Stable systolic category of the product of spheres

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The stable systolic category of a closed manifold M indicates the complexity in the sense of volume. This is a homotopy invariant, even though it is defined by some relations between homological volumes on M. We show an equality of the stable systolic category and the real cup-length for the product of arbitrary finite dimensional real homology spheres. Also we prove the invariance of the stable systolic category under the rational equivalences for orientable 0–universal manifolds.

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1 Introduction

In this paper, a manifold is assumed to be closed, connected, orientable and smooth. The *systole* of a manifold M is the least length of non-contractible closed loops in M. One can generalize this concept to the least volume of k–dimensional non-zero homology classes, called the *homology systole*. Now we can imagine such systoles have some kind of relations with the entire volume of M, and it is natural to ask what kind of relationship exists.

As an answer, Gromov proved a theorem which says that the existence of non-trivial cup product implies the existence of the stable isosystolic inequality as follows.

Gromov's Theorem [7, 7.4.C] *Let M be an n–manifold. If there exist some reduced real cohomology classes $\alpha_1^*, \dots, \alpha_k^*$ with α_i^* in $\widetilde{H}^{d_i}(M;\mathbb{R})$ and a non-zero cup product $\alpha_1^* \smile \cdots \smile \alpha_k^*$ in $\widetilde{H}^n(M;\mathbb{R})$, then there exists $C > 0$ satisfying*

$$\prod_{i=1}^{k} \operatorname{stsys}_{d_i}(M, \mathcal{G}) \le C \cdot \operatorname{mass}\big([M], \mathcal{G}\big)$$

for all Riemannian metrics $\mathcal{G}$ on M where $\operatorname{stsys}_{d_i}$ is the stable d_i–systole and $[M]$ is the fundamental class of M with coefficients in $\mathbb{Z}/2\mathbb{Z}$.

The greatest k satisfying the stable isosystolic inequality is called the *stable systolic category* of M, which was introduced by Katz and Rudyak [8], and is known to

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be a homotopy invariant by Katz and Rudyak [9]. We will show the stable systolic category of a 0-universal manifold is also invariant under the rational equivalences in Corollary 4.3.

For an orientable manifold M, Gromov's Theorem implies that the stable systolic category is not smaller than the real cup-length. So, is there some manifold M such that the stable systolic category is greater than the real cup-length? If such M exists, then the inversion of Gromov's Theorem will fail for M. This interesting question is not answered yet, but equality is known for some manifolds, see Dranishnikov and Rudyak [3] for example. In this paper, we also show more equality later in Theorem 3.6 and Theorem 3.8.

1.1 Definition of the stable systolic category

To define the stable systolic category, we need to consider the flat homology theory as a metric space whose metric structure is induced by the integration on the space. One can see the details about currents and homological integration in Federer [4; 5], Federer and Fleming [6], Serre [10] and White [11]. Since we use the integration theory to define the norm on real homology vector space, we consider the local Lipschitz category $\mathfrak{L}$ whose objects are pairs of local Lipschitz neighborhood retracts in some finite dimensional Euclidean space and whose morphisms are locally Lipschitzian maps. One can find the formal definition of $\mathfrak{L}$ in Federer [4, 4.1.29 and 4.4.1]. In this section, we define some notations of flat homology theory on $\mathfrak{L}$ briefly and define systoles and systolic category for a manifold.

Let (X, A) be an object of $\mathfrak{L}$. Then we can assume that X and A possess the restricted metrics of $\mathbb{R}^n$. Let G be a $\mathbb{Z}$–module with a norm $|\cdot|$ which makes G a complete metric space. If G is $\mathbb{Z}$ or $\mathbb{R}$, we assume that norm of G is the standard norm. The *comass* of a differential form ω on X is defined as

$$\operatorname{comass}(\omega) := \sup\bigl\{|\omega_x(\tau)| : x \in X, \text{orthonormal } q\text{–frame } \tau\bigr\}.$$

Also, the *mass* of a q–current T in X is the dual norm of comass, that is,

$$\operatorname{mass}(T) := \sup\bigl\{T(\omega) : \text{differential } q\text{–form } \omega, \operatorname{comass}(\omega) \le 1\bigr\}.$$

A Lipschitzian singular q–cube $\kappa\colon I^q \to X$, induces a homomorphism $\kappa_\flat$ from the module of polyhedral chains $\mathcal{P}_q(X; G)$ to the module of rectifiable currents $\mathcal{R}_q(X; G)$. Then the mass of κ is defined by the mass of the image $\kappa_\flat I^q$ where I^q is the corresponding polyhedral q–current of the unit rectangular parallelepiped I^q. This correspondence of κ to $\kappa_\flat I^q$ gives a chain map Φ of degree 0 from the chain complex of all Lipschitzian singular cubes into the chain complex of flat chains $\mathcal{F}_*(\mathbb{R}^n | X; G)$.

Here $\mathcal{F}_*(\mathbb{R}^n|X;G)$ denotes the submodule of the flat chains $\mathcal{F}_*(\mathbb{R}^n;G)$ in $\mathbb{R}^n$ which consists of all flat chains supported in X. Then one can verify that Φ induces an isomorphism Φ_* from the singular homology module $H_q(X,A;G)$ to the homology module $H_q^\flat(X,A;G)$ of the flat chains which is called the *flat homology*.

For a Lipschitzian singular chain c, there exists a representation $\sum_i \kappa_i \otimes g_i$ where g_i is contained in G and κ_i is a Lipschitzian singular q–cube which is not overlapping each other (subdivide if necessary). Then the *mass* of c is defined as

$$\mathrm{mass}(c) := \sum_i |g_i| \cdot \mathrm{mass}(\kappa_i)\,.$$

The *mass* or *volume* of a singular homology class η in $H_q(X,A;G)$ is defined by

$$\mathrm{mass}(\eta;G) := \inf\bigl\{\mathrm{mass}(c) : \eta = [c],\ c \text{ is a Lipschitzian cycle}\bigr\}\,.$$

If G is $\mathbb{R}$, the mass is a norm on the homology vector spaces. We will omit G in the case of $\mathbb{Z}$.

