Homotopy algebra structures on twisted tensor products and string topology operations

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Given a C_∞ coalgebra C_*, a strict dg Hopf algebra H_* and a twisting cochain $\tau\colon C_* \to H_*$ such that $\mathrm{Im}(\tau) \subset \mathrm{Prim}(H_*)$, we describe a procedure for obtaining an A_∞ coalgebra on $C_* \otimes H_*$. This is an extension of Brown's work on twisted tensor products. We apply this procedure to obtain an A_∞ coalgebra model of the chains on the free loop space LM based on the C_∞ coalgebra structure of $H_*(M)$ induced by the diagonal map $M \to M \times M$ and the Hopf algebra model of the based loop space given by $T(H_*(M)[-1])$. When C_* has cyclic C_∞ coalgebra structure, we describe an A_∞ algebra on $C_* \otimes H_*$. This is used to give an explicit (nonminimal) A_∞ algebra model of the string topology loop product. Finally, we discuss a representation of the loop product in principal G–bundles.

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1 Introduction

Brown's theory of twisting cochains, outlined in [2], provides a way to model the total space of a bundle in terms of the base and fiber. Given a principal bundle $G \to P \to M$ and a twisting cochain $\tau\colon C_*(M) \to C_*(G)$, Brown constructs a complex $(C_*(M) \otimes C_*(G), \partial_\tau)$ whose homology is isomorphic to $H_*(P)$. If Y is a G space and $Y \to P \times_G Y \to M$ is the associated bundle, then there is a complex $(C_*(M) \otimes C_*(Y), \partial_\tau)$ whose homology is isomorphic to $H_*(P \times_G Y)$. Quillen [21] shows that when $\mathrm{Im}(\tau) \subset \mathrm{Prim}(H_*)$, the isomorphism is one of coalgebras. There is an extensive literature on twisting cochains due to their wide ranging applications. We have focused on these two results immediately related to this discussion.

In Section 3, we push Brown's theory to homotopy algebras. That is, given a C_∞ coalgebra C_*, a dg bialgebra H_* and a twisting cochain $\tau\colon C_* \to H_*$ where $\mathrm{Im}(\tau) \subset \mathrm{Prim}(H_*)$, we define a twisted A_∞ coalgebra on $C_* \otimes H_*$. The twisted coalgebra structure is denoted $\{c_n^\tau\colon C_* \otimes H_* \to (C_* \otimes H_*)^{\otimes n}\}$. The twisted term in Brown's differential is described by applying the coproduct on C_*, then applying τ to one of the factors, and finally using the multiplication in H_*. The same idea is used for c_1^τ, except we use the higher homotopies $\{c_n\colon C_* \to C_*^{\otimes n}\}$ of the C_∞ coalgebra structure

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as well as the coproduct. We use the same process to obtain c_2^τ, except we use the maps $c_{n>2}$. And the process continues for all c_n^τ. If C_* is a strict differential graded coalgebra with $c_n = 0$ for $n > 2$, then the complex reduces to Brown's complex. For this reason, we denote c_1^τ by ∂_τ. The following theorem is proved in Section 3.

Theorem 3.9 *Let C_* be a C_∞ coalgebra, H_* a dg bialgebra and $\tau\colon C_* \to H_*$ a twisting cochain such that $\mathrm{Im}(\tau) \subset \mathrm{Prim}(H_*)$. The maps $\{\partial_\tau, c_2^\tau, c_3^\tau, \ldots\}$ define an A_∞ coalgebra on $C_* \otimes H_*$.*

We then define the conjugation action of H_* on itself. The action of a primitive element on H_* is both a derivation and a coderivation. If we go through the process of defining $\{c_n^\tau\}$ as above, except instead of using the multiplication in H_*, we use the conjugation action, the resulting maps also define an A_∞ coalgebra structure. Because the conjugation action involves the antipode map, we require H_* to be a dg Hopf algebra, as opposed to a dg bialgebra found in the Theorem 3.9.

Theorem 3.17 *Let C_* be a C_∞ coalgebra, H_* a dg Hopf algebra and $\tau\colon C_* \to H_*$ a twisting cochain such that $\mathrm{Im}(\tau) \subset \mathrm{Prim}(H_*)$. The maps $\{\partial_\tau, c_2^\tau, c_3^\tau, \ldots\}$ obtained using the conjugation action define an A_∞ coalgebra on $C_* \otimes H_*$.*

Since the conjugation action is a derivation, if C_* also has a multiplication, it is reasonable to ask for an A_∞ algebra on $C_* \otimes H_*$. When C_* is a cyclic C_∞ coalgebra, there is a twisted A_∞ algebra on $C_* \otimes H_*$.

Theorem 3.18 *Let C_* be a cyclic C_∞ coalgebra, H_* be a Hopf algebra and $\tau\colon C_* \to H_*$ be a twisting cochain with $\mathrm{Im}(\tau) \subset \mathrm{Prim}(H_*)$. The maps $\{\partial_\tau, m_2, m_3, \ldots\}$ defined using the conjugation action in H_* give $C_* \otimes_\tau H_*$ the structure of an A_∞ algebra.*

The A_∞ algebra and A_∞ coalgebra share the same differential ∂_τ, so they compute the same linear homology. We still do not know what the further compatibilities are.

Section 4 applies this work to the path space fibration $\Omega_b(M) \to P_b(M) \to M$. Since $\Omega_b(M)$ is homotopy equivalent to a topological group, we consider $P_b(M) \to M$ to be a principal bundle. The first step is to construct a twisting cochain $H_*(M) \to T(H_*(M)[-1])$, whose image is in $\mathcal{L}(H_*(M)[-1])$. We obtain such a map by considering the construction of a power series connection. Then we apply Theorem 3.9 to get an A_∞ coalgebra model of the based path space.

We also get a description of string topology operations from the path space fibration. Any group acts on itself by conjugation. The conjugate bundle is defined to be the associated bundle of a principal G bundle with respect to the conjugation action. The conjugate bundle obtained from the path space fibration is a model of the free loop space. Applying Theorem 3.17 gives an A_∞ coalgebra structure modeling the coalgebra on $H_*(LM)$ induced by the diagonal map. Applying Theorem 3.18 gives an A_∞ algebra structure modeling the algebra on $H_*(LM)$ given by the loop product.

The final section applies the work in Section 3 to the case of a principal G bundle $G \to P \to M$, where G is a connected Lie group. The A_∞ coalgebra on $H_*(M) \otimes H_*(G)$ given by applying Theorem 3.9 can be expressed in terms of the characteristic classes of the bundle. We can also consider the conjugate bundle, denoted $\mathrm{Conj}(P) \to M$. Then Theorem 3.17 gives an A_∞ coalgebra model for $H_*(\mathrm{Conj}(P))$ and Theorem 3.18 gives an A_∞ algebra model.

Given a connection on $P \to M$, there is a map of bundles $P_b(M) \to P$, which induces a map on associated bundles with respect to the conjugation action $\mathrm{Conj}(P_b(M)) \to \mathrm{Conj}(P)$. Then the algebraic structures we get modeling the total space $\mathrm{Conj}(P)$ are representations of algebraic structures on $\mathrm{Conj}(P_b(M))$. In this way, we get representations of string topology.

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2 Background material

Algebras and coalgebras are taken over $\mathbb{Q}$. Homology and cohomology are taken with coefficients in $\mathbb{Q}$.

2.1 Twisting cochains

We first describe Brown's theory of twisting cochains in a purely algebraic setting. Let C_* be a differential graded coalgebra and A_* a differential graded algebra. Then $(\mathrm{Hom}(C_*, A_*), \partial_C \otimes 1 + 1 \otimes \partial_A)$ is a differential graded algebra, and a twisting cochain is an element $\tau \in \mathrm{Hom}(C_*, A_*)$ satisfying the Maurer–Cartan equation

$$\partial_A \circ \tau + \tau \circ \partial_C + \tau \cdot \tau = 0.$$

The Maurer–Cartan equation makes sense for any differential graded algebra, and a twisting cochain is a Maurer–Cartan element in a differential graded algebra of the form $\mathrm{Hom}(C_*, A_*)$. The tensor differential $\partial_C \otimes 1 + 1 \otimes \partial_A$ on $C_* \otimes A_*$ is twisted by adding a term

$$C_* \otimes A_* \xrightarrow{\Delta\otimes 1} C_* \otimes C_* \otimes A_* \xrightarrow{1\otimes\tau\otimes 1} C_* \otimes A_* \otimes A_* \xrightarrow{1\otimes m} C_* \otimes A_*.$$

We refer to this term as the twisted term, and ∂_τ is the sum of the tensor differential and twisted term. The coproduct on C_* defines a comodule on the tensor $C_* \otimes A_* \to C_* \otimes C_* \otimes A_*$. The coalgebra C_* is a comodule over itself.

Theorem 2.1 [2] *Let C_* be a coalgebra, A_* an algebra and τ a twisting cochain. Then $(C_* \otimes A_*, \partial_\tau)$ is a chain complex. If $C_1 = 0$ and $\epsilon\colon A_* \to k$ is an augmentation, then $\mathrm{Id} \otimes \epsilon\colon C_* \otimes A_* \to C_*$ is a map of comodules.*

Proof In [2, page 229], ∂_τ is shown to square to zero. We give a diagrammatic proof of that $\partial^2 = 0$ in Remark 3.8.

The map $1 \otimes \epsilon$ obviously commutes with the comodule map, since the comodule map on $C_* \otimes A_*$ is given by the coproduct on C_* and the coproduct on C_* is the comodule structure for C_*. To show that $1 \otimes \epsilon$ commutes with the differential, it suffices to show that $1 \otimes \epsilon$ vanishes on the twisted term. To see this, note that ϵ is zero on any element of positive degree in A_*. Let $c \otimes h \in C_* \otimes A_*$. If h is in positive degree, then the twisted term will have positive degree in the A_* factor and will map to zero under $1 \otimes \epsilon$. Consider $C_* \otimes 1$ in $C_* \otimes A_*$ and $\Delta(c) = \sum c_{(1i)} \otimes c_{(2i)}$. Since τ is a degree -1 map, $\tau(c_{(2i)})$ will have positive dimension for $|c_{(2i)}| > 1$ and be zero for $|c_{(2i)}| = 0$. Since $C_1 = 0$, $1 \otimes \epsilon$ will vanish on the twisted term. □

We write $C_* \otimes_\tau A_*$ for the twisted complex $(C_* \otimes A_*, \partial_\tau)$.

This theory can be applied to principal bundles $G \to P \to M$. The chain complex $C_*(M)$ is a differential graded coalgebra, where the coproduct is induced by the diagonal map $M \to M \times M$. The group multiplication of G provides an algebra structure on $C_*(G)$. A twisting cochain is then a map $\tau\colon C_*(M) \to C_*(G)$ satisfying the Maurer–Cartan equation.

The complex $(C_*(M) \otimes C_*(G), \partial_M \otimes \mathrm{Id} + \mathrm{Id} \otimes \partial_G)$ will not, in general, compute the homology of P. However, when we twist the differential by a suitable twisting cochain $\tau\colon C_*(M) \to C_*(G)$, the complex $(C_*(M) \otimes C_*(G), \partial_\tau)$ will compute $H_*(P)$.

Theorem 2.2 [2, Theorem (4.2)] *The chain complex $(C_*(M) \otimes C_*(G), \partial_\tau)$ is chain equivalent to $C_*(P)$.*

The equivalence of the above theorem is of chain complexes and not of dg coalgebras, despite the fact that both complexes have coproducts. A further assumption is needed on τ to obtain an equivalence of dg coalgebras.

We return to the general setting. Let C_* be a dg coalgebra and H_* a dg bialgebra. The primitive elements $\mathrm{Prim}(H_*) = \{h \in H_* \mid \Delta(h) = h \otimes 1 + 1 \otimes h\}$ is a Lie algebra whose universal enveloping algebra is H_*. The following lemma is a reformulation of Quillen [21, Appendix B].

Lemma 2.3 *Let $\tau\colon C_* \to H_*$ be a twisting cochain from a cocommutative coalgebra to a dg bialgebra. Then $(C_* \otimes H_*, \partial_\tau)$ is a differential graded coalgebra if and only if* $\mathrm{Im}(\tau) \subset \mathrm{Prim}(H_*)$.

Proof To show that ∂_τ is a coderivation we need to show that

$$(\Delta_{C\otimes H})\partial_\tau = (\partial_\tau \otimes 1 + 1 \otimes \partial_\tau)\Delta_{C\otimes H}.$$

The key is that multiplication by a primitive element is a coderivation. We give a diagrammatic proof in Remark 3.8. The reader can find the computation in [21, page 289] □

3 Algebraic setting for twisted tensor products

In this section, we extend the discussion of Brown's theory of twisting cochains to the homotopy algebra setting. Let $(C_*, \{c_n\})$ be a C_∞ coalgebra and H_* a strict dg bialgebra. Given a twisting cochain $\tau\colon C_* \to H_*$, we define a twisted A_∞ coalgebra structure on $C_* \otimes H_*$.

There are three properties that are used in Brown's setting. For C_* a strict dg coalgebra and A_* a strict dg algebra, the following properties are used.

(1) $\mathrm{Hom}(C_*, A_*)$ is a differential graded algebra.

(2) twisting cochains $\tau\colon C_* \to A_*$ are in one-to-one correspondence with chain maps $\mathcal{F}(C_*) \to A_*$, where $\mathcal{F}$ is the cobar functor.

(3) a twisting cochain $\tau \in \mathrm{Hom}(C_*, A_*)$ defines a twisted differential on $C_* \otimes A_*$.

We address the analogs of these properties in the following subsections.

3.1 Maurer–Cartan equation in the homotopy algebra Setting

We review some definitions. An A_∞ algebra consists of a vector space V and maps $\{m_n\colon V[-1]^{\otimes n} \to V[-1]\}$ satisfying

$$\sum_{k=1}^{n}\sum_{j=0}^{n-1} m_{n-k+1}\circ(\mathrm{Id}^{\otimes j}\otimes m_k\otimes \mathrm{Id}^{n-j-k}) = 0.$$

The maps $\{m_n\}$ define a coderivation of square zero on $T(V[-1])$. The shuffle product is a map $T(V[-1])\otimes T(V[-1]) \to T(V[-1])$. If m_n vanishes on the image of the shuffle product for every n, then $(V,\{m_n\})$ is a C_∞ algebra.

An A_∞ coalgebra and C_∞ coalgebra are the dual notions of A_∞ and C_∞ algebras. So V is an A_∞ coalgebra if there are maps $\{c_n\colon V[-1]\to V[-1]^{\otimes n}\}$ defining a derivation of square zero on $T(V[-1])$. If the unshuffle product $T(V[-1]) \to T(V[-1])\otimes T(V[-1])$ vanishes on the image of each c_n, then $(V,\{c_n\})$ is a C_∞ coalgebra.

To deal with issues of convergence, we will make use of the completed tensor product. For a vector space V, let

$$\widehat{T}(V) = \prod_{i=0}^{\infty} V^{\otimes i}.$$

In our applications, V will be a finite dimensional vector space. So V has a unique topology making it a topological vector space. There is an induced topology on $\widehat{T}(V)$, known the inverse limit topology.

