Pinwheels and nullhomologous surgery on 4–manifolds with $b^+ = 1$

RONALD FINTUSHEL
RONALD J STERN

We present a method for finding embedded nullhomologous tori in standard 4–manifolds which can be utilized to change their smooth structure. As an application, we show how to obtain infinite families of simply connected smooth 4–manifolds with $b^+ = 1$ and $b^- = 2, \dots, 7$, via surgery on nullhomologous tori embedded in the standard manifolds $\mathbb{CP}^2 \# k\,\overline{\mathbb{CP}}^2$, $k = 2, \dots, 7$.

57R55; 57R57, 14J26, 53D05

1 Introduction

A primary goal of smooth 4–manifold theory is to understand the classification up to diffeomorphism of 4–manifolds in a fixed homeomorphism type. Unfortunately, there is not yet a single smooth 4–manifold for which this has been accomplished. A more modest, but still unachieved, goal is to understand whether or not the following conjecture is true:

Conjecture Let X be a simply connected smooth 4–manifold with $b^+ \geq 1$. Then there are infinitely many mutually nondiffeomorphic 4–manifolds homeomorphic to X.

Given a specific X, there are several approaches to finding such an infinite family. It is the authors' contention that the most useful and straightforward approach is to produce "exotic" manifolds by surgery on X itself. In this paper we promote this point of view by producing infinite families of mutually nondiffeomorphic manifolds homeomorphic to $\mathbb{CP}^2 \# k\,\overline{\mathbb{CP}}^2$ for $k = 2, \dots, 7$ by means of surgeries on nullhomologous tori embedded in $\mathbb{CP}^2 \# k\,\overline{\mathbb{CP}}^2$. (See Section 8.6 and the theorems that lead to it.)

In a previous paper [10], we showed that there are manifolds R_k homeomorphic to $\mathbb{CP}^2 \# k\,\overline{\mathbb{CP}}^2$, for $k = 5, \dots, 8$, such that (like $\mathbb{CP}^2 \# k\,\overline{\mathbb{CP}}^2$), the manifolds R_k have trivial Seiberg–Witten invariants, and we saw that there are nullhomologous tori in R_k such that surgery on them gives rise to infinite families of distinct smooth 4–manifolds

Published: 3 June 2011 DOI: 10.2140/agt.2011.11.1649

homeomorphic to $\mathbb{CP}^2 \# k\, \overline{\mathbb{CP}}^2$. However, we were never able to show that the R_k are actually diffeomorphic to the standard manifolds $\mathbb{CP}^2 \# k\, \overline{\mathbb{CP}}^2$.

The main new input of this paper is our technique for finding useful nullhomologous tori in standard 4–manifolds in terms of "pinwheel structures". Roughly speaking, our goal is to cover the given 4–manifold by 4–balls C_i meeting along solid tori $S^1 \times D^2$ in their boundary, then ambiently push out pairs of 2–handles from one 4–ball C_i into an adjacent ball C_j (ie adding 1–handles to C_j) so as to decompose the manifold into components C_j' that meet along $S^1 \times$(punctured torus). We then aim to identify nullhomologous tori in these C_j' upon which there are surgeries altering the smooth structure on X.

More specifically, a pinwheel structure is a generalization of the idea of a k–fold symplectic sum which was introduced by Symington [14]. The basic idea is this: One has a sequence $\{(X_i; S_i, T_i)\}$ of 4–manifolds X_i with embedded surfaces S_i, T_i, which intersect transversely at a single point and with the genus $g(T_i) = g(S_{i+1})$. Let C_i denote the complement in X_i of a regular neighborhood of the configuration $S_i \cup T_i$. We wish to glue the C_i's together so that the normal circle bundle of T_i is identified with the normal circle bundle of S_{i+1}. This can't be done unless the sum of the Euler numbers of the two bundles is 0. However, we can remove a 4–ball around the intersection point $S_i \cap T_i$ leaving the normal bundles over $S_i \smallsetminus D^2$ and $T_i \smallsetminus D^2$ which can be trivialized. It is then possible to glue each $(T_i \smallsetminus D^2) \times S^1$ to $(S_{i+1} \smallsetminus D^2) \times S^1$ to obtain a manifold whose boundary is a torus bundle over the circle. (Each $\partial B^4 \cap C_i \cong S^3 \smallsetminus$ (Hopf link) $\cong T^2 \times I$.) If this boundary is T^3, then one can glue in $T^2 \times D^2$ to obtain a closed 4–manifold X with a "pinwheel structure". We use this terminology because the components C_i fan out around a central torus $T_c = T^2 \times \{0\} \subset T^2 \times D^2$ like a pinwheel.

Our approach is then to find a pinwheel structure on the standard manifolds $X_k = \mathbb{CP}^2 \# k\, \overline{\mathbb{CP}}^2$ where the surfaces in question are spheres. Then by ambiently pushing out a pair of 2–handles from each C_i into C_{i+1}, we aim to obtain a new pinwheel structure on X_k where the interface surfaces are now tori and the new components C_i' contain nullhomologous tori upon which surgery changes smooth structures on X_k.

Here is the idea for identifying these tori in the C_i': Suppose a 4–manifold X contains an embedding of $T^2 \times D^2 = S^1 \times (S^1 \times D^2)$, with the central torus $T = T^2 \times \{0\}$ representing a nontrivial homology class in X. Define the Bing double B_T of the central torus $T = T^2 \times \{0\}$ to be $B_T = S^1 \times$ {Bing double of the core of $S^1 \times D^2$}. (Of course this depends on the splitting $T^2 = S^1 \times S^1$ and a choice of framing.) Then these Bing tori are a pair of nullhomologous tori in $T^2 \times D^2$, and hence in X. In earlier work, we showed that there are surgeries on these tori which alter the smooth structure of X provided that the Seiberg–Witten invariant of X is nontrivial. The utility of pinwheel structures on a manifold X arises from the realization that there is a manifold A properly embedded $T^2 \times D^2$ that contains the Bing tori B_T, that embeds in many of the pinwheel components C_i' (that do not contain any homologically essential tori), and such that there are surgeries on these tori altering the smooth structure of X.

The key to this surgery construction is to find surgeries on these Bing tori T_i in copies of B_T which give us a symplectic 4–manifold Q in which the images of the T_i become Lagrangian tori Λ_i and for which $b_1(Q)$ is the number of surgeries. We call these surgeries "standard". The manifold Q is a "model" in the sense of our paper with Park [6]. This puts us in a situation where we can employ our reverse engineering process [6] to surger the Λ_i and obtain a family of distinct smooth manifolds X_n homeomorphic to the given manifold X. Since the composition of the standard surgery on T_i with the surgery on Λ_i is again a surgery performed on T_i, the manifolds X_n can all be constructed by surgeries directly on the Bing tori in X.

We wish to emphasize that the families of manifolds constructed in this paper are presumably not new (see, for example, Akhmedov and Park [2; 3], Akhmedov, Baykur and Park [1], Baldridge and Kirk [5], Fintushel and Stern [9], and Fintushel, Park and Stern [6] and the references therein), although this is currently unproved. The fact that they can all be obtained via surgery on fixed standard manifolds is new.

Surprisingly, it is the construction of exotic manifolds by surgering $\mathbb{CP}^2 \# k\, \overline{\mathbb{CP}}^2$ with larger k via pinwheel structures that is the most challenging. As our abstract implies, we have not yet been able to accomplish this for $\mathbb{CP}^2 \# 8\, \overline{\mathbb{CP}}^2$ (but of course we have for $E(1) = \mathbb{CP}^2 \# 9\, \overline{\mathbb{CP}}^2$). We explain the cause of the difficulties near the end of Section 8.

