Small dilatation mapping classes coming from the simplest hyperbolic braid

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In this paper we study the small dilatation pseudo-Anosov mapping classes arising from fibrations over the circle of a single 3–manifold, the mapping torus for the "simplest hyperbolic braid". The dilatations that occur include the minimum dilatations for orientable pseudo-Anosov mapping classes for genus $g = 2, 3, 4, 5$ and 8. We obtain the "Lehmer example" in genus $g = 5$, and Lanneau and Thiffeault's conjectural minima in the orientable case for all genus g satisfying $g = 2$ or $4 \pmod 6$. Our examples show that the minimum dilatation for orientable mapping classes is strictly greater than the minimum dilatation for non-orientable ones when $g = 4, 6$ or 8. We also prove that if δ_g is the minimum dilatation of pseudo-Anosov mapping classes on a genus g surface, then

$$\limsup_{g\to\infty} (\delta_g)^g \leq \frac{3+\sqrt{5}}{2}.$$

57M50; 57M25

1 Introduction

Let S_g be a closed oriented surface of genus $g \geq 1$, and let Mod_g be the *mapping class group* of isotopy classes of orientation preserving self-homeomorphisms of S_g. A mapping class $\phi \in \mathrm{Mod}_g$ is called *pseudo-Anosov* if S_g admits a pair of ϕ–invariant, transverse measured, singular foliations on which ϕ acts by stretching transverse to one foliation by a constant $\lambda(\phi) > 1$ and contracting transverse to the other by $\lambda(\phi)^{-1}$. The constant $\lambda(\phi)$ is called the *(geometric) dilatation* of ϕ. A mapping class is pseudo-Anosov if it is neither periodic nor reducible (see Thurston [26], Fathi, Laudenbach and Poenaru [7], and Casson and Bleiler [3]). Denote by $\mathrm{Mod}_g^{\mathrm{pA}}$ the set of pseudo-Anosov mapping classes in Mod_g.

A pseudo-Anosov mapping class ϕ is defined to be *orientable* if its invariant foliations are orientable. We will denote the set of orientable pseudo-Anosov mapping classes by $\mathrm{Mod}_g^{\mathrm{pA}+}$. Let $\lambda_{\mathrm{hom}}(\phi)$ be the spectral radius of the action of ϕ on the first homology of S. Then

$$\lambda_{\mathrm{hom}}(\phi) \leq \lambda(\phi),$$

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with equality if and only if ϕ is orientable (see, for example, Lanneau and Thiffeault [16], and Koberda and Silberstein [15]).

The dilatations $\lambda(\phi)$ satisfy reciprocal monic integer polynomials of degree bounded from above by $6g-6$ (see Thurston [26]). If ϕ is orientable the degree is bounded by $2g$. For fixed g, it follows that $\lambda(\phi)$ achieves a minimum $\delta_g > 1$ on $\mathrm{Mod}_g^{\mathrm{pA}}$ (see also, Arnoux and Yoccoz [2] and Ivanov [12]). Let

$$\mathrm{Mod}_g^{\mathrm{pA}+} \subset \mathrm{Mod}_g^{\mathrm{pA}}$$

be the subset of orientable pseudo-Anosov mapping classes, and let δ_g^+ be the minimum dilatation among elements of $\mathrm{Mod}_g^{\mathrm{pA}+}$.

In this paper, we address the following question (see Penner [23], McMullen [20] and Farb [5]):

Question 1.1 *What is the behavior of δ_g and δ_g^+ as functions of g?*

So far, exact values of δ_g have only been found for $g \leq 2$. For $g = 1$, the derivative map determines an identification $\mathrm{Mod}_1 = \mathrm{SL}(2;\mathbb{Z})$, and

$$\delta_1 = \frac{3+\sqrt{5}}{2}.$$

For a monic integer polynomial $p(x)$, the *house* of $p(x)$, written $|p|$, is the absolute value of the largest root of p. For $g = 2$, Cho and Ham [4] show that δ_2 is given by

$$\left|t^4 - t^3 - t^2 - t + 1\right| \approx 1.72208.$$

In the orientable case more is known due to recent results of Lanneau and Thiffeault [16]. Given $(a,b) \in \mathbb{Z} \oplus \mathbb{Z}$ with $0 < a < b$, let

$$LT_{(a,b)}(t) = t^{2b} - t^b\left(1 + t^a + t^{-a}\right) + 1,$$

and let

$$\lambda_{(a,b)} = |LT_{(a,b)}(t)|.$$

Theorem 1.2 (Lanneau–Thiffeault [16, Theorem 1.2 and 1.3]) *For $g = 2, 3, 4, 6$ and 8,*

$$\lambda_{(1,g)} \leq \delta_g^+$$

with equality when $g = 2, 3$ or 4.

For $g=2$, the value of δ_2^+ was first determined by Zhirov [27]. For $g=5$, Lanneau and Thiffeault show that δ_5^+ equals Lehmer's number (≈ 1.17628) [17]. This dilatation is realized as a product of multi-twists along a curve arrangement dual to the E_{10} Coxeter graph (see Leininger [18] and Hironaka [10]), and as the monodromy of the $(-2,3,7)$–pretzel knot (see Hironaka [9]). Lanneau and Thiffeault also find a lower bound for δ_7^+. An example realizing this bound can be found in Aaber and Dunfield [1, page 4] and Kin and Takasawa [14, Theorem 1.12].

Based on their results, Lanneau and Thiffeault ask:

Question 1.3 (Lanneau–Thiffeault [16, Question 6.1]) *Is $\delta_g^+=\lambda_{(1,g)}$ for all even g?*

For convenience, we will call the affirmative answer to their question the *LT-conjecture*.