In order to say τ is a twisting cochain, the vector space $\mathrm{Hom}(C_*,H_*)$ must have at least an A_∞ algebra structure. Moreover, we need the Lie algebra version of the Maurer–Cartan equation, so we need an L_∞ algebra on $\mathrm{Hom}(C_*,\mathrm{Prim}(H_*))$.

Lemma 3.1 *Let $(C_*,\{c_n\})$ be a C_∞ coalgebra and A_* a differential graded algebra. The vector space $\mathrm{Hom}(C_*,A_*)$ is an A_∞ algebra.*

Since $\mathrm{Hom}(C_*,A_*)\cong C^*\otimes A_*$, the lemma is just the statement the tensor product of an A_∞ algebra with an associative algebra is an A_∞ algebra. We omit the proof, but define the maps m_n. Let

$$m_1^{\mathrm{Hom}}(f) = \partial_A\circ f + f\circ\partial_C,$$

where $\partial_C = c_1$ of the C_∞ coalgebra structure. For $n>1$, let

$$\begin{aligned} m_n^{\mathrm{Hom}}(f_1,\dots,f_n)\colon C_* &\to A_*\\ c &\mapsto m_A(f_1\otimes\cdots\otimes f_n))c_n(c), \end{aligned}$$

where by m_A we mean multiply all the terms using multiplication of A_*. Since the multiplication in A_* is associative, m_n^{Hom} is well-defined.

The Maurer–Cartan equation is then

$$\partial \circ \tau + \tau \circ \partial + m_2^{\mathrm{Hom}}(\tau,\tau) + m_3^{\mathrm{Hom}}(\tau,\tau,\tau) + m_4^{\mathrm{Hom}}(\tau,\tau,\tau,\tau) + \cdots = 0.$$

Since we have an infinite sum, a note on convergence is in order. In our application, $A_* = \widehat{T}(H_*(M)[-1])$. The twisting cochain we construct will have the property that

$$\mathrm{Im}(m_n(\tau,\ldots,\tau)) \subset (H_*(M)[-1])^{\otimes n}.$$

So the infinite sum can be expressed as a finite sum in different tensors. This is well defined in the completed tensor product.

For the Lie version of the Maurer–Cartan equation, we will need the following fact about L_∞ algebras.

The Koszul sign convention says that when two elements x and y of degree p and q are commuted, a sign of $(-1)^{pq}$ is obtained. For $x_1,\ldots,x_n$ and a permutation $\sigma \in S_n$, let $\epsilon(\sigma; x_1,\ldots x_n)$ be the sign so that in the free graded commutative algebra $\bigwedge(x_1,\ldots,x_n)$,

$$x_1 \wedge \cdots \wedge x_n = \epsilon(\sigma; x_1,\ldots,x_n) x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(n)}.$$

Let $\xi(\sigma) = \mathrm{sgn}(\sigma)\cdot\epsilon(\sigma; x_1,\ldots,x_n)$.

Theorem 3.2 (Lada–Markl [17, Theorem 3.1]) *Let $(V,\{m_n\})$ be an A_∞ algebra. Then there is an L_∞ algebra on V given by symmetrizing m_n. That is, if*

$$l_n(v_1,\ldots,v_n) = \sum_{\sigma\in S_n} \xi(\sigma) m_n(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)})$$

then $(V,\{l_n\})$ is an L_∞ algebra.

We denote the L_∞ algebra by $[V]$ to distinguish it from the A_∞ algebra V.

Lemma 3.3 *Let $(C_*,\{c_n\})$ be a C_∞ coalgebra and L_* be a differential graded Lie algebra. Then $\mathrm{Hom}(C_*,L_*)$ is an L_∞ algebra.*

Proof Our proof proceeds as follows. Let $U(L_*)$ be the universal enveloping algebra of L_*. The previous lemma shows that the space $\mathrm{Hom}(C_*,U(L_*))$ is an A_∞ algebra, with structure maps $\{m_n\}$. Symmetrizing each m_n defines an L_∞ algebra, with structure maps denoted $\{l_n\}$. Let $c_n(x) = x_{n,1} \otimes \cdots \otimes x_{n,n}$. Then the L_∞ algebra is given by

$$l_n(f_1 \cdots f_n)(x) = \sum_{\sigma \in S_n} \xi(\sigma) f_1(x_{n,\sigma(1)}) \cdots f_n(x_{n,\sigma(n)}).$$

To prove the lemma, it suffices to show that the maps $\{l_n\}$ restricts to $\mathrm{Hom}(C_*, L_*) \subset \mathrm{Hom}(C_*, U(L_*))$.

Suppose $f_i \in \mathrm{Hom}(C_*, L_*)$. This implies $\Delta(f_i(x)) = f_i(x) \otimes 1 + 1 \otimes f_i(x)$, where the coproduct is in $U(L_*)$. Since Δ is an algebra map, we see that

$$\begin{aligned}
&\Delta \circ l_n(f_1, \dots, f_n)(x) \\
&= \sum_{\sigma \in S_n} \Delta(f_1(x_{n,\sigma(1)})) \cdots \Delta(f_n(x_{n,\sigma(n)})) \\
&= \sum_{\sigma \in S_n} (f_1(x_{n,\sigma(1)} \otimes 1 + 1 \otimes f_1(x_{n,\sigma(1)}))) \cdots (f_n(x_{n,\sigma(n)}) \otimes 1 + 1 \otimes f_n(x_{n,\sigma(n)})) \\
&= \sum_{\sigma \in S_n} f_1(x_{n,\sigma(1)}) \cdots f_n(x_{n,\sigma(n)}) \otimes 1 + 1 \otimes f_1(x_{n,\sigma(1)}) \cdots f_n(x_{n,\sigma(n)}) \\
&\qquad\qquad + \sum_{\sigma \in S_n} \sum_j f_1(x_{n,\sigma(1)}) \cdots f(x_{n,\sigma(j)}) \otimes f(x_{n,\sigma(j+1)}) \cdots f_n(x_{(n,\sigma(n))}).
\end{aligned}$$

We need to show that the cross terms cancel. The composition

$$C_* \xrightarrow{c_n} C_*^{\otimes n} \hookrightarrow T(C_*) \xrightarrow{\text{unshuffle}} T(C) \otimes T(C)$$

is zero by definition of a C_∞ coalgebra. Each permutation σ is an (i, j) unshuffle of some linear order of the $\{x_{n,i}\}$. For example, for S_3, the collection of all the $(2,1)$ unshufflings of $x_{3,1} \otimes x_{3,2} \otimes x_{3,3}$ and $x'_{3,1} \otimes x'_{3,2} \otimes x'_{3,3} = x_{3,2} \otimes x_{3,1} \otimes x_{3,3}$ exhausts all combinations of $x_{3,\sigma(1)} \otimes x_{3,\sigma(2)} \otimes x_{3,\sigma(3)}$.

The L_∞ algebra on $\mathrm{Hom}(C_*, L_*)$ is then given by

$$l_n(f_1, \dots f_n)(x) = \sum_{\sigma \in S_n} \xi(\sigma) f(x_{1,\sigma(1)}) \cdots f(x_{n,\sigma(n)}),$$

where the multiplications are in $U(L_*)$. □

Let A_* and B_* be two A_∞ algebras and $\{f_n \colon A_*^{\otimes n} \to B_*\}$ an A_∞ algebra morphism. Suppose the Maurer–Cartan equation is well defined for A_* and B_* (so either there are only finitely many maps defining the A_∞ algebra or a suitable notion of convergence of the infinite sum holds). Let $\tau \in A_*$ be a Maurer–Cartan element. That is,

$$\partial_A \tau + m_2^A(\tau \otimes \tau) + m_3^A(\tau \otimes \tau \otimes \tau) + \cdots = 0.$$

The following well-known lemma shows how to obtain a Maurer–Cartan element in B_* from τ and $\{f_n\}$.

Lemma 3.4 *Let A_*, B_* be two A_∞ algebras and $\{f_n\colon A_*^{\otimes n} \to B_*\}$ be an A_∞ algebra morphism between them. If τ is a Maurer–Cartan element in A_* then*

$$\tau' = f(\tau) + f_2(\tau \otimes \tau) + \cdots + f_n(\tau^{\otimes n}) + \cdots$$

is a Maurer–Cartan element in B_*.

3.2 Maurer–Cartan equation and differential graded algebra maps

The following lemmas will be used to construct twisting cochains.

Lemma 3.5 *Let C_* be an A_∞ coalgebra and A_* an associative algebra. There is a one-to-one correspondence between twisting cochains $\tau\colon C_* \to A_*$ and differential graded algebra maps $\tau_T\colon T(C_*[-1]) \to A_*$.*

Proof Let $\partial^{T(C)}\colon T(C_*[-1]) \to T(C_*[-1])$ be the derivation of square zero given by the A_∞ coalgebra on C_*. Given a twisting cochain $\tau\colon C_* \to A_*$, let $\tau_T(c_1 \otimes \cdots \otimes c_n) = \tau(c_1)\cdots\tau(c_n)$. Then by construction, τ_T is an algebra map. It is a chain map, because

$$\begin{aligned}\partial^H(\tau(c)) &= \tau\partial^C(c) + m_2^A \circ (\tau \otimes \tau) \circ c_2(c) + m_3^A \circ \tau^{\otimes 3} \circ c_3(c)\\ &= \tau\partial^{T(C)}(c),\end{aligned}$$

where the first equality is due to the Maurer–Cartan equation for τ and the second equality is the definition of $\partial^{T(C)}$ in terms of the maps $c_n\colon C_*[-1] \to C_*[-1]^{\otimes n}$. Conversely, given a map of differential graded algebras $\tau_T\colon T(C_*) \to A_*$ restricting τ to C_* defines a twisting cochain. $\square$

Lemma 3.6 *Let C_* be a C_∞ coalgebra and H_* a Hopf algebra. There is a one-to-one correspondence between twisting cochains $\tau\colon C_* \to \mathrm{Prim}(H_*)$ and differential graded Lie algebra maps $\mathcal{L}(C_*[-1]) \to \mathrm{Prim}(H_*)$.*

Proof This lemma is proved in the same way as that of the previous. Note that a C_∞ coalgebra defines a derivation of square zero on the free Lie algebra $\mathcal{L}(C_*[-1])$. $\square$

3.3 C_∞ coalg $\otimes_\tau$ bialg as an A_∞ coalgebra using left multiplication

Given a twisting cochain $\tau\colon C_* \to H_*$, we want to define a twisted A_∞ coalgebra structure on $C_* \otimes H_*$. First, we define the untwisted A_∞ coalgebra.

Lemma 3.7 *Let $(C_*, \{c_n\})$ be an A_∞ coalgebra and H_* be an algebra with a strictly coassociative comultiplication. Then $C_* \otimes H_*$ is an A_∞ coalgebra with structure maps*

$$c_n^{\otimes} = c_n \otimes \big((\Delta \otimes \mathrm{Id}^{\otimes n-1}) \circ \cdots \circ \Delta\big)\colon C_* \otimes H_* \to (C_* \otimes H_*)^{\otimes n}.$$

Proof The proof is straightforward, using the A_∞ coalgebra relations for C_* terms and that H_* is a strict coassociative coalgebra. □

Remark 3.8 Before we define an A_∞ coalgebra structure on $C_* \otimes H_*$, we return to the classical setting of Brown's twisting cochains. We introduce a graphical picture of ∂_τ and a graphical proof that $\partial_\tau^2 = 0$. This technique will be used to define the twisted A_∞ coalgebra later on. Let C_* be a differential graded coalgebra and H_* a differential graded bialgebra. Let τ be a twisting cochain and ∂_τ be the twisted differential.

To represent $\partial_\tau\colon C_* \otimes H_* \to C_* \otimes H_*$, we draw two vertical lines, one to represent C_* the other to represent H_*. We draw a horizontal dash to denote the differential. The twisting term applies the coproduct on C_* and τ to one of the factors. We represent the twisting cochain $\tau\colon C_* \to H_*$ by connecting the lines representing C_* and H_* with a line. The resulting vertex on C_* of valence three can be thought of as the coproduct and the vertex of valence three on H_* can be thought of as the product. We refer the reader to Figure 1 for a picture of ∂_τ.

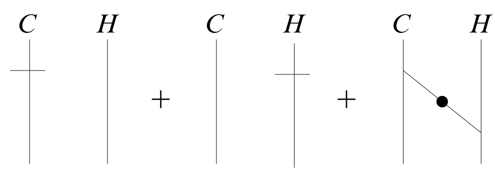

Figure 1: A graphical representation of the differential $\partial_\tau = \partial_C \otimes 1 + 1 \otimes \partial_H + (1 \otimes m_A \otimes \tau \otimes 1)\Delta_C \otimes 1$. A vertical line with a dash represents the differential. The diagonal line with a vertex represents the map $\tau\colon C \to H$.

We can prove that $\partial_\tau^2 = 0$ by analyzing the diagrams. The top row in Figure 2 are the terms that remain after canceling the terms in ∂_τ^2 that correspond to the tensor differential, which is well known to square to zero. Note that because ∂_C is a coderivation, the first and third terms in this row are equal to the first term in the second row of the figure. Similarly, since ∂_H is a derivation, the second and fourth terms on the first row equal the second term in the second row. The coassociativity of Δ_C and the associativity of m_H imply the last term of the first row is equal to the last term of the second row. The bottom row then is equal to zero, because the middle lines describe the Maurer–Cartan equation $\partial_H \tau + \tau \partial_C + \tau \cdot \tau$, which is zero by assumption.

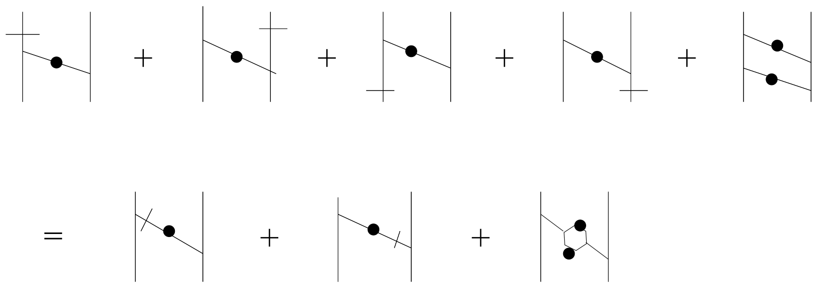

Figure 2: A graphical representation of $\partial_\tau^2 = 0$. The top row represents the five terms that remain in ∂_τ^2 when we cancel the terms corresponding to the tensor differential. The bottom row is zero because the middle lines represent the Maurer–Cartan equation $\partial_H \tau + \tau \partial_C + \tau \cdot \tau$.

There is a similar argument showing that if $\mathrm{Im}(\tau) \subset \mathrm{Prim}(H_*)$, then $(C_* \otimes H_*, \partial_\tau)$ is a differential graded coalgebra. The argument requires C_* to be a cocommutative coalgebra. We refer the reader to Figure 3.