The *q–dimensional homology systole* of (X,A) is defined by infimum of mass of non-trivial qth integral homology classes. However Gromov [2, page 301] claims that Gromov's Theorem will fail for $S^1 \times S^3$, if we consider the homology systoles instead of the stable systoles. Briefly, we can consider the stable systole as a systole in the real homology vector spaces. Here we give formal definition for the stable systole. The inclusion $\iota\colon \mathbb{Z} \to \mathbb{R}$ induces the coefficient homomorphism ι_* on homology. The *stable mass* on $H_q(X,A;\mathbb{Z})$ is defined as the mass of the image $\iota_*\eta$. Then we can define the *q–dimensional stable systole* of (X,A) as

$$\mathrm{stsys}_q(X,A) := \inf\{\mathrm{stmass}(\eta) : \eta \in H_q(X,A;\mathbb{Z}),\ \iota_*\eta \neq 0\}.$$

A homology q–systole or a stable q–systole is called *trivial*, if it is infinite. If the qth real homology vector space $H_q(X,A;\mathbb{R})$ is zero, then the stable q–systole is trivial for all Riemannian metrics on (X,A). Hence if the qth integral homology module $H_q(X,A;\mathbb{Z})$ is a torsion module, then the stable q–systole is trivial for every metric on (X,A).

For a given positive integer $n > 0$, a k–tuple $P = (p_1,\dots,p_k)$ of positive integers is called a *partition* of n if $n = p_1 + \cdots + p_k$ and $p_1 \le \cdots \le p_k \le n$. A partition P is called *positive* (or *non-negative*) if $p_i > 0$ (or $p_i \ge 0$) for all i. The *size* of a partition which denoted by $\mathrm{size}(P)$ is defined by the cardinality of positive integers contained in the partition. Hence if a k–tuple P is a positive partition, then the size of partition is k. From now on, we suppose a partition is positive unless otherwise stated. For a partition P, the *duplicated number* of p_i is the cardinality number of elements in P who are equal to p_i.

Now we define concepts for an n–manifold M. A partition P of n is called *stable systolic categorical* for M, if there exists a real number $C > 0$ and non-trivial stable p_i–systoles such that

$$\prod_{i=1}^{\mathrm{size}(P)} \mathrm{stsys}_{p_i}(M, \mathcal{G}) \leq C \cdot \mathrm{mass}\big([M], \mathcal{G}; \mathbb{Z}/2\mathbb{Z}\big)$$

for every Riemannian metric $\mathcal{G}$ on M with fundamental class $[M] \in H_n(M; \mathbb{Z}/2\mathbb{Z})$.

Definition 1.1 The *stable systolic category* of M is defined by

$\mathrm{cat}_{\mathrm{stsys}}(M) := \sup(\{\mathrm{size}(P) : P$ is stable systolic categorical partition for $M\} \cup \{0\})$.

As we said before, the real cup-length is a lower estimate for the stable systolic category from Gromov's Theorem, where the *real cup-length* of M is defined by

$$\mathrm{cup}_{\mathbb{R}}(M) := \min\big\{k \geq 0 : \alpha_0 \smile \alpha_1 \smile \cdots \smile \alpha_k = 0 \text{ for all } \alpha_i \in \widetilde{H}^*(M; \mathbb{R})\big\}$$

and $\widetilde{H}^*(M; \mathbb{R})$ denotes the reduced real cohomology ring of M.

If M is non-orientable, then the top dimensional real cohomology vector space $H^n(M; \mathbb{R})$ vanishes. So every cohomology class in $H^n(M; \mathbb{R})$ vanishes, we can not apply Gromov's Theorem for top dimension. This is a reason to consider only orientable manifolds in this paper.

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2 Preliminaries on the stable systoles

Many equations and inequalities for mass are studied. One can find those results at Babenko [1], Federer [4] and Whitney [12]. Here we state or recall some of them for the stable systoles, with some appropriate modifications applied. Through this section, we suppose U and V be open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively.

Proposition 2.1 *For a non-empty local Lipschitz neighborhood retract X in $\mathbb{R}^n$, the stable 0–systole is 1.*

Proof Let $\mathcal{D}_0(X)$ be the vector space of 0–currents. A map $\mathfrak{d}\colon X \to \mathcal{D}_0(X)$ can be defined as $\mathfrak{d}(x)(\omega) = \mathfrak{d}_x(\omega) := \omega(x)$ for a point x of X and a differential 0–form ω on X. Then $\mathfrak{d}_x$ is a polyhedral 0–current with $\mathrm{mass}(\mathfrak{d}_x) = 1$. This implies that $\mathfrak{d}_x$ is a normal 0–cycle with coefficients $\mathbb{Z}$. Furthermore, the image $\iota_*\Phi_*^{-1}[\mathfrak{d}_x]$ is not vanished in $H_0(X;\mathbb{R})$. So we have

$$\mathrm{stsys}_0(X) = \mathrm{mass}\big(\iota_*\Phi_*^{-1}[\mathfrak{d}_x]\big) = 1$$

for an arbitrary point x in X. □

Lemma 2.2 *For a local Lipschitz neighborhood retract X in $\mathbb{R}^n$, if one rescale the standard metric $\mathcal{G}$ on $\mathbb{R}^n$ by the square of a real number $t > 0$, then the quotient mass of a homology class $\eta \in H_q(X;G)$ increase by the t^q times. Furthermore, the stable q–systole satisfies*

$$\mathrm{stsys}_q(X, t^2\mathcal{G}|X) = t^q \cdot \mathrm{stsys}_q(X, \mathcal{G}|X)$$

where $\mathcal{G}|X$ is the restriction of $\mathcal{G}$ on X.

Proof A similar result was introduced by Whitney [12] for the real flat chains. So the first result is satisfied for an arbitrary homology class. Also the definition of the stable systole implies

$$\mathrm{stsys}_q(X, t^2\mathcal{G}|X) = \inf\big\{t^q \cdot \mathrm{mass}(\iota_*\eta, \mathcal{G}|X;\mathbb{R}) : \eta \in H_q(X, A;\mathbb{Z}),\ \iota_*\eta \neq 0\big\}$$

which means the equality for the stable systoles. □

Proposition 2.3 (Whitney [12, X.6 and X.7]) *For a locally Lipschitzian map $f\colon U \to V$ and an integral rectifiable q–current T whose support is contained in a compact subset K of U, there exists an inequality*

$$\mathrm{mass}(f_\flat T) \le \mathrm{Lip}(f|K)^q \cdot \mathrm{mass}(T)$$

where $\mathrm{Lip}(f|K)$ is the lower bound of Lipschitz constants of the restriction $f|K$.

Proposition 2.4 *If $f\colon (X, A) \to (Y, B)$ is a locally Lipschitzian map, then for any homology class η of $H_q(X, A; G)$, there is a compact subset K of $\mathbb{R}^m$ which satisfies*

$$0 \le \mathrm{mass}(f_*\eta; G) \le \mathrm{Lip}(f|K)^q \cdot \mathrm{mass}(\eta; G)$$

where $f_*\colon H_q(X, A; G) \to H_q(Y, B; G)$ is the induced homomorphism.

