

# Stems and spectral sequences

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We introduce the category  $\mathcal{Pstem}[n]$  of  $n$ -stems, with a functor  $\mathcal{P}[n]$  from spaces to  $\mathcal{Pstem}[n]$ . This can be thought of as the  $n$ -th order homotopy groups of a space. We show how to associate to each simplicial  $n$ -stem  $Q_\bullet$  an  $(n+1)$ -truncated spectral sequence. Moreover, if  $Q_\bullet = \mathcal{P}[n]X_\bullet$  is the Postnikov  $n$ -stem of a simplicial space  $X_\bullet$ , the truncated spectral sequence for  $Q_\bullet$  is the truncation of the usual homotopy spectral sequence of  $X_\bullet$ . Similar results are also proven for cosimplicial  $n$ -stems. They are helpful for computations, since  $n$ -stems in low degrees have good algebraic models.

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## 1 Introduction

Many of the spectral sequences of algebraic topology arise as the homotopy spectral sequence of a (co)simplicial space – including the spectral sequence of a double complex, the (stable or unstable) Adams spectral sequence, the Eilenberg–Moore spectral sequence, and so on (see [Examples 5.14](#)). Given a simplicial space  $X_\bullet$ , the  $E^2$ -term of its homotopy spectral sequence has the form  $E_{s,t}^2 = \pi_s \pi_t X_\bullet$ , so it may be computed by applying the homotopy group functor dimensionwise to  $X_\bullet$ .

In this paper we show that the higher terms of this spectral sequence are obtained analogously by applying “higher homotopy group” functors to  $X_\bullet$ . These functors are given explicitly in the form of certain *Postnikov stems*, defined in [Section 2](#); the Postnikov 0-stem of a space is equivalent to its homotopy groups.

We then show how the  $E^r$ -term of the homotopy spectral sequence of a simplicial space  $X_\bullet$  can be described in terms of the  $(r-2)$ -Postnikov stem of  $X_\bullet$ , for each  $r \geq 2$  (see [Theorem 4.13](#)) – and similarly for the homotopy spectral sequence of a cosimplicial space  $X^\bullet$  (see [Theorem 5.12](#)).

As an application for the present paper, in [\[5\]](#) we generalize the first author’s result with Mamuka Jibladze in [\[6\]](#), which shows that the  $E^3$ -term of the stable Adams spectral sequence can be identified as a certain secondary derived functor  $\text{Ext}$ . We do this by

showing how to define in general the *higher order derived functors* of a continuous functor  $F: \mathcal{C} \rightarrow \mathcal{T}_*$ , by applying  $F$  to a simplicial resolution  $W_\bullet$  in  $\mathcal{C}$ , and taking Postnikov  $n$ -stems of  $FW_\bullet$ .

**1.1 Notation and conventions** The category of pointed connected topological spaces will be denoted by  $\mathcal{T}_*$ ; that of pointed sets by  $\mathcal{S}et_*$ ; that of groups by  $\mathcal{G}p$ . For any category  $\mathcal{C}$ ,  $s\mathcal{C}$  denotes the category of simplicial objects over  $\mathcal{C}$ , and  $c\mathcal{C}$  that of cosimplicial objects over  $\mathcal{C}$ . We abbreviate  $s\mathcal{S}et$  to  $\mathcal{S}$ ,  $s\mathcal{S}et_*$  to  $\mathcal{S}_*$ , and  $s\mathcal{G}p$  to  $\mathcal{G}$ . For any small indexing category  $I$ , the category of functors  $I \rightarrow \mathcal{C}$  is denoted by  $\mathcal{C}^I$ .

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## 2 Postnikov stems

The Postnikov system of a topological space (or simplicial set)  $X$  is the tower of fibrations

$$(2.1) \quad \dots \rightarrow P^{n+1}X \xrightarrow{p^{n+1}} P^nX \xrightarrow{p^n} P^{n-1}X \dots P^1X \xrightarrow{p^1} P^0X,$$

equipped with maps  $q^n: X \rightarrow P^nX$  (with  $p^n \circ q^n = q^{n-1}$ ), which induce isomorphisms on homotopy groups in degrees  $\leq n$ . Here  $P^nX$  is  $n$ -coconnected (that is,  $\pi_i P^nX = 0$  for  $i > n$ ) and  $\pi_i p^n$  is an isomorphism for  $i < n$ . The fiber of the map  $p^n: P^nX \rightarrow P^{n-1}X$  is the Eilenberg–Mac Lane space  $K(\pi_n X, n)$ , so the fibers are determined up to homotopy by  $\pi_* X$ . Thus a generalization of the homotopy groups of  $X$  is provided by the following notion:

**2.2 Definition** For any  $n \geq 0$ , a *Postnikov  $n$ -stem* in  $\mathcal{T}_*$  is a tower

$$(2.3) \quad \mathcal{Q} := \left( \dots \rightarrow \mathcal{Q}_{k+1} \xrightarrow{q_{k+1}} \mathcal{Q}_k \xrightarrow{q_k} \mathcal{Q}_{k-1} \dots \mathcal{Q}_0 \right)$$

in  $\mathcal{T}_*^{(\mathbb{N}, \leq)}$ , in which  $\mathcal{Q}_k$  is  $(k-1)$ -connected and  $(n+k)$ -coconnected (so that  $\pi_i(\mathcal{Q}_k) = 0$  for  $i < k$  or  $i > n+k$ ) and  $\pi_i(q_k)$  is an isomorphism for  $k \leq i < n+k$ . Here  $(\mathbb{N}, \leq)$  is the usual linearly ordered category of the natural numbers. The space  $\mathcal{Q}_k$  is called the  $k$ -th  $n$ -window of  $\mathcal{Q}$ .

Such an  $n$ -stem is thus a collection of overlapping  $(k-1)$ -connected  $n+k$ -types, which may be depicted for  $n = 2$  as follows:



where each row exhibits the  $n + 1$  nontrivial homotopy groups (denoted by  $*$ ) of one  $n$ -window, and all those in the  $i$ -th column (corresponding to  $\pi_i$ ) are isomorphic.

We denote by  $\mathcal{P}stem[n]$  the full subcategory of Postnikov  $n$ -stems in the functor category  $\mathcal{T}_*^{(\mathbb{N}, \leq)}$  (with model category structure on the latter as in Hirschhorn [20, 11.6]). Thus the morphisms in  $\mathcal{P}stem[n]$  are given by strictly commuting maps of towers, and  $f: \mathcal{Q} \rightarrow \mathcal{Q}'$  is a weak equivalence (respectively, a fibration) if each  $f_k: \mathcal{Q}_k \rightarrow \mathcal{Q}'_k$  is such. This lets us define the homotopy category of Postnikov  $n$ -stems,  $ho \mathcal{P}stem[n]$ , as a full subcategory of  $ho \mathcal{T}_*^{(\mathbb{N}, \leq)}$ .

The category  $\mathcal{P}stem[n]$  is pointed, has products, and is equipped with canonical functors

$$(2.4) \quad \begin{array}{ccccccc} \mathcal{T}_* & & & & & & \\ & \searrow & & \searrow & & & \searrow \\ & \mathcal{P}[n] & & \mathcal{P}[n-1] & & & \mathcal{P}[0] \\ & \searrow & & \searrow & & & \searrow \\ \dots & \mathcal{P}stem[n] & \xrightarrow{\overline{\mathcal{P}}[n-1]} & \mathcal{P}stem[n-1] & \xrightarrow{\overline{\mathcal{P}}[n-2]} & \dots & \xrightarrow{\overline{\mathcal{P}}[0]} & \mathcal{P}stem[0] \end{array}$$

which preserve products and weak equivalences.