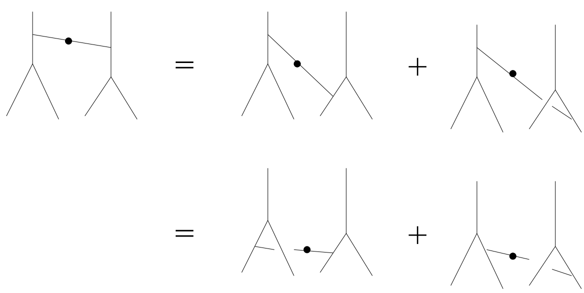

Figure 3: A graphical representation that ∂_τ is a coderivation of the coproduct of $C_* \otimes H_*$. The first equality is a result of the fact that multiplication by a primitive element is a coderivation. The second equality is a result of the coproduct in C_* being coassociative and cocommutative.

We can now describe how to twist the A_∞ coalgebra. Let $\tau\colon C_* \to \mathrm{Prim}(H_*)$ satisfy the Lie Maurer–Cartan equation. First consider $c_1^{\mathrm{Hom}}\colon C_* \otimes H_* \to C_* \otimes H_*$. As in the strict setting, there is a twisting term of the form

$$C_* \otimes H_* \xrightarrow{c_2} C_*^{\otimes 2} \otimes H_* \xrightarrow{1 \otimes \tau \otimes 1} C_* \otimes H_*^{\otimes 2} \xrightarrow{1 \otimes m_H} C_* \otimes H_*.$$

But this twisting only takes c_2 into account and ignores all of the higher c_n maps in the C_∞ coalgebra structure on C_*. To account for these maps, first apply c_n to C_* and apply $\tau^{\otimes n-1}$ to the last $n-1$ factors in $C_*^{\otimes n}$. Since $\mathrm{Im}(\tau) \subset \mathrm{Prim}(H_*)$, we can bracket these $n-1$ terms in all possible ways to get another primitive element. Then we multiply $\mathrm{Prim}(H_*)$ and H_* terms. To sum up, c_1^τ consists of terms

$$C_*\otimes H_* \xrightarrow{c_3\otimes 1} C_*^{\otimes 3}\otimes H_* \xrightarrow{1\otimes\tau^{\otimes 2}\otimes 1} C_*\otimes H_*^{\otimes 2}\otimes H \xrightarrow{1\otimes[,]\otimes 1} C_*\otimes H_*\otimes H_* \xrightarrow{1\otimes m} C_*\otimes H_*$$

$$C_*\otimes H_* \xrightarrow{c_4\otimes 1} C_*^{\otimes 4}\otimes H_* \xrightarrow{1\otimes\tau^{\otimes 3}\otimes 1} C_*\otimes H_*^{\otimes 3}\otimes H \xrightarrow{1\otimes[,]\otimes 1} C_*\otimes H_*\otimes H_* \xrightarrow{1\otimes m} C_*\otimes H_*$$

$$C_*\otimes H_* \xrightarrow{c_5\otimes 1} C_*^{\otimes 5}\otimes H_* \xrightarrow{1\otimes\tau^{\otimes 4}\otimes 1} C_*\otimes H_*^{\otimes 4}\otimes H \xrightarrow{1\otimes[,]\otimes 1} C_*\otimes H_*\otimes H_* \xrightarrow{1\otimes m} C_*\otimes H_*$$

and continue for all n in this way. By $[,]$ for three or more terms, we mean

$$[x_1,\dots,x_n] = \sum_{\sigma\in S_n}\Big[x_{\sigma(1)},\big[x_{\sigma(2)},\dots[x_{\sigma(n-1)},x_{\sigma(n)}]\big]\Big].$$

Note the similarity of the twisted terms with the L_∞ algebra on $\mathrm{Hom}(C_*,\mathrm{Prim}(H_*))$. Since c_1^τ is an infinite sum, we need to address the issue of convergence in $C_*\otimes H_*$. In our application, $H_* = \widehat{T}(H_*(M)[-1])$, with the multiplication given by concatenation of tensors. Let $x\in C_*\otimes H_*(M)[-1]$. When c_n is used to twist the differential, the corresponding term in $c_1^\tau(x)$ will be an element in $C_*\otimes(H_*(M)[-1])^{\otimes n}$. Then c_1^τ consists of finite sums in different tensor products. So in the completed tensor product, $c_1^\tau(x)$ is well defined.

When C_* is a strict dg coalgebra, then c_1^τ is the same as the twisted differential ∂_τ in Brown's construction. So we write c_1^τ by ∂_τ.

The higher maps c_n can be twisted in the same manner as c_1. To twist $c_2\colon C_*\otimes H_*\to C_*^{\otimes 2}\otimes H_*^{\otimes 2}$, we apply c_n for $n>2$, then τ^{n-1} to the last $n-2$ factors of $C_*^{\otimes n}$, and bracketing these $n-2$ terms in all possible ways, multiplying the result with the element in H_*, and finally applying the coproduct in H_*. For $n=3$, the process is the composition of

$$C_*\otimes H_* \xrightarrow{c_3\otimes 1} C_*^{\otimes 3}\otimes H_* \xrightarrow{1^{\otimes 2}\otimes\tau\otimes 1} C_*^{\otimes 2}\otimes H_*\otimes H_*$$
$$\xrightarrow{1\otimes m} C_*^{\otimes 2}\otimes H_* \xrightarrow{1^{\otimes 2}\otimes\Delta} C_*^{\otimes 2}\otimes H_*^{\otimes 2}.$$

The resulting map is denoted $c_2^\tau\colon C_*\otimes H_*\to(C_*\otimes H_*)^{\otimes 2}$.

For $n>3$, we must use the Lie bracket, and the composition of maps is

$$C_*\otimes H_* \xrightarrow{c_n\otimes 1} C_*^{\otimes n}\otimes H_* \xrightarrow{1^{\otimes 2}\otimes\tau^{\otimes n-2}\otimes 1} C_*^{\otimes 2}\otimes H^{\otimes n-2}\otimes H_*$$
$$\xrightarrow{1^{\otimes 2}\otimes[,]^{n-2}\otimes 1} 1\otimes H_*\otimes H_* \xrightarrow{1\otimes m} C_*^{\otimes 2}\otimes H_* \xrightarrow{1^{\otimes 2}\otimes\Delta} C_*^{\otimes 2}\otimes H_*^{\otimes 2}.$$

To show that $\{c_n^\tau\}$ defines an A_∞ coalgebra on $C_* \otimes H_*$, we use the diagrams as in Remark 3.8. For a picture of ∂_τ we refer the reader to Figure 4. For a picture of c_2^τ, we refer the reader to Figure 5. Since multiplying by a primitive element is a coderivation, we have some identities for the terms in c_2^τ. These identities are described in Figure 6.

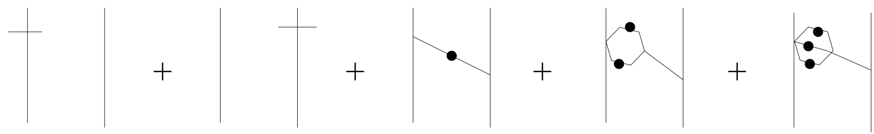

Figure 4: A graphical representation of ∂_τ. The terms are $\partial_C \otimes 1 + 1 \otimes \partial_H + (1 \otimes m)(1 \otimes \tau \otimes 1)c_2 \otimes 1 + (1 \otimes 1 \otimes \tau)c_3$

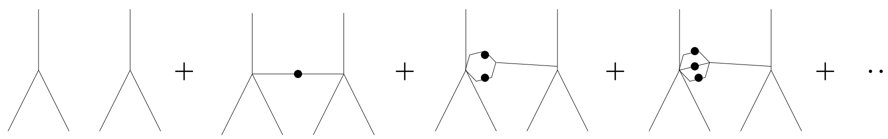

Figure 5: A graphical representation of c_2^τ

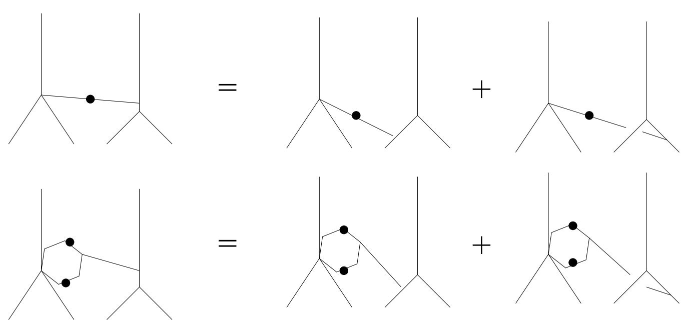

Figure 6: The above identities hold because $\mathrm{Im}(\tau) \subset \mathrm{Prim}(H_*)$ and multiplying by a primitive element is a coderivation. The same holds true for the other terms of c_2^τ and also for c_n^τ.

We can now show that $\{\partial_\tau, c_2^\tau, c_3^\tau, \ldots\}$ define an A_∞ coalgebra. The proof of the theorem uses a graphical approach.

Theorem 3.9 *Let C_* be a C_∞ coalgebra, H_* a dg bialgebra and $\tau\colon C_* \to H_*$ a twisting cochain such that $\mathrm{Im}(\tau) \subset \mathrm{Prim}(H_*)$. The maps $\{\partial_\tau, c_2^\tau, c_3^\tau, \ldots\}$ define an A_∞ coalgebra on $C_* \otimes H_*$.*

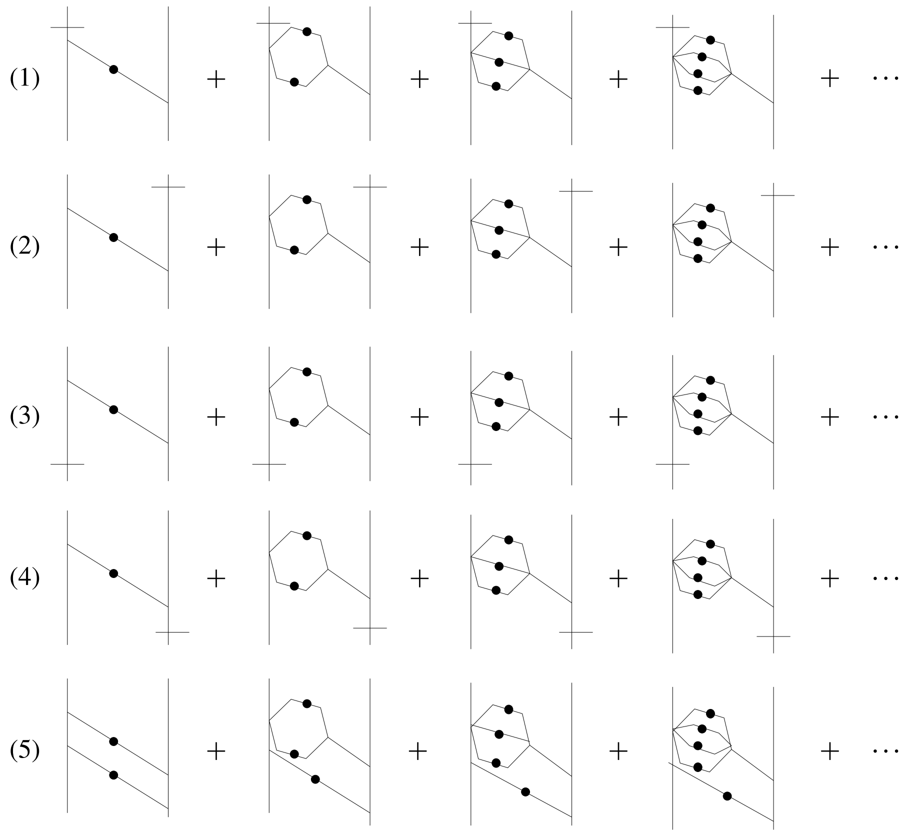

Figure 7: Some of the terms of ∂_τ^2. We have left out the terms in the tensor part, as these are known to square to zero.

Proof We first show that ∂_τ is a differential. To show that $\partial_\tau^2 = 0$ we will show that expanding the terms yield many occurrences of the Maurer–Cartan equation.

We list some of the terms of ∂_τ^2 in Figure 7. The fact that ∂_H is a derivation is expressed diagrammatically as in Figure 8. This relation can be used to add diagrams in the second and fourth rows of Figure 7. In place of the coderivation relations, we must use the C_∞ coalgebra relations for $(C_*, \{c_n\})$. The relation for $n = 3$ is expressed in Figure 9. We use these relations to add figures in the first and third rows of Figure 7. Some of the resulting diagrams will either cancel with diagrams in rows five or higher. The rest of the diagrams are shown in Figure 10. The Maurer–Cartan equation is present in each row. Since τ is a twisting cochain, the sum to zero.

Next, we show that c_2^τ is a coderivation of ∂_τ. In Figure 11, the graphs representing $c_2^\tau \circ \partial_\tau$ are drawn and in Figure 12 the graphs representing $(\partial_\tau \otimes 1) \circ c_2^\tau$ are drawn. The

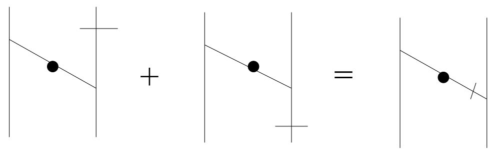

Figure 8: The equality here comes from the fact that H_* is a differential graded algebra.

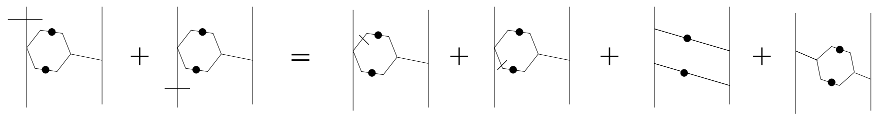

Figure 9: The equality here comes from the fact that $(C_*, \{c_n\})$ is a C_∞ coalgebra.

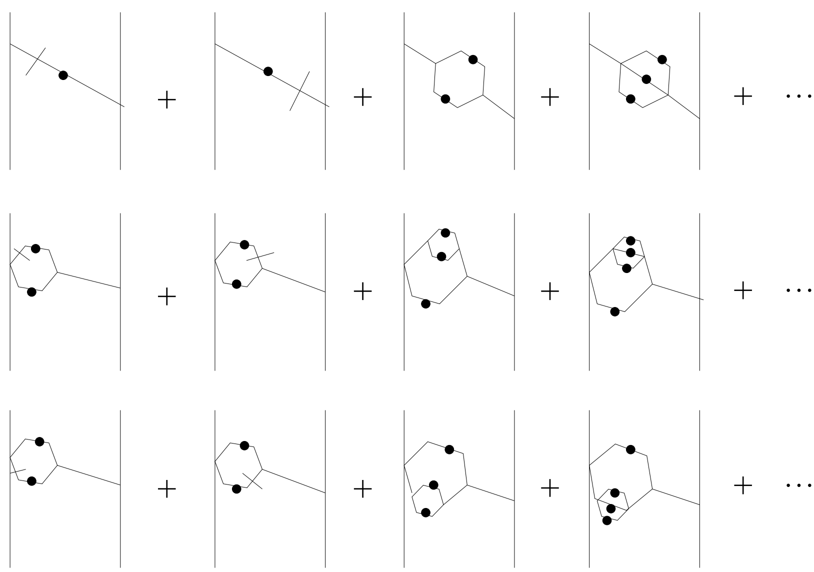

Figure 10: These remaining terms in $(\partial_\tau)^2$ sum to zero because $\partial_H \tau + \tau \partial_H + m_2^{\mathrm{Hom}}(\tau,\tau) + m_3^{\mathrm{Hom}}(\tau,\tau,\tau) + \cdots = 0$.

graphs representing $(1 \otimes \partial_\tau) \circ c_2^\tau$ are the same as the graphs representing $(\partial_\tau \otimes 1) \circ c_2^\tau$ except the graphs are connected by the right output edge of each tree as opposed to the left output edge.

