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The beta elements $\beta_{tp^2/r}$ in the homotopy of spheres

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In [1], Miller, Ravenel and Wilson defined generalized beta elements in the E_2–term of the Adams–Novikov spectral sequence converging to the stable homotopy groups of spheres, and in [4], Oka showed that the beta elements of the form $\beta_{tp^2/r}$ for positive integers t and r survive to the homotopy of spheres at a prime $p>3$, when $r \le 2p-2$ and $r \le 2p$ if $t>1$. In this paper, for $p>5$, we expand the condition so that $\beta_{tp^2/r}$ for $t \ge 1$ and $r \le p^2-2$ survives to the stable homotopy groups.

55Q45; 55Q10

1 Introduction

Let BP be the Brown–Peterson spectrum at a prime p, and consider the Adams–Novikov spectral sequence converging to homotopy groups $\pi_*(X)$ of a spectrum X with E_2–term $E_2^{s,t}(X) = \mathrm{Ext}^{s,t}_{\mathrm{BP}_*(\mathrm{BP})}(\mathrm{BP}_*, \mathrm{BP}_*(X))$. Here,

$$\mathrm{BP}_* = \mathbb{Z}_{(p)}[v_1, v_2, \dots] \quad \text{and} \quad \mathrm{BP}_*(\mathrm{BP}) = \mathrm{BP}_*[t_1, t_2, \dots]$$

for $v_i \in \mathrm{BP}_{2p^i-2}$ and $t_i \in \mathrm{BP}_{2p^i-2}(\mathrm{BP})$. In [1], Miller, Ravenel and Wilson defined generalized Greek letter elements in the E_2–term of the Adams–Novikov spectral sequence converging to the homotopy groups $\pi_*(S^0)$ of the sphere spectrum S^0 at each prime p. For the beta elements, we consider the mod p Moore spectrum M and finite spectra V_a for $a>0$ defined by the cofiber sequences

$$S^0 \xrightarrow{p} S^0 \xrightarrow{i} M \xrightarrow{j} S^1 \quad \text{and} \quad \Sigma^{aq} M \xrightarrow{\alpha^a} M \xrightarrow{i_a} V_a \xrightarrow{j_a} \Sigma^{aq+1} M, \tag{1.1}$$

where $p \in \pi_0(S^0) = \mathbb{Z}_{(p)}$, $\alpha \in [M,M]_q$ is the Adams map, and

$$q = 2p-2.$$

Since the maps j and j_a induce trivial homomorphisms on the BP_*–homologies, these cofiber sequences yield short exact sequences

$$\begin{aligned} &0 \to \mathrm{BP}_* \xrightarrow{p} \mathrm{BP}_* \xrightarrow{i_*} \mathrm{BP}_*/(p) \to 0, \\ &0 \to \mathrm{BP}_*/(p) \xrightarrow{v_1^a} \mathrm{BP}_*/(p) \xrightarrow{i_{a*}} \mathrm{BP}_*/(p, v_1^a) \to 0, \end{aligned} \tag{1.2}$$

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where

$$\text{(1.3)} \qquad \mathrm{BP}_*(M) = \mathrm{BP}_*/(p) \quad \text{and} \quad \mathrm{BP}_*(V_a) = \mathrm{BP}_*/(p, v_1^a).$$

The beta elements of the E_2–terms are now defined by

$$\text{(1.4)} \qquad \begin{aligned} \overline{\beta}'_{s/a-b} &= \delta_a(v_1^b v_2^s) \in E_2^{1,(sp+s-a+b)q}(M), \\ \overline{\beta}_{s/a-b} &= \delta(\overline{\beta}'_{s/a-b}) \in E_2^{2,(sp+s-a+b)q}(S^0) \end{aligned}$$

for $s>0$ and $a>b\geq 0$, if $v_1^b v_2^s \in E_2^{0,(sp+s+b)q}(V_a)$, where δ and δ_a are the connecting homomorphisms associated to the short exact sequences (1.2). We abbreviate $\overline{\beta}_{s/1}$ to $\overline{\beta}_s$ as usual. Now assume that the prime p is greater than three. Then L Smith [7] showed that every $\overline{\beta}_s$ for $s>0$ survives to a homotopy element $\beta_s \in \pi_{(sp+s-1)q-2}(S^0)$, and S Oka showed the following beta elements survive:

$$\begin{aligned} &\beta_{tp/r} && \text{for } t>0 \text{ and } r \leq p \text{ with } (t,r) \neq (1,p) \text{ in [2; 3]}, \\ &\beta_{tp^2/r} && \text{for } t>0 \text{ and } r \leq 2p-2 \text{ in [2]}, \\ &\beta_{tp^2/r} && \text{for } t>1 \text{ and } r \leq 2p \text{ in [4]}. \end{aligned}$$