2 Reverse engineering

In this section we review the notion of reverse engineering families of 4–manifolds as in [6]. The idea here is for any given smooth simply connected 4–manifold X to find a *model* Y for X; ie Y has the same Euler characteristic and signature as X, and Y has nontrivial Seiberg–Witten invariants, positive b_1, and essential tori that can be surgered to reduce b_1. (When $b_X^+ = 1$, these Seiberg–Witten invariants are the "small-perturbation invariants" – those corresponding to the same chamber as the solutions of the unperturbed equations. They are defined unambiguously when $b^- \leq 9$. See eg [9].) We begin by introducing the relevant notation. Suppose that T is a torus of self-intersection 0 with tubular neighborhood N_T. Let α and β be generators of $\pi_1(T^2)$ and let S_α^1 and S_β^1 be loops in $T^3 = \partial N_T$ homologous in N_T to α and β

respectively. Let μ_T denote a meridional circle to T in X. By *p/q–surgery* on T with respect to β we mean

$$X_{T,\beta}(p/q) = (X \smallsetminus N_T) \cup_\varphi (S^1 \times S^1 \times D^2),$$
$$\varphi\colon S^1 \times S^1 \times \partial D^2 \to \partial(X \smallsetminus N_T)$$

where the gluing map satisfies $\varphi_*([\partial D^2]) = q[S^1_\beta] + p[\mu_T]$ in $H_1(\partial(X \smallsetminus N_T);\mathbb{Z})$. We denote the "core torus" $S^1 \times S^1 \times \{0\} \subset X_{T,\beta}(p/q)$ by $T_{p/q}$. This notation depends on the given trivialization $\{S^1_\alpha, S^1_\beta, \mu_T\}$ for $T^3 = \partial N_T$. When the curve S^1_β is nullhomologous in $X \smallsetminus N_T$, then $H_1(X_{T,\beta}(1/q);\mathbb{Z}) = H_1(X;\mathbb{Z})$. In addition, when T itself is nullhomologous, then $H_1(X_{T,\beta}(p/q);\mathbb{Z}) = H_1(X;\mathbb{Z}) \oplus \mathbb{Z}/p\mathbb{Z}$.

If the homology class of T is primitive in $H_2(X;\mathbb{Z})$ then the meridian μ_T is nullhomologous in $X \smallsetminus N_T$. If also S^1_β represents a nontrivial class in $H_1(X;\mathbb{R})$, then for any integer p, in $X_{T,\beta}(p)$ the meridian to $T_p = T_{p/1}$ is $S^1_\beta + p\,\mu_T$ which is homologous to S^1_β in $X \smallsetminus N_T$. But S^1_β is not trivial in $H_1(X \smallsetminus N_T;\mathbb{R})$. This means that T_p is a nullhomologous torus in $X_{T,\beta}(p)$. The meridian μ_T to T becomes a loop on ∂N_{T_p} and it is nullhomologous in $X_{T,\beta}(p) \smallsetminus N_{T_p} = X \smallsetminus N_T$ and has a preferred pushoff $S^1_{\mu_T}$ on ∂N_{T_p}. Notice that 0–surgery on μ_T in $X_{T,\beta}(p)$ gives $(X_{T,\beta}(p))_{T_p,\mu_T}(0) = X$.

If X is a symplectic manifold and Λ is any Lagrangian torus, there is a canonical framing, called the Lagrangian framing, of N_Λ. This framing is uniquely determined by the property that pushoffs of Λ in this framing remain Lagrangian. If one performs $1/n$ surgeries ($n \in \mathbb{Z}$) with respect to the pushoff in this framing of any curve λ on Λ, then $X_{\Lambda,\lambda}(1/n)$ is also a symplectic manifold, and the core torus $\Lambda_{1/n}$ is a Lagrangian torus in the resultant manifold. We refer the reader to Auroux, Donaldson and Katzarkov [4] for a full discussion of this phenomenon, which is referred to there as *Luttinger surgery*.

Theorem 1 *Let X be a symplectic 4–manifold which contains $b = b_1(X)$ disjoint Lagrangian tori Λ_i which are primitive in $H_2(X;\mathbb{Z})$. Suppose that each Λ_i contains a simple loop λ_i such that the collection $\{\lambda_i\}$ generates $H_1(X;\mathbb{R})$. Let X' be the symplectic manifold which is the result of $\epsilon_i = \pm 1$ surgery on each Λ_i with respect to λ_i and the Lagrangian framing for N_{Λ_i}, and let $\Lambda' = (\Lambda_b)_{\epsilon_b}$, which is a Lagrangian torus in X'. Let $X_n = X'_{\Lambda',\mu_{\Lambda'}}(1/n)$ be the result of $1/n$–surgery on Λ' with respect to the loop $S^1_{\mu_{\Lambda'}}$ on $\partial N_{\Lambda'}$. Then among the manifolds $\{X_n\}$, infinitely many are pairwise nondiffeomorphic.*

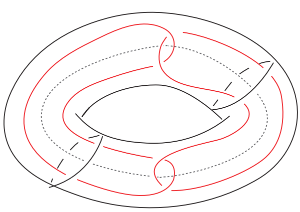
Note that $b_1(X') = 0$ and since $S^1_{\mu_T}$ is nullhomologous in $X' \smallsetminus N_{\Lambda'}$, $b_1(X_n) = 0$ for all n. We need also to point out that the preferred pushoff $S^1_{\mu_{L'}}$ on $\partial N_{\Lambda'}$ is not the

Lagrangian pushoff, and indeed, we shall see that the manifolds X_n, $n \geq 2$ admit no symplectic structure.

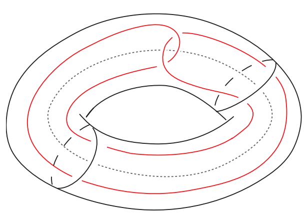
Proof (Compare [6].) Let X'' denote the symplectic manifold obtained from X via Lagrangian surgeries on (Λ_i, λ_i) with Lagrangian surgery coefficient ϵ_i for $i = 1, \dots, b-1$. Then $X' = X''_{\Lambda_b,\lambda_b}(\epsilon_b)$ and $X'' = X'_{\Lambda',\mu_{\Lambda'}}(0)$. Let k' and k'' denote the canonical classes of X' and X''; so the Seiberg–Witten invariants are $\mathrm{SW}_{X'}(k') = \pm 1$ and $\mathrm{SW}_{X''}(k'') = \pm 1$. Since Λ_b is Lagrangian, it follows from a theorem of Taubes [15] that there is no other basic class whose restriction to $H_2(X'' \smallsetminus N_{\Lambda_b}, \partial)$ agrees with that of k'', and the same is true for X' and k'. Furthermore, $X' \smallsetminus N_{\Lambda'} = X'' \smallsetminus N_{\Lambda_b}$, and the restriction of k'' to $H_2(X'' \smallsetminus N_{\Lambda_b}, \partial)$ agrees with that of k'. In $H_2(X_n)$, there is just one class which restricts to a fixed class in $H_2(X_n \smallsetminus N_{\Lambda'_{1/n}}, \partial)$ because $\Lambda'_{1/n}$ is nullhomologous. It now follows from Morgan, Mrowka, and Szabó [12] that there is a basic class $k_n \in H_2(X_n)$ satisfying $\mathrm{SW}_{X_n}(k_n) = \pm 1 \pm n$. Hence the integer invariants $\max\{|\mathrm{SW}_{X_n}(k)|,\ k \text{ basic for } X_n\}$ will distinguish an infinite family of pairwise nondiffeomorphic manifolds among the X_n. □

3 Bing doubling tori in 4–manifolds

Our goal in this section is to see how to find interesting nullhomologous tori in many common 4–manifolds. As a first step, consider a smooth 4–manifold X which contains an embedded smooth torus T of self-intersection 0. Choose local coordinates in which a tubular neighborhood $T \times D^2$ of T is $S^1 \times (S^1 \times D^2)$. The *Bing double* B_T of T consists of the pair of tori $S^1\times$ (Bing double of the core circle $S^1 \times \{0\}$). The solid torus $S^1 \times D^2$ is shown in Figure 1(a). This description (including the splitting of T^2



(a) Bing double



(b) Whitehead double

Figure 1

into the product $S^1 \times S^1$ and a fixed framing, ie a fixed trivialization of the normal bundle of T) determines this pair of tori up to isotopy. The component tori in B_T are nullhomologous in $T \times D^2$ and therefore also in X.