Multiplication by a primitive element is a coderivation, which gives us identities expressed in Figure 6. This allows us to compare the graphs from $c_2^\tau \circ \partial_\tau$ with the graphs from $(\partial_\tau \otimes 1 + 1 \otimes \partial_\tau) \circ c_2^\tau$. Note that on the left hand side of each pairing, we have many compositions of the form $(1 \otimes \cdots \otimes c_j \otimes \cdots \otimes 1) \circ c_i$, where c_i, c_j

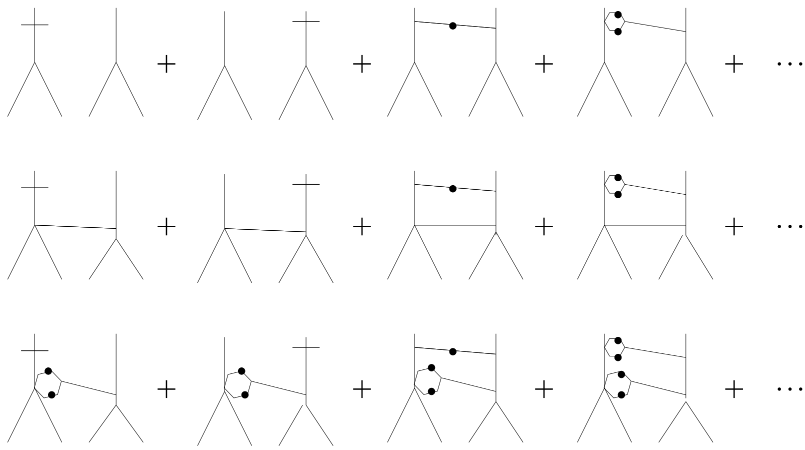

Figure 11: The graphs representing $c_2^\tau \circ \partial_\tau$

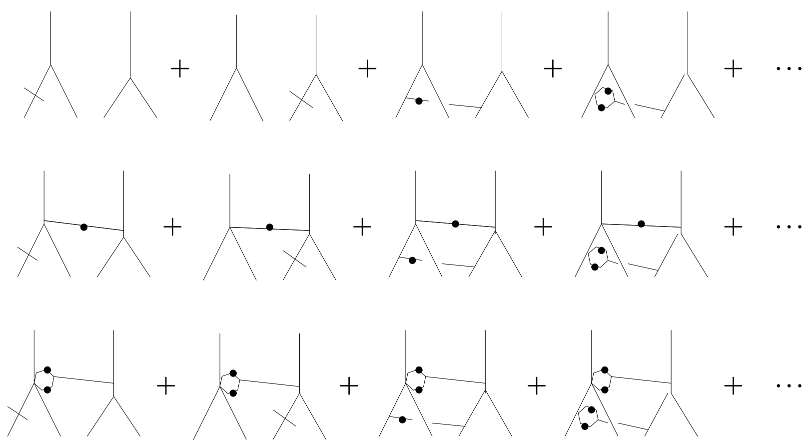

Figure 12: The graphs representing $\partial_\tau \otimes 1 \circ c_2^\tau$

are maps of the C_∞ coalgebra on C_*. The relations in the C_∞ coalgebra state that $\sum_{i+j+1=n}(1 \otimes \cdots \otimes c_j \otimes \cdots \otimes 1) \circ c_i = 0$. Noting which maps in our graphs appear in the sum and which graphs do not appear, we can apply the C_∞ coalgebra relation to obtain many identities. When this is done, we obtain graphs which involve the Maurer–Cartan equation for τ, just as we did in showing $\partial_\tau^2 = 0$. Since τ is a twisting cochain, this sum is zero and c_2^τ is a coderivation of ∂_τ. In Figure 13 we organize the graphs in $c_2^\tau \circ \partial_\tau + (\partial_\tau \otimes 1 + 1 \otimes \partial_\tau) \circ c_2^\tau$. The relations for the coalgebra structure on C_* state that the sum of these graphs are equal to the graphs in Figure 14. The sum of these graphs is zero, because of the Maurer–Cartan equation.

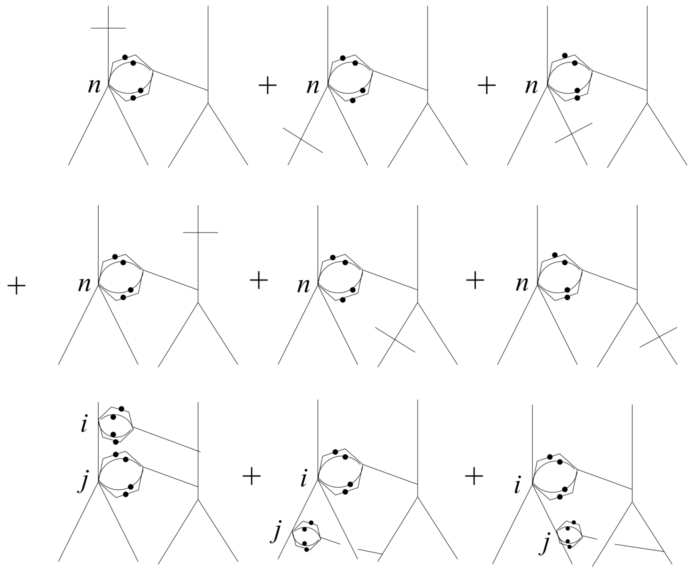

Figure 13: The graphs of $c_2^\tau \circ \partial_\tau + (\partial_\tau \otimes 1 + 1 \otimes \partial_\tau)c_2^\tau$ organized to show how the C_∞ coalgebra on C_* is used

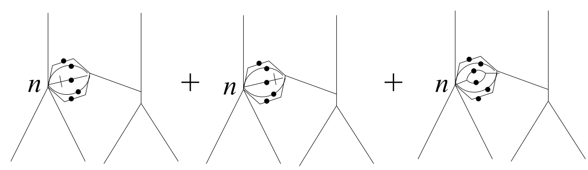

Figure 14: These graphs are equal to the graphs in Figure 13 using the C_∞ coalgebra on C_*. Note that these terms involve $\partial_H \tau + \tau \partial_C + \tau \cdot \tau + \tau \cdot \tau \cdot \tau + \cdots = 0$.

The reader can see that this situation generalizes for $n > 2$. In each of these cases, we have many compositions involved in the C_∞ coalgebra relation for C_*. When we replace these graphs, using the coalgebra structure, we obtain graphs involving Maurer–Cartan equation. We summarize the relation in Figure 15. $\square$

3.4 C_∞ coalg $\otimes_\tau$ bialgebra as an A_∞ coalgebra using bracket action

In the previous section, we used the twisting cochain and left multiplication in H_* to twist the A_∞ coalgebra structure on $C_* \otimes H_*$. In this section, we consider another action. For $a \in H_*$, the bracket action of a on H_* is defined by $[a, x] = ax - xa$. Note that $[a, -]$ is a derivation. If a is a primitive element, then $[a, -]$ is also a coderivation.

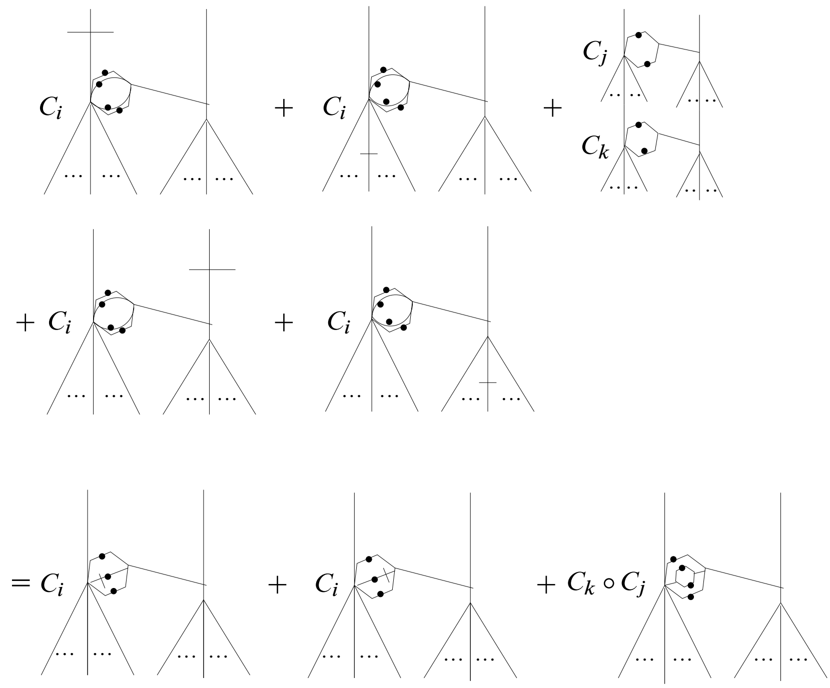

Figure 15: To show that c_n^τ form a coalgebra structure, use the relation above to get a sequence of graphs involving the Maurer–Cartan equation. The equality is due to the fact that C_* is a C_∞ coalgebra.

Given a twisting cochain $\tau\colon C_* \to H_*$ such that $\mathrm{Im}(\tau) \subset \mathrm{Prim}(H_*)$, we define a twisted A_∞ coalgebra structure on $C_* \otimes H_*$. The process is the same as the one defining the previous twisted A_∞ coalgebra, except we replace the multiplication in H_* with the bracket action. We use the same notation $\{\partial_\tau, c_2^\tau, c_3^\tau, \dots\}$ and so we will be explicit when to use left multiplication and when to use the bracket action.

Theorem 3.10 *Let C_* be a C_∞ coalgebra, H_* a dg bialgebra and $\tau\colon C_* \to H_*$ a twisting cochain such that $\mathrm{Im}(\tau) \subset \mathrm{Prim}(H_*)$. The maps $\{\partial_\tau, c_2^\tau, c_3^\tau, \dots\}$, obtained from the bracket action, define an A_∞ coalgebra on $C_* \otimes H_*$.*

Proof The only property of left multiplication used in the proof of Theorem 3.9 is that left multiplication by a primitive element is a coderivation. Since conjugation by a primitive element is a coderivation, the proof applies to this theorem as well. □

3.5 Cyclic C_∞ coalg $\otimes_\tau$ bialg as an A_∞ algebra using bracket action

Sometimes a C_∞ coalgebra has extra structure on it, allowing one to define an algebra structure on $C_* \otimes H_*$. We consider the case when the coalgebra has a nondegenerate bilinear form that is compatible with the coalgebra structure, ie, a cyclic C_∞ coalgebra. We review the relevant definitions.

A cyclic A_∞ algebra consists of a finite dimensional A_∞ algebra $(A_*, \{m_n\})$ and a nondegenerate bilinear form $\langle\, ,\rangle\colon A_* \otimes A_* \to k$ such that

$$\langle m_n(x_1, \dots, x_n), x_0\rangle = (-1)^N \langle m_n(x_0, \dots, x_{n-1}), x_n\rangle,$$

where $N = -1 + |x_0|(|x_1| + \dots + |x_n|)$. The bilinear form defines an isomorphism between A and its dual. The maps m_n can then be viewed as elements in $A[-1]^{*\otimes n} \otimes A[-1] \cong A[-1]^{\otimes n+1}$.

Lemma 3.11 *Let $(A_*, \{m_n\}, \langle\, ,\rangle)$ define a cyclic A_∞ algebra. Then $m_n \in A[-1]^{\otimes n+1}$ is cyclically invariant.*

Proof Let $m_n = \sum x_1 \otimes \cdots \otimes x_{n+1} \in A[-1]^{\otimes n+1}$. It suffices to show $x_1 \otimes \cdots \otimes x_{n+1} = x_2 \otimes \cdots \otimes x_{n+1} \otimes x_1$. This is seen to be the case by expressing $\langle\, ,\rangle$ as an element in $A_* \otimes A_*$ and writing the conditions for a cyclic A_∞ algebra in terms of elements in the tensor algebra. $\square$

Viewing the maps $\{m_n\}$ as elements in the tensor and using the Koszul sign rule, one can determine the sign $(-1)^N$ found in the definition of a cyclic A_∞ algebra. We define a cyclic A_∞ coalgebra viewing c_n as cyclically invariant elements in the tensor product.

Definition 3.12 $(C_*, \{c_n\}, \langle\, ,\rangle)$ is a *cyclic A_∞ coalgebra* if

(1) C_* is finite dimensional,

(2) $(C_*, \{c_n\})$ is an A_∞ coalgebra,

(3) $\langle\, ,\rangle$ is a nondegenerate bilinear form,

(4) the maps c_n when identified as elements $C_*^{\otimes n+1}$ using the bilinear form, are cyclically invariant.

The condition that C_* is finite dimensional implies that $\langle, \rangle$ defines an isomorphism between C_* and its dual C^*. A cyclic C_∞ coalgebra is defined in the obvious way. Given a cyclic C_∞ coalgebra C_*, the bilinear form and maps $\{c_n\}$ can be used to define a C_∞ algebra $\{m_n\colon C_*[-1]^{\otimes n} \to C_*[-1]\}$. So $C_* \otimes H_*$ has an A_∞ algebra structure given by combining the C_∞ algebra on C_* with the strict algebra structure on H_*. Does the twisting cochain $\tau\colon C_* \to H_*$ define a twisted A_∞ algebra on $C_* \otimes H_*$? We show that it does and unlike in the previous cases, we do not need to twist the higher maps. In Theorem 3.22, we prove the case when C_* is a strict cyclic coalgebra.

Also, note that since bracketing is always a derivation, whether by a primitive element or not, we do not require $\mathrm{Im}(\tau) \subset \mathrm{Prim}(H_*)$. If $\mathrm{Im}(\tau) \subset \mathrm{Prim}(H_*)$ and H_* is a dg Hopf algebra, and not just a dg bialgebra, then the bracket action agrees with another action, which we call the conjugation action. We use this action in Theorem 3.18.

Theorem 3.13 *Let C_* be a cyclic C_∞ coalgebra, H_* be a dg bialgebra and $\tau\colon C_* \to H_*$ be a twisting cochain. The maps $\{\partial_\tau, m_2, m_3, \dots\}$ defined using the bracket action in H_* give $C_* \otimes_\tau H_*$ the structure of an A_∞ algebra.*

Proof Since $\{\partial, m_2, m_3, \dots\}$ defines an (untwisted) A_∞ algebra, it suffices to show that the twisted terms in ∂_τ all cancel. We first show that ∂_τ is a derivation of m_2,

$$\partial_\tau \circ m_2 = m_2 \circ (\partial_\tau \otimes 1 + 1 \otimes \partial_\tau). \tag{3-1}$$

We refer the reader to Figures 16 and 17 for graphs representing the left-hand side and RHS of Equation (3-1). Since the bracket action is a derivation, the diagrams in Figure 16 are equal to the diagrams in Figure 18. We need to show that Figure 17 is equal to Figure 18.