Letting W denote the cofiber of the beta element $\beta_1 \in \pi_{pq-2}(S^0)$, we have a cofiber sequence

$$\text{(1.5)} \qquad S^{pq-2} \xrightarrow{\beta_1} S^0 \xrightarrow{i_W} W \xrightarrow{j_W} S^{pq-1}.$$

Then $E_2^{s,tq}(W \wedge V_a) = E_2^{s,tq}(V_a)$. In [6], we showed the following:

Theorem 1.6 [6, Theorem 1.4] *Suppose that $v_2^s \in E_2^{0,s(p+1)q}(W \wedge V_a)$. If the element v_2^s survives to $\pi_*(W \wedge V_a)$, then $\overline{\beta}_{st/r}$ for $t>0$ and $0<r<a-1$ survives to $\pi_*(S^0)$.*

In this paper, we show the following theorem:

Theorem 1.7 *Let p be a prime greater than five. Then the element $v_2^{p^2} \in E_2^0(W \wedge V_{p^2})$ is a permanent cycle.*

We work at a prime p greater than three throughout the paper except for Lemma 3.8, which requires us to exclude the case $p=5$.

Corollary 1.8 *Let p be a prime greater than five. Then the beta elements $\overline{\beta}_{tp^2/r} \in E_2^{2,(tp^2(p+1)-r)q}(S^0)$ for $t>0$ and $0<r<p^2-1$ are permanent cycles.*

2 Vanishing lines for Adams–Novikov E_3–terms for W

Ravenel constructed a ring spectrum $T(m)$ for each integer $m \geq 0$ characterized by $\mathrm{BP}_*(T(m)) = \mathrm{BP}_*[t_1, \dots, t_m]$ [5]. He then showed the change of rings theorem $E_2^{s,t}(T(m)\wedge U) = \mathrm{Ext}^{s,t}_{\Gamma(m+1)}(\mathrm{BP}_*, \mathrm{BP}_*(U))$ for a spectrum U and the Hopf algebroid $\Gamma(m+1) = \mathrm{BP}_*(\mathrm{BP})/(t_1, \dots, t_m)$. It follows from the Cartan–Eilenberg spectral sequence that

(2.1) $E_2^{s,t}(T(1)\wedge U)$ is a subquotient of $\mathrm{BP}_*(U) \otimes \bigotimes_{i\geq 2,\ j\geq 0} \big(E(h_{i,j}) \otimes P(b_{i,j})\big)$,

where $E(h_{i,j})$ and $P(b_{i,j})$ denote an exterior and a polynomial algebras on the generators $h_{i,j}$ and $b_{i,j}$, which have bidegrees $(1, 2p^j(p^i-1))$ and $(2, 2p^{j+1}(p^i-1))$. Ravenel further constructed a spectrum X_k, which is denoted by $T(0)_{(k)}$ in [5], characterized by BP_*–homology $\mathrm{BP}_*(X_k) = \mathrm{BP}_*[t_1]/(t_1^{p^k})$ as a $\mathrm{BP}_*(\mathrm{BP})$–comodule, and a diagram

(2.2)
$$\begin{tikzcd} X_{k-1} \arrow[d, "\iota_k"'] & \Sigma^{p^{k-1}q}\overline{X}_k \arrow[l, dotted, "\lambda_k"'] \arrow[d, "\iota'_k"] & \Sigma^{p^k q}X_{k-1} \arrow[l, dotted, "\lambda'_k"'] \\ X_k \arrow[ur, "\kappa_k"'] & \Sigma^{p^{k-1}q}X_k \arrow[ur, "\kappa'_k"'] & \end{tikzcd}$$

in which each triangle is a cofiber sequence with inclusion ι_k or ι'_k. Hereafter, we abbreviate X_1 to X. Since λ_k and λ'_k induce the zero homomorphisms on BP_*–homologies, applying the Adams–Novikov E_2–terms $E_M^*(-) = E_2^*(- \wedge M)$ to the diagram gives rise to an exact couple (D_1^s, E_1^s) with $D_1^{2s} = E_M^*(X_{k-1})$, $D_1^{2s+1} = E_M^*(\overline{X}_k)$ and $E_1^s = E_M^*(X_k)$, which defines the small descent spectral sequence (see [5, 7.1.13] with $k = \infty$):

(2.3)
$$^{\mathrm{SD}}E_1^* = E(h_{k-1}) \otimes P(b_{k-1}) \otimes E_M^*(X_k) \Longrightarrow E_M^*(X_{k-1}),$$

where $h_{k-1} \in {}^{\mathrm{SD}}E_1^{1,0,p^{k-1}q}$ and $b_{k-1} \in {}^{\mathrm{SD}}E_1^{2,0,p^k q}$ are represented by the cocycles $t_1^{p^{k-1}}$ and

$$y_{k-1} = \sum_{k=1}^{p-1} \frac{1}{p}\binom{p}{k} t_1^{kp^{k-1}} \otimes t_1^{(p-k)p^{k-1}},$$

respectively, of the cobar complex

$$\Omega^* = \Omega^*_{\mathrm{BP}_*(\mathrm{BP})}\mathrm{BP}_*/(p)$$

for computing $E_M^*(S^0) = E_2^*(M)$. Note that

(2.4)
$$\overline{\delta}'_k\overline{\delta}_k(x) = b_{k-1}x \quad \text{for } x \in E_M^*(X_{k-1}),$$

where $\overline{\delta}_k$ and $\overline{\delta}'_k$ denote the connecting homomorphisms corresponding to λ_k and λ'_k, respectively. Besides,

$$b_0 = \overline{\beta}_1. \tag{2.5}$$

Hereafter, we abbreviate $E_M^*(S^0)$ to E_M^*.

