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The stable 4–genus of knots

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We define the stable 4–genus of a knot $K \subset S^3$, $g_{\mathrm{st}}(K)$, to be the limiting value of $g_4(nK)/n$, where g_4 denotes the 4–genus and n goes to infinity. This induces a seminorm on the rationalized knot concordance group, $\mathcal{C}_{\mathbf{Q}} = \mathcal{C} \otimes \mathbf{Q}$. Basic properties of g_{st} are developed, as are examples focused on understanding the unit ball for g_{st} on specified subspaces of $\mathcal{C}_{\mathbf{Q}}$. Subspaces spanned by torus knots are used to illustrate the distinction between the smooth and topological categories. A final example is given in which Casson–Gordon invariants are used to demonstrate that $g_{\mathrm{st}}(K)$ can be a noninteger.

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1 Summary

In order to better understand the smooth 4–genus of knots $K \subset S^3$, denoted $g_4(K)$, we introduce and study here the *stable* 4*–genus*,

$$g_{\mathrm{st}}(K) = \lim_{n\to\infty} g_4(nK)/n.$$

As will be seen in Section 2, the existence of the limit and its basic properties follow from the subadditivity of g_4 as a function on the classical knot concordance group $\mathcal{C}$; that is, $g_4(K \# J) \le g_4(K) + g_4(J)$ for all K and J.

Neither classical knot invariants nor the invariants that arise from Heegaard Floer theory as in Oszváth and Szabó [12] or Khovanov homology as Rasmussen [13] can be used to demonstrate that $g_{\mathrm{st}}(K) \notin \mathbf{Z}$ for some K. One result of this paper is the construction of a knot K for which $g_{\mathrm{st}}(K)$ is close to $1/2$. Perhaps of greater interest is the exploration of the new perspective on the 4–genus and knot concordance offered from the stable viewpoint. In particular, a number of interesting and challenging new questions arise naturally. For example, we note that finding a knot K with $0 < g_{\mathrm{st}}(K) < 1/2$ is closely related to the existence of torsion in $\mathcal{C}$ of order greater than 2. We will also consider the distinction between the smooth and topological categories from the perspective of the stable genus.

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2 Algebraic preliminaries

The existence of the limiting value and its basic properties are summarized in the following general theorem.

Theorem 1 *Let $\nu\colon G \to \mathbf{R}_{\geq 0}$ be a subadditive function on an abelian group G. Then:*

(1) *The limit $\nu_{\mathrm{st}}(g) = \lim_{n\to\infty} \nu(ng)/n$ exists for all $g \in G$.*

(2) *The function $\nu_{\mathrm{st}}\colon G \to \mathbf{R}_{\geq 0}$ is subadditive and multiplicative: $\nu_{\mathrm{st}}(ng) = n\nu_{\mathrm{st}}(g)$ for $n \in \mathbf{Z}_{\geq 0}$. If $\nu(g) = \nu(-g)$ for all g, then $\nu_{\mathrm{st}}(g) = \nu_{\mathrm{st}}(-g)$ for all g.*

(3) *There is a factorization of ν_{st} through $G_{\mathbf{Q}} = G \otimes \mathbf{Q}$. That is, there is a multiplicative, subadditive function $\overline{\nu}_{\mathrm{st}}\colon G_{\mathbf{Q}} \to \mathbf{R}_{\geq 0}$ such that $\nu_{\mathrm{st}} = \overline{\nu}_{\mathrm{st}} \circ i$ where $i\colon G \to G_{\mathbf{Q}}$ is the map $g \to g \otimes 1$.*

Proof The proof of (1) is a standard elementary exercise using the consequence of subadditivity, $\nu(ng) \leq n\nu(g)$ for all g. In the appendix to this paper we summarize a proof. The rest of the theorem follows easily. □

A *seminorm* on a vector space is a nonnegative multiplicative and subadditive function. Thus, $\overline{\nu}_{\mathrm{st}}$ is a seminorm on $G_{\mathbf{Q}}$.