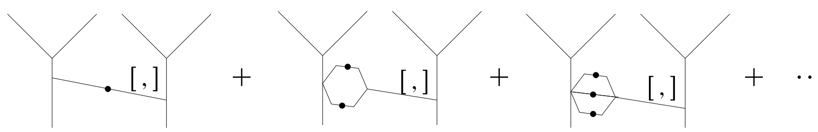

Figure 16: A graphical representation of $\partial_\tau \circ m_2$. The label $[\,,]$ is to remind the reader that the bracket action is applied on $T(H_*(M)[-1])$, and not the product in $T(H_*(M)[-1])$.

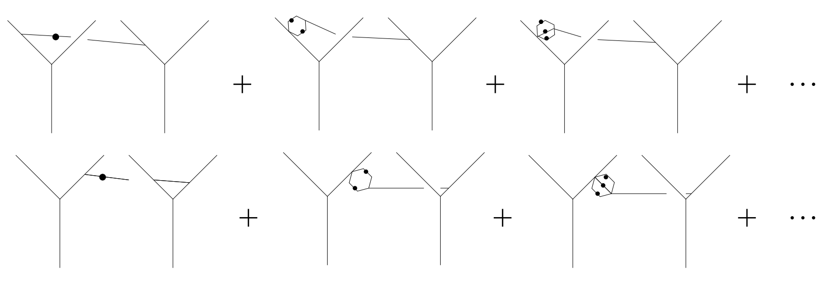

Figure 17: A graphical representation of $m_2 \circ (\partial_\tau \otimes 1 + 1 \otimes \partial_\tau)$

The left-hand side of Equation (3-1) has compositions $c_n \circ m_2\colon C_*[-1]^{\otimes 2} \to C_*[-1]^{\otimes n}$. The right-hand side maps have compositions $(1^{\otimes i} \otimes m_2 \otimes 1^{\otimes j}) \circ (c_n \otimes 1)\colon C_*[-1]^{\otimes 2} \to C_*[-1]^{\otimes n}$. We show these two maps are equal by writing the compositions as elements in $C_*[-1]^{\otimes n+2}$ and using Lemma 3.11.

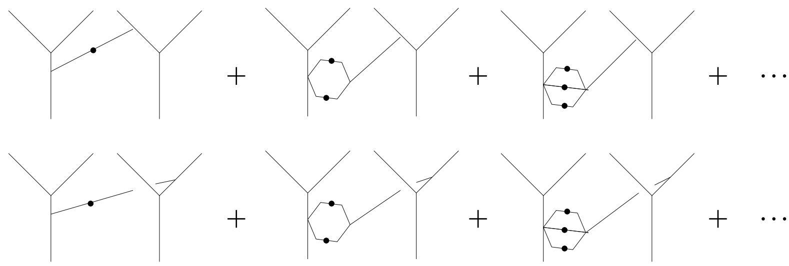

Figure 18: Because the bracket action is a derivation, these diagrams are equal to the one found in Figure 16.

The map c_n can be written as $\sum x_1 \otimes \cdots \otimes x_{n+1} \in C_*[-1]^{\otimes n+1}$ and m_2 as an $\sum y_1 \otimes y_2 \otimes y_3 \in C_*[-1]^{\otimes 3}$. Their composition $c_n \circ m_2$ is expressed as

$$\sum \langle x_1, y_3 \rangle x_2 \otimes \cdots x_n \otimes y_1 \otimes y_2 \in C_*[-1]^{\otimes 4}.$$

The composition on the right-hand side of the equation, $(1^{\otimes i} \otimes m_2 \otimes 1^{\otimes j}) \circ (c_2 \otimes 1)$ is described in the same way except for a different pairing $\langle x_i, y_j \rangle$. However, since c_n and m_2 are cyclically invariant, the compositions are equal.

The higher compatibilities for the A_∞ algebra proceed in exactly the same way, with m_2 replaced by m_l. □

Given the A_∞ algebra $C_* \otimes H_*$, we can symmetrize the maps to obtain an L_∞ algebra $([C_* \otimes H_*], \{\partial_\tau, l_2, l_3, \ldots\})$. This restricts to an L_∞ algebra structure on $C_* \otimes \mathrm{Prim}(H_*)$.

Theorem 3.14 *Let* $(C_* \otimes H_*, \{\partial_\tau, m_2, m_3, \ldots\})$ *be the* A_∞ *algebra described in Theorem 3.13. Then* $(C_* \otimes \mathrm{Prim}(H_*), \{\partial_\tau, l_2, l_3, \ldots\})$*, obtained by symmetrizing* $\{m_n\}$*, is an* L_∞ *algebra.*

Proof Since C_* is finite dimensional, we can identify $C_* \otimes H_* \cong \mathrm{Hom}(C^*, H_*)$, where C^* is a C_∞ coalgebra. Then the statement follows from Lemma 3.3. □

This gives an A_∞ algebra structure on $C_* \otimes H_*$ and an L_∞ algebra structure on $C_* \otimes \mathrm{Prim}(H_*)$. More can be said when C_* is a strict unital commutative algebra. In this situation, $C_* \otimes \mathrm{Prim}(H_*)$ can be viewed as a Lie algebra over C_*. Its universal enveloping algebra over C_*, denoted $U_{C_*}(C_* \otimes \mathrm{Prim}(H_*))$ is $C_* \otimes H_*$. Note if we take the universal enveloping algebra of $C_* \otimes \mathrm{Prim}(H_*)$ (viewed as a Lie algebra over the ground field), we obtain $U(C_* \otimes H_*)$ which is not equal to $U_{C_*}(C_* \otimes \mathrm{Prim}(H_*))$. We are not aware of the corresponding notion for C_∞ algebras to make the analogous statement. This seems to be a useful notion. We discuss the strict case in more detail in Theorem 3.23.

3.6 C_∞ coalg $\otimes_\tau$ Hopf alg as an A_∞ coalgebra using conjugation action

In our applications of the previous results, we would like to relate the twisted algebraic structures to the total space of some bundle. Let $G \to P \to M$ be a principal G bundle and $G \to \mathrm{Conj}(P) \to M$ be the associated bundle with respect to the conjugation action. Note that $H_*(M)$ is a cyclic C_∞ coalgebra and $H_*(G)$ a bialgebra, and moreover, a Hopf algebra. Then given a suitable twisting cochain $\tau\colon H_*(M) \to H_*(G)$, we can form the twisted algebraic structures using the methods described above. The homology of the total space, $H_*(\mathrm{Conj}(P))$ can be identified with linear homology of the twisted algebraic structures, that is homology the homology $H_*(M) \otimes H_*(G)$ with respect to ∂_τ. However, the argument uses Brown's theory of twisting cochains, which requires using the conjugation action. Because the conjugation action uses the inverse operation in G, the algebraic setup in this situation requires H_* to be a dg Hopf algebra. We will see that when $\mathrm{Im}(\tau) \subset \mathrm{Prim}(H_*)$, the conjugation action agrees with the bracket action.

Let H_* be a Hopf algebra. Denote the antipode map of H_* by $s\colon H_* \to H_*$. Given an element $a \in H_*$, we define the conjugation action of a on H_* by

$$\begin{aligned} \mathrm{conj}_a\colon H_* &\to H_* \\ x &\mapsto \sum a_{(1i)} x s(a_{(2i)}). \end{aligned}$$

The homology of a topological group $H_*(G)$ is a Hopf algebra. The group acts on itself by conjugation, and so induces an action on $H_*(G)$. The following lemma shows that this action is the same as the conjugation action of the Hopf algebra.

Lemma 3.15 *Let G be a topological group. The conjugation action in G induces a map*

$$\begin{aligned} H_*(G) \otimes H_*(G) &\to H_*(G) \\ a \otimes x &\mapsto \sum a_{(1i)} x s(a_{(2i)}). \end{aligned}$$

Proof Conjugation is described by the composition

$$\begin{array}{ccccccccc} G \times G & \xrightarrow{\mathrm{Diag}\times 1} & G \times G \times G & \xrightarrow{1\times\mathrm{inv}\times 1} & G \times G \times G & \to & G \times G \times G & \to & G \\ (x,y) & \mapsto & (x,x,y) & \mapsto & (x,x^{-1},y) & \mapsto & (x,y,x^{-1}) & \mapsto & xyx^{-1}. \end{array}$$

The diagonal map in G induces the coproduct Δ on $H_*(G)$ and the inverse map in G induces the antipode s. This proves the lemma. □

The following lemma shows that conjugation by a primitive element is a coderivation and a derivation.

Lemma 3.16 *Let H_* be a Hopf algebra.*

(1) *Conjugation by a primitive element in a Hopf algebra is a coderivation.*

(2) *Conjugation by a primitive element in a Hopf algebra is a derivation.*

Proof (1) Let a be a primitive element of H_*. The antipode has to satisfy $m\circ(1\otimes s)\circ\Delta(a)=0$, which means $s(a)=-a$. Then $\mathrm{conj}_a(x)=\sum a_{(1i)}xs(a_{(2i)})=ax-xa$. This is a coderivation because multiplying by a primitive element is a coderivation.

(2) Let a be a primitive element. Then

$$\begin{aligned}\mathrm{conj}_a(x)\cdot y+x\cdot\mathrm{conj}_a(y)&=axy-xay+xay-xya\\&=axy-xya\\&=\mathrm{conj}_a(xy).\end{aligned}$$ □

Theorem 3.17 *Let C_* be a C_∞ coalgebra, H_* a dg Hopf algebra and $\tau\colon C_*\to H_*$ a twisting cochain such that $\mathrm{Im}(\tau)\subset\mathrm{Prim}(H_*)$. The maps $\{\partial_\tau,c_2^\tau,c_3^\tau,\dots\}$, obtained from the conjugation action, define an A_∞ coalgebra on $C_*\otimes H_*$.*

Proof For $a\in\mathrm{Prim}(H_*)$, the conjugation action, conj_a, and bracket action $[a,]$ agree. Since $\mathrm{Im}(\tau)\subset\mathrm{Prim}(H_*)$, the maps $\{\partial_\tau,c_2^\tau,c_3^\tau,\dots\}$ defined using the conjugation action are equal to the maps defined using the bracket action. The statement then follows from Theorem 3.10 □

Theorem 3.18 *Let C_* be a cyclic C_∞ coalgebra, H_* be a dg Hopf algebra and $\tau\colon C_*\to H_*$ be a twisting cochain with $\mathrm{Im}(\tau)\subset\mathrm{Prim}(H_*)$. The maps $\{\partial_\tau,m_2,m_3,\dots\}$ defined using the conjugation action in H_* give $C_*\otimes_\tau H_*$ the structure of an A_∞ algebra.*

Proof Since $\mathrm{Im}(\tau)\subset\mathrm{Prim}(H_*)$, the twisted A_∞ algebra structure defined using the conjugation action agrees with the twisted A_∞ algebra structure defined using the bracket action.. The proof then follows from Theorem 3.13. □

3.7 Addendum

The graphical approach taken above can obscure some sign issues. In this section, we show that ∂_τ is a differential without appealing to graphs. We also look at the strict (noninfinity) versions of the proofs, with the idea that this will also shed some light on the constructions.

We first show that the twisted differential ∂_τ is indeed a differential, by referencing the work of Chuang and Lazarev [7]. They also define a twisted A_∞ algebra given a Maurer–Cartan element. While their construction is different on the higher maps, it agrees with the twisted differential described in this paper.

Let C^* be an A_∞ algebra and A_* a strict dg associative algebra. Then $C^* \otimes A_*$ is an A_∞ algebra, and a twisting cochain is an element $\tau \in C^* \otimes A_*$ satisfying the Maurer–Cartan equation

$$\partial_C \tau + \partial_H \tau + m_2(\tau,\tau) + m_3(\tau,\tau,\tau) + \cdots = 0.$$

The twisted differential is then

$$\partial_\tau(x) = \partial_C(x) + \partial_H(x) + m_2(\tau,x) + m_3(\tau,\tau,x) + \cdots .$$

This is related to our construction as follows. Let C_* be an A_∞ coalgebra, A_* a strict dga and $\tau\colon C_* \to A_*$. We are only looking to define a differential, which is why we do not require a C_* coalgebra and a dg Hopf algebra. Then the A_∞ algebra C^* used above is the linear dual of the A_∞ coalgebra. The twisting cochain $\tau\colon C_* \to A_*$ can be viewed as an element in $C^* \otimes A_*$ satisfying the Maurer–Cartan equation. The complex $C_* \otimes A_*$ is the A_*–dual of $C^* \otimes A_*$. The two definitions of the twisted differentials can then be related in this way.

Lemma 3.19 *Let C^* be an A_∞ algebra, A_* a differential graded algebra and $\tau \in C^* \otimes A_*$ a twisting cochain. Then $\partial_\tau^2 = 0$.*

Proof This is a special case of Theorem 2.6 $(2)a$ in [7].

We write out some terms in $\partial_\tau^2(x)$. The elements $\tau, x \in C_* \otimes A_*$ can be written as $\tau = \sum \tau_C \otimes \tau_A$ and $x = \sum x_C \otimes x_A$. Then

$$\begin{aligned}\partial_\tau(x) = (\partial_C x_C) \otimes x_A + (-1)^{|x_C|} x_C \otimes (\partial_A x_A) + m_2(\tau_C, x_C) \otimes \tau_A \cdot x_A \\ + m_3(\tau_C,\tau_C,x_C) \otimes \tau_A \cdot \tau_A \cdot x_A + \cdots ,\end{aligned}$$

where we dropped the summation for ease of notation. Applying ∂_τ a second time yields compositions of the A_∞ algebra maps $\{m_n\}$. Using the relations for an A_∞ algebra and strict dg algebra, we obtain terms involving the Maurer–Cartan equation for τ. The argument is similar to the one used to prove Theorem 3.9. □

Theorem 3.18 asserted the existence of a twisted A_∞ algebra on the tensor product. We review some definitions and then discuss the strict case of the theorem.

Definition 3.20 A Frobenius algebra structure on V consists of a commutative multiplication and a nondegenerate inner product such that

$$\langle a, bc\rangle = \langle ab, c\rangle.$$

Note that a Frobenius algebra is a cyclic C_∞ algebra with $m_n = 0$ for $n > 2$.

Using the nondegenerate inner product of a Frobenius algebra, one can turn the multiplication into a comultiplication. The multiplication and comultiplication satisfy a certain compatibility, which brings us to the notion of what some authors refer to as an open Frobenius algebra; see Chen, Eshmatov and Gan [5].