Lemma 2.6 *The homomorphism $\overline{\beta}_1\colon E_M^{s-2,t-pq} \to E_M^{s,t}$ is a monomorphism if*

$$E_M^{s-1,t}(X) = 0 \quad \textit{and} \quad E_M^{s-2,t-q}(X) = 0,$$

and an epimorphism if

$$E_M^{s,t}(X) = 0 \quad \textit{and} \quad E_M^{s-1,t-q}(X) = 0.$$

Proof This follows immediately from the exact sequences

$$\begin{gathered} E_M^{s-1,t}(X) \xrightarrow{\kappa_{1*}} E_M^{s-1,t-q}(\overline{X}) \xrightarrow{\overline{\delta}_1} E_M^{s,t} \xrightarrow{\iota_{1*}} E_M^{s,t}(X), \\ E_M^{s-2,t-q}(X) \xrightarrow{\kappa'_{1*}} E_M^{s-2,t-pq} \xrightarrow{\overline{\delta}'_1} E_M^{s-1,t-q}(\overline{X}) \xrightarrow{\iota'_{1*}} E_M^{s-1,t-q}(X) \end{gathered} \tag{2.7}$$

associated to the cofiber sequences in (2.2) for $k = 1$. □

For a non-negative integer s, we consider the integer $\tau(s)$ defined by

$$\tau(s) = \mu(s)p^2 + \varepsilon(s)p = \begin{cases} (s/2)p^2 & \text{if } s \text{ is even,} \\ ((s-1)/2)p^2 + p & \text{if } s \text{ is odd,} \end{cases} \tag{2.8}$$

where $\varepsilon(s)$ and $\mu(s)$ are the integers given by

$$2\varepsilon(s) = 1 - (-1)^s \quad \text{and} \quad 2\mu(s) = s - \varepsilon(s). \tag{2.9}$$

Lemma 2.10 *$E_M^{s,t}(X) = 0$ if $t < \tau(s)q$.*

Proof By an iterate use of the small descent spectral sequences (2.3) for k, we see that $E_M^{s,t}(X)$ is a subquotient of $E(h_j : j > 0) \otimes P(b_j : j > 0) \otimes E_M^*(T(1))$. For each dimension s, minding (2.1), the (additive) generator with the smallest internal degree is $h_1^{\varepsilon(s)} b_1^{\mu(s)}$, whose bidegree is $(s, \tau(s)q)$. □

Let $\widetilde{E}_M^{s,t}(U)$ denote the Adams–Novikov E_3–term $E_3^{s,t}(U \wedge M)$. Since the Adams–Novikov spectral sequence has the sparseness: $E_M^{s,t} = 0$ unless $q \mid t$, we see that $\widetilde{E}_M^{s,t}(S^0) = E_M^{s,t}$.

Lemma 2.11 $\widetilde{E}_M^{s,t}(W)=0$ *if one of the following conditions holds:*

(1) $q\nmid t(t+1)$.

(2) $q\mid t$ *and* $t<(\tau(s-1)+1)q$.

(3) $q\mid (t+1)$ *and* $t+1<(\tau(s)+1)q$.

Proof The cofiber sequence (1.5) induces the short exact sequence

$$0\longrightarrow E_M^{s,t}\xrightarrow{i_{W*}}E_M^{s,t}(W)\xrightarrow{j_{W*}}E_M^{s,t-pq+1}\longrightarrow 0$$

of the E_2–terms. Therefore, $E_M^{s,t}(W)=E_M^{s,t}\oplus gE_M^{s,t-pq+1}$ for an element $g\in E_M^{0,pq-1}(W)$. Since $d_2(g)=i_{W*}(\overline{\beta}_1)$ in the Adams–Novikov spectral sequence, we have the long exact sequence

$$E_M^{s-2,t-pq}\xrightarrow{\overline{\beta}_1}E_M^{s,t}\xrightarrow{i_{W*}}\widetilde{E}_M^{s,t}(W)\xrightarrow{j_{W*}}E_M^{s,t-pq+1}\xrightarrow{\overline{\beta}_1}E_M^{s+2,t+1}\tag{2.12}$$

of the E_3–terms. The sparseness of the spectral sequence implies that i_{W*} and j_{W*} in (2.12) are zero if $q\nmid t$ and $q\nmid(t+1)$, respectively. This immediately shows the lemma under the first condition. If the second (resp. third) condition holds, then Lemma 2.10 and Lemma 2.6 imply that the left (resp. right) $\overline{\beta}_1$ in (2.12) is an epimorphism (resp. a monomorphism). □

Remark Lemma 2.10 and Lemma 2.6 hold by the same proof after replacing $E_M(-)$ and $\widetilde{E}_M(-)$ by $E_2(-)$ and $E_3(-)$.