In order to illustrate the efficacy of surgery on these Bing doubles in changing the smooth structure of X, we present a simple example. Consider the case where X is $E(1)$, the rational elliptic surface, and T is a fiber of a given elliptic fibration on X. After splitting off an S^1, we choose a framing of the normal bundle of T so that $T \times D^2 = S^1 \times (S^1 \times D^2)$, and each $S^1 \times S^1 \times \{\mathrm{pt}\}$ is an elliptic fiber. We wish to perform -1 surgery on one component of B_T and $1/n$–surgery on the other component to obtain a manifold X_n. It is not difficult to see that this manifold is homeomorphic to X. The result of the -1–surgery on the first component of B_T is to turn the other component into the Whitehead double Λ_W of the core torus, namely, S^1 times the Whitehead double of the core circle shown in Figure 1(b). Thus X_n is obtained from surgery on Λ_W in $X = E(1)$.

We may reinterpret this construction as follows: X_n is obtained by removing a neighborhood $N \cong S^1 \times (S^1 \times D^2)$ of a fiber torus from $E(1)$, performing $1/n$–surgery on the Whitehead double torus Λ_W, and gluing back in. This is the same as forming the fiber sum of $E(1)$ and the result of $1/n$–surgery on Λ_W in $S^1 \times (S^1 \times S^2) = S^1 \times (S^1 \times D^2 \cup S^1 \times D^2)$. The particular gluing used for the fiber sum will not matter because the complement of a fiber in $E(1)$ has the property that each diffeomorphism of its boundary extends to a diffeomorphism of its interior. The result of $1/n$–surgery on Λ_W in $S^1 \times (S^1 \times S^2)$ is obtained from S^1 times a pair of surgeries on the components of the Whitehead link in S^3 with framings 0 and $1/n$. Performing $1/n$–surgery on one component of the Whitehead link turns the other component into the n–twist knot in S^3. The upshot of this analysis is that X_n is the fiber sum of $E(1)$ with S^1 times 0–surgery on the n–twist knot in S^3 – ie X_n is the result of knot surgery on $E(1)$ using the n–twist knot. (This shows that X_n is indeed homeomorphic to X.) In [8] we calculated the (small perturbation) Seiberg–Witten invariant to be $\mathcal{SW}_{X_n} = n\,(t^{-1} - t)$. Thus surgeries on the Bing double B_T give us an infinite family of distinct smooth 4–manifolds all homeomorphic to $E(1)$. For a different approach to these same examples, see our paper [10].

It is the case, however, that there are many 4–manifolds which do not have any self-intersection 0 minimal genus tori which one can Bing double; in particular, $\mathbb{CP}^2 \# n\,\overline{\mathbb{CP}}^2$, $n \leq 8$, and $S^2 \times S^2$. Here, by *minimal genus* we mean that there are no spheres that represent the nontrivial homology class of the torus. We are thus led to ask whether "Bing doubles" can appear without being embedded in $T^2 \times D^2$. Let $T_0 = T^2 \setminus \mathrm{Int}(D^2)$, a punctured torus. (Here, and throughout this paper, $\mathrm{Int}(S)$ denotes the interior of S.) View $B_T \subset S^1 \times (S^1 \times D^2)$, and write the first S^1 as $I_1 \cup I_2$ ($I_1 \cap I_2 \cong S^0$), and similarly write the second S^1 as $J_1 \cup J_2$, and consider $T_0 = S^1 \times S^1 \setminus (I_2 \times J_2)$. (See Figure 2.) Then $B_T \cap (T_0 \times D^2)$ consists of a pair of punctured tori, and $\partial(B_T \cap (T_0 \times D^2))$ is a link in $\partial(T_0 \times D^2) \cong S^1 \times S^2 \# S^1 \times S^2$.

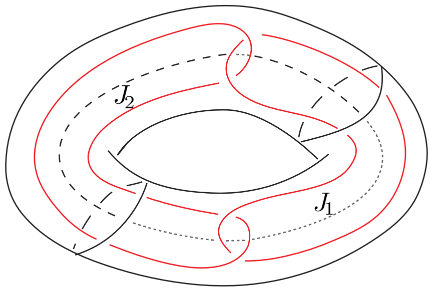
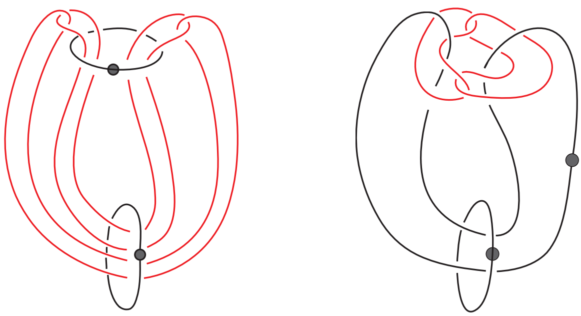


Figure 2

The intersection of B_T with $(I_2 \times J_2) \times D^2$ is a pair of disks $I_2 \times$ (intersection of the Bing double link with $J_2 \times D^2$). Its boundary is the double of the intersection of the Bing double link with $J_2 \times D^2$; ie the (1–dimensional) Bing double of $\partial(I_2 \times J_2 \times \{0\})$. In $\partial(T_0 \times D^2) \cong S^1 \times S^2 \# S^1 \times S^2$ this boundary is shown in Figure 3(a) or equivalently, Figure 3(b).



(a) (b)

Figure 3

Performing 0–framed surgeries on these boundary circles (with respect to the framing in Figure 3, we obtain a manifold, shown in Figure 4(a) which contains a pair of self-intersection 0 tori. Call this manifold A. It is given equivalently by Figure 4(b). (One can see that 4(a) and 4(b) are equivalent by twice sliding the bigger Borromean 0 over the large 0 in 4(b) and then cancelling a 1–2 handle pair to obtain 4(a).) Figure 4(b) points out that A is obtained from the 4–ball by attaching a pair of 2–handles and then carving out a pair of 2–handles. The Euler characteristic of A is $e = 1$ and its betti numbers are $b_1(A) = 2 = b_2(A)$.

Our above discussion shows that A embeds in $T^2 \times D^2$ and contains the Bing double B_T of the core torus. We continue to refer to the pair of tori in A as B_T, even though, in general, they do not constitute a Bing double of some other torus.