Notation We will usually drop the overbar notation; that is, we will denote both the functions ν_{st} on G and $\overline{\nu}_{\mathrm{st}}$ on $G_{\mathbf{Q}}$ by ν_{st} and be clear as to what domain we are using.

In our applications we will want to bound g_{st} using homomorphisms on the concordance group, in particular signatures, the Ozsváth–Szabó invariant τ, and the Khovanov-Rasmussen invariant s. The needed algebraic observation is the following, the proof of which the reader can readily provide.

Theorem 2 *If $\sigma\colon G \to \mathbf{R}$ is a homomorphism and $\nu(g) \geq |\sigma(g)|$ for all $g \in G$, then:*

(1) *$|\sigma|\colon G \to \mathbf{R}_{\geq 0}$ is subadditive.*

(2) *The stable function $|\sigma|_{\mathrm{st}}$ satisfies $|\sigma|_{\mathrm{st}} = |\sigma|$ and is a seminorm on $G_{\mathbf{Q}}$.*

(3) *$\nu_{\mathrm{st}}(x) \geq |\sigma(x)|$ for all $x \in G_{\mathbf{Q}}$.*

A seminorm can be completely understood via its unit ball.

Definition 3 If ν is a seminorm on a vector space V, then $B_\nu = \{x \in V \mid \nu(x) \le 1\}$.

Theorem 4 *Let ν be a subadditive nonnegative function on an abelian group G and let σ be a real-valued homomorphism on G.*

(1) *$B_{\nu_{\mathrm{st}}}$ and $B_{|\sigma|}$ are convex subsets of $G_{\mathbf{Q}}$.*

(2) *If $\nu(g) \ge |\sigma(g)|$ for all $g \in G$, then $B_{\nu_{\mathrm{st}}} \subset B_{|\sigma|}$.*

3 Elementary examples

We begin exploring the stable genus by computing its value for a few simple examples.

3.1 $g_{\mathrm{st}}(4_1) = 0$

The first example of a nonslice knot is the figure eight knot, 4_1, as originally proved by Fox and Milnor [5]. Since 4_1 is amphicheiral, $2(4_1)$ is slice, meaning that $g_4(2(4_1)) = 0$. It follows immediately that in taking limits, $g_{\mathrm{st}}(4_1) = 0$.

3.2 $g_{\mathrm{st}}(3_1) = 1$

The first knot of infinite order in $\mathcal{C}$ is the trefoil, 3_1, as originally proved by Murasugi [11]. Let $\sigma(K)$ denote of the classical signature of K: the signature of $V + V^{\mathrm{T}}$ where V is a Seifert matrix for K and V^{T} its transpose. Then we have the Murasugi bound, $g_4(K) \ge \frac{1}{2}|\sigma(K)|$. Hence Theorem 2 applies to show that $g_{\mathrm{st}}(3_1) \ge \frac{1}{2}|\sigma(3_1)| = 1$. On the other hand, $g_4(3_1) = 1$, so $g_{\mathrm{st}}(3_1) \le 1$.

3.3 $g_{\mathrm{st}}(3T_{2,7} - 2T_{2,11}) = 2$

As a final example that illustrates a simple application of Tristram–Levine signatures [9; 14], we consider the knot $3T_{2,7} - 2T_{2,11}$, where $T_{p,q}$ denotes the (p,q)–torus knot. We will now apply Theorem 2 to σ_t for appropriate t, where σ_t is the Tristram–Levine signature [9; 14], defined by

$$\sigma_t(K) = \text{signature}((1 - e^{2i\pi t})V + (1 - e^{-2i\pi t})V^{\mathrm{T}}).$$

(Formally, to achieve a concordance invariant one forms the two-sided limit $\sigma'_t(K) = \lim_{\epsilon \to 0} \frac{1}{2}(\sigma_{t-\epsilon}(K) + \sigma_{t+\epsilon}(K))$: then σ' is a homomorphism on the concordance group for any specific value of t.) For the knot $3T_{2,7} - 2T_{2,11}$ this signature function