We state here relations in the E_2–term $E_M^*=E_2^*(M)$:

Lemma 2.13 *In the Adams–Novikov* E_2*–term* E_M^2, $v_1^2b_0=0$ *and* $v_1^{p-1}b_1=0$.

Proof Note that $d(t_2)=-t_1\otimes t_1^p+v_1y_0$ in Ω^2 (see [5, 4.3.15]). Then $v_1^2y_0$ cobounds $c_0=-t_1\eta_R(v_2)+v_1t_2-(1/2)v_1^pt_1^2$, since v_1 and t_1 are primitive, and $\eta_R(v_2)\equiv v_2+v_1t_1^p-v_1^pt_1 \bmod (p)$ in $\mathrm{BP}_*\,\mathrm{BP}$ (see [5, 4.3.21]).

Consider the cobar complex $\Omega_2^*=\Omega^*_{\mathrm{BP}_*(\mathrm{BP})}\mathrm{BP}_*/(p^2)$. We define the element $w\in\Omega^1$ by

$$d(v_2^p)=v_1^pt_1^{p^2}-v_1^{p^2}t_1^p+pv_1w\in\Omega_2^1.\tag{2.14}$$

It is well defined, since $pv_1\colon\Omega^s\to\Omega_2^s$ is a monomorphism. Noticing that $d(t_1^{p^{i+1}})=-py_i$ and $d(v_1)=pt_1$ in Ω_2^*, send the equation (2.14) to Ω_2^2 under the differential d, and we obtain $0=-pv_1^py_1+pv_1^{p^2}y_0+pv_1d(w)\in\Omega_2^2$, which is pulled back to Ω^2 under pv_1 to give $d(w)=v_1^{p-1}y_1-v_1^{p^2-1}y_0\in\Omega^2$. It follows that $v_1^{p-1}y_1$ cobounds $w+v_1^{p^2-3}c_0$. □

3 Adams–Novikov E_2–terms for $X \wedge M$

Ravenel computed the small descent spectral sequences to determine $E_2^{s,t}(T(m))$ in [5, 7.2.6, 7.2.7] below internal degree $2(p^{m+3} - p^2)$. In particular, below internal degree $(p^3 + p^2)q$,

$$\bigoplus_{s\geq 2} E_2^{s,*}(T(1)) = k(2)_* \left\{v_3^s b_{20} : s \geq 0\right\} \otimes E(h_{20}) \otimes P(b_{20}). \tag{3.1}$$

Here, E and P denote an exterior and a polynomial algebras over $\mathbb{Z}/p$,

$$k(m)_* = \mathbb{Z}/p[v_m] \tag{3.2}$$

and $v_3^s b_{20}$ denotes the element corresponding to $\widehat{v}_2^{s+1}/pv_1$ in [5, 7.2.6]. We here read off the following formulas on the differential of the cobar complex $C_2^* = \Omega^*_{\Gamma(2)}\mathrm{BP}_*$ from the Hazewinkel and the Quillen formulas (see [1, (1.1), (1.2), (1.3)]):

$$\begin{aligned} &d(v_1) = 0, && d(v_2) = pt_2, \\ &d(v_3) = v_1 t_2^p - v_1^{p^2} t_2 + pt_3 - p^{-1} v_1 d(v_2^p), \quad && d(t_2) = 0. \end{aligned} \tag{3.3}$$

By virtue of these, we see that the generators v_1, h_{20} and $v_3^s b_{20}$ are represented by v_1, t_2 and $y_{2,s} = p^{-1} d(\overline{y}_{2,s})$ for

$$\overline{y}_{2,s} = -\sum_{i=1}^{s+1} \binom{s+1}{i} v_1^{i-1} v_3^{s+1-i} (t_2^p - v_1^{p^2-1} t_2)^i,$$

respectively, in the cobar complex C_2^* for computing $E_2^*(T(1))$.