Proof Note that f induces a homomorphism $f_\flat \colon Z_q(X, A; G) \to Z_q(Y, B; G)$ on flat cycles as well as $f_\flat \mathcal{F}_q(\mathbb{R}^m | A; G) \subset \mathcal{F}_q(\mathbb{R}^n | B; G)$. For a given flat homology class $\Phi_* \eta$, let T be a representative normal q–cycle in $Z_q(X, A; G)$. The naturality of Φ_* implies $\Phi_* f_* \eta = f_* \Phi_* \eta = f_*[T] = [f_\flat T]$. Also the relation of cosets $[f_\flat T] = [f_\flat T + f_\flat \mathcal{F}_q(\mathbb{R}^m | A; G)] = [f_\flat T + \mathcal{F}_q(\mathbb{R}^n | B; G)]$ implies that the relation of the sets

$$\{f_\flat T : [T] = \Phi_* \eta\} \subset \{S : [S] = \Phi_* f_* \eta\} \subset Z_q(Y, B; G)\,.$$

With the definition of the mass of homology class, we obtain

$$\operatorname{mass}(f_* \eta; G) \le \inf\{\operatorname{mass}(f_\flat T) : [T] = \Phi_* \eta\}.$$

Because of T is compact supported, there is a compact subset K of $\mathbb{R}^m$ with $\operatorname{supp}(T) \subset \operatorname{Int}(K)$. Here we can apply Proposition 2.3 for T, so we have

$$\operatorname{mass}(f_* \eta; G) \le \operatorname{Lip}(f | K)^q \cdot \inf\{\operatorname{mass}(T) : [T] = \Phi_* \eta\}$$

which implies the result. □

Lemma 2.5 *Let (X, A) and (Y, B) are local Lipschitz neighborhood retract pairs. If a locally Lipschitzian map $f \colon (X, A) \to (Y, B)$ induces a monomorphism*

$$f_* \colon H_q(X, A; \mathbb{R}) \to H_q(Y, B; \mathbb{R}),$$

then there is a compact subset K in the ambient space of X satisfying

$$\operatorname{stsys}_q(Y, B) \le \operatorname{Lip}(f | K)^q \cdot \operatorname{stsys}_q(X, A).$$

Furthermore, if $H_q(X, A; \mathbb{R})$ is non-zero, then $\operatorname{stsys}_q(Y, B)$ is a positive real number.

Proof Proposition 2.4 and $f_*\big(H_q(X, A; \mathbb{R}) \setminus \{0\}\big) \subset \big(H_q(Y, B; \mathbb{R}) \setminus \{0\}\big)$ imply the existence of inequality in the stable systole level.

For integral homology class η with $\iota_* \eta$ is non-zero, the image $f_* \iota_* \eta$ does not vanish, since f_* is a monomorphism. Recall that the mass of real homology classes is a norm, hence $\operatorname{mass}(f_* \iota_* \eta)$ is a positive real number. Furthermore, the stable q–systole does not converge to zero, since $\mathbb{Z}$ is discrete. □

Let $\mathcal{K}(U)$ be the set of all real valued compact supported continuous functions on U. We denote $\mathcal{K}^+(U)$ the subset of non-negative valued functions. For a subset A of U, we say a sequence of functions $f_1, f_2, \ldots$ in $\mathcal{K}(U)$ *suits* A, if $f_i(x) \le f_{i+1}(x)$ and $\lim_{i \to \infty} f_i(x) \ge 1$ for every x in A.

For a rectifiable current T in $\mathcal{R}_q(U)$ and a function f in $\mathcal{K}^+(U)$, a monotone Daniell integral $\|T\|$ can be defined by

$$\|T\|(f) := \sup\big\{T(\omega) : \text{comass}(\omega_x) \leq f(x) \text{ for all } x \in U\big\}$$

where the supremum is taken over all compact supported differential q–forms ω on U. In addition, there is associated Radon measure

$$\rho_T(A) := \inf\{\lim_{i\to\infty} \|T\|(f_i) : f_1, f_2, \dots \text{ suits } A\}$$

for a subset A of U, which satisfying

$$\|T\|(f) = \int_U f\, d\rho_T.$$

If we consider a function 1_U which is defined by $1_U(x) = 1$ for all x, the mass is obtained by ρ_T as

$$\rho_T(U) = \|T\|(1_U) = \text{mass}(T).$$

One can find more details about these arguments in Federer [4, 2.5 and 4.1].

Proposition 2.6 *For rectifiable currents S in $\mathcal{R}_p(U)$ and T in $\mathcal{R}_q(V)$, the mass of their cross product is equal to the multiplication of their masses, that is,*

$$\text{mass}(S \times T) = \text{mass}(S) \cdot \text{mass}(T)$$

with respect to the product metric on $U \times V$.

Proof Since S and T are rectifiable currents, mass can be written by associated Radon measures ρ_S, ρ_T and $\rho_{S\times T}$. Therefore Fubini's Theorem (see Federer [4, 2.6.2.(2)]) implies

$$\text{mass}(S \times T) = \rho_{S\times T}(U \times V) = \rho_S(U) \cdot \rho_T(V) = \text{mass}(S) \cdot \text{mass}(T). \qquad \square$$

Lemma 2.7 *Let (X, A) and (Y, B) are local Lipschitz neighborhood retract pairs. For homology classes $\xi \in H_p(X, A; G)$ and $\eta \in H_q(Y, B; G)$, we can estimate*

$$\text{mass}(\xi \times \eta; G) \leq \text{mass}(\xi; G) \cdot \text{mass}(\eta; G)$$

and
$$\text{stsys}_{p+q}\big((X, A) \times (Y, B)\big) \leq \text{stsys}_p(X, A) \cdot \text{stsys}_q(Y, B)$$

with respect to the product metric on $(X, A) \times (Y, B)$.