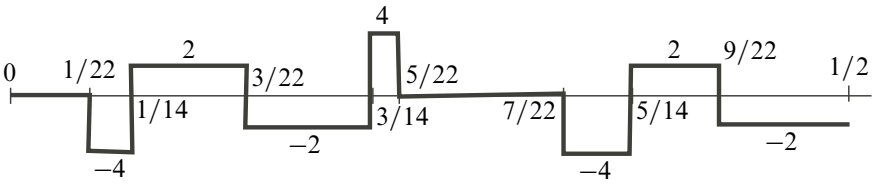

Figure 1: Signature function for $3T_{2,7} - 2T_{2,11}$

is graphed in Figure 1. Since the function is symmetric about $1/2$, we have graphed the portion of the function on the interval $[0, 1/2]$.

If we let x be any number between $3/14$ and $5/22$, then the Tristram–Levine bound $g_4(K) \geq \frac{1}{2}|\sigma_x(K)|$ implies $g_{\mathrm{st}}(K) \geq \frac{1}{2}|\sigma_x(K)|$. Thus $g_{\mathrm{st}}(3T_{2,7} - 2T_{2,11}) \geq 2$. On the other hand, the reader should have no trouble finding four band moves in the schematic diagram of $3T_{2,7} - 2T_{2,11}$ (Figure 2) that converts it into the torus knot $T_{2,1}$ which is the unknot and in particular bounds a disk. The corresponding surface in the 4–ball constructed by performing these band moves and capping off with the disk is of genus 2. Thus $g_{\mathrm{st}}(3T_{2,7} - 2T_{2,11}) \leq 2$.

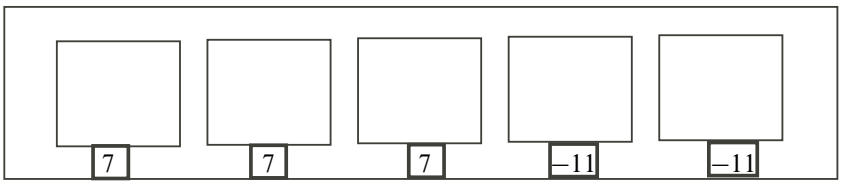

Figure 2: Schematic diagram for $3T_{2,7} - 2T_{2,11}$

4 Families of knots: $xT_{2,7} + yT_{2,11}$

A nice illustrative example is given by restricting to the subspace S of $\mathcal{C}_{\mathbf{Q}}$ spanned by the torus knots $T_{2,7}$ and $T_{2,11}$. We want to understand the unit ball of g_{st} on S in terms of the unit ball associated to the function $\mathrm{Max}_{0 \leq t \leq 1}\{\sigma_t\}$; for any particular example it is more straightforward to directly analyze the signature function. In the present case, the signature functions for $T_{2,7}$ and $T_{2,11}$ are zero near $t = 0$ and increase by two at each of the jumps at the points $\{1/14, 3/14, 5/14\}$ and $\{1/22, 3/22, 5/22, 7/22, 9/22\}$, respectively. For the readers convenience, we order the union of these two sets:

$$\{1/22, 1/14, 3/22, 3/14, 5/22, 7/22, 5/14, 9/22\}.$$

Evaluating the signature functions $\sigma_t(xT_{2,7}+yT_{2,11})$ at values between each of these numbers and for some t close to $1/2$ yields the following set of inequalities:

$$\begin{aligned}
g_{\mathrm{st}}(xT_{2,7}+yT_{2,11}) &\geq |y| \\
g_{\mathrm{st}}(xT_{2,7}+yT_{2,11}) &\geq |x+y| \\
g_{\mathrm{st}}(xT_{2,7}+yT_{2,11}) &\geq |x+2y| \\
g_{\mathrm{st}}(xT_{2,7}+yT_{2,11}) &\geq |2x+2y| \\
g_{\mathrm{st}}(xT_{2,7}+yT_{2,11}) &\geq |2x+3y| \\
g_{\mathrm{st}}(xT_{2,7}+yT_{2,11}) &\geq |2x+4y| \\
g_{\mathrm{st}}(xT_{2,7}+yT_{2,11}) &\geq |3x+4y| \\
g_{\mathrm{st}}(xT_{2,7}+yT_{2,11}) &\geq |3x+5y|.
\end{aligned}$$