Corollary 3.4 *The Adams–Novikov E_2–terms $E_M^{s,t}(T(1))$ below internal degree $(p^3 + p^2)q$ are given as follows:*

$$\bigoplus_{s\geq 2} E_M^{s,*}(T(1)) = b_{20} k(2)_*[v_3] \otimes E(h_{20}, h_{21}) \otimes P(b_{20})$$

Here, the generators have the following bidegrees:

$$|v_2| = (0, (p+1)q), \quad |v_3| = (0, (p^2 + p + 1)q),$$
$$|h_{20}| = (1, (p+1)q), \quad |h_{21}| = (1, (p^2 + p)q) \quad \text{and} \quad |b_{20}| = (2, (p^2 + p)q).$$

Proof Consider the long exact sequence

$$E_2^{s,t}(T(1)) \xrightarrow{p} E_2^{s,t}(T(1)) \xrightarrow{i_*} E_M^{s,t}(T(1)) \xrightarrow{\delta} E_2^{s+1,t}(T(1)) \xrightarrow{p} E_2^{s+1,t}(T(1))$$

associated to the first cofiber sequence in (1.1). Note that this is a sequence of $\mathbb{Z}[v_1]$–modules.

The s–th line $E_M^{s,*}(T(1))$ for $s \geq 2$ is the direct sum of the image $i_*E_2^{s,*}(T(1)) = E^s$ of i_* and the module isomorphic to the image of δ. Here $E^s = h_{20}^{\varepsilon(s)} b_{20}^{\mu(s)} k(2)_*[v_3]$ for the integers of (2.9). Since $v_1\overline{y}_{2,s} = d(v_3^{s+1}) \in \Omega^*_{\Gamma(2)}\mathrm{BP}_*/(p)$, we see that $\overline{y}_{2,s}$ is a cocycle that represents $v_3^s h_{21}$, and $\delta(v_3^s h_{21}) = v_3^s b_{20}$ by definition. Therefore, the image of δ is $b_{20}E^{s-1} = E_2^{s+1,*}(T(1))$, which is isomorphic to $h_{21}E^{s-1}$. □

By (2.4) for $k = 2$, we have a homomorphism $b_1\colon E_M^{s-2,t-p^2q}(X) \to E_M^{s,t}(X)$. As Lemma 2.6, the following lemma follows from the exact sequences

$$E_M^{s-1,t-pq}(\overline{X}_2) \xrightarrow{\overline{\delta}_2} E_M^{s,t}(X) \xrightarrow{\iota_{2*}} E_M^{s,t}(X_2),$$

$$E_M^{s-2,t-p^2q}(X) \xrightarrow{\overline{\delta}'_2} E_M^{s-1,t-pq}(\overline{X}_2) \xrightarrow{\iota'_{2*}} E_M^{s-1,t-pq}(X_2)$$

associated to the cofiber sequences in (2.2):

Lemma 3.5 *The homomorphism $b_1\colon E_M^{s-2,t-p^2q}(X) \to E_M^{s,t}(X)$ is an epimorphism if*

$$E_M^{s,t}(X_2) = 0 \quad \textit{and} \quad E_M^{s-1,t-pq}(X_2) = 0.$$

For each integer s and t, we consider the set

(3.6) $$S(s,t) = \{(s,t), (s-1, t-pq), (s-1, t+(p-2)q), (s-2, t-2q)\}$$

Corollary 3.7 *If $E_M^{s,t}(X_2) = 0$ for $(s,t) \in S(a,b)$, then* (see (2.5))

$$v_1^{2p-2}E_M^{a,b} \subset \overline{\beta}_1 E_M^{a-2,b+(p-2)q}.$$

Proof Consider the diagram (2.2) for $k = 1$ smashing with M. Then for any element $x \in E_M^{a,b}$,

$$\iota_{1*}(x) = b_1x_1 \in E_M^{a,b}(X) \quad \text{for some } x_1 \in E_M^*(X)$$

by Lemma 3.5. Since

$$\iota_{1*}(v_1^{p-1}x) = v_1^{p-1}b_1x_1 = 0$$

by Lemma 2.13, there is an element $x_2 \in E_M^{a-1,b+(p-2)q}(\overline{X})$ such that

$$\overline{\delta}_1(x_2) = v_1^{p-1}x.$$

In the same manner, we have an element $x_3 \in E_M^{a-2,b+(p-2)q}$ such that

$$\overline{\delta}'_1(x_3) = v_1^{p-1}x_2.$$

It follows that

$$v_1^{2p-2}x = v_1^{p-1}\overline{\delta}_1(x_2) = \overline{\delta}_1\overline{\delta}'_1(x_3) = \overline{\beta}_1(x_3). \qquad \square$$

We now consider the integer

$$u = p^3 + p^2 - 2p + 2.$$

Lemma 3.8 *If $p > 5$, then the E_2–terms $E_M^{s,t}(X_2) = 0$ for $(s,t) \in S(q+1,(u+1)q)$.*