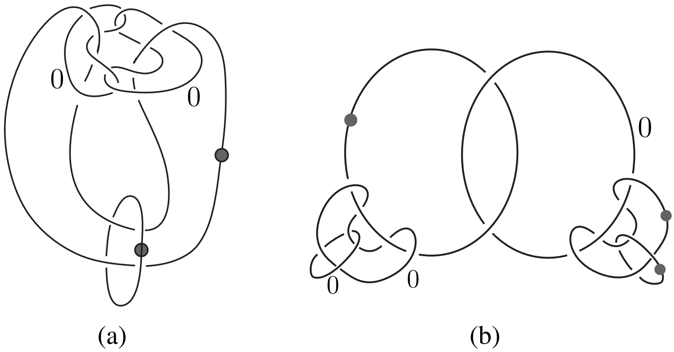


Figure 4: The manifold A

Lemma 1 *The manifold A embeds in $T^2 \times S^2$ as the complement of a pair of transversely intersecting tori of self-intersection 0.*

Proof Write $T^2 \times S^2$ as $(S^1 \times (S^1 \times D^2)) \cup (S^1 \times (S^1 \times D^2))$. We know that A embeds in, say, the second $T^2 \times D^2$. In Figure 4(a), if we remove the two 2–handles, we obtain $T_0 \times D^2$. The two 2–handles are attached along the Bing double of the circle $\beta = \partial T_0 \times \{0\}$. If instead, we attach a 0–framed 2–handle along β we obtain $T^2 \times D^2$. This implies that $(T^2 \times D^2) \smallsetminus A$ is the complement in the 2–handle, $D^2 \times D^2$, of the core disks of the 2–handles attached to obtain A. This complement is thus the result of attaching two 1–handles to the 4–ball. This is precisely the boundary connected sum of two copies of $S^1 \times B^3$, ie $T_0 \times D^2$. Using the notation in Figure 2 and above, $(T^2 \times D^2) \smallsetminus (T_0 \times D^2) = I_2 \times J_2 \times D^2$. The complement of the two 2–handles dug out of this is a neighborhood of $\{\text{pt}\}\times$ the shaded punctured torus in Figure 5(b).

Thus the two tori referred to in the lemma are illustrated in Figure 5. One of these tori, T, is S^1 times the core circle in Figure 5(a), and the second torus, $S_T = D_T \cup T_0'$, where D_T is $\{\text{pt}\}$ times the shaded meridional disk in Figure 5(a), and T_0' is $\{\text{pt}\}$ times the shaded punctured torus in Figure 5(b). Note that S_T represents the homology class of $\{\text{pt}\} \times S^2$.

From a Kirby calculus point of view, a depiction of a neighborhood N of these two tori is shown in Figure 13(a) below. Take its union with A as seen in Figure 4(b). The Borromean triple on the left side of Figure 4(b) cancels with the corresponding triple in Figure 13(a). We are left with the double of $T^2 \times D^2$, ie $T^2 \times S^2$. □

Note that we see $B_T \subset S^1 \times$ (the solid torus in Figure 5(b)) $= T^2 \times D^2 \subset T^2 \times S^2$. View $S^1 \times S^2$ as 0–framed surgery on an unknot in S^3. The Bing double of the core circle in Figure 5(b) is the Bing double of the meridian to the 0–framed unknot.

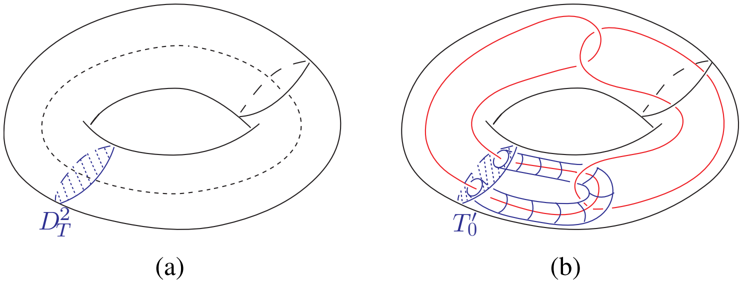


Figure 5

Performing 0–framed surgery on the two components of this Bing double gives us $\{0,0,0\}$–surgery on the Borromean rings, viz T^3. Thus performing S^1 times these surgeries gives:

Proposition 1 *One can perform surgery on the tori $B_T \subset A \subset T^2 \times S^2$ to obtain the 4–torus T^4.* □

Later we will be interested in other surgeries on B_T. We call the surgeries of Proposition 1 the *standard* surgeries on B_T. Conversely, *standard* surgeries on the corresponding pair of tori $\widetilde{B}_T \subset T^4$ yields $T^2 \times S^2$. Furthermore, $\widetilde{B}_T$ is a pair of disjoint Lagrangian tori in T^4, S^1 times two of the generating circles of T^3. The pair of tori of Lemma 1 can also be identified in T^4 after the surgeries. The first torus, S^1 times the core circle in Figure 5(a) becomes S^1 times the third generating circle of T^3. Call this torus T_T. The other torus intersects T_T once and is disjoint from B_T. We call it T_S. It is the dual generating torus of T^4. The complement of these tori in T^4 is $T_0 \times T_0$. We thus have:

Proposition 2 *The standard surgeries on the pair of Lagrangian tori $\widetilde{B}_T$ in $T_0 \times T_0$ give rise to A, and conversely, the standard surgeries on $B_T \subset A$ yield $T_0 \times T_0$.* □

We wish to emphasize a important point which follows from our discussion.

The standard surgeries on the Bing tori B_T transform $T^2 \times D^2$ into $T^2 \times T_0$.

Thus the result of the standard surgeries on $T^2 \times S^2$ is to transform A into the complement of a transverse pair of generating tori $T_T = T^2 \times \{\mathrm{pt}\}$ and $T_S = \{\mathrm{pt}\} \times T^2$ in T^4. The reason for this notation is that T_S is the torus in T^4 which is sent to S_T in $T^2 \times S^2$ after standard surgeries on $\widetilde{B}_T$, and T_T is the torus that is sent to T.

These surgeries also transform the Bing tori in B_T into the Lagrangian tori $\Lambda_1 = S^1_1 \times S^1_3$ and $\Lambda_2 = S^1_1 \times S^1_4$ in $T^4 = T^2 \times T^2 = (S^1_1 \times S^1_2) \times (S^1_3 \times S^1_4)$. The surgeries on Λ_1 and Λ_2 are not Lagrangian surgeries in the sense of [4], and so one does not get an induced symplectic structure. Indeed, $T_0 \times T_0$ is the complement of transversely intersecting symplectic tori in T^4, but after surgery, in $T^2 \times S^2$, the complement of A is the regular neighborhood of a pair of tori, one of which is not minimal genus and so cannot be symplectically embedded.

4 Pinwheels

In her thesis (cf [14]), Symington discussed the operation of symplectic summing 4–manifolds along surfaces embedded with normal crossings – the k–fold sum. We study this operation now from a topological point of view. Suppose that we are given smooth 4–manifolds X_i, $i = 1, \dots, k$, and that each X_i contains a pair of smoothly embedded surfaces, S_i, T_i, with genus $g(S_i)$, $g(T_i)$, and self-intersection m_i, n_i, and suppose that $S_i \cap T_i = \{x_i\}$ is a single transverse intersection. Let $N(S_i)$ and $N(T_i)$ be tubular neighborhoods and N_i a 4–ball neighborhood of x_i large enough so that it is not contained in $N(S_i) \cup N(T_i)$ and such that S_i and T_i intersect it in disks. Then we have

$$S_i = S_i' \cup S_i'', \quad T_i = T_i' \cup T_i''$$

where S_i'' is a disk, $S_i'' = N_i \cap S_i$, and $S_i' = S_i \smallsetminus S_i''$ is a punctured surface, and similarly for T_i. The rough idea of the k–fold sum is to remove all the tubular neighborhoods $N(S_i)$ and $N(T_i)$ and glue $\partial N(S_i')$ to $\partial N(T_{i+1}')$, $i = 1, \dots, k$ (the subscripts thought of mod k), to obtain a manifold whose boundary is a torus bundle over S^1. Thus we assume that $g(S_i) = g(T_{i+1})$ for all i, and our goal is to point out conditions on the self-intersections m_i, n_i under which the boundary of the resulting 4–manifold is the 3-torus T^3. We then fill in with $T^2 \times D^2$ to obtain a closed 4–manifold.