Based on these, we find that the unit ball $B_{g_{\mathrm{st}}}$ restricted to the span of $T_{2,7}$ and $T_{2,11}$ is contained in the set illustrated in Figure 3.

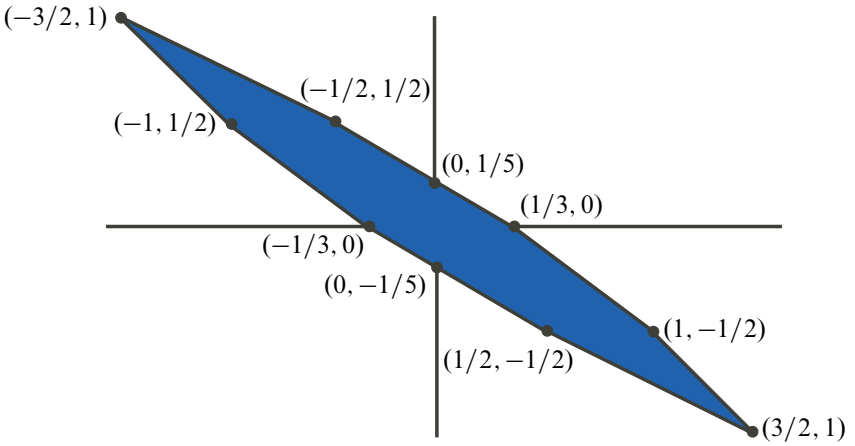

Figure 3: The unit ball for g_{st} in the span of $T_{2,7}$ and $T_{2,11}$

By convexity, to show that this set is actually the unit ball for g_{st}, we need to check only the vertices. For instance, we want to see that $g_{\mathrm{st}}(\frac{3}{2}T_{2,7}-T_{2,11})=1$. That is, we need to show $g_{\mathrm{st}}(3T_{2,7}-2T_{2,11})=2$. That calculation was done in the previous section. The other vertices are handled similarly. (That is, one shows that $g_4(T_{2,7})=3$, $g_4(T_{2,7}-T_{2,11})=2$, $g_4(3T_{2,7}-2T_{2,11})=2$ and $g_4(2T_{2,7}-T_{2,11})=2$. The point $(0,1/5)$ is not a vertex so need not be considered.)

Note Rick Litherland [10] has proved that for any pair of two stranded torus knots, the 4–genus of a linear combination $xT_{2,k} + yT_{2,j}$ is determined by its signature function.

5 A smooth versus topological comparison: $xT_{3,7} + yT_{2,5}$

We now summarize a more complicated example of the computation of the g_{st} unit ball on the 2–dimensional subspace spanned by $T_{3,7}$ and $T_{2,5}$. The added complexity occurs because the signature function of $T_{3,7}$ does not determine its smooth 4–genus; this signature function has positive jumps at $1/21, 2/21, 4/21, 5/21$ and $8/21$ but a negative jump at $10/21$. Thus its maximum value is 5, and its value at $t = 1/2$ is 4. On the other hand, the Ozsváth–Szabó τ or Khovanov–Rasmussen invariant s both take value 6, and thus determine the smooth 4–genus of $T_{3,7}$ to be 6. (See [12; 13] for details.)

Considering only the signature function, we can show that the unit g_{st} ball is contained within the entire shaded region. Using either τ or s places additional bounds which eliminate the two thin darker triangles. The innermost parallelogram represents points that we know are in the unit ball.