In our first result, we improve on the following previous best bounds for the minimum dilatation of infinite families

$$(\delta_g)^g \le (\delta_g^+)^g \le 2+\sqrt{3}$$

found in Minakawa [22], and Hironaka–Kin [11].

Theorem 1.4 *If $g = 0, 1, 3$ or $4 \pmod 6$, $g \ge 3$, then*

$$\delta_g \le \lambda_{(3,g+1)},$$

and if $g = 2$ or $5 \pmod 6$ and $g \ge 5$, then

$$\delta_g \le \lambda_{(1,g+1)}.$$

For the orientable case, our results complement those of Lanneau and Thiffeault for $g = 2$ or $4 \pmod 6$.

Theorem 1.5 *Let $g \ge 3$. Then*

$$\begin{array}{ll} \delta_g^+ \le \lambda_{(3,g+1)} & \text{if } g = 1 \text{ or } 3 \pmod 6, \\ \delta_g^+ \le \lambda_{(1,g)} & \text{if } g = 2 \text{ or } 4 \pmod 6, \text{ and} \\ \delta_g^+ \le \lambda_{(1,g+1)} & \text{if } g = 5 \pmod 6. \end{array}$$

Putting Theorem 1.5 together with Lanneau and Thiffeault's lower bound for $g = 8$ gives:

Corollary 1.6 *The minimal dilatation for orientable pseudo-Anosov mapping classes for genus 8 is given by*

$$\delta_8^+ = \lambda_{(1,8)}.$$

The following is a table of the minimal dilatations that arise in this paper's examples for genus 1 through 12. All numbers in the table are truncated to 5 decimal places. An asterisk $*$ marks the numbers that have been verified to equal δ_g^+ (resp., δ_g). For singularity-type, we use the convention that $(a_1, \dots, a_k)$ means that the singularities of the invariant foliations have degrees $a_1, \dots, a_k$ (see Lanneau and Thiffeault's notation [16, page 3]). The singularity-types for our examples are derived from the formula given in Corollary 3.6.

g	orientable	degrees of singularities	unconstrained	degrees of singularities
1	2.61803*	no singularities	2.61803*	no singularities
2	1.72208*	(4)	1.72208*	(4)
3	1.40127*	$(2, 2, 2, 2)$	1.40127	(2,2,2,2)
4	1.28064*	(10,2)	1.26123	(3,3,3,3)
5	1.17628*	(16)	1.17628	(16)
6	–	–	1.1617	(5,5,5,5)
7	1.13694	(6,6,6,6)	1.13694	(6,6,6,6)
8	1.12876*	(22,6)	1.1135	(25,1,1,1)
9	1.1054	(8.8.8.8)	1.1054	(8,8,8,8)
10	1.10149	(28,8)	1.09466	(9,9,9,9)
11	1.08377	(34,2,2,2)	1.08377	(34,2,2,2)
12	–	–	1.07874	(11,11,11,11)

Table 1: Minimal orientable and unconstrained dilatations coming from M_{sb}

For $g = 2, 3, 4$ and 5, our orientable examples agree both in dilatation and in singularity-type with the previously known minimizing examples (see [16, Sections 3, 4 and 6]). For $g = 8$, our example agrees with the singularity-type anticipated by Lanneau and Thiffeault [16, (6.4)]. We prove that the known minimal dilatation examples for $g = 2, 3, 4, 5$ and 8 arise as the monodromy of fibrations of a single 3–manifold M_{sb}. For $g = 7$, our minimal example gives a larger dilatation than δ_7^+. (The dilatation δ_7^+ is realized by Kin and Takasawa [14], and by Aaber and Dunfield [1].)

Lanneau and Thiffeault show that $\delta_5^+ \le \delta_6^+$, and hence δ_g^+ is not strictly monotone decreasing (see Farb [5, Question 7.2]). Theorem 1.5 implies the following stronger statement.

Proposition 1.7 *If the LT-conjecture is true, then* $\delta_g^+ \le \delta_{g+1}^+$, *whenever* $g = 5 \pmod 6$.

Another consequence concerns the question of when the inequality $\delta_g \leq \delta_g^+$ is strict. In [14] and [1] it is shown that $\delta_5 < \delta_5^+$. Table 1 shows the following.

Corollary 1.8 *For $g = 4, 6$ and 8 we have*

$$\delta_g < \delta_g^+.$$

Theorem 1.4 and Proposition 4.3 imply the following.

Proposition 1.9 *If the LT-conjecture is true, then for all even $g \geq 4$ we have*

$$\delta_g < \delta_g^+.$$

For large g, it is known that δ_g and δ_g^+ converges to 1. Furthermore,

$$\log(\delta_g) \asymp \frac{1}{g} \qquad \text{and} \qquad \log(\delta_g^+) \asymp \frac{1}{g} \tag{1}$$

(see Penner [23], McMullen [20], Minakawa [22] and Hironaka–Kin [11]). The LT-conjecture together with (1) leads to the natural question:

Question 1.10 (See McMullen [20, page 551], Farb [5, Problem 7.1]) *Do the sequences*

$$(\delta_g)^g \qquad \textit{and} \qquad (\delta_g^+)^g$$

converge as g grows? What is the limit?

Theorem 1.4 and Theorem 1.5 imply the following.