To determine the conditions on the self-intersections m_i, n_i, first note that the normal bundle of S_i in X_i restricted over the punctured surface S_i' can be trivialized. Let σ_i' denote the homology class of the normal circle $\{\mathrm{pt}\} \times S^1$ in $H_1(S_i' \times S^1; \mathbb{Z})$. Similarly let τ_i' denote the homology class of the normal circle in $H_1(T_i' \times S^1; \mathbb{Z})$. We begin the process of forming the k–fold sum by first gluing each $T_i' \times S^1$ to $S_{i+1}' \times S^1$ via a diffeomorphism sending T_i' to S_{i+1}' and τ_i' to σ_{i+1}'. Since the boundary of T_i' represents σ_i' and the boundary of S_{i+1} represents τ_{i+1}', in terms of the bases $\{\sigma_i', \tau_i'\}$ and $\{\sigma_{i+1}', \tau_{i+1}'\}$, the gluing map restricted to $\partial T_i' \times S^1 \to \partial S_{i+1}' \times S^1$ is given by the matrix

$$\varphi_i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{1}$$

The tubular neighborhood $N(S_i) = S_i'' \times D^2 \cup_{\alpha_i} S_i' \times D^2$, where $\alpha_i \colon \partial S_i'' \times S^1 \to \partial S_i' \times S^1$ is then given by the matrix

$$\alpha_i = \begin{pmatrix} 1 & m_i \\ 0 & -1 \end{pmatrix} \tag{2}$$

using bases $\{\sigma_i'', \tau_i''\}$ and $\{\sigma_i', \tau_i'\}$ where σ_i'' and τ_i'' are defined in the obvious way. Also $N(T_i) = T_i'' \times D^2 \cup_{\beta_i} T_i' \times D^2$ with

$$\beta_i = \begin{pmatrix} -1 & 0 \\ n_i & 1 \end{pmatrix} \tag{3}$$

again using bases $\{\sigma_i'', \tau_i''\}$ and $\{\sigma_i', \tau_i'\}$.

There is a self-diffeomorphism r of the punctured surface S_i' given by reflection through a plane which restricts to reflection in the boundary circle of S_i'. If, instead of gluing $T_i' \times S^1$ to $S_{i+1}' \times S^1$ via the diffeomorphism inducing φ_i we precompose with r and identify τ_i' to $-\sigma_{i+1}'$, we instead get

$$\varphi_i' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Set $V_i = \partial N_i \smallsetminus (\partial N_i \cap (\mathrm{Int}(N(S_i) \cup N(T_i)))$. (See Figure 6 for a schematic.) Each V_i is diffeomorphic to the complement of a neighborhood of a Hopf link in S^3; ie $V_i \cong T^2 \times I$. The homology of the factor T^2 is generated by σ_i'' and τ_i''. After gluing together the manifolds $X_i \smallsetminus (N_i \cup S_i \cup T_i)$, $i = 1, \dots, k$, along $T_i' = S_{i+1}'$ as above, we obtain a manifold whose boundary is the union of the V_i. This manifold is the torus bundle over the circle whose monodromy is the map $\vartheta_k \circ \cdots \circ \vartheta_1$ where $\vartheta_i = \alpha_{i+1}^{-1} \circ \varphi_i \circ \beta_i$. Thus

$$\vartheta_i = \begin{pmatrix} n_i + m_{i+1} & 1 \\ -1 & 0 \end{pmatrix}$$

(compare [14]). If $\vartheta_k \circ \cdots \circ \vartheta_1$ is the identity, then the boundary $\bigcup_{i=1}^k V_i \cong T^2 \times S^1$, where T^2 is the torus fiber generated by the σ_i'' and τ_i''. This trivial T^2 bundle over S^1 can then be extended over the trivial T^2 bundle over the disk, $T^2 \times D^2$. The manifold thus constructed is called the k–fold sum of the $\{X_i; S_i, T_i\}$. More precisely, the k–fold sum X is

$$X = \bigcup_{\varphi_i} (X_i \smallsetminus (N \cup N(S_i) \cup N(T_i))) \cup (T^2 \times D^2)$$

whenever $\vartheta_k \circ \cdots \circ \vartheta_1$ is the identity with respect to the given trivializations. If we replace, say, ϑ_k, with $\vartheta_k' = \alpha_1^{-1} \circ \varphi_k' \circ \beta_k = -\vartheta_k$, then we see that X can also be constructed when $\vartheta_k \circ \cdots \circ \vartheta_1$ is minus the identity (after changing one of the gluings).

In Figure 6, $\bar{S}_i$ denotes the annulus $\partial S_i' \times [0,1]$ which we think of as the disk S_i'' with a smaller disk about the origin removed. The $\bar{T}_i$ are defined similarly. We further discuss these annuli below. It is slightly misleading to think of S_i', $\bar{S}_i$, etc. as actually in X. Since the normal bundles of these manifolds can be trivialized, they do embed, but in Figure 6, for example, the text S_i' is only meant to illustrate the boundary of its normal circle bundle, $S_i' \times S^1$, and similarly for other variants of "S" and "T".

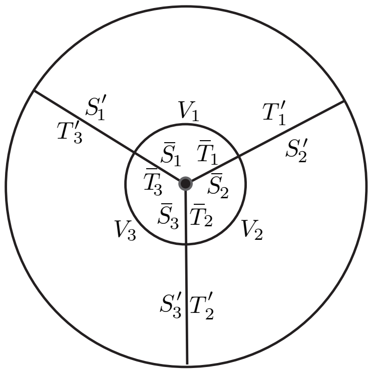


Figure 6: A 3-fold sum

Note that the tori $T^2 \times \{\mathrm{pt}\} \subset T^2 \times D^2$ are nullhomologous in X, since $T^2 \times \{\mathrm{pt}\}$ is isotopic to $\partial(S_i' \times S^1)$ and $\partial(T_i' \times S^1)$.

Proposition 3 (Symington [14]) *If $n_i + m_{i+1} = -1$ for $i = 1, 2, 3$ then the 3–fold sum exists.*

Proof In this case

$$\vartheta_3 \circ \vartheta_2 \circ \vartheta_1 = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \qquad \square$$

In [14], Symington shows that if all manifolds are symplectic and the surfaces are symplectically embedded then one can arrange for the 3–fold sum to be symplectic, and similarly in the case of the next proposition.

Proposition 4 (McDuff and Symington [11]) *If $m_{i+1} = -n_i$ for $i = 1, \ldots, 4$ then the 4–fold sum exists.*

Proof In this case,

$$\vartheta_4 \circ \cdots \circ \vartheta_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \qquad \square$$

For the next proposition we use the following notation for continued fractions:

$$[c_1, c_2, \dots, c_p] = c_1 - 1/(c_2 - 1/(\cdots - 1/c_p)\dots)$$

In case $c_p = 0$, $[c_1, c_2, \dots, c_p]$ is defined to be $[c_1, c_2, \dots, c_{p-2}]$. For a sequence of integers $\{a_1, \dots, a_k\}$, we call the set of k continued fractions $[a_1, \dots, a_{k-1}]$, $[a_2, \dots, a_k]$, $[a_3, \dots, a_k, a_1]$, $\dots$, $[a_k, , a_1 \dots, a_{k-2}]$, the *cyclic continued fractions* of the sequence.