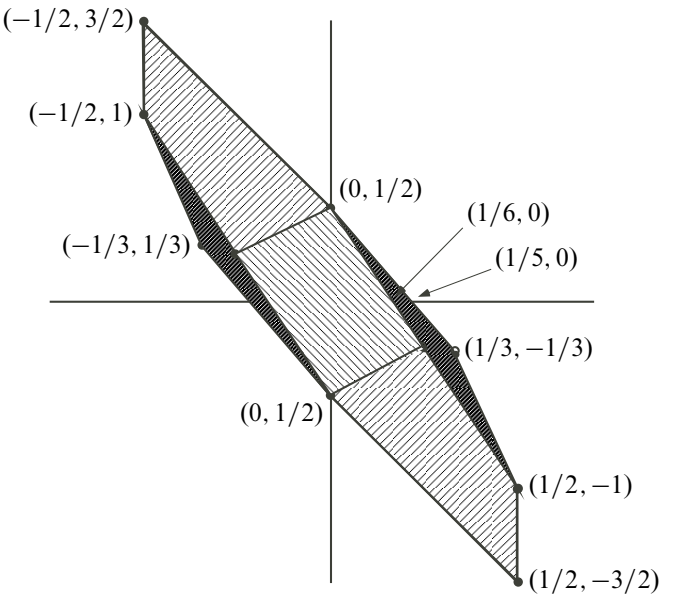

Figure 4: Bounds on the topological and smooth unit ball for g_{st} on the span $\langle T_{3,7} + yT_{2,5} \rangle$

Note Recent work has slightly enlarged the region which we know lies in the unit g_{st} ball for the span of these two knots, but most of the region remains unknown. We know of no knots in this span for which the topological and smooth 4–genus differ.

6 A 4–dimensional example

As our final example related to finding a g_{st} unit ball, we consider the span of the first four knots that are of infinite order in $\mathcal{C}$: 3_1, 5_1, 5_2 and 6_2. If we identify the span of these with $\mathbf{Q}^4$ via the coordinates $x_1(3_1)+x_2(5_1)+x_3(5_2)+x_4(6_2)$, then the unit ball determined by the maximum of the signature function turns out to be a polyhedron formed as the convex hull of 24 points that come in antipodal pairs. We list one from each pair:

(1) $(2,-1,0,0),(0,1,-2,0),(0,1,0,-1),(2,-1,0,-1),(0,0,1,0),(2,0,-1,0)$

(2) $(0,1,0,-2)$

(3) $(2,1,-2,-2),(2,1,-2,-1),(0,1,-2,1),(0,0,1,-2),(2,0,-1,-2)$.

Those in the first set of five have all been shown to have $g_4=1$. For those in the last set we have been unable to compute the genus or stable genus. For the second set, $(0,1,0,-2)$, we have been unable to compute the 4–genus, but we know that twice this knot has 4–genus 2, and hence its stable 4–genus is 1.

7 A knot K with $g_{\mathrm{st}}(K)$ near $1/2$. Gilmer, Casson–Gordon bounds

We begin by presenting Gilmer's result [7] bounding the 4–genus of a knot K in terms of Casson–Gordon signature invariants [4].

Let K be a knot and let $M_d(K)$ denote its d–fold branched cover, with d a prime power. To each prime p and character $\chi\colon H_1(M_d(K),\mathbf{Z})\to\mathbf{Z}_p$, there is the Casson–Gordon invariant $\sigma(K,\chi)\in\mathbf{Q}$. By [7], this invariant is additive under connected sum of knots and direct sums of characters. A special case of the main theorem of Gilmer [6] states the following:

Theorem 5 *If K is an algebraically slice knot for which $H_1(M_d(K),\mathbf{Z})\cong\mathbf{Z}_p^{2n}$ and $g_4(K)=g$, then there is a subspace $H\subset\mathrm{Hom}(H_1(M_d(K),\mathbf{Z}),\mathbf{Z}_p)\cong H^1(M_d(K),\mathbf{Z}_p)$ of dimension $\frac{1}{2}(2n-2(d-1)g)$ such that for all $\chi\in H$, $|\sigma(K,\chi)|\le 2dg$.*