Theorem 1.11

$$\limsup_{g\to\infty} (\delta_g)^g \leq \frac{3+\sqrt{5}}{2} \qquad \textit{and} \qquad \limsup_{g\not\equiv 0 \pmod 6} (\delta_g^+)^g \leq \frac{3+\sqrt{5}}{2}.$$

This leads to the question:

Question 1.12 (Golden Mean Question) *Do the sequences $(\delta_g)^g$ and $(\delta_g^+)^g$ satisfy*

$$\lim_{g\to\infty} (\delta_g)^g = \lim_{g\to\infty} (\delta_g^+)^g = \frac{3+\sqrt{5}}{2} = (\textit{golden mean})^2\ ?$$

For any pseudo-Anosov mapping class ϕ, let $M(\phi)$ be the mapping torus of ϕ. Conversely, given a compact hyperbolic 3–manifold with torus boundary components M, let $\Phi(M)$ be the collection of pseudo-Anosov mapping classes ϕ such that $M = M(\phi)$. Let Σ be the suspensions of singularities of the stable and unstable foliations of ϕ and let

$$M^*(\phi) = M(\phi) \setminus \Sigma.$$

Theorem 1.13 (Farb, Leininger and Margalit [6, Theorem 1.1]) *The set*

$$\mathcal{T}_P = \{M^*(\phi) \ : \ \phi \in \mathrm{Mod}_g^{\mathrm{pA}}, \lambda(\phi) \leq P^{1/g}\}$$

is finite for any $P > 1$.

The asymptotic equations (1) and Theorem 1.13 imply that

$$\begin{aligned} \mathcal{T} &= \{M^*(\phi) \ : \ \phi \in \mathrm{Mod}_g^{\mathrm{pA}}, \lambda(\phi) = \delta_g\} \\ \text{and} \qquad \mathcal{T}^+ &= \{M^*(\phi) \ : \ \phi \in \mathrm{Mod}_g^{\mathrm{pA}+} \lambda(\phi) = \delta_g^+\} \end{aligned}$$

are finite.

This leads to the question:

Question 1.14 *How large are the sets $\mathcal{T}$ and $\mathcal{T}^+$?*

If the LT-conjecture is true, then our results imply that a single 3–manifold M_{sb} would realize δ_g^+ for all $g = 2, 4 \pmod 6$. The manifold M_{sb} is the complement of the 6_2^2 braid (see Rolfsen's tables [24], and Figure 1). Another 3–manifold that produces small dilatation mapping classes is the complement $M_{-2,3,8}$ of the $(-2,3,8)$–pretzel link in S^3. These have been studied independently by Kin and Takasawa [14] and by Aaber and Dunfield [1]. For certain genera the mapping classes in $\Phi(M_{-2,3,8})$ have smaller dilatation than the minima realized by M_{sb}, but the asymptotic behavior of the minimal dilatations for large genus, supports the affirmative to Question 1.12. Both $M_{-2,3,8}$ and M_{sb} can be obtained from the *magic manifold* by Dehn fillings (see Martelli and Petronio [19]). The pseudo-Anosov braid monodromies with smallest known dilatations found in Hironaka–Kin [11] are also realized on the magic manifold (see Kin–Takasawa [13]).

Section 2 contains a brief review of Thurston norms, fibered faces and the Teichmüller polynomial. These are the basic tools used in this paper. In Section 3 we describe our family of examples, and in Section 4 we prove Theorem 1.4 and Theorem 1.5.

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2 Background and tools

In this section we give a brief review of invariants and properties of fibrations of a hyperbolic 3–manifold M, emphasizing the tools that we will use in the rest of the paper. For more details see, for example, Thurston [25], Fathi–Laudenbach–Poenaru [7], and McMullen [20; 21].

The theory of fibered faces of the Thurston norm ball and the existence of Teichmüller polynomials provides a way to study in a single picture a collection of pseudo-Anosov mapping classes defined on surfaces of different Euler characteristics and genera. Assume M is a compact hyperbolic 3–manifold with boundary. Given an embedded orientable surface S on M, let $\chi_-(S)$ be the sum of $|\chi(S_i)|$, where S_i are the connected components of S with negative Euler characteristic. The Thurston norm of $\psi \in \mathrm{H}^1(M;\mathbb{Z})$ is defined to be

$$\|\psi\|_T = \min \chi_-(S),$$

where the minimum is taken over oriented embedded surfaces $(S, \partial S) \subset (M, \partial M)$ such that the class of $(S, \partial S)$ in $\mathrm{H}_2(M, \partial M; \mathbb{Z})$ is the Poincaré dual of ψ.

Elements of $\mathrm{H}^1(M;\mathbb{Z})$ are canonically associated with epimorphisms

$$\pi_1(M;\mathbb{Z}) \to \mathbb{Z}.$$

We thus make the following natural identification:

$$\mathrm{H}^1(M;\mathbb{Z}) = \mathrm{Hom}(\pi_1(M), \mathbb{Z}) = \mathrm{Hom}(\mathrm{H}_1(M;\mathbb{Z}), \mathbb{Z})$$

We consider this as a lattice Λ_M inside $\mathbb{R}^{b_1(M)}$, where $b_1(M)$ is the first Betti number of M. If $\psi \in \Lambda_M$ corresponds to a fibration

$$\psi\colon M \to S^1$$

we say that ψ is *fibered*. In this case the Thurston norm of ψ is given by

$$\|\psi\|_T = \chi_-(S),$$

where S is homeomorphic to the fiber of ψ. Let

$$\Psi(M) = \big\{\psi\colon M \to S^1 \ :\ \psi \text{ is a fibration}\big\}.$$

The *monodromy* ϕ of $\psi \in \Psi(M)$ is the mapping class $\phi\colon S \to S$, such that M is the mapping torus of ϕ, and ψ is the natural projection to S^1. Since M is hyperbolic, ϕ is pseudo-Anosov.