Proposition 5 *Let $a_i = n_i + m_{i+1}$ (indices mod k), and consider the sequence $\{a_1, \dots, a_k\}$. If each of its cyclic continued fractions is 0, then the corresponding k–fold sum exists.*

Proof We prove this proposition in the case $k = 4$, which is the only case that we shall use, and we leave the general case as an exercise for the reader.

$$\vartheta_4 \circ \cdots \circ \vartheta_1 = \begin{pmatrix} 1 - a_2 a_3 & a_1 + a_3 - a_1 a_2 a_3 \\ -a_2 - a_4 + a_2 a_3 a_4 & 1 - a_1 a_2 - a_1 a_4 - a_3 a_4 + a_1 a_2 a_3 a_4 \end{pmatrix}$$

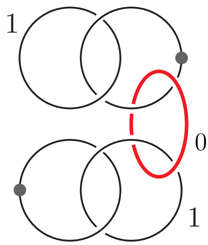
The cyclic continued fractions are:

$$\begin{array}{ll} (-a_1 - a_3 + a_1 a_2 a_3)/(-1 + a_2 a_3) & (-a_2 - a_4 + a_2 a_3 a_4)/(-1 + a_3 a_4) \\ (-a_1 - a_3 + a_1 a_3 a_4)/(-1 + a_1 a_4) & (-a_2 - a_4 + a_1 a_2 a_4)/(-1 + a_1 a_2) \end{array}$$

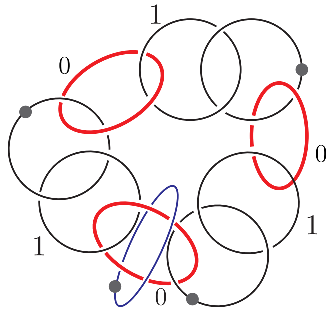
The hypothesis immediately implies that the off-diagonal entries of $\vartheta_4 \circ \cdots \circ \vartheta_1$ are 0. Furthermore, the determinant is $1 - a_1 a_2 - a_2 a_3 + a_1 a_2^2 a_3 - a_1 a_4 - a_3 a_4 + 2 a_1 a_2 a_3 a_4 + a_2 a_3^2 a_4 - a_1 a_2^2 a_3^2 a_4$ and it is not difficult to see that the hypothesis shows this $= 1$; so $\vartheta_4 \circ \cdots \circ \vartheta_1 = \pm \mathrm{Id}$. □

As mentioned above and shown [14; 11], if the (X_i, S_i, T_i) are all symplectic, then the conditions in Propositions 3 and 4 guarantee that the resulting k–fold sum is symplectic. However, the manifolds obtained as k–fold sums using the general criterion of Proposition 5 need not be symplectic. We give an example below.

The conditions given in Proposition 5 are sufficient for the existence of pinwheel structures with given components, but they are not necessary. A necessary and sufficient condition is, of course, that the boundary of the manifold that is formed by gluing the components by identifying the trivial bundles over punctured surfaces in their boundary should be T^3. The way to identify these trivial bundles over punctured surfaces in a Kirby calculus diagram is to take the Kirby calculus diagrams of the two components and add a simple loop with framing 0 around the two meridians that are to be identified. See Figure 7(a). After all the identifications are made in this way, we introduce a 1–handle as in Figure 7(b). This last figure can be seen to have boundary diffeomorphic to T^3. It gives an additional check that the pinwheel structure on $\mathbb{CP}^2$ described in Figure 9 exists.



(a) Connecting components



(b) $\mathbb{CP}^2 \setminus T^2 \times D^2$

Figure 7

Each k–fold sum has a central 2–torus $T_c = T^2 \times \{0\} \subset T^2 \times D^2$, and the various component pieces, $X_i \setminus (\mathrm{Int}(N(S_i) \cup N(T_i)))$ fan out around T_c like a pinwheel. In order to emphasize this structure, we henceforth refer to a k–fold sum as a k–*component pinwheel* and say that X has a *pinwheel structure*. Note that one can have a pinwheel structure without specifying the actual manifolds X_i of a k–fold sum. One only needs the complements of neighborhoods of the transversely intersecting surfaces (S_i, T_i) in X_i. Of course, given a manifold with the correct boundary, one can form an X_i by gluing in the correct neighborhood of transverse surfaces. We sometimes call S_i and T_i *interface surfaces*.

Consider the projection $\pi_i\colon V_i \cong T^2 \times I \to I = I_i$. The central $T^2 \times D^2$ extends the trivial T^2 bundle over the circle $\partial D^2 = \bigcup I_i$. Let $B_i = T^2 \times \mathrm{cone}(I_i)$ and $C_i = B_i \cup (X_i \setminus (\mathrm{Int}(N(S_i) \cup N(T_i))))$. This gives the "pinwheel structure"

$$X = \bigcup_{i=1}^{k} C_i, \qquad \bigcap_{i=1}^{k} C_i = T_c.$$

We refer to C_i as a *component* of the pinwheel. Note that C_i is diffeomorphic to $X_i \setminus (\mathrm{Int}(N(S_i) \cup N(T_i)))$. We have $\partial S_i' \times S^1 \subset V_i$, and in fact, if u_i and v_i are the beginning and endpoints of I_i, then $\pi_i^{-1}(v_i) = \partial S_i' \times S^1$ and $\pi_i^{-1}(u_i) = \partial T_i' \times S^1$. The annuli $\overline{S}_i$ and $\overline{T}_i$ discussed above are the products $\overline{S}_i = [c, v_i] \times \partial S_i \times \{\mathrm{pt}\}$ and $\overline{T}_i = [c, u_i] \times \partial T_i \times \{\mathrm{pt}\}$, where $[c, v_i]$ and $[c, u_i]$ are the intervals from the cone point to v_i and u_i in $\mathrm{cone}(I_i)$. Note that $\pi_i^{-1}[c, v_i] = \partial S_i \times S^1 \times [c, v_i] = \overline{S}_i \times S^1$.

Many examples of pinwheels can be obtained from torus actions. Torus actions on simply connected 4–manifolds are completely classified by Orlik and Raymond [13] in terms of their orbit data. Briefly, the orbit space must be a disk whose boundary circle is the union of arcs of constant isotropy type separated by (isolated) fixed point images. The isotropy groups which are not trivial or all of T^2 are circle subgroups described in polar coordinates by $G(p,q) = \{(\varphi, \vartheta) \mid p\varphi + q\vartheta = 0, gcd(p,q) = 1\}$. The orbit space data is a cyclic sequence of pairs (p_i, q_i) of relatively prime integers describing the orbit types over boundary segments. The preimage of each of these closed segments is a 2–sphere, A_i. Orlik and Raymond's theorem is that each such cyclic sequence gives rise to a simply connected 4–manifold with a T^2–action provided that each determinant

$$\begin{vmatrix} p_{i-1} & p_i \\ q_{i-1} & q_i \end{vmatrix} = \pm 1.$$

This condition ensures that the link of each fixed point will be S^3. The 2–sphere A_i which sits over the boundary component of the orbit space which is labelled (m_i, n_i) has self-intersection

$$A_i^2 = \begin{vmatrix} p_{i-1} & p_i \\ q_{i-1} & q_i \end{vmatrix} \cdot \begin{vmatrix} p_i & p_{i+1} \\ q_i & q_{i+1} \end{vmatrix} \cdot \begin{vmatrix} p_{i-1} & p_{i+1} \\ q_{i-1} & q_{i+1} \end{vmatrix} = \pm \begin{vmatrix} p_{i-1} & p_{i+1} \\ q_{i-1} & q_{i+1} \end{vmatrix}$$

and the intersection number of adjacent spheres is

$$A_{i-1} \cdot A_i = \begin{vmatrix} p_{i-1} & p_i \\ q_{i-1} & q_i \end{vmatrix}.$$

The second betti number of the 4–manifold is $b_2 = k - 2$ where k is the number of fixed points (equivalently the number of distinct segments in the boundary of the orbit space). Using this technology, Orlik and Raymond show that a simply connected 4–manifold which admits a T^2–action must be a connected sum of copies of S^4, $\mathbb{CP}^2$, $\overline{\mathbb{CP}}^2$, and $S^2 \times S^2$. So given the orbit data, we may determine the intersection form which tells us which manifold we have.