Proof Let S and T be representative rectifiable cycles corresponding to ξ and η respectively, that is, $\Phi_*\xi = [S]$ with $S \in Z_p^\flat(X, A; G)$ and $\Phi_*\eta = [T]$ with $T \in Z_q^\flat(Y, B; G)$. Then the naturality of a cross product implies that there is a representative rectifiable current with the form of a cross product $S \times T$ in the coset $[c] = \Phi_*(\xi \times \eta)$. Therefore

$$\begin{aligned}\big\{S \times T : [S] \times [T] = \Phi_*\xi \times \Phi_*\eta\big\} &= \big\{S \times T : [S \times T] = \Phi_*(\xi \times \eta)\big\} \\ &\subset \big\{c : [c] = \Phi_*(\xi \times \eta)\big\} \\ &\subset Z^\flat_{p+q}\big((X, A) \times (Y, B); G\big).\end{aligned}$$

Hence Proposition 2.6 implies an inequality

$$\begin{aligned}\mathrm{mass}(\xi \times \eta; G) &\leq \inf\big\{\mathrm{mass}(S \times T) : [S] \times [T] = \Phi_*\xi \times \Phi_*\eta)\big\} \\ &= \mathrm{mass}(\xi; G) \cdot \mathrm{mass}(\eta; G)\end{aligned}$$

on homology level. To show the inequality of the stable systoles, recall that the cross product homomorphism

$$H_p(X, A; \mathbb{R}) \otimes H_q(Y, B; \mathbb{R}) \to H_{p+q}\big((X, A) \times (Y, B); \mathbb{R}\big)$$

is a monomorphism. Therefore we can estimate the stable q–systole as

$$\begin{aligned}\mathrm{stsys}_{p+q}\big((X, A) \times (Y, B)\big) &\leq \inf\left\{\mathrm{mass}(\xi \times \eta) : \begin{array}{l}\xi \in H_p(X, A; \mathbb{Z}),\ \iota_*\xi \neq 0, \\ \eta \in H_q(Y, B; \mathbb{Z}),\ \iota_*\eta \neq 0\end{array}\right\} \\ &\leq \mathrm{stsys}_p(X, A) \cdot \mathrm{stsys}_q(Y, B)\,.\end{aligned}$$

where the second inequality is obtained by the result on homology level. □

Lemma 2.8 *Suppose X and Y are local Lipschitz neighborhood retracts. If Y is connected and the Künneth formula gives an isomorphism of non-trivial vector spaces*

$$H_q(X; \mathbb{R}) \otimes H_0(Y; \mathbb{R}) \cong H_q\big(X \times Y; \mathbb{R}\big) \neq \{0\}\,,$$

then the stable q–systole satisfies

$$0 < \mathrm{stsys}_q\big(X \times Y\big) = \mathrm{stsys}_q(X) < \infty.$$

with respect to the product metric on $X \times Y$.

Proof Let $\mathfrak{pr}_1 \colon X \times Y \to X$ be the first projection. From the assumption, for a non-zero homology class η in $H_q(X \times Y; \mathbb{R})$, there exist $[S] \neq 0$ in $H^\flat_q(X; \mathbb{R})$ and $[T] \neq 0$ in $H^\flat_0(Y; \mathbb{R})$ whose cross product is the image of η in $H^\flat_q\big(X \times Y; \mathbb{R}\big)$ with the same positive mass, that is,

$$\mathrm{mass}\big([S] \times [T]\big) = \mathrm{mass}(\eta) > 0.$$

Note that the vector space of normal 0–chains $\mathcal{N}_0(Y; \mathbb{R})$ is equal to the vector space of polyhedral 0–chains $\mathcal{P}_0(Y; \mathbb{R})$ which is generated by $\{\mathfrak{d}_y : y \in Y\}$ where $\mathfrak{d}$ is defined in the proof of Proposition 2.1. For all points y and y' in Y, $[\mathfrak{d}_y]=[\mathfrak{d}_{y'}]$ implies that there is a non-zero real number r such that $[T]=r[\mathfrak{d}_y]$ with $\mathrm{mass}[T]=|r|\cdot\mathfrak{d}_y(1_Y^*)=|r|$. Also, every $[S]\times[T]$ has representation of $[r\cdot S]\times[\mathfrak{d}_y]$, therefore $\mathfrak{pr}_{1*}$ is an isomorphism with $\mathfrak{pr}_{1*}\big([S]\times[T]\big)=[r\cdot S]$. Hence Lemma 2.5 implies

$$\mathrm{stsys}_q(X\times Y)\geq \mathrm{stsys}_q(X)>0$$

with the fact of $\mathfrak{pr}_1$ is a Lipschitzian map with $\mathrm{Lip}(\mathfrak{pr}_1)=1$. As a result, we obtain the equality by combining the result of Lemma 2.7. □

3 Calculation by dimension and constructing metrics

At first, we will calculate the stable systolic category from the dimensional information of homology. If the homology group is not so complex, such as in the case of a real homology sphere, we know the stable systolic category by only using dimensional information. If an oriented manifold has a relatively simple cup-product structure such as n–fold producted space of spheres, then the stable systolic category can be also calculated instantly. Such methods to calculate the stable systolic category can be generalized as follows.

For a topological space X, let $\mathrm{lpd}(X)$ denote the *least positive dimension* of real cohomology vector spaces of X. So $\mathrm{lpd}(X)=l$ if and only if $\widetilde{H}^i(X;\mathbb{R})=\{0\}$ for $0<i<l$ and $\widetilde{H}^l(X;\mathbb{R})\neq\{0\}$. If M is an m–manifold, then $\mathrm{lpd}(M)$ is less than or equal to m.

Definition 3.1 An n–dimensional CW space X is said to *have maximal real cup length*, if there exist some real cohomology classes $\alpha_1,\dots,\alpha_r$ with $\alpha_i\in\widetilde{H}^{d_i}(X;\mathbb{R})$, a non-zero cup-product $\alpha_1\smile\cdots\smile\alpha_r\in\widetilde{H}^n(X;\mathbb{R})$ and $r:=\lfloor n/\mathrm{lpd}(X)\rfloor$ where $\lfloor x\rfloor$ denotes the floor of a real number x.

Example Let S be a manifold which is a real homology sphere. Then S has maximal real cup length, because of $\mathrm{lpd}(S)=\dim(S)$. The n–fold direct product of S also has maximal real cup length. The direct product $S^2\times S^3$ of spheres has maximal real cup length.

Corollary 3.2 *If an m–manifold M has maximal real cup length, then the stable systolic category of M is equal to the real cup-length of M, that is,*

$$\mathrm{cat}_{\mathrm{stsys}}(M)=\mathrm{cup}_{\mathbb{R}}(M)=\lfloor m/\,\mathrm{lpd}(M)\rfloor.$$

Proof We need to verify that $\operatorname{cat}_{\mathrm{stsys}}(M) \le \operatorname{cup}_{\mathbb{R}}(M)$. Let $r := \lfloor m/\operatorname{lpd}(M) \rfloor$. If $(d_1, \dots, d_k)$ is a partition of m such that each stable d_i–systole is non-trivial, then $d_i \ge \operatorname{lpd}(M)$, so there is an inequality

$$k \cdot \operatorname{lpd}(M) \le m = d_1 + \cdots + d_k < (r+1) \cdot \operatorname{lpd}(M)$$

which implies $k \le r = \operatorname{cup}_{\mathbb{R}}(M)$. □

In general, the direct product $M \times N$ of manifolds does not have maximal real cup length even if M and N have maximal real cup-length. For example, the direct product of spheres $S^1 \times S^2$ does not have maximal real cup length.