Let B be the unit ball in $\mathbb{R}^{b_1(M)}$ with respect to the extended Thurston norm.

Theorem 2.1 (Thurston [25]) *The Thurston norm ball B is a convex polyhedron and for any top-dimensional open face F of B, $(F \cdot \mathbb{R}^+) \cap \Psi(M)$ is either empty or equal to $(F \cdot \mathbb{R}^+) \cap \Lambda_M$.*

If $(F \cdot \mathbb{R}^+) \cap \Psi(M) \neq \varnothing$, we say F is a *fibered face* of B. An element of $\Psi(M)$ is called *primitive* if its fiber is connected. The elements of Λ_M project to the rational points on the boundary of B. If F is a fibered face, then each rational point x on F corresponds to a unique primitive element $\psi_x \in \Psi(M)$, namely the element of $(x \cdot \mathbb{R}^+) \cap \Psi(M)$ that lies closest to the origin.

Theorem 2.2 (Fried [8, Theorem E]) *There is a continuous function $\mathcal{Y}$, homogeneous of degree one, defined on the fibered cone in $\mathbb{R}^{b_1(M)}$, so that if ψ is fibered with monodromy ϕ_ψ, then*

$$\mathcal{Y}(\psi) = \frac{1}{\log(\lambda(\phi_\psi))}.$$

The function $\mathcal{Y}$ is concave and tends to zero along the boundary of the cone.

Corollary 2.3 *For each fibered face F,*

$$\overline{\lambda}(\psi) = \lambda(\phi_\psi)^{\|\psi\|_T},$$

extends to a continuous function on $F \cdot \mathbb{R}^+$ that is constant on rays through the origin, and $\overline{\lambda}$ achieves a unique minimum on F.

Let G be a group and $\psi\colon G \to \mathbb{Z}$ a homomorphism. If $f \in \mathbb{Z}[G]$ is given by

$$f = \sum_{g \in G} \alpha_g g,$$

then the *specialization* of f at ψ is the polynomial in $\mathbb{Z}[t]$ defined by

$$f^\psi(t) = \sum_{g \in G} \alpha_g t^{\psi(g)}.$$

Theorem 2.4 (McMullen [20]) *Let F be a fibered face for a 3–manifold M, and let $G = H_1(M;\mathbb{Z})$. Then there is an element $\theta_F \in \mathbb{Z}[G]$ such that for all integral lattice points ψ in the fibered cone of F,*

$$\lambda(\phi_\psi) = \left|\theta_F^\psi\right|.$$

The polynomial θ_F is called the *Teichmüller polynomial* of M for the fibered face F.

3 The mapping torus for the simplest hyperbolic braid

We now look at a particular 3–manifold, and study properties of its fibrations. This example has also been studied by McMullen [20, Section 11], and the first part of this section will be a review of what is found there.

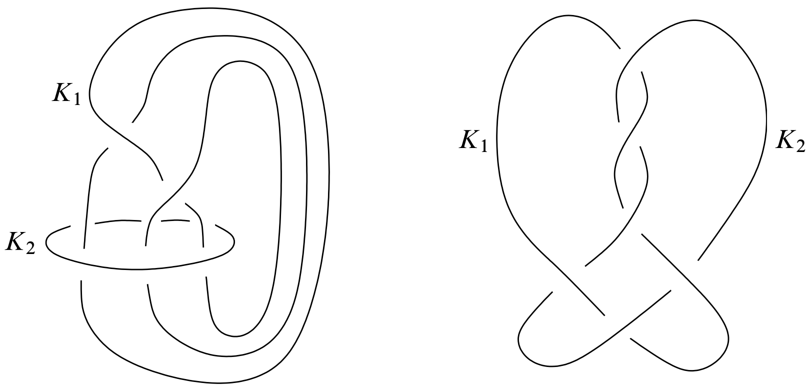

Figure 1: Two diagrams for the link 6_2^2

Let $M = S^3 \setminus N(L)$, where L is the link drawn in two ways in Figure 1, and $N(L)$ is a tubular neighborhood. As seen from the left diagram in Figure 1, M fibers over the circle with fiber a sphere with four boundary components $S_{0,4}$. Let ψ_0: $M \to S^1$ be the corresponding fibration, and let ϕ_0: $S_{0,4} \to S_{0,4}$ be the monodromy. Then ϕ_0 is the mapping class associated to the braid written with respect to standard generators as $\sigma_1\sigma_2^{-1}$ (see Figure 2) and its dilatation is given by

$$\lambda(\phi_0) = \frac{3+\sqrt{5}}{2}.$$

The braid $\sigma_1\sigma_2^{-1}$ has been called the "simplest hyperbolic braid" (see McMullen [20, Section 11]).