Example Consider the T^2–manifold X with orbit data

$$\{(1,-1), (0,1), (1,-1), (2,-1)\}.$$

The spheres A_2 and A_4 both have self-intersection 0 and $A_1^2 = -2$, $A_3^2 = +2$. Thus X is the ruled surface $\mathbb{F}_2 \cong S^2 \times S^2$ with A_3 the positive section, A_1 the negative section, and A_2 and A_4 fibers. Here $\mathbb{F}_n$ is the ruled surface with positive section having self-intersection n.

For our next set of examples, note that complement B_n of the negative section and a fiber in the ruled surface $\mathbb{F}_n$ is a 4–ball. This complement in $\mathbb{F}_n$ has a handle decomposition with a single cancelling 1– and 2–handle pair as shown in Figure 8.

The framing on the 2–handle is inherited from the positive section of $\mathbb{F}_n$; hence it is n (≥ 0).

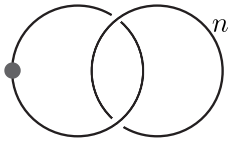
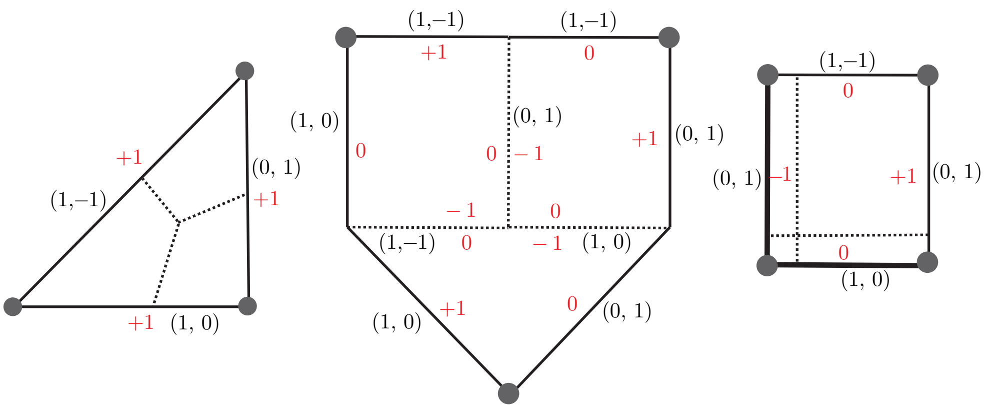


Figure 8: B_n

The next example exhibits $\mathbb{CP}^2$ as a 3–component pinwheel made up of three 4–balls (the standard coordinate neighborhoods in $\mathbb{CP}^2$). Our description is essentially that of Symington [14], using the language of Orlik and Raymond [13] rather than that of toric varieties.

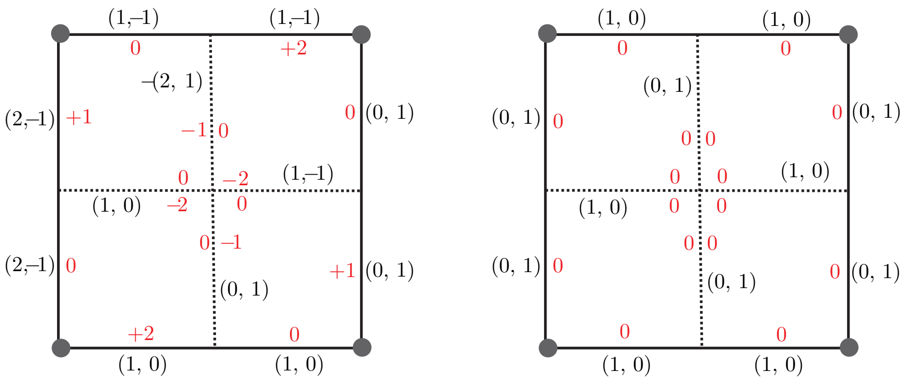


(a) $\mathbb{CP}^2$ (b) $\mathbb{CP}^2$ with pinwheel structure (c) $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$

Figure 9

Example This example is described by Figure 9. Figure 9(a) shows a torus action on $\mathbb{CP}^2$, and Figure 9(b) shows how it has the structure of a pinwheel with three B_1 components. (See Figure 9(c) where the two removed spheres are indicated by bold lines. This gives the upper right-hand component of Figure 9(b).) The dotted lines in this figure indicate that tubular neighborhoods of the corresponding invariant 2–spheres have been removed and boundaries glued together. (The point in Figure 9(c) where the two dotted lines meet is the image in the orbit space of the central torus T_c.) The single digits in the figures are self-intersection numbers. Each dotted line represents a solid torus, and the role of the self-intersection numbers assigned to them is to indicate how they get glued together. For example, the boundary of each component in Figure 9(b) (ie the union of two solid tori) is S^3. This example illustrates Proposition 3.

Example This example is described by Figure 10(a). It gives $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ with a T^2–action. We use the same notation as in the previous example. This pinwheel structure illustrates Proposition 5. For two of the components we use B_2 and for the other two, B_1.



(a) $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ (b) $S^2 \times S^2$

Figure 10

Example This final example is simpler. It gives $S^2 \times S^2$ as the union of four copies of B_0, each the complement of a section and a fiber in $\mathbb{F}_0 \cong S^2 \times S^2$, and it illustrates Proposition 4. The pinwheel structure is given in Figure 10(b).

One can deduce a general technique for constructing pinwheel structures on smooth simply connected manifolds which carry T^2–actions: As explained above, the orbit space is a disk which may be viewed as a polygon P whose open edges have a fixed isotropy subgroup $G(p_i, q_i)$ and whose vertices are the images of fixed points. The pinwheel structure will be built from n ruled surfaces $\mathbb{F}_{|r_i|}$ where r_i is the self-intersection number

$$r_i = \begin{vmatrix} p_{i-1} & p_{i+1} \\ q_{i-1} & q_{i+1} \end{vmatrix}$$

of the sphere A_i which is the preimage of the i–th edge under the orbit map.