Lemma 3.3 *If manifolds $M_1^{m_1}, \dots, M_n^{m_n}$ have maximal real cup length, then the stable systolic category of their n–fold direct product $M_1 \times \cdots \times M_n$ is greater than the sum of stable systolic categories for each M_i, that is,*

$$\operatorname{cat}_{\mathrm{stsys}}(M_1 \times \cdots \times M_n) \ge \operatorname{cat}_{\mathrm{stsys}}(M_1) + \cdots + \operatorname{cat}_{\mathrm{stsys}}(M_n).$$

Proof Since M_i has maximal real cup length, there is non-zero cup product $\alpha_{i,1} \smile \cdots \smile \alpha_{i,r_i}$ in $H^{m_i}(M_i; \mathbb{R})$ where $r_i := \lfloor m_i/\operatorname{lpd}(M_i) \rfloor = \operatorname{cat}_{\mathrm{stsys}}(M_i)$ for $1 \le i \le n$.

By the Künneth formula, the n–fold cross product on the top dimensions induces an isomorphism

$$\bigotimes_{i=1}^{n} H^{m_i}(M_i; \mathbb{R}) \cong H^m(M_1 \times \cdots \times M_n; \mathbb{R}) \quad \text{where} \quad m := \sum_{i=1}^{n} m_i.$$

This implies that the cross product of all $\alpha_{i,1} \smile \cdots \smile \alpha_{i,r_i}$ is non-zero which can be written as a cup product

$$\smile_{i=1}^{n} \mathfrak{pr}_i^*\big(\alpha_{i,1} \smile \cdots \smile \alpha_{i,r_i}\big) = \mathfrak{pr}_1^*\alpha_{1,1} \smile \cdots \smile \mathfrak{pr}_i^*\alpha_{i,j_i} \smile \cdots \smile \mathfrak{pr}_n^*\alpha_{n,r_n}$$

in the top-dimensional real cohomology vector space $H^m(M_1 \times \cdots \times M_n; \mathbb{R})$, where $\mathfrak{pr}_i \colon M_1 \times \cdots \times M_n \to M_i$ is the ith projection, $1 \le i \le n$ and $1 \le j_i \le r_i$. This cup product implies that $r_1 + \cdots + r_n$ is a lower estimate for the stable systolic category of $M_1 \times \cdots \times M_n$ from Gromov's Theorem. □

Proposition 3.4 *For manifolds M and N, the least positive dimension of cohomology of $M \times N$ is the minimum of $\operatorname{lpd}(M)$ and $\operatorname{lpd}(N)$.*

Proof From the Künneth formula, the cohomology $H^i(M \times N; \mathbb{R}) = \{0\}$ for $0 < i < \min\big(\operatorname{lpd}(M), \operatorname{lpd}(N)\big)$. If $l := \min\big(\operatorname{lpd}(M), \operatorname{lpd}(N)\big) = \operatorname{lpd}(M)$, then $H^l(M; \mathbb{R})$ is non-zero and the cross product homomorphism

$$H^l(M; \mathbb{R}) \otimes H^0(N; \mathbb{R}) \to H^l(M \times N; \mathbb{R})$$

is a monomorphism. Therefore $H^l(M \times N;\mathbb{R})$ is non-zero. The case of $\mathrm{lpd}(M) > \mathrm{lpd}(N)$ is shown by using the same arguments. $\square$

For integers i and $j \neq 0$, let $\mathrm{mod}(i,j)$ denotes the remainder from the division of i by j.

Corollary 3.5 *Suppose manifolds M^m and N^n have maximal real cup length, and an integer $l := \mathrm{lpd}(M \times N)$. If M and N satisfy the conditions*

$$\lfloor m/\,\mathrm{lpd}(M)\rfloor = \lfloor m/l\rfloor, \qquad \lfloor n/\,\mathrm{lpd}(N)\rfloor = \lfloor n/l\rfloor$$

and
$$\mathrm{mod}(m,l) + \mathrm{mod}(n,l) < l,$$

then $M \times N$ has maximal real cup length. Therefore,

$$\mathrm{cat}_{\mathrm{stsys}}(M \times N) = \mathrm{cat}_{\mathrm{stsys}}(M) + \mathrm{cat}_{\mathrm{stsys}}(N).$$

Proof Let integers $r := \lfloor m/l\rfloor$ and $s := \lfloor n/l\rfloor$.

Proposition 3.4 implies that $l = \min\big(\mathrm{lpd}(M), \mathrm{lpd}(N)\big) = \mathrm{lpd}(M \times N)$. So we can formulate $\lfloor (m+n)/\,\mathrm{lpd}(M \times N)\rfloor = r + s + \lfloor \mathrm{mod}(m,l) + \mathrm{mod}(n,l)\rfloor$. By the assumption, $\lfloor \mathrm{mod}(m, \mathrm{lpd}(M)) + \mathrm{mod}(n, \mathrm{lpd}(N))\rfloor$ is zero, so we have

$$\lfloor (m+n)/\,\mathrm{lpd}(M \times N)\rfloor = r + s.$$

Thus it is sufficient to show that there is a non-zero cup product with the length of $r+s$.

Since M and N have maximal real cup length, there are cohomology classes $\alpha_1, \dots, \alpha_r$ and $\beta_1, \dots, \beta_s$ with their cup products are non-zero cohomology classes $\alpha_1 \smile \cdots \smile \alpha_r$ in $H^m(M;\mathbb{R})$ and $\beta_1 \smile \cdots \smile \beta_s$ in $H^n(M;\mathbb{R})$. From the proof of Lemma 3.3, there is a non-zero cup product $\mathfrak{pr}_1^*\alpha_1 \smile \cdots \smile \mathfrak{pr}_1^*\alpha_r \smile \mathfrak{pr}_2^*\beta_1 \smile \cdots \smile \mathfrak{pr}_2^*\beta_s$ in the top dimensional cohomology vector space $H^{m+n}(M \times N;\mathbb{R})$. $\square$

Without the condition of the product $M \times N$ has maximal real cup length, we can generalize this corollary as follow.