Definition 3.21 An open Frobenius algebra structure on V consists of a commutative multiplication and a cocommutative comultiplication such that the comultiplication is a map of bimodules. That is,

$$\Delta(ab) = \sum a_{(1i)} \otimes a_{(2i)}b = \sum ab_{(1i)} \otimes b_{(2i)}.$$

Abrams [1] proved that unital Frobenius algebras and unital, counital open Frobenius algebras are equivalent.

Theorem 3.22 *Let C_* be a dg Frobenius algebra and H_* a dg bialgebra. Let $\tau\colon C_* \to H_*$ be a twisting cochain such that* $\mathrm{Im}(\tau) \subset \mathrm{Prim}(H_*)$. *Then $(C_* \otimes H_*, \partial_\tau)$ is a differential graded algebra.*

Proof To prove the theorem, we need to show that the twisted term is a derivation. Let $a \otimes b, c \otimes d \in C_* \otimes H_*$, and let $\mathrm{conj}_b\colon H_* \to H_*$ be the conjugation action by $b \in H_*$. Then we need to show that

$$(ac)_{(1i)} \otimes \mathrm{conj}_{\tau(ac)_{(2i)}} bd = a_{(1i)}c \otimes (\mathrm{conj}_{\tau(a_{(2i)})} b)d + ac_{(1i)} \otimes b(\mathrm{conj}_{\tau(c_{(2i)})} d).$$

Since $\mathrm{Im}(\tau) \subset \mathrm{Prim}(H_*)$ and conjugating by a primitive element is a derivation, the left-hand side of the equation is

$$(ac)_{(1i)} \otimes \mathrm{conj}_{\tau(ac)_{(2i)}} bd = (ac)_{(1i)} \otimes (\mathrm{conj}_{\tau(ac)_{(2i)}} b)d + ac_{(1i)} \otimes b(\mathrm{conj}_{\tau(ac)_{(2i)}} d).$$

We need to show that $(ac)_{(1i)} \otimes (ac)_{2i} = a_{(1i)} \otimes a_{(2i)}c = ac_{(1i)} \otimes c_{(2i)}$.

Note that this is the condition that the coproduct is a map of bimodules, ie, an open Frobenius algebra. If we use the result that Frobenius algebras and open Frobenius algebras are equivalent, we are done.

We use another argument which follows the proof of Theorem 3.18. Using the nondegenerate inner product, we express the coproduct Δ as an element in $C_*^{\otimes 3}$. The multiplication $m_2\colon C_* \otimes C_* \to C_*$ is obtained by dualizing the coproduct $C^* \otimes C^* \to C^*$

and using the isomorphism between C^* and C_*. So m_2 is represented by the same element in $C_*^{\otimes 3}$. Write this element as $m_2 = \Delta = \sum x_{(1i)} \otimes x_{(2i)} \otimes x_{(3i)} \in C_*^{\otimes 3}$.

We need to show that certain compositions of Δ and m_2 are equal. In writing the compositions of Δ and m_2, we use the subscript i to represent $m_2\ (x_{(1i)} \otimes x_{(2i)} \otimes x_{(3i)})$ and the subscript j to represent Δ. Then compositions are then given by

$$\begin{aligned}
\Delta \circ m_2 &= \sum_{i,j} \langle x_{(3i)}, x_{(1j)} \rangle x_{(1i)} \otimes x_{(2i)} \otimes x_{(2j)} \otimes x_{(3j)} \\
(m_2 \otimes 1) \circ (\Delta \otimes 1) &= \sum_{i,j} \langle x_{(2j)}, x_{(1i)} \rangle x_{(1j)} \otimes x_{(3j)} \otimes x_{(2i)} \otimes x_{(3i)} \\
(m_2 \otimes 1) \circ (1 \otimes \Delta) &= \sum_{i,j} \langle x_{(3j)}, x_{(2i)} \rangle x_{(1j)} \otimes x_{(2j)} \otimes x_{(1i)} \otimes x_{(3i)}.
\end{aligned}$$

Since $m_2 = \Delta$ are cyclically invariant, we get the necessary equalities. □

In our construction of a twisted A_∞ algebra structure on $C_* \otimes H_*$, we used a cyclic C_∞ coalgebra. A cyclic C_∞ algebra is the homotopy version of a Frobenius algebra. It should be possible to define a twisted A_∞ algebra using the homotopy version of an open Frobenius algebra. The Koszul Duality theory for dioperads described by Gan [9] and for properads described by Vallette [22] provide a definition for such an object. The dioperad describing Lie bialgebras, denoted BiLie, and the dioperad describing open Frobenius algebras, denoted $\mathrm{BiLie}^!$, are Koszul dual [9, Corollary 5.10]. So a resolution for $\mathrm{BiLie}^!$ is obtained by taking the cobar dual of BiLie, denoted $D(\mathrm{BiLie})$, and an open Frob_∞ algebra structure on V is a map of differential graded dioperads $D(\mathrm{BiLie}) \to \mathrm{End}(V)$, where $\mathrm{End}(V)$ is the endomorphism dioperad.

The cohomology of a Poincaré Duality space is a cyclic C_∞ algebra. An open manifold is not a Poincaré Duality space, but its cohomology is an open Frobenius algebra. The constructions using cyclic C_∞ algebra would define string topology operations for Poincaré Duality spaces, and the constructions using open Frob_∞ algebras would define string topology operations for open manifolds.

Theorem 3.14 said that the L_∞ algebra structure on $C_* \otimes H_*$ restricts to $C_* \otimes \mathrm{Prim}(H_*)$. In the strict case, more can be said about the relation between the associative algebra $C_* \otimes H_*$ and the Lie algebra $C_* \otimes \mathrm{Prim}(H_*)$. Let $U_{C_*}(C_* \otimes \mathrm{Prim}(H_*))$ be the universal enveloping algebra of $C_* \otimes \mathrm{Prim}(H_*)$ viewed as a Lie algebra over C_*. Recall, if A_* is an associative algebra, then $[A_*]$ is the Lie algebra obtained by symmetrizing the multiplication.

Theorem 3.23 *The Lie bracket on $[C_* \otimes H_*]$ restricts to $C_* \otimes \mathrm{Prim}(H_*)$. Moreover, if C_* is unital, $U_{C_*}(C_* \otimes \mathrm{Prim}(H_*)) = C_* \otimes H_*$.*

Proof We first show that the Lie bracket on $[C_* \otimes H_*]$ fixes $C_* \otimes \mathrm{Prim}(H_*)$. This is a simple computation

$$\begin{aligned}[a_1 \otimes b_1, a_2 \otimes b_2] &= a_1a_2 \otimes b_1b_2 - a_2a_1 \otimes b_2b_1 \\ &= a_1a_2 \otimes (b_1b_2 - b_2b_1) \\ &= a_1a_2 \otimes [b_1, b_2],\end{aligned}$$

where the bracket is in $[H_*]$. Since $\mathrm{Prim}(H_*)$ is a Lie subalgebra of $[H_*]$, this proves the claim.

For the second part, suppose C_* is unital. Then an element in $U_{C_*}(C_* \otimes \mathrm{Prim}(H_*))$ can be rewritten

$$(c_1 \otimes h_1) \otimes_{C_*} \cdots \otimes_{C_*} (c_n \otimes h_n) = (c_1 \cdots c_n \otimes h_1) \otimes_{C_*} (1 \otimes h_2) \otimes_{C_*} \cdots \otimes_{C_*} (1 \otimes h_n).$$

The claim then follows from the construction of the universal enveloping algebra as a quotient of the tensor algebra. □

4 Application to spaces

To describe string topology operations, we start with the path space fibration $\Omega_b(M) \to P_b(M) \to M$. The based loop space $\Omega_b(M)$ is homotopy equivalent to a topological group, so we view $\Omega_b(X)$ as a topological group and the path space fibration as a principal $\Omega_b(M)$ bundle. The group acts on itself by conjugation and the associated bundle with respect to this bundle, which we refer to as the conjugate bundle, is a model for the free loop space.

Lemma 4.1 *The conjugate bundle $\Omega_b(M) \to \mathrm{Conj}(P_b(M)) \to M$ is equivalent to the free loop space bundle $\Omega_b(M) \to LM \to M$.*

Proof The total space $\mathrm{Conj}(P_b(M))$ is $P_b(M) \times_{\Omega_b(M)} \Omega_b(M)$. We define a bundle map from $\mathrm{Conj}(P_b(M)) \to LM$. Let $[p, a]$ be an element in $P_b(M) \times_{\Omega_b(M)} \Omega_b(M)$ and choose a representative (p, a), where $p\colon [0, 1] \to M$ and $a\colon S^1 \to M$. Then consider the map $f\colon [p, a] \mapsto pap^{-1}$. This map is well defined since a different representative will be of the form $(pg, g^{-1}ag)$, which gets sent to

$$(pg)(g^{-1}ag)(pg)^{-1} = pap^{-1}.$$

If f maps fibers isomorphically onto fibers, then f will be a homeomorphism (see for example Milnor and Stasheff [20, Lemma 2.3]). Let $F_x(\mathrm{Conj})$ be the fiber of $\mathrm{Conj}(P_b(M))$ above the point $x \in M$. An element in the fiber is of the form $[p, a]$

where p is a path from b to x and a is a loop at b. Let $\alpha \in F_x(LM)$ be an element in the fiber of the free loop space bundle. Then letting p be any path from b to x and $a = p^{-1}\alpha p$, then $f[p, p^{-1}\alpha p] = \alpha$. $\square$

4.1 Power series connection

To apply the theorems proved in Section 3, we need to construct a twisting cochain. There are several different constructions available for this purpose. The commutative algebra structure on $\Omega^*(M)$ defines a C_∞ algebra on $H^*(M)$, (see Cheng and Getzler [6] for a description of how to transfer structure). The C_∞ algebra defines a derivation of square zero on $\mathcal{L}(H_*(M)[-1])$ and the inclusion $H_*(M) \hookrightarrow \mathcal{L}(H_*(M)[-1])$ defines a twisting cochain. Note that the C_∞ algebra on $H^*(M)$ is a minimal model for $\Omega^*(M)$. Kadeishvili's Minimal Model Theorem [15] provides another construction of a twisting cochain.

We choose to review the work of Chen [4] and Hain [12] on power series connections, which gives an equivalent construction of the minimal model for $\Omega^*(M)$ as the one described above. A power series connection will be a twisting cochain from $H_*(M) \to \mathcal{L}(H_*(M)[-1])$ in slightly different terminology. The equivalence of Kadeishvili's construction and Hain's construction is described in Huebschmann [14]. The construction is explicit and self contained, which is why we have chosen to include it.

Let M be a simply connected manifold. We introduce some notation. If L is a Lie algebra, let $I^2L = [L, L]$, and for $s > 2$, $I^sL = [L, I^{s-1}L]$. Also, for $w \in \Omega^*(M)$, let $J(w) = (-1)^{|w|}w$.

Hain [12] defines a *power series connection* to be a pair consisting of an element $\omega \in \Omega^*(M) \otimes \mathcal{L}(H_*(M;\mathbb{R})[-1])$ and derivation ∂ on $\mathcal{L}(M_*(X;\mathbb{R})[-1])$, such that

(1) $\partial^2 = 0$,

(2) if $\omega \equiv \sum W_i X_i (\operatorname{mod} \Omega^*(M) \otimes I^2\mathcal{L}(H_*(M)[-1]))$, then W_i are closed forms whose cohomology classes form a basis for $H^*(M;\mathbb{R})$,

(3) $\partial\omega + d\omega - \frac{1}{2}[J\omega, \omega] = 0$.

The last condition for ω is referred to as the *twisting cochain condition*.

We go through Hain's construction of a power series connection, which requires the next lemma. The statement can be found in [11], where a dual statement is proved.

Lemma 4.2 [11, Lemma 3.8] *Let L be a graded Lie algebra and ∂ be a derivation of L such that $\partial(L) \subset [L, L]$. Suppose ω is an element of $\Omega^*(M) \otimes L$ such that*

(1) $\omega \equiv \sum W_i X_i (\operatorname{mod} \Omega^*(M) \otimes I^2 L)$, *where W_i are closed forms whose cohomology classes form a linear basis for $H^*(M)$,*

(2) $\partial\omega + d\omega - \frac{1}{2}[J\omega, \omega] \equiv 0 (\operatorname{mod} \Omega^*(M) \otimes I^n L)$,

Then

(1) $\partial^2 \equiv 0 (\operatorname{mod} I^{n+1} L)$,

(2) $d(\partial\omega + d\omega - \frac{1}{2}[J\omega, \omega]) \equiv 0 (\operatorname{mod} \Omega^*(M) \otimes I^{n+1} L)$.

Theorem 4.3 [12, Theorem 2.6] *There exists a pair (ω, ∂) such that*

(1) $\omega \in \Omega^*(M) \otimes \mathcal{L}(H_*(M)[-1])$,

(2) ∂ *is a derivation of $\mathcal{L}(H_*(M)[-1])$ of square zero,*

(3) $\partial\omega + d\omega - \frac{1}{2}[J\omega, \omega] = 0$.

Proof The proof can be found in [12]. But we go over it, because this construction will be referred to later on. Let (X_i) be a basis of $H_*(M)$. Suppose (W_i) are closed forms in $\Omega^*(M)$ whose cohomology classes form a basis of $H^*(M)$ dual to (X_i). We construct ∂ and ω inductively and simultaneously. For ease of notation, let $L = \mathcal{L}(H_*(M)[-1])$.

The first step is to let

$$\omega_1 = \sum_i W_i X_i$$
$$\partial_1 X_i = 0 \quad \text{for all } i.$$

Then the Maurer–Cartan equation is partially satisfied:

$$\partial_1\omega_1 + d\omega_1 - \frac{1}{2}[J\omega_1, \omega_1] \equiv 0 \,(\operatorname{mod} \Omega^*(M) \otimes I^2 L).$$

Now, suppose that ∂_r and ω_r for $r < s$ are defined so that

(1) ∂_r is a derivation of L,

(2) $\partial_{s-1} X_i \equiv \partial_r X_i \ (\operatorname{mod} I^{r+1} L)$,

(3) $\omega_{s-1} \equiv \omega_r \ (\operatorname{mod} \Omega^*(M) \otimes I^{r+1} L)$,

(4) $\partial_r\omega_r + d\omega_r - \frac{1}{2}[J\omega_r, \omega_r] \equiv 0 \ (\operatorname{mod} \Omega^*(M) \otimes I^{r+1} L)$.