It was observed by Gilmer and the author in [8] that H can be assumed to be invariant under the deck transformation. Applying this and specializing to the case of $d = 3$, we have:

Corollary 6 *If K is an algebraically slice knot for which $H_1(M_3(K), \mathbf{Z}) \cong \mathbf{Z}_p^{2n}$ and $g_4(K) = g$, then there is a $\mathbf{Z}_3$–invariant subspace $H \subset \mathrm{Hom}(H_1(M_3(K), \mathbf{Z}), \mathbf{Z}_p) \cong H^1(M_3(K), \mathbf{Z}_p)$ of dimension $n - 2g$ such that for all $\chi \in H$, $|\sigma(K, \chi)| \leq 6g$.*

7.1 The example

Consider the knot illustrated in Figure 5, which we denote $K(J_1, J_2)$. This family of knots has been used throughout the study of knot concordance; a detailed description can be found, for instance, in [8], in which the details of the results we now summarize can be found. First, the homology of the 3–fold branched cover is the direct sum of cyclic groups of order seven: $H_1(M_3(K(J_1, J_2))) \cong \mathbf{Z}_7 \oplus \mathbf{Z}_7$. Furthermore, the homology splits as the direct sum of $E_2 \cong \mathbf{Z}_7$ and $E_4 \cong \mathbf{Z}_7$, the 2–eigenspace and 4–eigenspace of the deck transformation. (Note that $2^3 = 4^3 = 1 \mod 7$.)

Similarly, $H_1^*(M_3(K(J_1, J_2))) = \mathrm{Hom}(H_1(M_3(K(J_1, J_2))), \mathbf{Z}_p)$ splits as a direct sum of eigenspaces, which we denote E_2^* and E_4^*. Using two eigenvectors as a basis for $H_1^*(M_3(K(J_1, J_2)))$ and letting $\chi_{a,b}$ be the character corresponding to (a, b) via this identification, as proved in [8] we have:

Theorem 7 *The invariants $\sigma(K(J_1, J_2), \chi_{a,0}) = \sigma_{a/7}(J_1) + \sigma_{2a/7}(J_1) + \sigma_{4a/7}(J_1)$ and $\sigma(K(J_1, J_2), \chi_{0,b}) = \sigma_{b/7}(J_2) + \sigma_{2b/7}(J_2) + \sigma_{4b/7}(J_2)$. In particular, it follows that $\sigma(K(J_1, J_2), \chi_{0,0}) = 0$.*

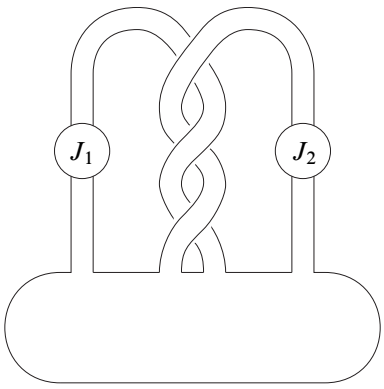

Figure 5: The knot $K(J_1, J_2)$

We can now demonstrate that particular knots in this family have $g_{\mathrm{st}}(K(J_1, J_2))$ near $1/2$.

Theorem 8 *For any $\epsilon > 0$, there is a knot J so that $\frac{1}{2}(1-\epsilon) \leq g_{\mathrm{st}}(K(J,-J)) \leq 1/2$.*

Proof By the additivity of 3–genus, for any knot J we have $g_3(2K(J,-J)) = 2$. On the evident Seifert surface for $2K(J,-J)$ there is a curve on the surface with framing 0 representing the knot $J\#-J$, which is slice. Thus, the Seifert surface can be surgered in the 4–ball to give a surface of genus one bounded by $2K(J,-J)$. Therefore, $g_4(2K(J,-J)) \leq 1$ and $g_{\mathrm{st}}(K(J,-J)) \leq 1/2$.