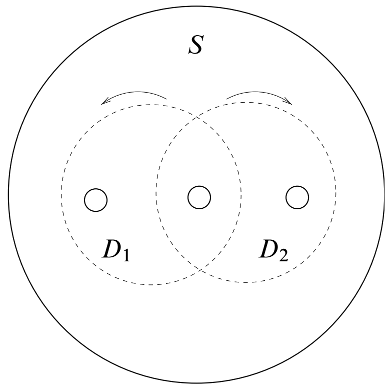

Figure 2: Braid monodromy associated to $\sigma_1\sigma_2^{-1}$

Let K_1 and K_2 be the components of L as drawn in Figure 1. Let μ_1 be the meridian of K_1 and μ_2 be the meridian of K_2. These determine coordinate functions for $H^1(M;\mathbb{Z})$

$$(\mu_1,\mu_2)(\psi) = (\psi(\mu_1),\psi(\mu_2)) \in \mathbb{Z}\times\mathbb{Z}.$$

With respect to these coordinates, the Thurston norm and the Alexander norm both are given by

$$\|(a,b)\| = \max\{2|a|, 2|b|\}. \tag{2}$$

The lattice points Λ_M in the fibered cone $F\cdot\mathbb{R}^+$ defined by $\psi = (0,1)$ is the set

$$\Psi = \{(a,b)\in\mathbb{Z}\times\mathbb{Z} \;:\; b>0, -b<a<b\}$$

as shown in Figure 3. For the rest of this paper, we will only be concerned with the subset $\Psi_{\mathrm{prim}} \subset \Psi$ consisting of elements of Ψ with connected fibers, i.e., the *primitive elements*. Thus,

$$\Psi_{\mathrm{prim}} = \{(a,b)\in\mathbb{Z}\times\mathbb{Z} \;:\; b>0, -b<a<b, \gcd(a,b)=1\}.$$

The Alexander polynomial for L is given by

$$\Delta_L(x,u) = u^2 - u(1-x-x^{-1}) + 1 \tag{3}$$

(see Rolfsen's table [24]), and the Teichmüller polynomial is given by

$$\Theta_L(x,u) = u^2 - u(1+x+x^{-1}) + 1 \tag{4}$$

(see [20, page 47]).

Specialization to the element $(a,b)\in \mathrm{H}^1(M;\mathbb{Z})$ is the same as plugging (t^a,t^b) into the equations for the Alexander and Teichmüller polynomials (see Section 2).

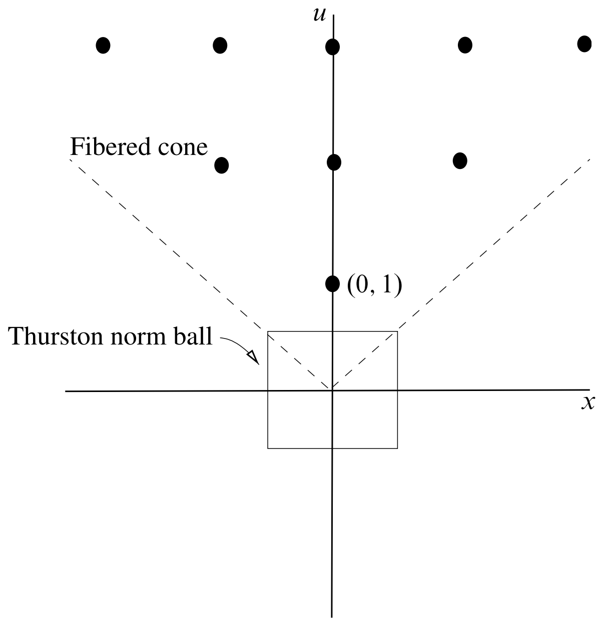

Figure 3: Fibered cone Ψ containing $\psi = (0,1)$

Proposition 3.1 *If $(a,b) \in \Psi_{prim}$, then the associated monodromy $\phi_{(a,b)}$ is pseudo-Anosov with geometric dilatation given by*

$$\lambda_{(a,b)} = \left|\Theta_L(t^a, t^b)\right| = \left|t^{2b} - t^b(1 + t^a + t^{-a}) + 1\right|,$$

and homological dilatation given by

$$\lambda^{\mathrm{hom}}_{(a,b)} = \left|\Delta_L(t^a, t^b)\right| = \left|t^{2b} - t^b(1 - t^a - t^{-a}) + 1\right|.$$

Corollary 3.2 *If $(a,b) \in \Psi_{prim}$, then the associated monodromy $\phi_{(a,b)}$ is orientable if a is odd and b is even.*

Proof If a is odd and b is even, then the roots of $\Theta_L(t^a, t^b)$ are the negatives of the roots of $\Delta_L(t^a, t^b)$. This implies that the geometric and homological dilatations of $\phi_{(a,b)}$ are equal, and therefore $\phi_{(a,b)}$ is orientable. □

Later in this section, we prove the converse of Corollary 3.2. First we consider how the monodromy behaves near the boundary of $S_{(a,b)}$.

Proposition 3.3 *Let $\phi_{(a,b)}\colon S_{(a,b)} \to S_{(a,b)}$ be the monodromy associated to $(a,b) \in \Psi_{prim}$. The boundary components of $S_{(a,b)}$ has $\gcd(3,a)$ components coming from*

$T(K_1)$ and $\gcd(3,b)$ coming from $T(K_2)$. Thus, the total number of boundary components of $S_{(a,b)}$ is given by

$$\begin{cases} 2 & \textit{if } \gcd(3,ab) = 1 \\ 4 & \textit{if } \gcd(3,ab) = 3 \end{cases}$$

Proof The number of components in $T(K_i) \cap S_{(a,b)}$ is the index of the image of $\pi_1(T(K_i))$ in $\mathbb{Z}$ under the composition of maps

$$\pi_1(T(K_i)) \to \pi_1(M) \to \mathbb{Z}$$

induced by inclusion and $\psi_{(a,b)}$.