Proof By use of the small descent spectral sequences (2.3) for $k \geq 2$, we see that our $E_M^{s,t}(X_2)$ is a subquotient of the module $A^{s,t} = E_M^{s,t}(T(1)) \oplus h_2 E_M^{s-1,t-pq}(T(1))$ by degree reason. It suffices to show that $A^{s,t} = 0$ for $(s,t) \in S(q+1,(u+1)q)$. The integers t fit in the table:

t/q	$u+1$	$u+1-p$	$u+p-1$	$u-1$
t/q mod $(p+1)$	5	6	2	3
t/q mod (p)	3	3	1	1

Corollary 3.4 implies that the module $A^{s,t}$ is generated by elements of the form $v_2^i v_3^j h_2^k h_{20}^l h_{21}^m b_{20}^n$ with $k,l,m \in \{0,1\}$ and $i,j,n \geq 0$. The internal degree of it is q times

$$a = (p^2+p+1)j + p^2 k + (p+1)(i+l+p(m+n)), \tag{3.9}$$

which is congruent to $j+k$ modulo $(p+1)$ and $i+j+l$ modulo (p). Since $s \geq q-1$ and $s = k+l+m+2n$, we see that $n \geq p-3$. Then $a \geq (p^2+p+1)j + p^3 - 2p^2 - 3p > u+p-1$ if $j \geq 3$. It follows that $j+k \leq 3$, and the first two cases in the above table are excluded if $p > 5$. The last case is also excluded. Indeed, in this case, $j = 2$ and $k = 1$, which shows $a \geq 3p^2 + 2p + 2 + p^3 - 2p^2 - 3p > u-1$.

In the third case, $j+k = 2$, and $i+j+l = rp+1$ for some $r \geq 0$. Then $a = 2p^2 + (p+1)(rp+1+p(m+n))$, which equals $u+p-1$ if and only if $r = 0$, $m = 0$ and $n = p-2$, since $n \geq p-2$ in this case. The solution $m = 0$ implies $k = l = 1$ and so $j = 1$. Then $1 = i+j+l = i+2$, which contradicts to $i \geq 0$. □

Remark If $p = 5$, we have elements $v_2^3 b_{20}^4$ and $v_2^2 h_{20} h_{21} b_{20}^3$ in $A^{q,u+1-5}$.

Lemma 3.10 *Suppose that $\xi \in \pi_{uq-1}(M)$ is detected by an element of $E_M^{q+1,(u+1)q}$. Then $i_{W*}(\alpha^{2p-2}\xi) = 0 \in \pi_{(u+2p-2)q-1}(W \wedge M)$.*

Proof Let x be an element that detects ξ. Then by Corollary 3.7 with Lemma 3.8, we see that $v_1^{2p-2}x = \overline{\beta}_1 y$ for some $y \in E_M^{q-1,(u+p-1)q}$, and so $i_{W*}(v_1^{2p-2}x) = 0 \in \widetilde{E}_M^{q+1,(u+2p-1)q}(W)$. The lemma now follows from Lemma 2.11. □

4 The beta element $\beta'_{p^2/p^2} \in \pi_{p^3q-1}(W \wedge M)$

Consider the set

$$S'(s,t) = \{(s+1,t), (s,t), (s,t-q), (s-1,t-q)\}.$$

Lemma 4.1 *If $E_2^{s,t}(X) = 0$ for $(s,t) \in S'(a,b)$, then $\overline{\beta}_1\colon E_M^{a-2,b-pq} \to E_M^{a,b}$ is an epimorphism.*

Proof The condition on (s,t) implies that $E_M^{a,b}(X) = 0 = E_M^{a-1,b-q}(X)$ by the exact sequence associated to the first cofiber sequence (1.1). The lemma follows from Lemma 2.6. □

In [5, 7.5.1], Ravenel determined $E_2^{s,t}(X)$ for $t < (p^3+p)q$. In particular, he showed

$$\textbf{(4.2)} \qquad \begin{aligned} E_2^{s,t}(X) &= 0 \quad \text{for } (s,t) \in S'(q+2, (p^3+1)q), \\ E_2^{s,t}(X) &= 0 \quad \text{for } (s,t) \in S'(q, (p^3-p+2)q). \end{aligned}$$

Remark A preferable condition for the second equation is $(s,t) \in S'(q,(p^3-p+1)q)$, but $h_1 b_{20}^{p-3}\gamma_2 \in E_2^{q,(p^3-p)q}(X)$.

Proposition 4.3 *The element $i_{W*}(\overline{\beta}'_{p^2/p^2}) \in E_2^{1,p^3q}(W \wedge M)$ for the beta element $\overline{\beta}'_{p^2/p^2} \in E_2^{1,p^3q}(M)$ survives to a homotopy element $\beta'_{p^2/p^2} \in \pi_{p^3q-1}(W \wedge M)$.*

Proof The E_3–terms $\widetilde{E}_M^{rq+2,(p^3+r)q}(W)$ are all trivial by Lemma 2.11 for $r > 1$. We also see that $d_{q+1}(i_{W*}(\overline{\beta}'_{p^2/p^2})) = i_{W*}d_{q+1}(\overline{\beta}'_{p^2/p^2}) = 0 \in \widetilde{E}_M^{q+2,(p^3+1)q}(W)$ by Lemma 4.1 with the first equation of (4.2). □

Hereafter, for an element $f \in [X,Y]_t$, we abbreviate $f \wedge Z \in [X \wedge Z, Y \wedge Z]_t$ to f. Since $\alpha^2\beta_1 = 0 \in [M,M]_{(p+2)q-2}$ [8], we have elements $\sigma \in [W \wedge M, M]_{2q}$ and $\sigma^* \in [M, W \wedge M]_{(p+2)q-1}$ such that $\sigma i_W = \alpha^2 = j_W\sigma^*$.