We now proceed to show that for each ϵ there is some J for which $g_{\mathrm{st}}(K(J,-J)) \geq \frac{1}{2}(1-\epsilon)$. For a given J, if this is inequality is false, then for some $n > 0$, we have $g_4(nK(J,-J)) < \frac{1}{2}(1-\epsilon)n$. (Since this holds for some n, it holds for all n sufficiently large.) For this n, we have $H_1(M_3(nK(J,-J))) = \mathbf{Z}_7^{2n}$. Applying Corollary 6 we find the relevant subgroup H has dimension $\dim(H) > n - 2(\frac{1}{2}(1-\epsilon)n)$. Simplifying, we have $\dim(H) > \epsilon n$.

As $H_1(M_3(K(J,-J)))$ splits as the direct sum of a 2–eigenspace and a 4–eigenspace, the same is true for $H_1(M_3(nK(J,-J)))$. Thus, we also have an eigenspace splitting of $H_1^*(M_3(nK(J,-J)))$. The subspace H given by Corollary 6 is invariant under the deck transformation, so it too must split as the sum of eigenspaces, $H = H_2 \oplus H_4$. Given that $\dim(H) > \epsilon n$, one of these must have dimension at least $\frac{1}{2}\epsilon n$. We will assume $\dim(H_2) > \frac{1}{2}\epsilon n$; the case $\dim(H_4) > \frac{1}{2}\epsilon n$ is similar.

We next use the fact, easily established using the Gauss–Jordan algorithm, that a subspace of dimension a in $\mathbf{Z}_p^b$ contains some vector with at least a nonzero coordinates. Thus, H_2 contains a vector h with at least $\frac{1}{2}\epsilon n$ nonzero coordinates.

For the character χ_h given by h, by the additivity of Casson–Gordon invariants and Theorem 7,

$$\sigma(K, \chi_h) = \sum \sigma(K(J,-J), \chi_{a_i,0}) = \sum \sigma_{a_i/7}(J)$$

where the sum has at least $\frac{1}{2}\epsilon n$ elements and each $a_i = 1, 2$, or 3. Now, letting $M > 0$ be a fixed constant assume that $\sigma_{a_i/7}(J) > M$ for $a_i = 1, 2, 3$. Such a J is easily constructed using the connected sum of $(2,k)$–torus knots. Then $\sigma(nK(J,-J), \chi_h) > \frac{1}{2}\epsilon nM$. Thus, we will have a contradiction to Corollary 6 if $\frac{1}{2}\epsilon nM \geq 6(\frac{1}{2}(1-\epsilon)n)$. Simplifying, we find that there is a contradiction if $M \geq 6((1-\epsilon)/\epsilon)$. In conclusion, if $\sigma_{a/7} \geq 6((1-\epsilon)/\epsilon)$ then $g_{\mathrm{st}}(K) \geq (1-\epsilon)\frac{1}{2}$.

In the case that we are working with the 4–eigenspace instead of the 2–eigenspace, the same condition appears, since Corollary 6 concerns the absolute value of the Casson–Gordon invariant, and switching eigenspaces simply interchanges $\sigma_{a/7}(J)$ with $\sigma_{a/7}(-J)$. □

7.2 Other noninteger examples

The 4–genus of the knot $K(J,-nJ)$ illustrated in Figure 6 can be shown to satisfy $g_4((n+1)K(J,-nJ)) \le n/(n+1)$, in much the same way as the special case of $n=1$, $K(J,-J)$. Thus, $g_{\mathrm{st}}(K(J,-nJ) \le n/(n+1)$. The argument used above, based on the 3–fold cover, cannot be successfully applied to find a lower bound. However, using the 2–fold cover we have been able to prove a weaker result. Given n, there is a J so that $(n-1)/n \le g_{\mathrm{st}}(K(J,-nJ)) \le n/(n+1)$.