We need to define ∂_s and ω_s to continue the induction step. By Lemma 4.2,

$$d\left(\partial_{s-1}\omega_{s-1} + d\omega_{s-1} - \frac{1}{2}[J\omega_{s-1}, \omega_{s-1}]\right) = 0.$$

But since the cohomology classes of (W_i) form a basis, we have the identity

$$\begin{aligned}\partial_{s-1}\omega_{s-1} &+ d\omega_{s-1} - \frac{1}{2}[J\omega_{s-1}, \omega_{s-1}] \\ &= \sum_{i_1\cdots i_s}\left(\sum_i a_i^{i_1\cdots i_s} W_i + dW_{i_1\cdots i_s}\right)[X_{i_1}, [X_{i_2}, \dots [X_{i_{s-1}}, X_{i_s}]]].\end{aligned}$$

Then let
$$\omega_s = \omega_{s-1} + \sum_{i_1\cdots i_s} W_{i_1\cdots i_s}[X_{i_1}, [X_{i_2}, \dots, [X_{i_{s-1}}, X_{i_s}]]]$$
$$\partial_s X_i = \partial_{s-1} X_i + \sum_{i_1\cdots i_s} a_i^{i_1\cdots i_s}[X_{i_1}, [X_{i_2}, \dots [X_{i_{s-1}}, X_{i_s}]]].$$

Looking at the Maurer–Cartan equation modulo $\Omega^*(M) \otimes I^{s+1}L$,

$$\begin{aligned}\partial_s\omega_s &+ d\omega_s - \frac{1}{2}[J\omega_s, \omega_s] \\ &\equiv \partial_{s-1}\omega_{s-1} + d\omega_{s-1} - \frac{1}{2}[J\omega_{s-1}, \omega_{s-1}] \\ &\qquad + \sum_i\left(\sum_{i_1\cdots i_s} a_i^{i_1\cdots i_s} W_i + dW_{i_1\cdots i_s}\right)[X_{i_1}, [X_{i_2}, \dots [X_{i_{s-1}}, X_{i_s}]]] \\ &\equiv 0.\end{aligned}$$

This allows us to continue our induction. Define ω and ∂ by the equations

$$\partial X_i \equiv \partial_s \ (\mathrm{mod}\, I^{s+1}L)$$
$$\omega \equiv \omega_s \ (\mathrm{mod}\, \Omega^*(M) \otimes I^{s+1}L). \qquad \square$$

It is a result of rational homotopy theory that the homology of $(\mathcal{L}(H_*(M)[-1]), \partial)$ is isomorphic to $\pi_*(M) \otimes \mathbb{Q}$ and the homology of $(U(\mathcal{L}(H_*(M)[-1])), \partial)$ is isomorphic to $H_*(\Omega_b(M))$ as a Hopf algebra.

The twisting cochain will be the inclusion $H_*(M) \hookrightarrow \mathcal{L}(H_*(M)[-1])$. The power series connection defines the differential on $\mathcal{L}(H_*(M)[-1])$ to be used in the Maurer–Cartan equation and the twisting cochain condition implies that the inclusion is indeed a twisting cochain. The power series connection also has the following consequence.

Theorem 4.4 [10] *The power series connection ω defines a dg coalgebra map $T(H^*(M)[1]) \to T(\Omega^*(M)[1])$. There is map $T(\Omega^*(M)[1]) \to T(H^*(M)[1])$ such* *that the composition of the two maps is homotopic to the identity on $T(\Omega^*(M)[1])$ and equal to the identity on $T(H^*(M)[1])$.*

Proof The element ω defines a map $T(H^*(M)[1]) \to \Omega^*(M)$, using the adjunction between tensor and Hom. The twisting cochain condition on ω implies that the map satisfies the Maurer–Cartan equation. The relations between power series connections and twisting cochains is described in [10, Section 1.3]. Using the correspondence between twisting cochains and coalgebra maps then implies that extending the map as a coalgebra respects the differentials.

The second claim about the map $T(\Omega^*(M)[1]) \to T(H^*(M)[1])$ is a consequence of the map being a deformation retraction. This result can be found in [19]. □

4.2 A_∞ coalgebra modeling the homology of the principal path space

With a twisting cochain $H_*(M) \to \mathcal{L}(H_*(M)[-1])$ at our disposal, we can apply the theorems of Section 3 to the path space fibration and its conjugate bundle. This gives us three structures, a twisted A_∞ coalgebra on $H_*(M) \otimes T(H_*(M)[-1])$ modeling the coproduct on $H_*(P_b(M))$, a twisted A_∞ coalgebra on $H_*(M) \otimes T(H_*(M)[-1])$ with the conjugation action modeling $H_*(LM)$ modeling the coproduct on $H_*(LM)$, and a twisted A_∞ algebra on $H_*(M) \otimes T(H_*(M)[-1])$ modeling the loop product.

Theorem 4.5 *Let M be a simply connected manifold, $\Omega_b(M) \to P_b(M) \to M$ be the path space fibration, and $H_*(M) \hookrightarrow \mathcal{L}(H_*(M)[-1])$ be the twisting cochain given by the inclusion. Then $(H_*(M) \otimes T(H_*(M)[-1]), \{c_n^\tau\})$ defines an A_∞ coalgebra model $H_*(P)$.*

Proof The diagonal map $M \to M \times M$ defines a C_∞ coalgebra on $H_*(M)$ and $T(H_*(M)[-1])$ is a Hopf algebra model for $H_*(\Omega_b(M))$. The theorem is then a consequence of Theorem 3.9. □

4.3 A_∞ coalgebra modeling the homology of the free loop space

This brings us to defining operations in string topology. The tensor product $H_*(M) \otimes T(H_*(M)[-1])$ is an A_∞ coalgebra given by combining the C_∞ coalgebra on $H_*(M)$ and the strict associative algebra on $T(H_*(M)[-1])$. Using our twisting cochain, we twist the A_∞ coalgebra as described in Section 3.6.

Theorem 4.6 *Let $H_*(M)$ be a simply connected manifold. Consider the C_∞ coalgebra on $H_*(M)$, the Hopf algebra on $T(H_*(M)[-1])$, and the conjugation action on $T(H_*(M)[-1])$. The maps*

$$\begin{aligned}\partial_\tau &: H_*(M) \otimes T(H_*(M)[-1]) \to H_*(M) \otimes T(H_*(M)[-1])\\ c_n^\tau &: H_*(M) \otimes T(H_*(M)[-1]) \to (H_*(M) \otimes T(H_*(M)[-1])^{\otimes n}\end{aligned}$$

define an A_∞ coalgebra. The linear homology, $(H_*(M) \otimes T(H_*(M)[-1]), \partial_\tau)$, is the homology of the free loop space of the manifold $H_*(LM)$.

Proof The proof follows from the application of Theorem 3.9. □

4.4 A_∞ algebra modeling the homology of the free loop space

The loop product in $H_*(LM)$, first described by Chas and Sullivan [3], is intuitively defined as combining the intersection product of $H_*(M)$ with loop concatenation in $H_*(\Omega_b(M))$. The set-up of twisted tensor products accommodates such a description. The tensor product $H_*(M) \otimes T(H_*(M)[-1])$ is an A_∞ algebra. The map

$$m_2 : (H_*(M) \otimes T(H_*(M)[-1]))^{\otimes 2} \to H_*(M) \otimes T(H_*(M)[-1])$$

is a combination of the intersection product and loop concatenation. However, its linear homology is not $H_*(LM)$ so it does not define an operation in $H_*(LM)$. For this we need to take the twisted differential ∂_τ. Unlike the coalgebra case, we do not need to twist the higher multiplication maps.

Theorem 4.7 *Let M be a simply connected manifold. Consider the cyclic C_∞ coalgebra on $H_*(M)$, the Hopf algebra on $T(H_*(M)[-1])$, and the conjugation action on $T(H_*(M)[-1])$. The maps*

$$\begin{aligned}\partial_\tau &: H_*(M) \otimes T(H_*(M)[-1]) \to H_*(M) \otimes T(H_*(M)[-1])\\ m_n &: (H_*(M) \otimes T(H_*(M)[-1]))^{\otimes n} \to H_*(M) \otimes T(H_*(M)[-1])\end{aligned}$$

define an A_∞ algebra on $H_*(M) \otimes T(H_*(M)[-1])$.

Proof The proof is an application of Theorem 3.18. □

Example 4.8 Let $M = G$ be a connected Lie group and consider the path space fibration, $\Omega_b(G) \to P_b(G) \to G$. We claim that the conjugation action of $\Omega_b(G)$ is trivial, and so there is no twisting given by the twisting cochain $H_*(G) \hookrightarrow \mathcal{L}(H_*(G)[-1])$. Consequently, the string topology operations are given by the untwisted tensor $H_*(G) \otimes T(H_*(G)[-1])$.

To see that the conjugation action is trivial, recall that a Hopf algebra H_* is commutative if the Lie bracket on $\mathrm{Prim}(H_*)$ is zero. In this case, the Hopf algebra is $H_*(\Omega_b(G))$. There is a homotopy equivalence, $\Omega_b(G) \cong \Omega_b^2(BG)$. The Lie bracket is the same as the Samelson bracket on $\pi_*(\Omega_b^2(BG))$ which is equal to the Whitehead bracket on $\pi_*(\Omega_b(BG))$. This bracket is zero because the Whitehead bracket is trivial on H–spaces. Since the multiplication is commutative, the conjugation action is trivial and there is no twisting coming from a twisting cochain. This computation agrees with that in [13]. In that paper, Hepworth uses the isomorphism between LG and $G \times \Omega_b(G)$ to determine the Batalin-Vilkovisky algebra on $H_*(\Omega_b(G))$. Menichi [18] investigates the BV structure on $H_*(\Omega_b^2(BG)) \otimes H_*(M)$, and also considers the case when $M = G$. In that paper, he constructs a BV algebra morphism $H_*(\Omega_b(G)) \to H_*(\Omega_b(G) \otimes H_*(M) \to H_*(LM)$.

The argument that the conjugation action is trivial can be applied to any manifold M that is an H–space.

5 Application to principal G bundles

We are interested in applying the results in Section 3 to the case of a principal G bundle $G \to P \to M$. This will turn out to be representations of the algebraic structures on $H_*(M) \otimes_\tau T(H_*(M)[-1])$ given in the previous section. Given a connection on a bundle $G \to P \to M$, we get a map of bundles $P_b(M) \to M$ to $P \to M$ in the following way. Choose a basepoint above the fiber in $P \to M$, and denote it by $e \in F_b(M)$. Then the fiber can be identified with G, and e is identified with the identity element. Using the lifting property for connections gives us maps

$$\Omega_b(M) \to G$$
$$P_b(M) \to P.$$

The map $\Omega_b(M) \to G$ is often referred to as the holonomy map.

Lemma 5.1 *Let $G \to P \to M$ be a principal bundle with connection and $\Omega_b(M) \to P_b(M) \to M$ be the path space fibration. The diagram*

$$\begin{tikzcd} P_b(M) \arrow[r] \arrow[d] & P \arrow[d] \\ M \arrow[r, "\mathrm{Id}"] & M \end{tikzcd}$$

commutes. Furthermore, the map $P_b(M) \to P$ commutes with the $\Omega_b(M)$ action on $P_b(M)$ and the G action on P.

Proof This first part is the definition of lifting paths. See Kobayashi and Nomizu [16, Proposition 3.2] for the second statement. □

This bundle map induces a map on the conjugate bundles

$$\begin{array}{ccc} \mathrm{Conj}(P_b(M)) & \longrightarrow & \mathrm{Conj}(P) \\ \downarrow & & \downarrow \\ M & \longrightarrow & M. \end{array}$$

An element in $\mathrm{Conj}(P_b(M))$ is represented by an element $(p_t, \alpha) \in P_b(M) \times \Omega_b(M)$. The induced map is defined by taking a representative (p_t, α) and sending it by the map

$$(p_t, \alpha) \mapsto [\widetilde{p_t}(1), \widetilde{\alpha}] \in \mathrm{Conj}(P) = P \times_G G.$$

A loop $\alpha \in \Omega_b(M)$ lifts to a path $\widetilde{\alpha}$ starting at $e \in F_b(M)$ and ending in $F_b(M)$. This path corresponds to an element in G. A path $p_t \in P_b(M)$ lifts to a path $\widetilde{p}_t$ in P starting at e. Then $p_t \mapsto \widetilde{p}_t(1) \in P$.

Proposition 5.2 *Let $G \to P \to M$ be a principal G bundle with connection and $\Omega_b(M) \to P_b(M) \to M$ be the path space fibration. The map*

$$\mathrm{Conj}(P_b(M)) \to \mathrm{Conj}(P)$$
$$(p_t, \alpha) \mapsto (\widetilde{p}_t(1), \widetilde{\alpha})$$

is well defined and independent of choice of basepoint $e \in F_b(M)$.

Proof Let $\beta \in \Omega_b(M)$. For the map to be well defined, $(\widetilde{p_t\beta}, \widetilde{\beta\alpha\beta^{-1}})$ and $(\widetilde{p}_1, \widetilde{\alpha})$ must be in the same equivalence class in $P \times_G G$. We see that conjugating $(\widetilde{p}_1, \widetilde{\alpha})$ by $\widetilde{\beta} \in G$ is $(\widetilde{p_t\beta}, \widetilde{\beta\alpha\beta^{-1}})$. So the map is well defined.

Choosing a different point $e' \in P_b(M)$ changes the map $P_b(M) \to P$ by the G action and changes the map $\Omega_b(M) \to G$ by a conjugation. In the conjugate bundle, the images belong to the same equivalence class. □

Given a bundle $G \to P \to M$, with G a connected Lie group, we look to construct a twisting cochain $\tau \colon H_*(M) \to H_*(G)$. Then using the methods in Section 3, we obtain various structures on $H_*(M) \otimes H_*(G)$ modeling $H_*(P)$. The twisting cochain will be in terms of the characteristic classes of the bundle.

Proposition 5.3 [8, page 249] *Let G be a Lie group and R a ring. The cohomology, $H^*(BG; R)$ is a polynomial R–algebra of finite type on generators of even degree.*

For $H_*(BG)$, we need a separate argument.

Lemma 5.4 *Let G be a connected Lie group. Then $H_*(BG)$ is a free commutative algebra.*

Proof The classifying space BG is rationally equivalent to a product of Eilenberg–Mac Lane spaces. Furthermore, since G is connected, the long exact sequence in homotopy groups of $G \to EG \to BG$, implies $\pi_1(BG) = 0$. The Eilenberg–Mac Lane spaces here are then infinite loop spaces, and so BG is rationally an infinite loop space. This means $H_*(BG)$ is a Hopf algebra, which is commutative if the Lie bracket in $\mathrm{Prim}(H_*(BG))$ is zero. This bracket is equivalent to the Whitehead bracket on $\pi_*(Y)$ where $\Omega_b^2(Y) = BG$. But Y is a loop space, since BG is, rationally, an infinite loop space. And the Whitehead bracket on H–spaces is zero.

Hopf algebras are self dual, so $H^*(BG)$ is a Hopf algebra and $H_*(BG)$ is the dual Hopf algebra. We see that $H_*(BG)$ is also a polynomial algebra. □

5.1 Constructing the twisting cochain $H_*(M) \to H_*(G)$

The power series connection $\omega \in \Omega^*(M) \otimes \mathcal{L}(H_*(M)[-1])$, constructed in Section 4.1 will be used once more. Theorem 4.4 defines a dg coalgebra map $T(H^*(M)[1]) \to T(\Omega^*(M)[1])$, which has an inverse $T(\Omega^*(M)[1]) \to T(H^*(M)[1])$.