**2.5 Remark** The sequence of functors (2.6) is described by a commuting diagram, in which we may take all maps to be fibrations:

$$(2.6) \quad \begin{array}{ccccccc} & & & & & & \\ & & & & & & \\ & & & & & & \\ \dots & \longrightarrow & \mathcal{Q}_{k+1}^{n+k+1} & \xrightarrow{q_{k+1}^n} & \mathcal{Q}_k^{n+k} & \xrightarrow{q_k^n} & \mathcal{Q}_{k-1}^{n+k-1} & \longrightarrow & \dots & \longrightarrow & \mathcal{Q}_0^n & \\ & & \downarrow p_{k+1}^n & \nearrow r_{k+1}^{n-1} & \downarrow p_k^n & \nearrow r_k^{n-1} & \downarrow p_{k-1}^n & & & & \downarrow p_0^n & \\ \dots & \longrightarrow & \mathcal{Q}_{k+1}^{n+k} & \xrightarrow{q_{k+1}^{n-1}} & \mathcal{Q}_k^{n+k-1} & \xrightarrow{q_k^{n-1}} & \mathcal{Q}_{k-1}^{n+k-2} & \longrightarrow & \dots & \longrightarrow & \mathcal{Q}_0^{n-1} & \\ & & \downarrow p_{k+1}^{n-1} & \nearrow r_{k+1}^{n-2} & \downarrow p_k^{n-1} & \nearrow r_k^{n-2} & \downarrow p_{k-1}^{n-1} & & & & \downarrow p_0^{n-1} & \\ \dots & \longrightarrow & \mathcal{Q}_{k+1}^{n+k-1} & \xrightarrow{q_{k+1}^{n-2}} & \mathcal{Q}_k^{n+k-2} & \xrightarrow{q_k^{n-2}} & \mathcal{Q}_{k-1}^{n+k-3} & \longrightarrow & \dots & \longrightarrow & \mathcal{Q}_0^{n-2} & \\ & & \vdots & & \vdots & & \vdots & & & & \vdots & \\ & & \dots & \xrightarrow{q_{k+1}^0} & \mathcal{Q}_k^k & \xrightarrow{q_k^0} & \mathcal{Q}_{k-1}^{k-1} & \longrightarrow & \dots & \longrightarrow & \mathcal{Q}_0^0 & \end{array}$$

Here  $\pi_i Q_k^n = 0$  for  $i < k$  or  $i > n$ , and all maps induce isomorphisms in  $\pi_i$  whenever possible. Thus:

- (a) The  $k$ -th column (from the right) is the Postnikov tower for  $Q_k := \lim_n Q_k^n$ .
- (b) The diagonals are the dual Postnikov system of connected covers for  $Q_0^j$ .
- (c) The  $n$ -th row (from the bottom) is a Postnikov  $n$ -stem.
- (d) In particular, each space in the 0-stem (the bottom row) is an Eilenberg–Mac Lane space, and the maps  $q_k^0$  are nullhomotopic. Thus the homotopy type of the bottom line in  $\text{ho } \mathcal{P}stem[0]$  is determined by the collection of homotopy groups  $\{\pi_k Q_k^k\}_{k=0}^\infty$ .

**2.7 Definition** The motivating example of a Postnikov  $n$ -stem is a *realizable* one, associated to a space  $X \in \mathcal{T}_*$ , and denoted by  $\mathcal{P}[n]X$ , with  $(\mathcal{P}[n]X)_k := P^{n+k} X \langle k \rangle$ . As usual,  $Y \langle k \rangle$  denotes the  $(k-1)$ -connected cover of a space  $Y \in \mathcal{T}_*$ . Each fibration  $q_k: (\mathcal{P}[n]X)_k \rightarrow (\mathcal{P}[n]X)_{k-1}$  fits into a commuting triangle of fibrations

$$(2.8) \quad \begin{array}{ccc} P^{n+k+1} X \langle k+1 \rangle & \xrightarrow{q} & P^{n+k} X \langle k \rangle \\ & \searrow p & \nearrow r \\ & P^{n+k} X \langle k+1 \rangle & \end{array}$$

in which the maps  $p$  and  $r$  are the fibration of (2.1) and the covering map, respectively. See our paper [4, Section 10.5] for a natural context in which nonrealizable Postnikov  $n$ -stems arise.

**2.9 Examples of stems** The functor  $\mathcal{P}[0]_*: \mathcal{T}_* \rightarrow \text{ho } \mathcal{P}stem[0]$  induced by  $\mathcal{P}[0]$  is equivalent to the homotopy group functor: in fact, the homotopy groups of a space define a functor  $\pi_*: \mathcal{T}_* \rightarrow \mathcal{K}$  into the product category  $\mathcal{K} := \prod_{i=0}^\infty \mathcal{K}_i$ , where  $\mathcal{K}_0 = \mathcal{S}et_*$ ,  $\mathcal{K}_1 = \mathcal{G}p$ , and  $\mathcal{K}_i = \mathcal{A}b\mathcal{G}p$ , for  $i \geq 2$ . Moreover, there is an equivalence of categories  $\vartheta: \mathcal{K} \cong \text{ho } \mathcal{P}stem[0]$ , such that the functor  $\mathcal{P}[0]_*$  is equivalent to the composite functor  $\vartheta \circ \pi_*: \mathcal{T}_* \rightarrow \mathcal{K}$ .

Similarly, the functor  $\mathcal{T}_* \rightarrow \text{ho } \mathcal{P}stem[1]$  induced by  $\mathcal{P}[1]$  is equivalent to the secondary homotopy group functor of the first author and Muro [7, Section 4], in the sense that each secondary homotopy group  $\pi_{n,*} X$  completely determines the  $n$ -th 1-window of  $X$ . Using the results on secondary homotopy groups in [7], one obtains a homotopy category of algebraic 1-stems which is equivalent to  $\text{ho } \mathcal{P}stem[1]$ .

A category of algebraic models for 2-stems is only partially known. The homotopy classification of  $(k-1)$ -connected  $(k+2)$ -types is described for all  $k$  in [3]; this theory can be used to classify homotopy types of Postnikov 2-stems.

### 3 The spectral sequence of a simplicial space

We begin with the construction of the homotopy spectral sequence for a simplicial space (cf Quillen [21], Bousfield and Friedlander [14, Theorem B.5], and Bousfield and Kan [15, X, Section 6]), using the version given by Dwyer, Kan and Stover in [19, Section 8] (see also Bousfield [11; 12, Section 2.5] and Dwyer, Kan and Stover [18, Section 3.6]). For this purpose, we require some explicit constructions for the  $E^2$ -model category of simplicial spaces.

**3.1 Definition** Given a simplicial object  $X_\bullet \in s\mathcal{C}$ , over a complete pointed category  $\mathcal{C}$ , for each  $n \geq 1$  define its  $n$ -cycles object to be

$$Z_n X_\bullet := \{x \in X_n \mid d_i x = * \text{ for } i = 0, \dots, n\} .$$

Similarly, the  $n$ -chains object for  $X_\bullet$  is

$$C_n X_\bullet := \{x \in X_n \mid d_i x = * \text{ for } i = 1, \dots, n\} .$$

Set  $Z_0 X_\bullet := X_0$ . We denote the map  $d_0|_{C_n X_\bullet}: C_n X_\bullet \rightarrow Z_{n-1} X_\bullet$  by  $\mathbf{d}_0^{X_n}$ .