Lemma 4.4 (a) $[W \wedge M, W \wedge M,]_{2q} = \mathbb{Z}/p\{\alpha^2, \delta_W\delta\alpha^{p+2}, \delta_W\alpha^{p+2}\delta, \sigma^* j_W\}$, *where $\delta_W = i_W j_W$.*

(b) $[W \wedge M, M]_{(2-p)q+1} = \mathbb{Z}/p\{\alpha^2 j_W\}$.

(c) $[M, W \wedge M]_{2q} = \mathbb{Z}/p\{\alpha^2 i_W\}$.

Proof The homotopy groups $[M,M]_t$ for $t < p^2q-4$ are given in [8, Th.I]. In particular, the generators are given in the table:

t	$2q$	$2q+1$	$(p+2)q-2$	$(p+2)q-1$
$[M,M]_t$	α^2	0	$\delta\alpha^{p+2}\delta$	$\alpha^{p+2}\delta, \delta\alpha^{p+2}$

We have the exact sequence

$$[M,M]_{t-pq+2} \xrightarrow{\beta_1} [M,M]_t \xrightarrow{i_{W*}} [M, W\wedge M]_t \xrightarrow{j_{W*}} [M,M]_{t-pq+1} \xrightarrow{\beta_1} [M,M]_{t-1}$$

associated to the cofiber sequence (1.5). From this sequence and the previous table, we obtain the following:

t	$2q$	$2q+1$	$(p+2)q-1$
$[M, W\wedge M]_t$	$i_W\alpha^2$	0	$i_W\delta\alpha^{p+2}, i_W\alpha^{p+2}\delta, \sigma^*$

In particular, we have part (c). The cofiber sequence (1.5) also induces the exact sequence

$$[M, W\wedge M]_{2q+1} \xrightarrow{\beta_1^*} [M, W\wedge M]_{(p+2)q-1} \xrightarrow{j_W^*} [W\wedge M, W\wedge M]_{2q}$$
$$\xrightarrow{i_W^*} [M, W\wedge M]_{2q} \xrightarrow{\beta_1^*} [M, W\wedge M]_{(p+2)q-2},$$

from which we obtain part (a).

Part (b) is the Spanier–Whitehead dual of (c). □

Lemma 4.5 *$i_W\sigma + \sigma^* j_W \equiv \alpha^2$ modulo $\mathbb{Z}/p\{\delta_W\delta\alpha^{p+2}, \delta_W\alpha^{p+2}\delta\}$. In particular, $i_W\sigma = \alpha^2 + \varphi j_W$ for some φ.*

Proof By virtue of Lemma 4.4 (a), we put

$$i_W\sigma = a_1\alpha^2 + a_2\delta_W\delta\alpha^{p+2} + a_3\delta_W\alpha^{p+2}\delta + a_4\sigma^* j_W \in [W\wedge M, W\wedge M]_{2q}$$

for $a_i \in \mathbb{Z}/p$. Send this to $[W\wedge M, M]_{(2-p)q+1}$ by j_W to obtain

$$0 = j_W i_W\sigma = a_1 j_W\alpha^2 + a_4 j_W\sigma^* j_W = a_1\alpha^2 j_W + a_4\alpha^2 j_W$$

Since $\alpha^2 j_W$ is a generator by Lemma 4.4 (b), we have $a_1 = -a_4$. Next send the above equality to $[M, W\wedge M]_{2q}$ by i_W, and we have

$$i_W\sigma i_W = a_1\alpha^2 i_W.$$

It follows that $a_1 = 1$ by Lemma 4.4 (c). □

Proposition 4.6 *The element*

$$\sigma\beta'_{p^2/p^2} \in \pi_{(p^3+2)q-1}(M)$$

for $\beta'_{p^2/p^2} \in \pi_{p^3q-1}(W\wedge M)$ in Proposition 4.3 is detected by the beta element

$$\overline{\beta}'_{p^2/p^2-2} \in E_M^{1,(p^3-2)q}.$$

Proof The homomorphism on the E_2–term induced from $\sigma i_W = \alpha^2$ is multiplication by v_1^2, so $\sigma_*\overline{\beta}'_{p^2/p^2} = \sigma_* i_{W*}\overline{\beta}'_{p^2/p^2} = v_1^2\overline{\beta}'_{p^2/p^2} = \overline{\beta}'_{p^2/p^2-2}$ in the E_2–term. □

Lemma 4.7 $\alpha^5 i_{W*}(\sigma\beta'_{p^2/p^2}) = \alpha^7\beta'_{p^2/p^2} \in \pi_*(W\wedge M)$.

