{P}\boxtimes C_*(E\Sigma_\bullet))\circ\mathsf{P},\Delta+\delta)\xrightarrow{\sim}(\mathsf{P}\circ K\mathsf{P}\circ\mathsf{P},\partial).$$

We can compose it with the quasi-isomophism between $\mathsf{P}\circ K\mathsf{P}\circ\mathsf{P}$ and P, and finally this gives us a quasi-isomorphism $(\mathsf{P}\circ(K\mathsf{P}\boxtimes C_*(E\Sigma_\bullet))\circ\mathsf{P},\Delta+\delta)\xrightarrow{\sim}\mathsf{P}$ of P–bimodules.

3.5 Back to Γ–homology

We now prove that the P–bimodule constructed in the previous paragraphs satisfies all the required hypotheses so we can use it to compute Γ–homology.

It has the form $\mathsf{P}\circ M\circ P$, with M a Σ_*–module such that $M(0)=0$. We now have to prove that $(\mathsf{P}\circ(K\mathsf{P}\boxtimes C_*(E\Sigma_\bullet))\circ\mathsf{P},\Delta+\delta)$ is a cofibrant P–bimodule.

The P–bimodule $(\mathsf{P}\circ(K\mathsf{P}\boxtimes C_*(E\Sigma_\bullet))\circ\mathsf{P},\Delta+\delta)$ can be seen as

$$(\mathsf{P}\circ(K\mathsf{P}\boxtimes C_*(E\Sigma_\bullet),\delta)\circ\mathsf{P},\Delta).$$

The differential Δ is decomposable. We first prove that $(K\mathsf{P}\boxtimes C_*(E\Sigma_\bullet),\delta)$ is a cofibrant Σ_*–module, and then Proposition 1.6 will give us the result.

3.5.1 Lemma *The Σ_*–bimodule $(K\mathsf{P}\boxtimes C_*(E\Sigma_\bullet),\delta)$ is cofibrant.*

Proof We consider the map f of Σ_*–modules $0 \to (K\mathsf{P} \boxtimes C_*(E\Sigma_\bullet), \delta)$, which can be written as $f = (0 \otimes id_{\Sigma_r})_{r\in\mathbb{N}}$. According to the description of generating cofibrations in Section 1.4, we have to prove that $0 \to (K\mathsf{P}(r), 0)$ is a cofibration of dg-modules. But $(K\mathsf{P}(r), 0)$ is assumed to be free and its differential is 0. Hence the claim is immediate. □

Thus we have proved the following.

3.6 Proposition *The* P*–bimodule* $(\mathsf{P} \circ (K\mathsf{P} \boxtimes C_*(E\Sigma_\bullet)) \circ \mathsf{P}, \Delta + \delta)$ *is cofibrant.*

Besides, we have seen in the previous paragraph that $(\mathsf{P} \circ (K\mathsf{P} \boxtimes C_*(E\Sigma_\bullet)) \circ \mathsf{P}, \Delta + \delta)$ is weakly equivalent to P.

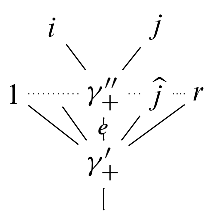
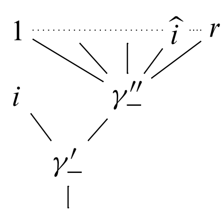
So we can use $K\mathsf{P} \boxtimes C_*(E\Sigma_\bullet)$ to compute Γ–homology of algebras over P. Explicitly, we have:

3.7 Theorem *Let* P *be a binary Koszul operad,* A *an algebra over* P *and* E *a right* $U_{\mathsf{P}}(A)$*–module.*

$$H\Gamma_*^{\mathsf{P}}(A, E) = H_*(E \otimes (K\mathsf{P} \boxtimes C_*(E\Sigma_\bullet)) \circ A, \partial'')$$

where ∂'' *is the differential induced by* $\Delta + \delta$ *in two steps, explained in the proofs of Lemmas 1.7.1 and 2.1.1.*

Explicitly, for $x \in E$, $\gamma \in K\mathsf{P}$ *such that*

$\Delta_+(\gamma) = \sum_{i<j}$  *and* $\Delta_-(\gamma) = \sum_{i}$  ,

$(w_0, \dots, w_*) \in C_*(E\Sigma_r)$ *and* $a_1, \dots, a_r$ *in* A*, we have:*

$$\partial''(x \otimes \gamma \otimes (w_0, \dots, w_*) \otimes (a_1, \dots, a_r)) =$$
$$\sum_{i<j} \pm x \otimes \gamma'_+ \otimes (c^e_{i,j}(w_0), \dots, c^e_{i,j}(w_*)) \otimes (a_1, \dots, \kappa(\gamma''_+)(a_i, a_j), \dots, \hat{a}_j, \dots a_r)$$
$$+ \sum_i \pm\kappa(\gamma'_-)(x, a_i) \otimes \gamma''_- \otimes (c_{\varnothing,i}(w_0), \dots, c_{\varnothing,i}(w_*)) \otimes (a_1, \dots, \hat{a}_i, \dots, a_r).$$

Signs are induced by the usual Koszul rule.

3.8 Examples

(1) For $\mathsf{P} = \mathsf{Com}$, we have $K\mathsf{P} = (\Lambda \mathsf{Lie})^{\#}$ where Λ denotes the operadic suspension and $\#$ denotes the linear duality. Here we retrieve Robinson's complex (up to a usual isomorphism on the bar construction).

(2) For $\mathsf{P} = \mathsf{Lie}$, we have $K\mathsf{P} = (\Lambda \mathsf{Com})^{\#}$. We denote γ_r the generator of $K\mathsf{P}$ in arity r (it is in degree $1-r$). Let A be a Lie algebra concentrated in degree 0.

$$\partial''(x \otimes \gamma_r \otimes (w_0, \dots, w_*) \otimes (a_1, \dots, a_r)) =$$

$$\sum_{i<j} (-1)^j x \otimes \gamma_{r-1} \otimes (c^e_{i,j}(w_0), \dots, c^e_{i,j}(w_*)) \otimes (a_1, \dots, \kappa(\gamma_2)(a_i, a_j), \dots, \widehat{a}_j, \dots a_r)$$

$$+ \sum_i (-1)^{i-1} \kappa(\gamma_2)(x, a_i) \otimes \gamma_{r-1} \otimes (c_{\varnothing,i}(w_0), \dots, c_{\varnothing,i}(w_*)) \otimes (a_1, \dots, \widehat{a}_i, \dots, a_r).$$

Note that we find the same signs as in the complex of Chevalley–Eilenberg.

Similarly for the cohomology, we have the following theorem:

3.9 Theorem *Let P be a binary Koszul operad, A an algebra over P and F a left $U_{\mathsf{P}}(A)$–module.*

$$H^*_{\mathsf{Q}}(A, F) = H^*(\mathrm{Hom}(K\mathsf{P} \boxtimes C_*(E\Sigma_\bullet) \circ A, F), \partial'').$$

where ∂'' is the differential induced by $\Delta + \delta$ in two steps, explained in the proofs of Lemmas 1.7.1 and 2.1.1.

Acknowledgements I am grateful to David Chataur and Benoit Fresse for many useful discussions on the matter of this article. I also would like to thank Joan Millès, Birgit Richter and Bruno Vallette for their careful reading.

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Received: 1 February 2010