**3.2 Notation** For any nonnegatively graded object  $T_*$ , we write  $\Omega T_*$  for the graded object with  $(\Omega T_*)_j := T_{j+1}$  for all  $j \geq 0$ . The notation is motivated by the natural isomorphism of graded groups  $\pi_* \Omega X \cong \Omega(\pi_* X)$  for  $X \in \mathcal{T}_*$ .

**3.3 Definition** Now assume that  $\mathcal{C}$  is a pointed model category of spaces, such as  $\mathcal{T}_*$  or  $\mathcal{G}$ , and  $X_\bullet$  is a Reedy fibrant simplicial object over  $\mathcal{C}$  – that is, for each  $n \geq 1$ , the universal face map  $\delta_n: X_n \rightarrow M_n X_\bullet$  into the  $n$ -th matching object of  $X_\bullet$  is a fibration (see [20, 15.3]). The map  $\mathbf{d}_0 = \mathbf{d}_0^{X_n}$  then fits into a fibration sequence in  $\mathcal{C}$ :

$$(3.4) \quad \dots \Omega Z_n X_\bullet \rightarrow Z_{n+1} X_\bullet \xrightarrow{j_{n+1}^{X_\bullet}} C_{n+1} X_\bullet \xrightarrow{\mathbf{d}_0^{X_{n+1}}} Z_n X_\bullet$$

(see [19, Proposition 5.7]).

For each  $n \geq 0$ , the  $n$ -th natural homotopy group of the simplicial space  $X_\bullet$ , denoted by  $\pi_n^{\natural} X_\bullet = \pi_{n,*}^{\natural} X_\bullet$ , the cokernel of the map  $(\mathbf{d}_0^{X_{n+1}})_{\#}$  (induced on homotopy groups by  $\mathbf{d}_0^{X_{n+1}}$ ). Note that the cokernel of a maps of groups or pointed sets is generally just a pointed set.

We thus have an exact sequence of graded groups:

$$(3.5) \quad \pi_* C_{n+1} X_\bullet \xrightarrow{(\mathbf{d}_0^{X_{n+1}})_{\#}} \pi_* Z_n X_\bullet \xrightarrow{\hat{\vartheta}_n} \pi_{n,*}^{\natural} X_\bullet \rightarrow 0 .$$

Together the groups  $(\pi_{n,k}^{\natural} X_\bullet)_{n,k=0}^{\infty}$  constitute the bigraded homotopy groups of [19, Section 5.1].

**3.6 Construction of the spiral sequence** Applying the functor  $\pi_*$  to the fibration sequence (3.4) yields a long exact sequence, with connecting homomorphism  $\partial_\# : \Omega \pi_* Z_n X_\bullet = \pi_* \Omega Z_n X_\bullet \rightarrow \pi_* Z_{n+1} X_\bullet$ . Note that the inclusion  $\iota : C_n X_\bullet \hookrightarrow X_n$  induces an isomorphism  $\iota_\star : \pi_* C_n X_\bullet \cong C_n(\pi_* X_\bullet)$  for each  $n \geq 0$  (see [10, Proposition 2.7]). From (3.5) we see that

$$\Omega \pi_n^{\natural} X_\bullet = \Omega \operatorname{Coker}(\mathbf{d}_0^{X_{n+1}})_\# \cong \operatorname{Im} \partial_\# \cong \operatorname{Ker}(j_{n+1}^{X_\bullet})_\# \subseteq \pi_* Z_{n+1} X_\bullet,$$

so we obtain a commutative diagram with exact rows and columns

$$(3.7) \quad \begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \operatorname{Ker}(j_n)_* & \hookrightarrow & B_{n+1} X_\bullet & \xrightarrow{(j_n)_*} & B_{n+1} \pi_* X_\bullet \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \Omega \pi_{n-1}^{\natural} X_\bullet & \xrightarrow{\ell_{n-1}} & \pi_* Z_n X_\bullet & \xrightarrow{(j_n^{X_\bullet})_\#} & Z_n \pi_* X_\bullet \longrightarrow \operatorname{Coker} h_n \longrightarrow 0 \\ & & \downarrow & \searrow s_n & \downarrow \hat{\vartheta}_n & & \downarrow \vartheta_n \\ 0 & \longrightarrow & \operatorname{Ker} h_n & \hookrightarrow & \pi_n^{\natural} X_\bullet & \xrightarrow{h_n} & \pi_n \pi_* X_\bullet \longrightarrow \operatorname{Coker} h_n \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

in which  $B_{n+1} X_\bullet := \operatorname{Im}(\mathbf{d}_0^{X_{n+2}})_\# \subseteq \pi_* Z_n X_\bullet$  and  $B_{n+1} \pi_* X_{n+2} := \operatorname{Im} \mathbf{d}_0^{\pi_* X_{n+2}}$  are the respective boundary objects. Note that the map

$$(j_n^{X_\bullet})_\# : \pi_* Z_n X_\bullet \rightarrow \pi_* C_n X_\bullet$$

induced by the inclusion  $j_n^{X_\bullet}$  of (3.4) above in fact factors through  $Z_n \pi_* X_\bullet$ , as indicated in the middle row of (3.7).

This defines the map of graded groups  $h_n : \pi_n^{\natural} X_\bullet \rightarrow \pi_n(\pi_* X_\bullet)$ . Note that for  $n = 0$  the map  $\hat{\iota}_\star$  is an isomorphism, so  $h_0$  is, too. The map

$$s_n : \Omega \pi_{n-1}^{\natural} X_\bullet \rightarrow \pi_n^{\natural} X_\bullet$$

is the composite of the inclusion

$$\ell_{n-1} : \operatorname{Ker}(j_n^{X_\bullet})_\# \hookrightarrow \pi_* Z_n X_\bullet$$

with the quotient map

$$\hat{\vartheta}_n : \pi_* Z_n X_\bullet \rightarrow \pi_n^{\natural} X_\bullet$$

of (3.5), using the natural identification of  $\Omega \pi_n^{\natural} X_\bullet$  with  $\operatorname{Ker}(j_{n+1}^{X_\bullet})_\#$ .

The map  $\partial_{n+2}: \pi_{n+2}\pi_*X_\bullet \rightarrow \Omega\pi_n^{\natural}X_\bullet$  is induced by the composite

$$(3.8) \quad Z_{n+2}\pi_*X_\bullet \subseteq C_{n+2}\pi_*X_\bullet \cong \pi_*C_{n+2}X_\bullet \xrightarrow{(d_0^{X_{n+2}})_\#} \pi_*Z_{n+1}X_\bullet,$$

which actually lands in  $\text{Ker}(j_{n+1}^{X_\bullet})_\#$  by the exactness of the long exact sequence for the fibration (3.4).

These maps  $s_n, h_n,$  and  $\partial_n$  fit into a *spiral long exact sequence*:

$$(3.9) \quad \begin{aligned} \cdots \rightarrow \Omega\pi_{n-1}^{\natural}X_\bullet &\xrightarrow{s_n} \pi_n^{\natural}X_\bullet \xrightarrow{h_n} \pi_n\pi_*X_\bullet \xrightarrow{\partial_n} \Omega\pi_{n-2}^{\natural}X_\bullet \\ &\xrightarrow{s_{n-1}} \pi_{n-1}^{\natural}X_\bullet \rightarrow \cdots \rightarrow \pi_0^{\natural}X_\bullet \xrightarrow{\cong} \pi_0\pi_*X_\bullet \end{aligned}$$

(cf [19, 8.1]).