Proof Since $j_{W*}(\overline{\beta}'_{p^2/p^2}) = 0$ in the E_2–term, the homotopy element $j_{W*}(\beta'_{p^2/p^2})$ is detected by an element x of $E_M^{rq,(p^3-p+r)q}$ for some $r>0$. If $r=1$, then $v_1x = \overline{\beta}_1 x'$ for some x' by Lemma 4.1 with the second equation of (4.2). Therefore, $v_1^3 x = v_1^2\overline{\beta}_1 x' = 0$ by Lemma 2.13. It follows that, in any case, $\alpha^3 j_{W*}(\beta'_{p^2/p^2})$ is detected by an element of $E_M^{rq,(p^3-p+r)q}$ for some $r>1$. Then $i_{W*}(\alpha^3 j_{W*}(\beta'_{p^2/p^2})) = 0$ by Lemma 2.11, and $\alpha^3 j_{W*}(\beta'_{p^2/p^2}) = \beta_1\xi'$ for some homotopy element ξ'. Now, we compute

$$\begin{aligned}\alpha^5 i_{W*}(\sigma\beta'_{p^2/p^2}) &= \alpha^7\beta'_{p^2/p^2} + \varphi_*(\alpha^5 j_{W*}(\beta'_{p^2/p^2}))\\ &= \alpha^7\beta'_{p^2/p^2} + \varphi_*(\alpha^2\beta_1\xi') = \alpha^7\beta'_{p^2/p^2}\end{aligned}$$

by Lemma 4.5 and Lemma 2.13. □

Lemma 4.8 $\alpha^{p^2}\beta'_{p^2/p^2} = 0 \in \pi_{(p^3+p^2)q-1}(W\wedge M)$.

Proof Oka [2] constructed the beta element $\beta'_{p^2/2p-2} \in \pi_{uq-1}(M)$ such that $\alpha^{2p-2}\times \beta'_{p^2/2p-2} = 0$ in homotopy, which is detected by $v_1^{p^2-2p+2}\overline{\beta}'_{p^2/p^2}$ in the E_2–term. Consider an element $\xi = \alpha^{p^2-2p}\sigma\beta'_{p^2/p^2} - \beta'_{p^2/2p-2} \in \pi_{uq-1}(M)$. Then it goes to zero in the E_2–term, and is detected by an element of $E_M^{rq+1,(u+r)q}$ for $r>0$. If $r>1$, $i_{W*}(\xi)$ is zero by Lemma 2.11. If $r=1$, then it satisfies the condition of Lemma 3.10, and so $\alpha^{2p-2} i_{W*}(\xi) = 0$. Therefore, by Lemma 4.7,

$$\alpha^{p^2}\beta'_{p^2/p^2} = \alpha^{p^2-2} i_{W*}(\sigma\beta'_{p^2/p^2}) = \alpha^{2p-2} i_{W*}(\xi + \beta'_{p^2/2p-2}) = 0. \qquad \square$$

Proof of Theorem 1.7 Consider the second cofiber sequence (1.1) for $a = p^2$. Then, by Lemma 4.8, we have an element $v \in \pi_*(W\wedge V_{p^2})$ such that $(j_{p^2})_*(v) = \beta'_{p^2/p^2}$. As v is detected by an element of $E_2^{0,(p^3+p^2)q}(W\wedge V_{p^2})$, we see $v = v_2^{p^2}$ by degree reasons. □

References

[1] **H R Miller**, **D C Ravenel**, **W S Wilson**, *Periodic phenomena in the Adams–Novikov spectral sequence*, Ann. Math. (2) 106 (1977) 469–516 MR0458423

[2] **S Oka**, *A new family in the stable homotopy groups of spheres*, Hiroshima Math. J. 5 (1975) 87–114 MR0380791

[3] **S Oka**, *A new family in the stable homotopy groups of spheres. II*, Hiroshima Math. J. 6 (1976) 331–342 MR0418096

[4] **S Oka**, *Realizing some cyclic* BP_**–modules and applications to stable homotopy of spheres*, Hiroshima Math. J. 7 (1977) 427–447 MR0474290

[5] **D C Ravenel**, *Complex cobordism and stable homotopy groups of spheres*, second edition, AMS Chelsea Publ. 347, Amer. Math. Soc. (2004) MR860042

[6] **K Shimomura**, *Note on beta elements in homotopy, and an application to the prime three case*, Proc. Amer. Math. Soc. 138 (2010) 1495–1499 MR2578544

[7] **L Smith**, *On realizing complex bordism modules. IV. Applications to the stable homotopy groups of spheres*, Amer. J. Math. 99 (1977) 418–436 MR0433450

[8] **N Yamamoto**, *Algebra of stable homotopy of Moore space*, J. Math. Osaka City Univ. 14 (1963) 45–67 MR0157382

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