For $i = 1, 2$, let ℓ_i be the longitude of K_i that is contractible in $S^3 \setminus K_i$. Then, for $T(K_1)$ we have

$$\psi_{(a,b)}(\mu_1) = a \quad \text{and} \quad \psi_{(a,b)}(\ell_1) = 3\psi_{(a,b)}(\mu_2) = 3b,$$

so the number of boundary components contributed by $T(K_1)$ is

$$\gcd(a, 3b) = \gcd(3, a),$$

since we are assuming that $\gcd(a,b) = 1$. The contribution of $T(K_2)$ is computed similarly. □

Proposition 3.4 *The genus of $S_{(a,b)}$, for $(a,b) \in \Psi_{prim}$ is given by*

$$\begin{aligned} g(S_{(a,b)}) &= |b| + \left(1 - \frac{\gcd(3,a) + \gcd(3,b)}{2}\right) \\ &= \begin{cases} |b| & \textit{if } \gcd(3,ab) = 1 \\ |b| - 1 & \textit{if } \gcd(3,ab) = 3. \end{cases} \end{aligned}$$

Proof Equation (2) gives

$$2|b| = \chi_-(S_{(a,b)}) = 2g - 2 + \gcd(3,a) + \gcd(3,b).$$ □

Proposition 3.5 *Let $(a,b) \in \Psi_{prim}$, and let $\mathcal{F}$ be a $\phi_{(a,b)}$–invariant foliation. Then $\mathcal{F}$*

(1) *has no interior singularities,*

(2) *is $(3b/\gcd(3,a))$–pronged at each of the $\gcd(3,a)$ boundary components coming from $T(K_1)$, and*

(3) *is $(b/\gcd(3,b))$–pronged at each of the $\gcd(3,b)$ boundary components coming from $T(K_2)$.*

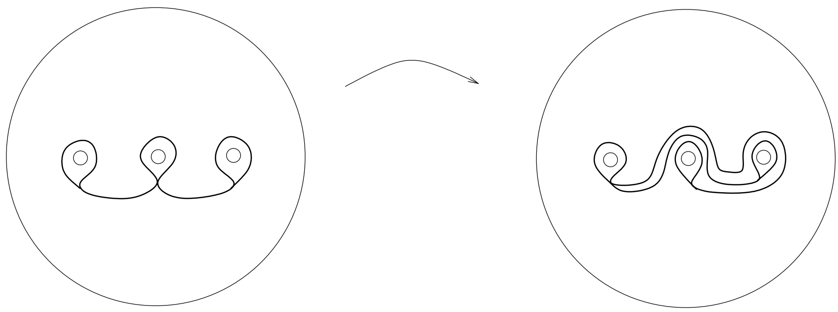

Figure 4: Train track for $\phi\colon S \to S$

Proof Let $\mathcal{L}$ be the lamination of M defined by suspending $\mathcal{F}$ over M considered as the mapping torus of ϕ. From the train track for ϕ (Figure 4), one sees that each of the boundary components of S are one-pronged, and that there are no other singularities. It follows that $\mathcal{L}$ has no singularities outside a neighborhood of the K_i, and near each K_i the leaves of $\mathcal{L}$ come together at a simple closed curve $\gamma_i \in \mathrm{H}_1(T(K_i))$. Write

$$\gamma_i = r_i\mu_i + s_i\ell_i$$

for $i = 1, 2$.

For $(a,b) \in \Psi_{\mathrm{prim}}$, the number of intersections of γ_i with $S_{(a,b)}$ is the image of γ_i under the epimorphism

$$\psi_{(a,b)}\colon \pi_1(M) \to \mathbb{Z}$$

defining the fibration. Figure 4 shows that $s_1 = 1$ and $r_2 = 1$. Using the identities

$$\begin{aligned} s_1 &= 1, & \ell_1 &= 3\mu_2, \\ r_2 &= 1, & \ell_2 &= 3\mu_1, \end{aligned}$$

we have

$$\begin{aligned} \psi_{(a,b)}(\gamma_1) &= r_1\psi_n(\mu_1) + 3\psi_n(\mu_2) = r_1a + 3b \\ \psi_{(a,b)}(\gamma_2) &= \psi_n(\mu_2) + 3s_2\psi_n(\mu_1) = 3s_2a + b. \end{aligned}$$

Let $m_1 = \gcd(3,a)$ and $m_2 = \gcd(3,b)$. Then $\phi_{(a,b)}$ is $(r_1a+3b)/m_1$–pronged at m_1 boundary components and $(3s_2a+b)/m_2$–pronged at m_2 boundary components. We find r_1 and s_2 by looking at some particular examples.

In general, if $f\colon \Sigma \to \Sigma$ is pseudo-Anosov on a compact oriented surface Σ with genus g and $n_1, \dots, n_k$ are the number of prongs at the singularities and boundary components, then by the Poincaré-Hopf theorem

$$\sum_{i=1}^{k}(n_i - 2) = 4g - 4. \tag{5}$$

For $(a,b) = (1,n)$, n not divisible by 3, we have two singularities with number of prongs given by:

$$\begin{aligned}\psi_n(\gamma_1) &= r_1 + 3n\\ \psi_n(\gamma_2) &= 3s_2 + n.\end{aligned}$$

Plugging into (5) gives

$$r_1 + 3s_2 = 0.$$

The mapping class $\phi_{(1,2)}$ is the unique genus 2 pseudo-Anosov mapping class with dilatation equal to λ_2 (see Cho and Ham [4], and Lanneau and Thiffeault [16]) and has one 6–pronged singularity (see Hironaka and Kin [11]). Thus, $r_1 = s_2 = 0$ and

$$\gamma_1 = \ell_1 = 3\mu_2 \qquad \text{and} \qquad \gamma_2 = \mu_2.$$

The claim follows. □

Corollary 3.6 *The map $\phi_{(a,b)}$ has singularities with number of prongs (or prong-type) given by*

$$\begin{cases}(3b, b) & \textit{if } \gcd(3, ab) = 1\\ (3b, b/3, b/3, b/3) & \textit{if } \gcd(3, b) = 3\\ (b, b, b, b) & \textit{if } \gcd(3, a) = 3\end{cases}$$

The degree of a singularity and the number of prongs differ by 2, yielding Table 1.