**3.10 The spectral sequence of a simplicial space** For any simplicial space  $X_\bullet \in s\mathcal{T}_*$  (or bisimplicial set), Bousfield and Friedlander showed that there is a first-quadrant spectral sequence of the form

$$(3.11) \quad E_{s,t}^2 = \pi_s\pi_t X_\bullet \Rightarrow \pi_{s+t} \|X_\bullet\|,$$

where  $\|X_\bullet\| \in \mathcal{T}_*$  is the realization (or the diagonal, in the case of  $X_\bullet \in s\mathcal{S}_*$ ). The spectral sequence is always defined, but  $X_\bullet$  must satisfy certain ‘‘Kan conditions’’ to guarantee *convergence* – see [14, Theorem B.5].

In [19, Section 8.4], Dwyer, Kan and Stover showed that (3.11) coincides up to sign, from the  $E^2$ -term on, with the spectral sequence associated to the exact couple of (3.4), which we call the *spiral spectral sequence* for  $X_\bullet$ .

If we assume that each  $X_n$  is connected, by taking loops (or applying Kan’s functor  $G$ , if  $X_\bullet \in s\mathcal{S}_*$ ), we may replace  $X_\bullet$  by a bisimplicial group  $GX_\bullet \in s\mathcal{G}$ , and then (3.11) becomes the spectral sequence of [21].

## 4 Simplicial stems and truncated spectral sequences

As noted in Section 2.9, the  $E^2$ -term of any of the above equivalent spectral sequences for a simplicial space  $X_\bullet$  is determined explicitly by the simplicial 0-stem of  $X_\bullet$ .

Our goal is to extend this description to the higher terms of the spectral sequence. For this purpose, fix  $n \geq 0$ , and consider a simplicial Postnikov  $n$ -stem  $\mathcal{Q}_\bullet$  (which need not be realizable as  $\mathcal{P}[n]X_\bullet$  for some simplicial space  $X_\bullet$ ). This is equivalent to having a collection of simplicial spaces  $\mathcal{Q}_\bullet^{n+k}(k)$  for each  $k \geq 0$ , equipped with maps as in (2.3), with  $\pi_i \mathcal{Q}_\bullet^{n+k}(k) = 0$  for  $i < k$  or  $i > n + k$ .

We assume that  $\mathcal{Q}_\bullet$  is *Reedy fibrant* in the sense that for each  $k \geq 0$ , the simplicial space  $\mathcal{Q}_\bullet^{n+k}(k)$  is Reedy fibrant. In this case, the “ $n$ -stem version” of the spiral long exact sequence is defined as follows: for each  $t, i, k \geq 0$ , set

$$(4.1) \quad \begin{aligned} \pi_{t,i}^{\natural(k,n)} \mathcal{Q}_\bullet &:= \pi_{t,i+k}^{\natural} \mathcal{Q}_\bullet^{n+k}(k), \\ \pi_i^{(k,n)} \mathcal{Q}_\bullet &:= \pi_{i+k} \mathcal{Q}_\bullet^{n+k}(k) = \begin{cases} \pi_{i+k} \mathcal{Q}_\bullet & \text{if } 0 \leq i \leq n, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Note that the  $(i+k)$ -th homotopy group  $\pi_{i+k} \mathcal{Q}_\bullet$  of a Postnikov  $n$ -stem  $\mathcal{Q}_\bullet$  is well-defined, and coincides with  $\pi_{i+k} X_\bullet$  for  $0 \leq i \leq n$  when  $\mathcal{Q}_\bullet = \mathcal{P}[n]X_\bullet$ .

**4.2 Definition** The collection of long exact sequences (3.9) for  $\mathcal{Q}_\bullet^{n+k}(k)$  (indexed by  $k \geq 0$ ):

$$(4.3) \quad \cdots \Omega \pi_{t-1,*}^{\natural(k,n)} \mathcal{Q}_\bullet \xrightarrow{s_t^{(k,n)}} \pi_{t,*}^{\natural(k,n)} \mathcal{Q}_\bullet \xrightarrow{h_t^{(k,n)}} \pi_t \pi_*^{(k,n)} \mathcal{Q}_\bullet \xrightarrow{\partial_t^{(k,n)}} \Omega \pi_{t-2,*}^{\natural(k,n)} \mathcal{Q}_\bullet \cdots,$$

together with the maps between adjacent  $k$ -windows induced by the map  $q$  in (2.6), will be called the *spiral  $n$ -system* of  $\mathcal{Q}_\bullet$ . When  $\mathcal{Q}_\bullet = \mathcal{P}[n]X_\bullet$ , we will refer to this simply as the spiral  $n$ -system of  $X_\bullet$ .

**4.4 Remark** Using the exactness of (4.3), definition (4.1) implies that

$$(4.5) \quad \pi_{t,i}^{\natural(k,n)} \mathcal{Q}_\bullet = \pi_{t,i}^{\natural} \mathcal{Q}_\bullet^{n+k}(k) = 0 \text{ for } i > n,$$

by induction on  $t \geq 0$ . Note, however, that while the groups  $\pi_i^{(k,n)} \mathcal{Q}_\bullet$  are explicitly described by (4.1), the dependence of  $\pi_{t,i}^{\natural(k,n)} \mathcal{Q}_\bullet$  on  $k$  and  $n$  requires more care.

**4.6 The  $E^2$ -term of the spectral sequence** The spiral 0-system of a simplicial Postnikov 0-stem  $\mathcal{Q}_\bullet$  reduces to a series of isomorphisms

$$h_t: \pi_{t,*}^{\natural(k,0)} \mathcal{Q}_\bullet \cong \pi_t \pi_*^{(k,0)} \mathcal{Q}_\bullet$$

(for each  $k \geq 0$ ). When  $\mathcal{Q}_\bullet = \mathcal{P}[0]X_\bullet$  is the Postnikov 0-stem of a simplicial space  $X_\bullet$ , this allows us to identify the  $E_{t,k}^2$ -term of the spiral spectral sequence for  $X_\bullet$ , which is

$$\pi_t \pi_k X_\bullet = \pi_t \pi_k P^{0+k} X_\bullet(k) = \pi_t \pi_k (P[0]X_\bullet)_k = \pi_t \pi_*^{(k,0)} \mathcal{P}[0]X_\bullet = \pi_t \pi_*^{(k,0)} \mathcal{Q}_\bullet,$$

with  $\pi_{t,*}^{\natural(k,0)} \mathcal{Q}_\bullet = \pi_{t,*}^{\natural(k,0)} \mathcal{P}[0]X_\bullet$ .

The first interesting case is the spiral 1-system, for which we have:

**4.7 Proposition** The  $E^3$ -term of the spiral spectral sequence for a simplicial space  $X_\bullet$  is determined by the spiral 1-system of  $X_\bullet$ . In fact,  $d_{t,k}^2$  may be identified with

$$\partial_t^{(k,1)}: \pi_t \pi_k X_\bullet \rightarrow \Omega \pi_{t-2,0}^{\natural(k,1)} X_\bullet,$$

while  $E_{t,k}^3$  is the image of the composite map

$$(4.8) \quad \pi_{t,0}^{\natural(k,1)} X_\bullet \xrightarrow{h_t^{(k,1)}} \pi_t \pi_k X_\bullet \cong \pi_t \pi_1^{(k-1,1)} X_\bullet \\ \xleftarrow[\cong]{h_t^{(k-1,1)}} \pi_{t,1}^{\natural(k-1,1)} X_\bullet \xrightarrow{s_{t+1}^{(k-1,1)}} \pi_{t+1,0}^{\natural(k-1,1)} X_\bullet.$$

Observe that (4.8) involves maps from different windows of the spiral 1-system, implicitly identified using the isomorphisms induced by the map  $q$  in (2.6).