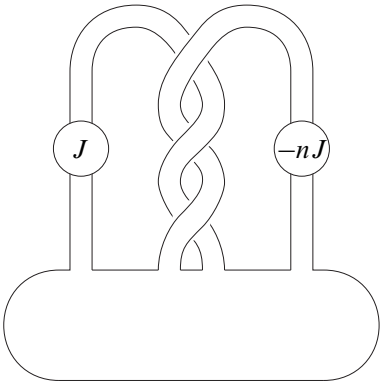

Figure 6: The knot $K(J_1, J_2)$

8 Questions

(1) Is g_{st} a norm on $\mathcal{C}_{\mathbf{Q}}$? That is, if $g_{\mathrm{st}}(K)=0$, does K represent torsion in $\mathcal{C}$?

(2) Is there a knot K such that $0 < g_{\mathrm{st}}(K) < \frac{1}{2}$? This question relates to that of finding torsion of order greater than 2 in $\mathcal{C}$. For instance, if there is a knot K of order three, then $g_4(3K)=0$. A simpler question than that of finding such a knot is to find a knot satisfying $g_4(3K)=1$ but $g_4(2K) \ge 2$.

(3) Is $g_{\mathrm{st}}(K) \in \mathbf{Q}$ for all K? Presumably the examples constructed in the previous section satisfy $g_{\mathrm{st}} = n/(n+1)$ for some n, though this seems difficult to prove.

(4) Related to this previous question, is there a knot for which $g_{\mathrm{st}}(K) \ne g_4(nK)/n$ for any n?

(5) Let $\{K_i\}$ be finite set of knots and let S be the span of these knots in $\mathcal{C}_{\mathbf{Q}}$. Is the g_{st} ball in S a finite sided polyhedron?

(6) For some pair of distinct nontrivial positive torus knots, $T_{p,q}$ and $T_{p',q'}$, with $p,q,p',q' > 2$, determine the unit g_{st} ball on their span in $\mathcal{C}_{\mathbf{Q}}$, in either the smooth or topological category.

8.1 Stable commutator length

If $g \in [G,G]$ is an element in the commutator subgroup of a group G, it can be expressed as a product of commutators. The shortest such expression for g is called the commutator length, $\mathrm{cl}(g)$. The limit $\lim_{n\to\infty} \mathrm{cl}(g^n)/n$ is called the stable commutator length. The notion was first studied by Bavard [1]. Although no formal connections between this and the stable 4–genus are known at this time, the possibility of such connections is provocative. We note that Calegari's work [3] has revealed much of the behavior of the stable commutator length for free groups. In particular, the stable commutator length is always rational for free groups, though this is not true for all groups by Zhuang [15]. Further details can be found in Calegari [2].

Appendix A Limits

We sketch the proof of Theorem 1, restated as follows.

Proposition 9 *Let $f\colon \mathbf{Z}_+ \to R_{\geq 0}$ satisfy $f(nm) \leq nf(m)$ for all n and m. Then $\lim_{n\to\infty} f(n)/n$ exists.*

Proof Let L be the greatest lower bound of $\{f(n)/n\}_{n\in\mathbf{Z}_+}$. For any ϵ there is an N such that $f(N)/N \leq L+\epsilon/2$. Any n can be written as $n = aN+b$ where $0 \leq b < N$. Also, $f(b) \leq B = \max\{f(b)\}_{0\leq b<N}$. By subadditivity we have $f(n) \leq af(N)+f(b)$. Dividing by n we have

$$\frac{f(n)}{n} \leq \frac{af(N)}{aN+b} + \frac{f(b)}{aN+b} \leq \frac{f(N)}{N} + \frac{B}{aN}.$$

Thus, if n is chosen large enough that $B/(aN) \leq \epsilon/2$, (for instance, choose $n \geq 2B/\epsilon + N$) we have $f(n)/n \leq L+\epsilon$. □

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