Theorem 3.6 *Let manifolds M^m and N^n have maximal real cup length. If*

$$\mathrm{mod}\big(m, \mathrm{lpd}(M)\big) + \mathrm{mod}\big(n, \mathrm{lpd}(N)\big) < \max\big(\mathrm{lpd}(M), \mathrm{lpd}(N)\big),$$

then the stable systolic category of their product $M \times N$ is the sum of each stable systolic category, that is,

$$\mathrm{cat}_{\mathrm{stsys}}(M \times N) = \mathrm{cat}_{\mathrm{stsys}}(M) + \mathrm{cat}_{\mathrm{stsys}}(N).$$

Proof Since M and N have maximal real cup length,

$$r := \lfloor m/\operatorname{lpd}(M)\rfloor = \operatorname{cat}_{\mathrm{stsys}}(M) \quad \text{and} \quad s := \lfloor n/\operatorname{lpd}(N)\rfloor = \operatorname{cat}_{\mathrm{stsys}}(N)\,.$$

In the case of $\operatorname{lpd}(M) = \operatorname{lpd}(N)$ is Corollary 3.5. So we will assume $\operatorname{lpd}(M) < \operatorname{lpd}(N)$.

From Lemma 3.3, $\operatorname{cat}_{\mathrm{stsys}}(M\times N) \geq \operatorname{cat}_{\mathrm{stsys}}(M) + \operatorname{cat}_{\mathrm{stsys}}(N) = r+s$. Therefore, it is sufficient to show that any partition of $m+n$ whose size is greater than $r+s$, is not a stable systolic categorical partition.

Suppose the partition $(d_1,\dots,d_k)$ of $m+n$ is a stable systolic categorical for $M\times N$ with some integer $1\leq r'\leq k$ and the condition $0<\operatorname{lpd}(M)\leq d_1\leq\cdots\leq d_{r'}<\operatorname{lpd}(N)$. For an arbitrary $t\geq 1$, let $\mathcal{G}_t := t^2\mathcal{G}_M + \mathcal{G}_N$ be a Riemannian metric on $M\times N$. Then Lemma 2.2 and Lemma 2.8 imply that the stable systoles for the partition $(d_1,\dots,d_k)$ satisfies

$$\begin{aligned}\prod_{i=1}^{k}\operatorname{stsys}_{d_i}(M\times N,\mathcal{G}_t) &\geq \prod_{i=1}^{r'}\operatorname{stsys}_{d_i}(M,t^2\mathcal{G}_M)\cdot\prod_{j=r'+1}^{k}\operatorname{stsys}_{d_j}(M\times N,\mathcal{G}_t)\\ &= t^{d_1+\cdots+d_{r'}}\cdot\prod_{i=1}^{r'}\operatorname{stsys}_{d_i}(M,\mathcal{G}_M)\cdot\prod_{j=r'+1}^{k}\operatorname{stsys}_{d_j}(M\times N,\mathcal{G}_t)\end{aligned}$$

Since $t\geq 1$, we can obtain the inequality $\operatorname{stsys}_{d_j}(M\times N,\mathcal{G}_t)\geq\operatorname{stsys}_{d_j}(M\times N,\mathcal{G}_1)$ for each $r'+1\leq j\leq k$. On the other hands, the mass of integral fundamental class $[M\times N]$ is characterized by Lemma 2.2 and Lemma 2.7 as

$$\begin{aligned}\operatorname{mass}\big([M\times N],\mathcal{G}_t\big) &\leq \operatorname{mass}\big([M],t^2\mathcal{G}_M\big)\cdot\operatorname{mass}\big([N],\mathcal{G}_N\big)\\ &= t^m\cdot\operatorname{mass}\big([M],\mathcal{G}_M\big)\cdot\operatorname{mass}\big([N],\mathcal{G}_N\big).\end{aligned}$$

Here if we assume that $d_1+\cdots+d_{r'}>m$, then we have

$$\frac{\prod_{i=1}^{k}\operatorname{stsys}_{d_i}(M\times N,\mathcal{G}_t)}{\operatorname{mass}\big([M\times N],\mathcal{G}_t\big)}\geq t^{(d_1+\cdots+d_{r'})-m}\cdot\frac{\prod_{i=1}^{r'}\operatorname{stsys}_{d_i}(M,\mathcal{G}_M)\cdot\prod_{j=r'+1}^{k}\operatorname{stsys}_{d_j}(M\times N,\mathcal{G}_1)}{\operatorname{mass}\big([M],\mathcal{G}_M\big)\cdot\operatorname{mass}\big([N],\mathcal{G}_N\big)}$$

where the right-hand side of the inequality diverges as $t\to\infty$. This contradicts to that $(d_1,\dots,d_k)$ is a stable systolic categorical partition. Hence we obtain $d_1+\cdots+d_{r'}\leq m$ and $d_{r'+1}+\cdots+d_k\geq n$. This condition for m implies

$$r'\leq\lfloor(d_1+\cdots+d_{r'})/\operatorname{lpd}(M)\rfloor\leq\lfloor m/\operatorname{lpd}(M)\rfloor\leq r\,.$$

Let $s' := k-r'$. From the assumption, $\operatorname{lpd}(M)/\operatorname{lpd}(N)<1$ and

$$\operatorname{mod}(m,\operatorname{lpd}(M))+\operatorname{mod}(n,\operatorname{lpd}(N))<\operatorname{lpd}(N),$$

so we can calculate as

$$k = r' + s' \leq r + s$$

which implies $\mathrm{cat}_{\mathrm{stsys}}(M \times N) \leq \mathrm{cat}_{\mathrm{stsys}}(M) + \mathrm{cat}_{\mathrm{stsys}}(N)$. □

Corollary 3.7 *Suppose manifolds $M_0 \times M_1 \times \cdots \times M_k$ and $M_{k+1} \times \cdots \times M_n \times M_{n+1}$ have maximal real cup length with*

$$\mathrm{lpd}(M_0) = \mathrm{lpd}(M_1) = \cdots = \mathrm{lpd}(M_k)$$

and $$\mathrm{lpd}(M_{k+1}) = \cdots = \mathrm{lpd}(M_n) = \mathrm{lpd}(M_{n+1})\,.$$

Let $r_i := \lfloor \dim(M_i)/\,\mathrm{lpd}(M_i) \rfloor$ for $0 \leq i \leq n+1$. If $M_0, \ldots, M_{n+1}$ satisfy conditions $\dim(M_i) = \mathrm{lpd}(M_i) \cdot r_i$ for $1 \leq i \leq n$ and

$$\dim(M_0) - \mathrm{lpd}(M_0) \cdot r_0 + \dim(M_{n+1}) - \mathrm{lpd}(M_{n+1}) \cdot r_{n+1} < \max\big(\mathrm{lpd}(M_0), \mathrm{lpd}(M_{n+1})\big),$$

then

$$\mathrm{cat}_{\mathrm{stsys}}\left(\prod_{i=0}^{n+1} M_i\right) = \sum_{i=0}^{n+1} \mathrm{cat}_{\mathrm{stsys}}(M_i) = \sum_{i=0}^{n+1} r_i.$$

Note that Theorem 3.6 is not applied for the product $S^1 \times S^2$ of spheres, but we will show the equality for such partial cases as follow.