**Proof** Because  $n=1$  throughout, we abbreviate  $\pi_{t,i}^{\natural(k,1)} \mathcal{Q}_\bullet$  to  $\pi_{t,i}^{\natural(k)} \mathcal{Q}_\bullet$  and  $\pi_i^{(k,1)} \mathcal{Q}_\bullet$  to  $\pi_i^{(k)} \mathcal{Q}_\bullet$  observing that  $\pi_i^{(k)} \mathcal{Q}_\bullet$  is simply  $\pi_{i+k} X_\bullet$  for  $i = 0, 1$ , and zero otherwise, since  $\mathcal{Q}_\bullet = \mathcal{P}[1]X_\bullet$ . Thus the spiral 1-system (4.3) is nontrivial for each  $t \geq 1$  in (internal) degrees  $i = 0, 1$  only, and we can write it in two rows:

$$0 \rightarrow \pi_{t,1}^{\natural(k)} \mathcal{Q}_\bullet \xrightarrow{\cong} \pi_t \pi_1^{(k)} \mathcal{Q}_\bullet \rightarrow 0 \rightarrow \pi_{t-1,1}^{\natural(k)} \mathcal{Q}_\bullet \xrightarrow{\cong} \pi_{t-1} \pi_1^{(k)} \mathcal{Q}_\bullet \\ \Omega \pi_{t-1,0}^{\natural(k)} \mathcal{Q}_\bullet \xrightarrow{s_t} \pi_{t,0}^{\natural(k)} \mathcal{Q}_\bullet \xrightarrow{h_t} \pi_t \pi_0^{(k)} \mathcal{Q}_\bullet \xrightarrow{\partial_t} \Omega \pi_{t-2,0}^{\natural(k)} \mathcal{Q}_\bullet \\ \xrightarrow{s_{t-1}} \pi_{t-1,0}^{\natural(k)} \mathcal{Q}_\bullet \xrightarrow{h_{t-1}} \pi_{t-1} \pi_0^{(k)} \mathcal{Q}_\bullet.$$

Since  $\mathcal{Q}_\bullet := \mathcal{P}[1]X_\bullet$  is the simplicial Postnikov 1-stem of  $X_\bullet$ , we actually have a collection of two-row long exact sequences, one for each  $k$ -window of  $\mathcal{P}[1]X_\bullet$ .

For each such  $k$ -window  $\mathcal{P}_k[1]X_\bullet$ , we can use the top row to identify

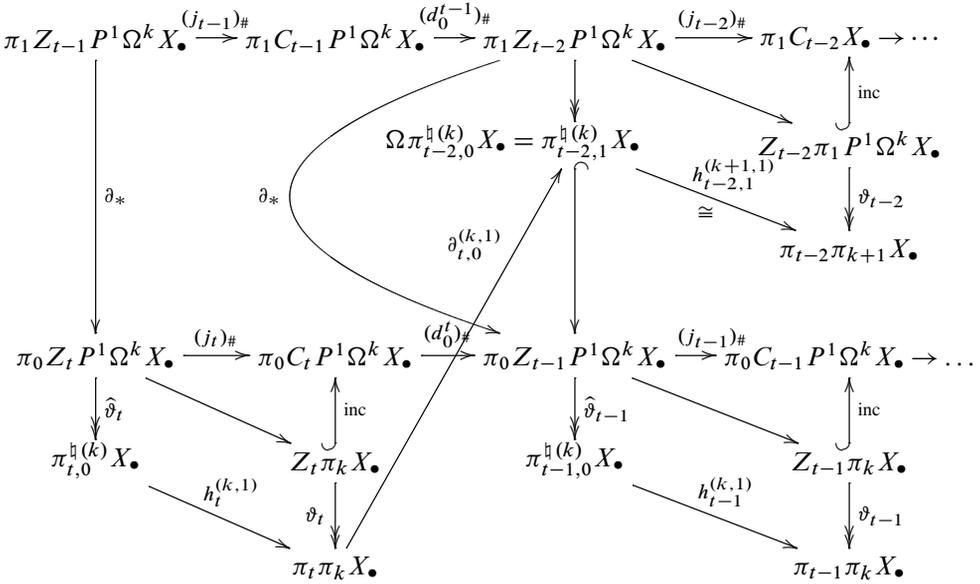
$$\Omega \pi_{t,0}^{\natural(k)} \mathcal{Q}_\bullet = \Omega \pi_{t,0}^{\natural(k)} \mathcal{P}_k[1]X_\bullet = \pi_{t,1}^{\natural(k)} \mathcal{P}_k[1]X_\bullet = \pi_{t,1}^{\natural(k)} \mathcal{Q}_\bullet$$

with 
$$\pi_t \pi_1^{(k)} \mathcal{Q}_\bullet = \pi_t \pi_1^{(1)} \mathcal{P}_k[1]X_\bullet = \pi_t \pi_{k+1} X_\bullet,$$

so the bottom row reduces to:

$$\begin{array}{ccccccc} \pi_{t-1} \pi_{k+1} X_\bullet & \xrightarrow{s_t^{(k,1)}} & \pi_{t,0}^{\natural(k)} \mathcal{Q}_\bullet & \xrightarrow{h_t^{(k,1)}} & \pi_t \pi_k X_\bullet & \xrightarrow{\partial_t^{(k,1)}} & \pi_{t-2} \pi_{k+1} X_\bullet \\ & \searrow & \nearrow & \searrow & \nearrow & \searrow & \nearrow \\ & & \text{Im}(s_t^{(k,1)}) & & \text{Im}(h_t^{(k,1)}) & & \text{Im}(\partial_t^{(k,1)}) \\ & & & & = \text{Ker}(\partial_t^{(k,1)}) & & \end{array}$$

Note that the following part of the  $E^1$ -term of the exact couple for the fibration sequence  $C_{n+1}P^1\Omega^i X_\bullet \rightarrow Z_n P^1\Omega^i X_\bullet$ , (as in (3.4)):



is naturally isomorphic to the exact couple for  $C_{n+1}\Omega^k X_\bullet \rightarrow Z_n\Omega^k X_\bullet$ , since  $C_{n+1}$  and  $Z_n$  are limits, so they commute with  $P^1$ , and then

$$\pi_1 P^1 Z_{t-1} \Omega^k X_\bullet \cong \pi_1 Z_{t-1} \Omega^k X_\bullet,$$

and so on. This does not imply, of course, that  $\pi_{t,1}^{\natural(k)} X_\bullet \cong \pi_{t,k+1}^{\natural(k)} X_\bullet$ .