Corollary 3.7 *If b is odd, then $\phi_{(a,b)}$ is not orientable.*

Corollary 3.8 *For $(a,b) \in \Psi_{\mathrm{prim}}$, $\phi_{(a,b)}$ is 1–pronged at one or more boundary components of $S_{(a,b)}$ if and only if $(a,b) \in \{(0,1), (\pm 1, 3), (\pm 2, 3)\}$.*

Corollary 3.9 *If $(a,b) \notin \{(0,1), (\pm 1, 3), (\pm 2, 3)\}$, then $\phi_{(a,b)}$ extends to the closure of $S_{(a,b)}$ over the boundary components to a mapping class $\overline{\phi}_{(a,b)}$ with the same dilatation as $\phi_{(a,b)}$.*

Table 2 describes the pairs $(a,b) \in \Psi_{\mathrm{prim}}$ that give rise to an orientable (or non-orientable) genus g pseudo-Anosov mapping class. (Here $g \geq 4$.)

$g \pmod 6$	orientable	non-orientable
0	no example	$b = g + 1$, $a = 0 (\mathrm{mod}\, 3)$
1	$b = g + 1$, $a = 3 (\mathrm{mod}\, 6)$	$b = g$, $a = 1, 2 (\mathrm{mod}\, 3)$
2	$b = g$, $a = 1, 5 (\mathrm{mod}\, 6)$	$b = g + 1$, $a = 1, 2 (\mathrm{mod}\, 3)$
3	$b = g + 1$, $a = 3 (\mathrm{mod}\, 6)$	no example
4	$b = g$, $a = 1, 5 (\mathrm{mod}\, 6)$	$b = g + 1$, $a = 0 (\mathrm{mod}\, 3)$
5	$b = g + 1$, $a = 1, 5 (\mathrm{mod}\, 6)$	$b = g$, $a = 1, 2 (\mathrm{mod}\, 3)$

Table 2: Fibrations of M according to genus

4 Minimal dilatations for the fibered face.

Let Ψ_{prim} be the primitive elements of the fibered cone discussed in Section 3. Let

$$d_g = \min\left\{\lambda(\psi) \ : \ \psi \in \Psi_{\text{prim}}, \ \text{genus of } \psi \text{ is } g\right\}, \ \text{and}$$
$$d_g^+ = \min\left\{\lambda(\psi) \ : \ \psi \in \Psi_{\text{prim}}, \ \text{genus of } \psi \text{ is } g, \ \text{the monodromy of } \psi \text{ is orientable}\right\}.$$

In this section, we finish the proofs of Theorem 1.4 and Theorem 1.5 and their consequences by determining d_g and d_g^+.

Proposition 4.1 *Let $(a, b) \in \Psi_{prim}$. Then*

$$\lambda_{(a,b)} < \lambda_{(a',b')}$$

if either

(1) $|a| < |a'|$ *and* $|b| = |b'|$; *or*

(2) $|a| = |a'|$ *and* $|b| > |b'|$.

Proof One compares the slopes of rays from the origin to (a, b) and (a', b'). The claim follows from Theorem 2.2. □

Proposition 4.2 *For $b \geq 3$, we have*

$$\lambda_{(1,b)} \geq \lambda_{(3,b+1)},$$

with equality when $b = 3$.

Proof Let $\lambda = \lambda_{(3,b+1)}$. We will show that $LT_{(1,b)}(\lambda) < 0$. Multiplying by λ^2 and using the fact that $LT_{(3,b+1)}(\lambda) = 0$ gives

$$\begin{aligned}
\lambda^2 LT_{(1,b)}(\lambda) &= \lambda^2 LT_{(1,b)}(\lambda) - LT_{(3,b+1)}(\lambda) \\
&= \lambda^{b+4} - \lambda^{b+3} - \lambda^{b+2} + \lambda^{b-2} + \lambda^2 - 1 \\
&= (\lambda - 1)(\lambda^{b+3} - \lambda^{b-2}(\lambda^3 + \lambda^2 + \lambda + 1) + \lambda + 1) \\
&= (\lambda - 1)\lambda^{b-2}[\lambda^5 - \lambda^3 - \lambda^2 - \lambda - 1 + \lambda^{2-b}(\lambda + 1)].
\end{aligned}$$

Thus, it is enough to show that for $\lambda > 1$ and $b > 3$

$$\lambda^5 - \lambda^3 - \lambda^2 - \lambda - 1 + \lambda^{2-b}(\lambda + 1) < 0.$$

Let C be the quantity on the left side of this inequality. Then

$$C < \lambda^5 - \lambda^3 - \lambda^2 = \lambda^2(\lambda^3 - \lambda - 1).$$

One can check that the right hand side is negative for

$$1 < \lambda < 1.3.$$

By Proposition 4.1, λ decreases as b increases. A check shows that

$$1 < \lambda_{(3,5)} < 1.3,$$

and hence $C < 0$ for $b \geq 4$. For $b = 3$, one checks directly that

$$\lambda_{(1,3)} = \lambda_{(3,4)}. \qquad \square$$

Remark *The mapping class $\phi_{(1,3)}$ is defined on a genus 2 surface with four boundary components, with prong-type (3,1,1,1) and is not orientable. The mapping class $\phi_{(3,4)}$ is defined on a genus 3 surface with prong-type (4,4,4,4) and is orientable. By Proposition 4.2 these two examples have the same dilatation.*

Proposition 4.1 and Proposition 4.2 imply the following.