Theorem 3.8 *If manifolds $S_1^{m_1}, \ldots, S_n^{m_n}$ are real homology spheres, then the stable systolic category of their n–fold direct product is the number of spheres.*

Proof Since every real homology spheres have maximal real cup length, Lemma 3.3 gives us a lower estimate $\mathrm{cat}_{\mathrm{stsys}}(S_1 \times \cdots \times S_n) \geq n$.

Suppose $m_i \leq m_{i+1}$ for each $1 \leq i \leq n$. Then a partition $(m_1, \ldots, m_n)$ of $\sum_i m_i$ can be rewritten as $(r_1, \ldots, r_1, r_2, \ldots, r_{l-1}, r_l, \ldots, r_l)$ where r_i is a range. This corresponding to rewrite

$$S_1^{m_1} \times \cdots \times S_n^{m_n} = \big(S_1^{r_1} \times \cdots \times S_{s_1}^{r_1}\big) \times \big(S_{s_1+1}^{r_2} \times \cdots \times S_{s_1+s_2}^{r_2}\big) \times \cdots \times \big(S_{s_1+\cdots+s_{l-1}+1}^{r_l} \times \cdots \times S_{s_1+\cdots+s_{l-1}+s_l}^{r_l}\big)$$

where $r_i := m_{s_1+\cdots+s_{i-1}+1} = \cdots = m_{s_1+\cdots+s_{i-1}+s_i}$ with $r_i < r_{i+1}$ and $s_i > 0$ is the duplicated number of r_i, so that $s_1 + \cdots + s_l = n$. For simplicity, let define

$$X_p := S_1 \times \cdots \times S_{s_1+\cdots+s_p} \quad \text{and} \quad Y_p := S_{s_1+\cdots+s_p+1} \times \cdots \times S_n$$

for $1 \le p \le n$. Then $S_1 \times \cdots \times S_n = X_p \times Y_p$ and we can observe that $\mathcal{G}_{p,t} := t^2 \mathcal{G}_{X_p} + \mathcal{G}_{Y_p}$ is a Riemannian metric on $X_p \times Y_p$ for $t > 0$ when $\mathcal{G}_{X_p} + \mathcal{G}_{Y_p}$ is a Riemannian metric on $X_p \times Y_p$. Now we can apply Lemma 2.8 and Lemma 2.2, so there exist equations

$$\operatorname{stsys}_q(X_p \times Y_p, \mathcal{G}_{p,t}) = \operatorname{stsys}_q(X_p, t^2 \mathcal{G}_{X_p}) = t^q \cdot \operatorname{stsys}_q(X_p, \mathcal{G}_{X_p})$$

for the non-trivial stable systoles in the dimension of $1 \le q \le s_1 + \cdots + s_p$.

Let $(d_1, \ldots, d_k)$ be the longest stable systolic categorical partition for $S_1 \times \cdots \times S_n$ with the condition $d_i \le d_{i+1}$. Then we can rewrite $(d_1, \ldots, d_k)$ by the ranges $\{r_1, \ldots, r_l\}$ with the duplicated number $s_i' \ge 0$ of r_i. We will show that the partition is not longer than n by induction on p for $1 \le p \le l$ and contradiction. Assume that $s_i' = s_i$ for $1 \le i \le p-1$. If $s_p' > s_p$, then using a similar argument in the proof of Theorem 3.6, we can observe that the right-hand side of the inequality

$$\frac{\prod_{i=1}^{k} \operatorname{stsys}_{d_i}(X_p \times Y_p, \mathcal{G}_{p,t})}{\operatorname{mass}\big([X_p \times Y_p], \mathcal{G}_{p,t}\big)} \ge t^w \cdot \frac{\prod_{i=1}^{p} \operatorname{stsys}_{r_i}(X_p, \mathcal{G}_{X_p})^{s_i'} \cdot \prod_{i=p+1}^{l} \operatorname{stsys}_{r_i}(X_p \times Y_p, \mathcal{G}_{p,1})^{s_i'}}{\operatorname{mass}\big([X_p], \mathcal{G}_{X_p}\big) \cdot \operatorname{mass}\big([Y_p], \mathcal{G}_{Y_p}\big)}$$

diverges as t approaches ∞ where $w := r_1(s_1' - s_1) + \cdots + r_p(s_p' - s_p) = r_p(s_p' - s_p) > 0$. This contradicts to that the partition $(d_1, \ldots, d_k)$ is stable systolic categorical, and hence we obtain $s_p' \le s_p$. However we must choose $s_p' = s_p$ to make the longest partition. As a result, the size of the longest stable systolic categorical partition can not exceed $n = s_1 + \cdots + s_l$. $\square$

4 Invariance under the rational equivalences

Let U be an open subset of some finite dimensional Euclidean space. For a compact subset C of U and a flat q–chain T in $\mathcal{F}_q(U|C; \mathbb{R})$, the *flat norm* is defined by

$$|T|_C^\flat := \inf\big\{\operatorname{mass}(T - \partial S) + \operatorname{mass}(S) : S \in \mathcal{F}_{q+1}(U|C; \mathbb{R})\big\}$$

where $\mathcal{F}_q(U|C)$ is the module of all flat q–chains in U whose support is contained in C.

Suppose M and N are n–manifolds. Let K and L be a triangulation of M and N respectively. In this section, K and L are subdivided if necessary, but we will use the same symbol. For a continuous map $f\colon M \to N$, there is a non-degenerate simplicial approximation $g\colon K \to L$ of f. For an open n–simplex e in L, consider a map $h\colon K \xrightarrow{g} L \to L/(L \setminus e)$. We will call $\deg(h)$ the *degree* of g at e which is denoted by $\deg_e(g)$. Let

$$D(g) := \sup\big\{|\deg_e(g)| : \text{open } n\text{-simplex } e \text{ in } L\big\}.$$

Here $D(g)$ is finite, because of we can assume that K and L are finite simplicial complexes.

For an arbitrary Riemannian metric $\mathcal{G}_N$ on N, consider an embedding in $\mathbb{R}^m$. Then a current $V_N(\omega) := \int_N \text{comass}(\omega_x)\, d\mathcal{L}^n x$ is defined for an arbitrary compact supported differential n–form ω where $\mathcal{L}^n$ is the n–dimensional Lebesgue measure. We can observe that V_N is contained in $\mathcal{F}_n(\mathbb{R}^m|N;\mathbb{R})$ and satisfying $\text{mass}(V_N) = \text{stsys}_n(N)$.