We therefore see from (3.7) and (3.8) that the differential  $d_{t,k}^2: E_{t,k}^2 \rightarrow E_{t-2,k+1}^2$  may be identified with:

$$(4.9) \quad \pi_t \pi_k X_\bullet \cong \pi_t \pi_0^{(k,1)} X_\bullet \xrightarrow{\vartheta_{t,0}^{(k,1)}} \Omega \pi_{t-2,0}^{\natural(k)} X_\bullet = \pi_{t-2,1}^{\natural(k)} X_\bullet \cong \pi_{t-2} \pi_1^{(k,1)} X_\bullet \cong \pi_{t-2} \pi_{k+1} X_\bullet.$$

Now by definition,  $E_{t,k}^3$  fits into a commutative diagram

$$(4.10) \quad \begin{array}{ccccc} E_{t+2,k-1}^2 & \xrightarrow{d_{t+2,k-1}^2} & E_{t,k}^2 & \xrightarrow{q} & \text{Coker}(d_{t+2,k-1}^2) \\ r \downarrow & & \uparrow j & & \uparrow \kappa \\ \text{Im}(d_{t+2,k-1}^2) & \xrightarrow{\ell} & \text{Ker}(d_{t,k}^2) & \xrightarrow{s} & E_{t,k}^3 \end{array}$$

with exact rows,  $\ell$ ,  $j$  and  $\kappa$  monic, and thus  $E_{t,k}^3 \cong \text{Im}(q \circ j)$ .

From the exactness of (4.3) we see that

$$\text{Coker}(d_{t+2,k-1}^2) = \text{Coker}(\partial_{t+2}^{(k-1,1)}) = \text{Im}(s_{t+1}^{(k-1,1)})$$

and 
$$\text{Ker}(d_{t,k}^2) = \text{Ker}(\partial_t^{(k,1)}) = \text{Im}(h_t^{(k,1)}) ,$$

so  $E_{t,k}^3 = \text{Im}(q \circ j)$  is indeed the image of the map in (4.8). □

**4.11 Definition** An  $r$ -truncated spectral sequence is one defined up to and including the  $E^r$ -term, together with the differential  $d^n: E_{t,i}^r \rightarrow E_{t-r-1,t+r}^r$ , but without requiring that  $d^r \circ d^r = 0$  (so the  $E^{r+1}$ -term is defined in terms of the  $r$ -truncated spectral sequence only if  $d^r d^r = 0$ ).

The main example is the  $n$ -truncation of an (ordinary) spectral sequence (such as that of a simplicial space). In this case we do have  $d^r \circ d^r = 0$ , of course.

**4.12 Corollary** Any Reedy fibrant simplicial Postnikov 1-stem has a well-defined 2-truncated spiral spectral sequence. Moreover, if  $\mathcal{Q}_\bullet = \mathcal{P}[1]X_\bullet$  for some simplicial space  $X_\bullet$ , this 2-truncated spectral sequence coincides with the 2-truncation of the Bousfield–Friedlander spectral sequence for  $X_\bullet$ .

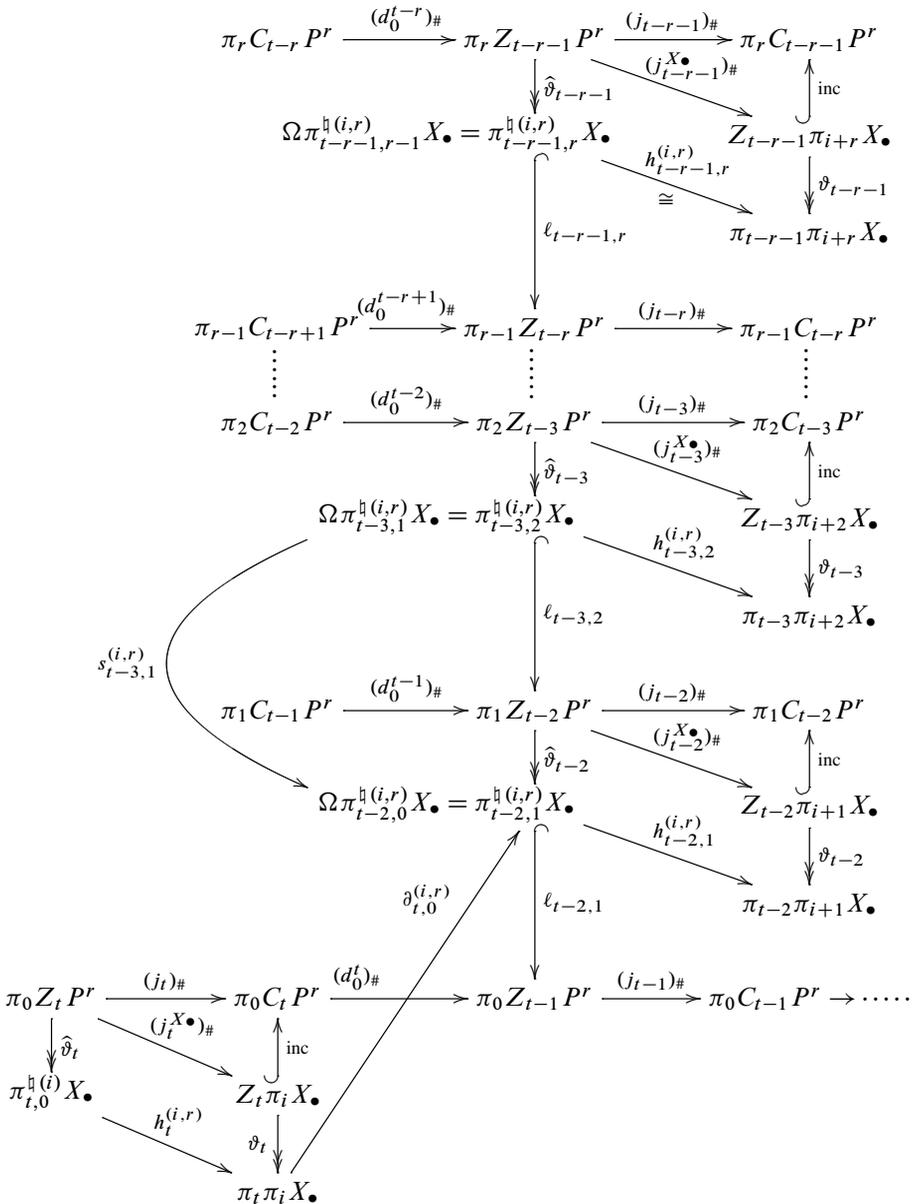
In general, we have a less explicit description of the higher terms in the spiral spectral sequence:

**4.13 Theorem** For each  $r \geq 0$ , the  $E^{r+2}$ -term of the spiral spectral sequence for a simplicial space  $X_\bullet$  is determined by the spiral  $r$ -system of  $X_\bullet$ . Moreover, for any  $\alpha \in E_{t,i}^{r+1}$ , we have  $d_{t,i}^{r+1}(\alpha) = \beta \in E_{t-r-1,i+r}^{r+1}$  if and only if  $\alpha$  and  $\beta$  have representatives  $\bar{a} \in \pi_t \pi_i X_\bullet$  and  $\bar{b} \in \pi_{t-r-1} \pi_{i+r} X_\bullet$ , respectively, such that:

$$(4.14) \quad (s_{t-2,1}^{(i,r)}) \circ (s_{t-3,2}^{(i,r)}) \circ \dots \circ (s_{t-r,r-1}^{(i,r)}) \circ (h_{t-r-1,r}^{(i,r)})^{-1}(\bar{b}) = \partial_{t,0}^{(i,r)}(\bar{a})$$

**Proof** We naturally identify  $\pi_{t,k}^{\natural(i,r)} X_\bullet$  with  $\pi_{t,k+s}^{\natural(i,r-s)} X_\bullet$  for  $k \geq s$ , and similarly for the maps in (4.3), so the spiral  $(r-1)$ -system embeds in the spiral  $r$ -system (with an index shift).