Proposition 4.3 *The sequences $\lambda_{(1,b)}$ and $\lambda_{(3,b)}$ satisfy:*

$$\lambda_{(1,b)} > \lambda_{(3,b+1)} > \lambda_{(1,b+1)}.$$

Table 3 describes the pairs $(a,b) \in \Psi_{\text{prim}}$ that give rise to the minima d_g and d_g^+ realized on M.

$g \mod 6$	$\lambda(\phi_{(a,b)}) = d_g^+$, $\phi_{(a,b)}$ orientable	$\lambda(\phi_{(a,b)}) = d_g$
0	no example	$(3, g+1)$
1	$(3, g+1)$	$(3, g+1)$
2	$(1, g)$	$(1, g+1)$
3	$(3, g+1)$	$(3, g+1)$
4	$(1, g)$	$(3, g+1)$
5	$(1, g+1)$	$(1, g+1)$

Table 3: Pairs (a, b) giving smallest dilatations for $\phi \in \Phi(M_{\mathrm{sb}})$

Proposition 4.4 *For $n \geq 2$,*

$$\lim_{n\to\infty} (\lambda_{(a,n)})^n = \frac{3+\sqrt{5}}{2},$$

for any fixed a.

Proof The rays through the lattice points $(a, n) \in \Lambda_M$ on the fibered face of ψ converge to the ray through $(0, 1)$. □

Corollary 4.5 *For the minimal dilatations d_g and d_g^+ that are realized on M, we have*

$$\lim_{g\to\infty} (d_g)^g = \frac{3+\sqrt{5}}{2}, \qquad \textit{and} \qquad \lim_{\substack{g\to\infty \\ g\not\equiv 0 \pmod 6}} (d_g^+)^g = \frac{3+\sqrt{5}}{2}.$$

Table 3 and Corollary 3.9 complete the proofs of Theorem 1.4 and Theorem 1.5. A pictorial view of how the elements of Ψ giving the least dilatations for each genus up to 12 lie on a fibered cone of M is shown in Figure 5.

The results of this paper and those in Aaber and Dunfield [1], Kin and Takasawa [14], and Lanneau and Thiffeault [16] imply that for genus $g = 2, 3, 4, 5, 7,$ and 8,

$$\delta_g^+ = \lambda_{(a,b)}$$

where

$$(a,b) = \begin{cases} (1, g) & \text{if } g = 2, 3, 4 \text{ or } 8 \\ (1, g+1) & \text{if } g = 5 \\ (2, g+2) & \text{if } g = 7 \end{cases}$$

and

$$\delta_6^+ \geq \lambda_{(1,6)}.$$

These results suggest the following generalization to Question 1.3.

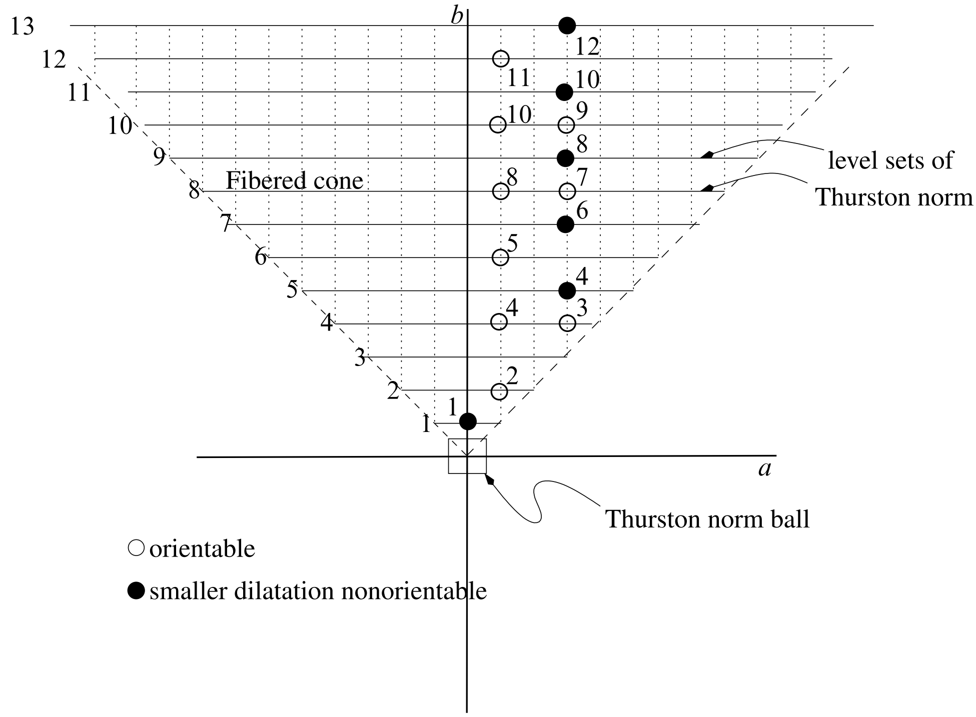

Figure 5: Minima for d_g and d_g^+ in genus $g = 1, \dots, 12$

Question 4.6 *For every $g \geq 2$, is it true that*

$$\delta_g^+ = \lambda_{(a,b)}$$

for some a, b with $b \geq g \geq a \geq 1$?

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