We take a closed m–ball C in $\mathbb{R}^m$ which contains N and L in its internal. For a sufficiently small $\varepsilon > 0$, there is a piecewise linear metric $\mathcal{G}_L = \mathcal{G}_L(\varepsilon)$ on L satisfying

$$|V_L - V_N|_C^\flat \le \varepsilon \quad \text{and} \quad \left|\text{stsys}_q(L,\mathcal{G}_L) - \text{stsys}_q(N,\mathcal{G}_N)\right| \le \varepsilon$$

for every non-trivial stable q–systoles (compare Federer [4, 4.1.22]) and the realization of L with $\mathcal{G}_L$ is a PL section of the normal bundle over N with $\mathcal{G}_N$ in $\mathbb{R}^m$. Such metric can be obtained by subdividing K and L, and translating vertices in L along the fiber of the normal bundle to do not degenerate any simplex. For $0 < \varepsilon' < \varepsilon$, a suitable metric $\mathcal{G}_L(\varepsilon')$ also can be acquired by the same way. Hence we can assume that $D(g)$ is not changed by ε and $\mathcal{G}_L$. As ε approaches to 0, each L, $\mathcal{G}_L$ and $g^*\mathcal{G}_L$ converges to N, $\mathcal{G}_N$ and a piecewise Riemannian metric on M respectively. Under this circumstance, we obtain following lemma.

Lemma 4.1 *Suppose q th real homology vector space of K and L are non-trivial. If $g\colon K \to L$ induces a monomorphism g_* between the q th real homology vector spaces, then*

$$\text{stsys}_q(L,\mathcal{G}_L) \le \text{stsys}_q(K, g^*\mathcal{G}_L) \le D(g)\cdot \text{stsys}_q(L,\mathcal{G}_L) < \infty$$

for every piecewise linear metric $\mathcal{G}_L$ on L.

Proof With the pullback PL metric $g^*\mathcal{G}_L$ on K, g is a distance decreasing map. Combining this with Lemma 2.5,

$$\text{stsys}_q(L,\mathcal{G}_L) \le \text{Lip}(g)^q \cdot \text{stsys}_q(K, g^*\mathcal{G}_L) \le \text{stsys}_q(K, g^*\mathcal{G}_L).$$

On the other hands, the inverse image of an arbitrary q–simplex of L is $D(g)$ of q–simplices as at most, since g is a non-degenerate simplicial map and every q–simplex is contained in the boundary of some n–simplex for $q < n$. Also each simplex in the inverse image has same mass of the preimage, since the restriction of g on each simplex is isometry. This implies that the mass of a q–chain c of K is not greater than $D(g)$ times of the mass of the image $g_\flat(c)$ which is not trivial. Therefore we can verify that

$$\text{stsys}_q(K, g^*\mathcal{G}_L) \le D(g)\cdot \text{stsys}_q(L,\mathcal{G}_L)$$

for an arbitrary PL metric $\mathcal{G}_L$. □

Remark If K is not a triangulation of a manifold, we can not sure that every q–simplex of K is contained in the boundary of some n–simplex for $q < n$. For example, a triangulation of the one-point union $S^1 \vee S^2$ has some 1–simplex in S^1 which is not contained in the boundary of any 2–simplex.

Since the stable systolic category is a homotopy invariant, here we obtain following proposition using similar techniques of Katz and Rudyak [9].

Proposition 4.2 *Let M and N are n–manifolds. If there exists a smooth map $f\colon M \to N$ which induces a monomorphism on every real homology vector space, then* $\mathrm{cat}_{\mathrm{stsys}}(M) \leq \mathrm{cat}_{\mathrm{stsys}}(N)$.

Proof We apply Lemma 4.1,

$$\mathrm{stsys}_q(N, \mathcal{G}_N) \leq \mathrm{stsys}_q(L, \mathcal{G}_L) + \varepsilon \leq \mathrm{stsys}_q(K, g^*\mathcal{G}_L) + \varepsilon$$

and

$$\mathrm{stsys}_q(N, \mathcal{G}_N) + \varepsilon \geq \mathrm{stsys}_q(L, \mathcal{G}_L) \geq 1/D(g) \cdot \mathrm{stsys}_q(K, g^*\mathcal{G}_L)$$

where L converges to N in some Euclidean space and $g^*\mathcal{G}_L$ converges to a piecewise Riemannian metric $\mathcal{G}_M$ on M as ε approaches to 0. Suppose there exists a stable systolic categorical partition $(d_1, \ldots, d_k)$ for M. Then there exist $C > 0$ and $\delta = \delta(\varepsilon) > 0$ such that δ converges to 0 as ε approaches to 0 and

$$\prod_{i=1}^{k} \mathrm{stsys}_{d_i}(K, g^*\mathcal{G}_L) \leq C \cdot \mathrm{mass}([K], g^*\mathcal{G}_L) + \delta,$$

because of each metric $g^*\mathcal{G}_L$ can be approximated by some Riemannian metrics on M. We can assume that $\varepsilon \leq \mathrm{stsys}_{d_i}(N, \mathcal{G}_N)$ for all i, so

$$\begin{aligned}
\prod_{i=1}^{k} \mathrm{stsys}_{d_i}(L, \mathcal{G}_L) &\leq 2^k \cdot \prod_{i=1}^{k} \mathrm{stsys}_{d_i}(K, g^*\mathcal{G}_L) \\
&\leq 2^k \cdot C \cdot \mathrm{mass}([K], g^*\mathcal{G}_L) + 2^k \delta \\
&\leq 2^k \cdot C \cdot D(g) \cdot \mathrm{mass}([L], \mathcal{G}_L) + 2^k (C \cdot D(g) \cdot \varepsilon + \delta).
\end{aligned}$$

This implies the partition $(d_1, \ldots, d_k)$ is also stable systolic categorical for N. Therefore we obtain the result $\mathrm{cat}_{\mathrm{stsys}}(M) \leq \mathrm{cat}_{\mathrm{stsys}}(N)$. □

Let X and Y are simply connected spaces. A continuous map $f\colon X \to Y$ is called a *rational equivalence*, if the induced map $f^*\colon H^*(Y; \mathbb{Q}) \to H^*(X; \mathbb{Q})$ is an isomorphism.

Corollary 4.3 *The stable systolic category of a* 0*–universal manifold is invariant under the rational equivalences.*

Proof For a 0–universal manifold M and a rational equivalence to a space X, there exists a rational equivalence from X to M. □

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