Again we write out the  $E^1$ -term of the spiral exact couple, letting  $P^r = P^r \Omega^i X_\bullet$ :



The differential  $d_{t,i}^{r+1}: E_{t,i}^{r+1} \rightarrow E_{t-r-1, i+r}^r$  may then be described as a “relation” (cf [17, Section 3.1]) in the usual way:

Given a class  $\alpha \in E_{t,i}^{r+1}$ , choose a representative for it  $a \in E_{t,i}^{r+1} = \pi_0 C_t P^r \Omega^i X_\bullet$ . Since it is a cycle for  $d_{t,i}^1 = (j_{t-1})\# \circ (d_0^t)\#$ , it lies in  $Z_t \pi_i X_\bullet$ , and thus represents an

element  $\bar{a} \in \pi_t \pi_i X_\bullet = E_{t,i}^2$ . From the exactness of the middle row of (3.7) we see that

$$(d_0^t)_\#(a) \in \text{Ker}((j_{t-1})_\#) = \Omega \pi_{t-2,0}^{\natural(i,r)} X_\bullet,$$

and in fact  $(d_0^t)_\#(a)$  represents  $\partial_{t,0}^{(i,r)}(\bar{a})$ . Since  $\widehat{\vartheta}_{t-2}$  is surjective, we can choose  $e_{t-2} \in \pi_1 Z_{t-2} P^r \Omega^i X_\bullet$  mapping to  $(d_0^t)_\#(a)$ . Because

$$d_{t,i}^2(\bar{a}) = h_{t-2,1}^{(i,r)} \circ \partial_{t,0}^{(i,r)}(\bar{a}),$$

as in the proof of Proposition 4.7 (though  $h_{t-2,1}^{(i,r)}$  need no longer be an isomorphism!), we see that it is represented by  $(j_{t-2})_*(e_{t-2})$ . If  $r = 1$ , we are done. Otherwise, we know that  $d_{t,i}^2(\bar{a}) = 0$ , so we can choose  $e_{t-2}$  so that  $(j_{t-2})_*(e_{t-2}) = 0$ , using exactness of the third column of (3.7). Again this implies that

$$e_{t-2} \in \text{Ker}((j_{t-2})_\#) = \Omega \pi_{t-3,1}^{\natural(i,r)} X_\bullet,$$

and  $d_{t,i}^3(\langle a \rangle)$  is represented by  $h_{t-3,2}^{(i,r)}(e_{t-2})$ . Moreover, we see from (3.7) that

$$s_{t-3,1}^{(i,r)}(e_{t-2}) = \partial_{t,0}^{(i,r)}(\bar{a}),$$

using the identification  $\Omega \pi_{t-2,0}^{\natural(i,r)} X_\bullet = \pi_{t-2,1}^{\natural(i,r)} X_\bullet$ .

Choosing a lift to  $e_{t-3} \in \pi_2 Z_{t-3} P^r \Omega^i X_\bullet$ , we may assume that  $(j_{t-3})_*(e_{t-3}) = 0$ , so

$$e_{t-3} \in \Omega \pi_{t-4,2}^{\natural(i,r)} X_\bullet \quad \text{and} \quad s_{t-4,2}^{(i,r)}(e_{t-3}) = e_{t-2}.$$

Continuing in this way, we finally reach

$$e_{t-r-1} \in \Omega \pi_{t-r-1,r-1}^{\natural(i,r)} X_\bullet \quad \text{with} \quad s_{t-r-2,r}^{(i,r)}(e_{t-r-1}) = e_{t-r},$$

and so on, and see that  $d_{t,i}^{r+1}(\langle a \rangle)$  is represented by  $h_{t-r-1,r}^{(i,r)}(e_{t-r-1})$ . Since (as in the proof of Proposition 4.7)  $h_{t-r-1,r}^{(i,r)}$  is an isomorphism, we deduce that  $d_{t,i}^{r+1}(\alpha)$  is as in (4.14). □

**4.15 Remark** From the exactness of (4.3) we have

$$\text{Im}(\partial_{t,0}^{(i,r)}) = \text{Ker}(s_{t-1,0}^{(i,r)}),$$

so the image of  $d_{t,i}^{r+1}$  as described in (4.14) is  $\text{Ker}(\sigma_{t,i}^{r+1})$ , where

$$\sigma_{t,i}^{r+1} := (s_{t-1,0}^{(i,r)}) \circ (s_{t-2,1}^{(i,r)}) \circ (s_{t-3,2}^{(i,r)}) \circ \cdots \circ (s_{t-r,r-1}^{(i,r)}).$$

Therefore,  $E_{t+r-1,i+r}^{r+1}$  embeds naturally in  $\text{Im}(\sigma_{t,i}^{r+1})$ .

**4.16 Corollary** Every Reedy fibrant simplicial Postnikov  $r$ -stem has a well-defined  $(r + 1)$ -truncated spiral spectral sequence. If  $\mathcal{Q}_\bullet = \mathcal{P}[r]X_\bullet$  for some simplicial space  $X_\bullet$ , this truncated spectral sequence coincides with the  $(r + 1)$ -truncation of the Bousfield–Friedlander spectral sequence for  $X_\bullet$ .

Thus the bigraded homomorphism

$$d^{r+1} \circ d^{r+1}: E_{t,i}^r \rightarrow E_{t-2r-2,i+2r}^{r+1} \quad (t \geq 2r + 2, i \geq 0)$$

serves as the first obstruction to the realizability of the simplicial Postnikov  $r$ -stem  $\mathcal{Q}_\bullet$  by a simplicial space  $X_\bullet$ .

## 5 A cosimplicial version

There are actually four variants of the above spectral sequence which we might consider, for a simplicial or cosimplicial object over simplicial or cosimplicial sets. The case of bicosimplicial sets is in principle strictly dual to that of bisimplicial sets, but because the category of cosimplicial *sets* has no (known) useful model category structure, we must restrict to bicosimplicial abelian groups – or equivalently, ordinary double complexes. Thus the main new case of interest is that of cosimplicial simplicial sets, or *cosimplicial spaces*.

**5.1 The spectral sequence of a cosimplicial space** If  $X^\bullet \in c\mathcal{S}_*$  is a fibrant cosimplicial pointed space with total space  $\text{Tot } X^\bullet$ , there are various constructions for the homotopy spectral sequence of  $X^\bullet$ :

- (a) Using the tower of fibrations for  $(\text{Tot}_n X^\bullet)_{n=0}^\infty$  (cf [15, X,Section 6]).
- (b) Using “relations” on the normalized cochains  $N^n \pi_t X^\bullet := \pi_t X^n \cap \text{Ker}(s^0) \cap \dots \cap \text{Ker}(s^{n-1})$  (cf [17, Section 7]).
- (c) Using a cofibration sequence dualizing (3.4) (cf [22, Section 3]).

Bousfield and Kan showed that the result is essentially unique (see [17]). Since the main ingredient needed for to define the spiral exact couple is the diagram (3.7), we use the first approach:

**5.2 Definition** For any Reedy fibrant cosimplicial pointed space  $X^\bullet \in c\mathcal{S}_*$ , consider the fibration sequence

$$(5.3) \quad F_n X^\bullet \xrightarrow{J_n} \text{Tot}_n X^\bullet \xrightarrow{P_n} \text{Tot}_{n-1} X^\bullet,$$

where  $\text{Tot}_n X^\bullet := \text{map}_{cS_*}(\text{sk}_n \Delta, X^\bullet)$  and the fibration  $p_n$  is induced by the inclusion of cosimplicial spaces  $\text{sk}_{n-1} \Delta \hookrightarrow \text{sk}_n \Delta$ .