Since G is a connected Lie group, $H^*(BG)$ is a polynomial algebra. This allows us to define maps from $H^*(BG)$ in terms of its polynomial generators. Let $\{p_i \in H^*(M\})$ be the characteristic classes of a bundle $G \to P \to M$. Then there is an algebra map $H^*(BG) \to \Omega^*(M)$ defined as follows. Let $\{P_i \in H^*(BG)\}$ be the polynomial generators which pullback to the characteristic classes $\{p_i\}$. Then define an algebra map by $P_i \mapsto \widehat{p}_i$, where $\widehat{p}_i \in \Omega^*(M)$ is a representative for p_i. Extend the map as an algebra map to all of $H^*(BG)$. The algebra map $H^*(BG) \to \Omega^*(M)$ defines a coalgebra map $T(H^*(BG)[1]) \to T(\Omega^*(M)[1])$.

Therefore we have a coalgebra map $T(H^*(BG)[1]) \to T(\Omega^*(M) \to T(H^*(M)[1])$, which defines an algebra map $T(H_*(M)[-1]) \to T(H_*(BG)[-1])$. To this algebra map, there is a corresponding twisting cochain $H_*(M) \to T(H_*(BG)[-1])$. Since $T(H_*(BG)[-1])$ is a model for $\Omega_b(BG)$, which is homotopy equivalent to G, we could do our work with twisting cochains now.

To replace $T(H^*(BG)[1])$ with $H^*(G)$ we need to find a coalgebra map $H^*(G) \to T(H^*(BG)[1])$. Recall that $H^*(G)$ is generated by odd dimensional generators U_i.

To each U_i there is a generator of $H^*(BG)$ one degree higher, which we denote by P_i. We define

$$\begin{aligned} f\colon H^*(G) &\to T(H^*(BG)[1]) \\ U_i &\mapsto P_i \\ U_{i_1}U_{i_2} &\mapsto P_{i_1} \otimes P_{i_2} + P_{i_2} \otimes P_{i_1} \end{aligned}$$

and extending the map as an algebra map. So $f(U_{i_1} \cdots U_{i_j}) = P_{i_1} \Cup \cdots \Cup P_{i_j}$, where $\Cup$ is the shuffle product.

Lemma 5.5 *The map $f\colon H^*(G) \to T(H^*(BG)[1])$ is a map of differential graded coalgebras. Therefore, f is a map of differential graded Hopf algebras.*

Proof The coproduct on $H^*(G)$ is given by

$$\Delta_G(U_{i_1}U_{i_2}) = U_{i_1}U_{i_2} \otimes 1 + U_{i_1} \otimes U_{i_2} + U_{i_2} \otimes U_{i_1} + 1 \otimes U_{i_1}U_{i_2},$$

and extended so that Δ_G is an algebra map. The coproduct on $T(H^*(BG)[1])$ is given by deconcatenation,

$$\Delta(P_{i_1} \otimes \cdots \otimes P_{i_k}) = \sum_j P_{i_1} \otimes \cdots P_{i_j} \bigotimes P_{i_{j+1}} \otimes \cdots \otimes P_{i_k}.$$

The following computation shows that f is a coalgebra map,

$$\begin{aligned} (f \otimes f) \circ \Delta(U_iU_j) &= (f \otimes f)(U_iU_j \otimes 1 + U_i \otimes U_j + U_j \otimes U_i + 1 \otimes U_iU_j) \\ &= (P_i \otimes P_j) \otimes 1 + (P_j \otimes P_i) \otimes 1 + P_i \otimes P_j \\ &\qquad + P_j \otimes P_i + 1 \otimes (P_i \otimes P_j) + 1 \otimes (P_j \otimes P_i) \\ &= \Delta(P_i \otimes P_j + P_j \otimes P_i) \\ &= \Delta f(U_iU_j). \end{aligned}$$

The differential on $H^*(G)$ is zero, so for f to be a chain map, f must map to cocycles in $T(H^*(BG)[1])$. We see that δ is zero on P_i. Then since f maps to shuffle products of P_i and δ is a derivation with respect to the shuffle product, f maps to cocycles. $\square$

To replace $T(H_*(BG)[-1])$ with $H_*(G)$, we take the dual of the above map to get a differential graded algebra map $T(H_*(BG)[-1]) \to H_*(G)$. So given a twisting cochain $\tau\colon H_*(M) \to T(H_*(BG)[-1])$, composing maps defines a twisting cochain $H_*(M) \to T(H_*(BG)[-1]) \to H_*(G)$. Similarly, $H^*(G) \to H^*(M)$ is a twisting cochain obtained by composing the twisting cochain $T(H^*(BG)[1]) \to H^*(M)$ and the coalgebra map $H^*(G) \to T(H^*(BG)[1])$.

We summarize the construction of the twisting cochain and give a formula for it. Let $G \to P \to M$ be a principal G bundle, where G is a connected Lie group and M is a simply connected manifold. Let $\{P_i\}$ be the multiplicative basis for $H^*(BG)$ where $p_i \in H^*(M)$ is the pullback of $P_i \in H^*(BG)$. The elements P_i are even dimensional and correspond to an element $U_i \in H^*(G)$ such that $\{U_i\}$ form a basis for $H^*(G)$. The following coalgebra maps are composed

(1) $T(\Omega^* M)[1]) \to T(H^*(M)[1])$

(2) $T(H^*(BG)[1]) \to T(\Omega^*(M)[1])$

(3) $H^*(G) \to T(H^*(BG)[1])$

to define a coalgebra map $H^*(G) \to T(H^*(M)[1])$ which corresponds to a twisting cochain $H^*(G) \to H^*(M)$. When this process is carried out, $\tau\colon H^*(G) \to H^*(M)$ is defined on generators by

$$\begin{aligned} H^*(G) &\to H^*(M) \\ U_i &\mapsto p_i, \end{aligned}$$

and zero on products of generators.

Proposition 5.6 *Consider the coalgebra structure on $H^*(G)$ given by group multiplication and the C_∞ algebra structure on $H^*(M)$ given by the cup product. Then the map $\tau\colon H^*(G) \to H^*(M)$ which on generators is $U_i \mapsto p_i$ and zero on products of generators is the twisting cochain coming from the twisting cochain $H_*(M) \hookrightarrow \mathcal{L}(H_*(M)[-1])$ given by the inclusion.*

Proof There are no differentials on $H^*(G)$ and $H^*(M)$, and so it suffices to show that $m_n^{\mathrm{Hom}}(\tau^{\otimes n}) = 0$ for each n. For $m_n^{\mathrm{Hom}}(\tau^{\otimes n})$ to be possibly nonzero, we need to consider the product of n generators $U_{i_1} \cdots U_{i_n}$. We look at terms in $\Delta^n(U_{i_1} \cdots U_{i_n})$ of the form

$$\sum U_{i_{\sigma(1)}} \otimes \cdots \otimes U_{i_{\sigma(n)}}.$$

Then we apply τ to each factor and apply $m_n\colon H^*(M)^{\otimes n} \to H^*(M)$ of the C_∞ algebra. But each m_n vanishes on shuffle products, so it is zero on products of these terms. □

5.2 A_∞ coalgebra of $H_*(M) \otimes_\tau H_*(G)$ for a principal G–bundle

We can now define the twisted A_∞ coalgebra structure on $H_*(M) \otimes H_*(G)$. We use the dual of $\tau\colon H^*(G) \to H^*(M)$, to get a twisting cochain. The map is also denoted τ and is defined as

$$\begin{aligned} \tau\colon H_*(M) &\to H_*(G) \\ p_i^* &\mapsto U_i^*, \end{aligned}$$

is zero on products $p^*_{i_1}\cdots p^*_{i_n}$. Note that $U^*_i\in\mathrm{Prim}(G)$, and $[U^*_{i_1},U^*_{i_2}]$ is defined. The tensor differential on $H_*(M)\otimes H_*(G)$ is zero, so ∂_τ consists only of twisted terms. These terms are obtained by applying $\{c_n\colon H_*(M)\to H_*(M)^{\otimes n}\}$, applying τ to the last $n-1$ terms, bracketing the results, and then multiplying the resulting bracket with the element in $H_*(G)$. The higher coproducts $c^\tau_2, c^\tau_3.\cdots$ are defined in the same way.

Theorem 5.7 *Let $\{p\}$ be the characteristic classes of a G bundle $G\to P\to M$, where G is a connected Lie group and M a simply connected manifold. The maps $\{\partial_\tau, c^\tau_2, c^\tau_3,\ldots\}$ define an A_∞ coalgebra on $H_*(M)\otimes H_*(G)$ whose linear homology is isomorphic to $H_*(P)$.*

Proof This is an application of Theorem 3.9. □

The twisting cochain is more easily defined as $\tau\colon H^*(G)\to H^*(M)$, so the dual A_∞ algebra can be made more explicit. Note that if C_* is a C_∞ coalgebra, H_* a Hopf algebra, and a twisting cochain $C_*\to H_*$ has its image in the primitives, then its dual map $\tau\colon H^*\to C^*$ has the property that $\ker(\tau)\cup\mathrm{Prim}(H_*)=H_*$. This property of τ implies the derivation property dual to the statement that multiplying by a primitive element is a coderivation. It is described in Figure 19.

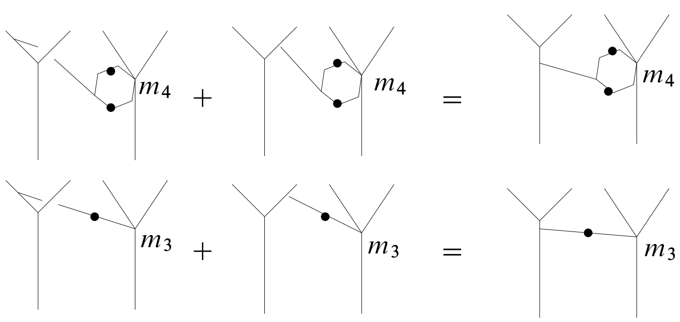

Figure 19: This identity is a consequence of the fact that $\ker(\tau)\cup\mathrm{Prim}(H_*)=H_*$. The figure is dual to Figure 6.

We define an A_∞ algebra on $H^*(G)\otimes H^*(M)$, where we view $H^*(G)$ as a Hopf algebra and $H^*(M)$ as a C_∞ algebra. The map $\partial_\tau\colon H^*(G)\otimes H^*(M)\to H^*(G)\otimes H^*(M)$ is given by:

$$\begin{aligned}\partial_\tau(U_{i_1}\cdots U_{i_n}\otimes a) &= \sum_{\sigma\in S_n} U_{i_{\sigma(1)}}\cdots U_{i_{\sigma(n-1)}}\otimes m_2(p_{i_{\sigma(n)}}\otimes a)\\ &\quad+\sum_{\sigma\in S_n} U_{i_{\sigma(1)}}\cdots U_{i_{\sigma(n-2)}}\otimes m_3(p_{i_{\sigma(n-1)}}\otimes p_{i_{\sigma(n)}}\otimes a)\\ &\quad\vdots\end{aligned}$$

The map $m_2^\tau \colon (H^*(G) \otimes H^*(M))^{\otimes 2} \to H^*(G) \otimes H^*(M)$ is given by:

$$
\begin{aligned}
m_2^\tau(U_{i_1} \cdots U_{i_k} \otimes a, U_{i_{k+1}} \cdots U_{i_n} \otimes b) \\
= U_{i_1} \cdots U_{i_n} \otimes m_2(a \otimes b) \\
+ \sum_{\sigma \in S_n} U_{i_{\sigma(1)}} \cdots U_{i_{\sigma(n-1)}} \otimes m_3(p_{i_{\sigma(n)}} \otimes a \otimes b) \\
+ \sum_{\sigma \in S_n} U_{i_{\sigma(1)}} \cdots U_{i_{\sigma(n-2)}} \otimes m_4(p_{i_{\sigma(n-1)}} \otimes p_{i_{\sigma(n)}} \otimes a \otimes b) \\
\vdots
\end{aligned}
$$

Proposition 5.8 *Let $\{p_i\}$ be the characteristic classes of a principal G bundle $P \to M$, with M simply connected and G a connected Lie group. The maps $\{\partial_\tau, m_2^\tau, \ldots\}$ define an A_∞ algebra on $H^*(G) \otimes H^*(M)$ whose linear cohomology is isomorphic to $H^*(P)$.*

Proof This is the algebraic dual of Theorem 3.9. One can see that $\partial_\tau^2 = 0$ directly, as well:

$$
\begin{aligned}
&\partial_\tau^2(U_{i_1} \cdots U_{i_n} \otimes a) \\
&= \sum_{\sigma'} \sum_{\sigma} U_{i_{\sigma'\sigma(1)}} \cdots U_{i_{\sigma'\sigma(n-2)}} \otimes m_2(p_{i_{\sigma'}} \otimes m_2(p_{i_{\sigma(n)}} \otimes a)) \\
&\quad + \sum_{\sigma'} \sum_{\sigma} U_{i_{\sigma'\sigma(1)}} \cdots U_{i_{\sigma'\sigma(n-3)}} \otimes m_3(p_{i_{\sigma'\sigma(n-2)}} \otimes p_{i_{\sigma'\sigma(n-1)}} \otimes m_2(p_{i_{\sigma(n)}} \otimes a)) \\
&\quad \vdots \\
&\quad + \sum_{\sigma'} \sum_{\sigma} U_{i_{\sigma'\sigma(1)}} \cdots U_{i_{\sigma'\sigma(n-3)}} \otimes m_2(p_{i_{\sigma'(n-2)}} \otimes m_3(p_{i_{\sigma(n-1)}} \otimes p_{i_{\sigma(n)}} \otimes a)) \\
&\quad + \sum_{\sigma'} \sum_{\sigma} U_{i_{\sigma'\sigma(1)}} \cdots U_{i_{\sigma'\sigma(n-4)}} \\
&\qquad\qquad \otimes m_3(p_{i_{\sigma'\sigma(n-3)}} \otimes p_{i_{\sigma'\sigma(n-2)}} \otimes m_3(p_{i_{\sigma(n-1)}} \otimes p_{i_{\sigma(n)}} \otimes a)) \\
&\quad \vdots
\end{aligned}
$$

Note that on the $H^*(M)$ side of the tensor, there are compositions of m_i and m_j. The C_∞ algebra relation on $H^*(M)$ states that such sums will be zero.

For the higher identities, we use the identity in Figure 19 and follow the same argument that was made in Theorem 3.9. □

5.3 A_∞ coalgebra on $H_*(M) \otimes_\tau H_*(G)$ using conjugation action

The conjugation action of $H_*(G)$ on itself is trivial when G is a connected Lie group. This shows that there is no twisting needed for the A_∞ coalgebra on $H_*(M) \otimes H_*(G)$.

That is, the coalgebra is given by $\{c_n \otimes \Delta^n\}$, where $\{c_n\}$ is the C_∞ coalgebra given by the diagonal map and Δ^n is the n–fold composition of the coproduct on $H_*(G)$.

5.4 A_∞ algebra on $H_*(M) \otimes_\tau H_*(G)$ using conjugation action

Since the conjugation action is trivial, the A_∞ algebra on $H_*(M) \otimes H_*(G)$ is given by $\{m_n \otimes m_G\}$, with no twisting terms. Here, $\{m_n\}$ is the C_∞ algebra on $H_*(M)$ given by the intersection product and m_G is the associative multiplication in $H_*(G)$.

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