The cokernel of  $(j_n)_\# : \pi_* F_n X^\bullet \hookrightarrow \pi_* \text{Tot}_n X^\bullet$  is called the  $n$ -th natural (graded) cohomotopy group of  $X^\bullet$ , and denoted by  $\pi_{\natural}^n X^\bullet$ .

**5.4 Remark** We may identify  $F_n X^\bullet$  with the looped normalized cochain object  $\Omega^n N^n X^\bullet$ , where

$$(5.5) \quad N^n X^\bullet := X^n \cap \text{Ker}(s^0) \cap \dots \cap \text{Ker}(s^{n-1}),$$

and  $\pi_* N^n X^\bullet$  with  $N^n \pi_* X^\bullet$  (see [15, X, Proposition 6.3]).

Moreover, the composite

$$\begin{aligned} \pi_{*+1} \Omega^n N^n X^\bullet &\cong \pi_{*+1} F_n X^\bullet \xrightarrow{(j_n)_\#} \pi_{*+1} \text{Tot}_n X^\bullet \\ &\xrightarrow{\partial_n} \pi_* F_{n+1} X^\bullet \cong \pi_* \Omega^{n+1} N^{n+1} X^\bullet \end{aligned}$$

(where  $\partial_n$  is the connecting homomorphism for the (5.3)), may then be identified with the differential

$$(5.6) \quad \delta^n := \sum_{i=0}^n (-1)^i d^i : N^n \pi_* X^\bullet \rightarrow N^{n+1} \pi_* X^\bullet,$$

for the normalized cochain complex  $N^* \pi_* X^\bullet$ , so that

$$(5.7) \quad \text{Ker}(\delta^n) / \text{Coker}(\delta^{n+1}) \cong \pi^n \pi_* X^\bullet$$

(cf [15, X, Section 7.2]).

**5.8 Proposition** For any pointed cosimplicial space  $X^\bullet$  there is a natural spiral long exact sequence:

$$(5.9) \quad \begin{aligned} \dots \rightarrow \Omega \pi_{\natural}^{n-1} X^\bullet &\xrightarrow{s^n} \pi_{\natural}^n X^\bullet \xrightarrow{h^n} \pi^n \pi_* X^\bullet \xrightarrow{\partial^n} \Omega \pi_{\natural}^{n-2} X^\bullet \\ &\xrightarrow{s^{n-1}} \pi_{\natural}^{n-1} X^\bullet \rightarrow \dots \rightarrow \pi_{\natural}^0 X^\bullet \xrightarrow{\cong} \pi^0 \pi_* X^\bullet \end{aligned}$$

**Proof** By choosing a fibrant replacement in the model category of cosimplicial simplicial sets defined in [15, X, Section 5], if necessary, we may assume that  $X^\bullet$  is

Reedy fibrant. We then obtain a commutative diagram as in (3.7) with exact rows and columns:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \text{Ker}(j_n)_* & \xrightarrow{\subset} & B^{n+1} X^\bullet & \xrightarrow{(j_n)_*} & B^{n+1} \pi_* X^\bullet \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 (5.10) \quad 0 & \longrightarrow & \Omega \pi_{\natural}^{n-1} X^\bullet & \xrightarrow{\ell_{n-1}} & \pi_* \text{Tot}_n X^\bullet & \xrightarrow{(j_n^{X^\bullet})_\#} & Z^n \pi_* X^\bullet \longrightarrow \text{Coker } h^n \longrightarrow 0 \\
 & & \downarrow & \searrow s_n & \downarrow \hat{\vartheta}_n & & \downarrow \vartheta_n \\
 0 & \longrightarrow & \text{Ker } h^n & \xrightarrow{\subset} & \pi_{\natural}^n X^\bullet & \xrightarrow{h_n} & \pi^n \pi_* X^\bullet \longrightarrow \text{Coker } h^n \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

in which  $B^{n+1} X^\bullet := \text{Im}(j_{n+1})_\# \subseteq \pi_* \text{Tot}_n X^\bullet$  and  $B^{n+1} \pi_* X^\bullet := \text{Im}(\delta^{n+1}) = \text{Im}(\partial_{n+1} \circ (j_{n+1})_\#)$  are the respective coboundary objects.

The construction of the maps  $h^n$ ,  $s^n$ , and  $\partial^n$ , and the proof of the exactness of (5.9), are then precisely as in Section 3.6. □

**5.11 Definition** The *spiral  $n$ -system* of a pointed cosimplicial space  $X^\bullet \in c\mathcal{S}_*$  is defined to be the collection of long exact sequences (5.9) for the Postnikov  $n$ -stem functor  $\mathcal{P}[n]$  applied to  $X^\bullet$ , one for each  $k$ -window of  $\mathcal{P}[n]X^\bullet$ .

As in Definition 4.2, this may actually be defined for a cosimplicial Postnikov  $n$ -stem  $\mathcal{P}^\bullet$ , not necessarily realizable as  $\mathcal{P}^\bullet = \mathcal{P}[n]X^\bullet$ .

By construction, the homotopy spectral sequence of a (fibrant) cosimplicial space  $X^\bullet$ , obtained as in 5.1, is associated to the spiral exact couple (5.9). The proofs of Proposition 4.7 and Theorem 4.13 use only the description of the spiral exact couple for  $X_\bullet$  derived from (5.10), so by using (5.10) instead we can prove their analogues in the cosimplicial case, and show:

**5.12 Theorem** The  $E_{r+2}$ -term of the homotopy spectral sequence for a cosimplicial space  $X^\bullet$  is determined by the spiral  $r$ -system of  $X^\bullet$ .

An analogue of Corollary 4.16 also holds, as well as:

**5.13 Proposition** The differential  $d_2^{t,i}: E_2^{t,i} \rightarrow E_2^{t+2,i+1}$  may be identified with  $\partial_{(i,1)}^t: \pi^t \pi_i X^\bullet \rightarrow \Omega \pi_{\natural}^{t+2,0} X^\bullet$ .

**5.14 Examples** As noted in the introduction, many commonly used spectral sequences arise as the spiral spectral sequence of an appropriate (co)simplicial space, so [Theorem 4.13](#) and [Theorem 5.12](#) allow us to extract their  $E^r$  – or  $E_r$  – terms from the appropriate spiral systems. For instance:

- (a) Segal’s homology spectral sequence (cf [\[23\]](#)), the van Kampen spectral sequence (cf [\[24\]](#)), and the Hurewicz spectral sequence (cf [\[9\]](#)) are constructed using bisimplicial sets.
- (b) The unstable Adams spectral sequences of [\[13; 16\]](#) and [\[8, Section 4\]](#), Rector’s version of the Eilenberg–Moore spectral sequence (cf [\[22\]](#)), and Anderson’s generalization of the latter (cf [\[2\]](#)) are all associated to cosimplicial spaces.
- (c) The usual construction of the stable Adams spectral sequence for  $\pi_*^s X \otimes \mathbb{Z}/p$  (cf [\[1, Section 3\]](#)) uses a tower of (co)fibrations, rather than a cosimplicial space, but when  $X$  is finite dimensional, it agrees in a range with the unstable version for  $\Sigma^N X$ , so [Theorem 5.12](#) applies stably, too.

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