Barycentrically divide P to obtain a union of 4–gons. Each of the edges will be divided in half and we label the first half-edge with r_i and the second half with 0. These will be self-intersections of sections or fibers in the ruled surfaces $\mathbb{F}_{|r_i|}$. Each (half) edge labelled 0 must be flanked by two edges with equal isotropy subgroups. (This follows from the rule for computing self-intersections.) Thus in each 4–gon, two of the isotropy types are given and one is determined by this rule. This determines the isotropy subgroup associated to each edge of the barycentric subdivision. See Figure 11 where

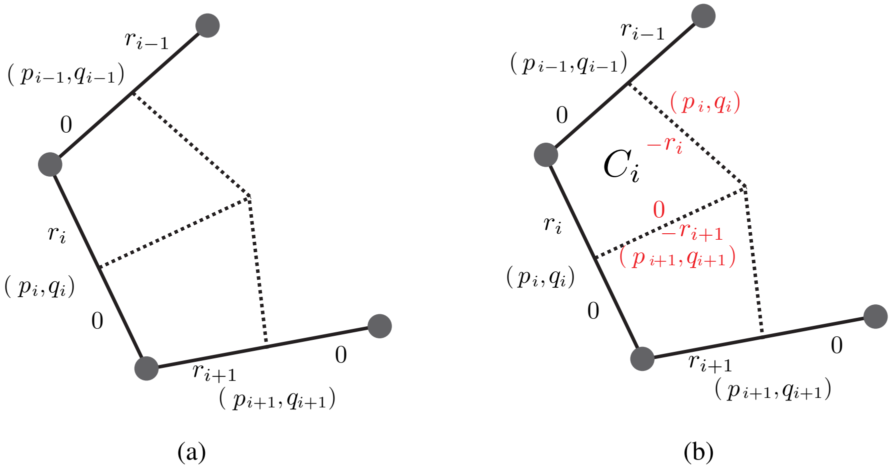


Figure 11

we are given a T^2–action, part of whose polygon is shown in 11(a), and we deduce the extra data on the barycentric subdivision in 11(b). The pinwheel component C_i in 11(b) is obtained from $\mathbb{F}_{|r_i|}$ by removing neighborhoods of a fiber and a section of self-intersection $-r_i$.

5 Pinwheel surgery

As we have seen in Section 3, when we can find an embedding of the manifold A inside a manifold X, we get a pair of tori B_T which can be useful to surger. In this section, we show how to accomplish this in some basic 4–manifolds with pinwheel structures. The general technique we present can be applied in many situations that go beyond the examples given in this paper.

Given a pinwheel structure, the first step is to cyclically push out a pair of 2–handles from each component into an adjacent component. This has the effect of subtracting a pair of 2–handles from the adjacent component. The result is that the genus of the interface increases by one, *eg* a solid torus interface becomes $S^1 \times T_0$, T_0 a punctured torus. In other words, we trade handles at the interface of each component. This gives the manifold a new pinwheel structure. The second step is to identify a copy of A inside the new components. The final steps are to compute the effect of the surgeries in A on the Seiberg–Witten invariants and to calculate the fundamental group of the ambient manifold. To compute the effect on Seiberg–Witten invariants, we shall show that in many cases there are (standard) surgeries on the Bing tori in each of the copies of A that result in a symplectic manifold, but with positive first betti number b_1. We then employ the techniques of Section 2 to find Lagrangian tori to surger to make $b_1 = 0$. It will follow that there are surgeries on all of the Bing tori in the pinwheel structure that result in distinct smooth manifolds. In order to show that the fundamental group is trivial (so that the resulting manifolds are homeomorphic), we are able to concentrate on the specific pinwheel components in the examples in Sections 6 and 7.

The first example to demonstrate this strategy is $S^2 \times S^2$ with the pinwheel structure exhibited in Figure 10(b). Each component is the complement B_0 of a section and a fiber in the ruled surface $\mathbb{F}_0$; it is a 4–ball with a handle decomposition as given in Figure 8 with $n = 0$. The dotted half-lines in Figure 10(b) each correspond to a solid torus, and the union of orthogonal half-lines corresponds to the link of a fixed point of the T^2–action, a 3–sphere. This pinwheel structure is represented by the left hand part of Figure 12. We now want to push out 2–handles from each component into an adjacent component to obtain a new pinwheel structure represented by the right hand part of Figure 12. To do this, consider the first, say upper right hand, component in the left hand part of Figure 12, a 4–ball denoted C_1. We can view C_1 as constructed from a copy of B^4 by attaching a 2–handle with framing 0 to one component of the Hopf link in Figure 8 and scooping out a tubular neighborhood of a disk in the 4–ball bounded by the other component of the Hopf link. (This "scooping out" is equivalent to attaching a 1–handle.) We thus identify the boundary of C_1 as the union of two solid tori joined together by a $T^2 \times I$; the first solid torus, U_1, is the normal circle bundle of the core of the attached 2–handle, the other solid torus, V_1, is the normal circle bundle neighborhood of the scooped out D^2, and the $T^2 \times I$ is the complement of the Hopf link in S^3. We similarly have $\partial C_i = U_i \cup V_i$ for all the components of the pinwheel. We will also use the notation $\partial B_n = U_{(n)} \cup V_{(n)}$, recalling from the previous section that B_n is the complement of the negative section and a fiber in the ruled surface $\mathbb{F}_n$. The framing on the 2–handle is inherited from the positive section of $\mathbb{F}_n$; hence it is n (≥ 0). The solid torus $U_{(0)}$ has a preferred framing coming from the fact that its core circle is the meridian of the 2–handle in Figure 8, and so it bounds a disk in ∂B_0. The Bing double of the core circle of $U_{(0)}$ with respect to this framing bounds disjoint 2–disks in B_0.

The components of the pinwheel structure are glued together so that each U_i is identified with a V_j in an adjoining component. We may assume that the components are ordered so that V_i is identified with U_{i+1}. This induces a preferred framing on V_i with respect to which the Bing double of the core circle of V_i is identified with the Bing double of the core circle of U_{i+1} and so bounds disjoint 2–disks in C_{i+1}. Add tubular neighborhoods of these disks to C_i and subtract them from C_{i+1}. This process has the effect of attaching a pair of 0–framed 2–handles to the components of the Bing double link in $V_i \subset \partial C_i$ and adding a pair of 1–handles to the components of the Bing double link in $U_{i+1} \subset \partial C_{i+1}$. Once we have applied this process to each pinwheel component C_i, we have traded handles so that $S^2 \times S^2 = \bigcup_{i=1}^4 \widetilde{C_i}$ where $\widetilde{C_i}$ is C_i with a pair of 1–handles and a pair of 2–handles attached exactly as in Figure 4(b). In other words, we have a new pinwheel representation for $S^2 \times S^2$ and each component is a copy of A. Figure 12 illustrates this process.

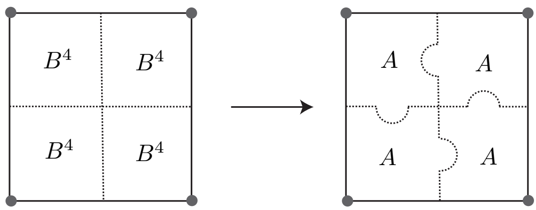
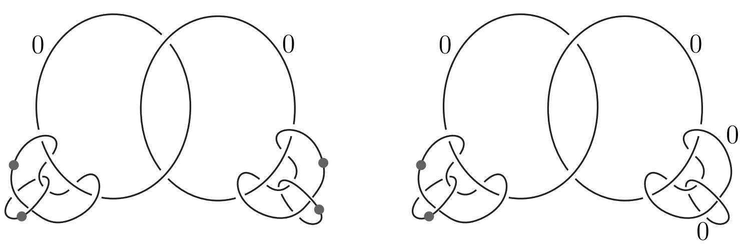


Figure 12

Proposition 6 *By performing the standard surgeries on the Bing tori B_T in each of the four copies of A in the above pinwheel structure for $S^2 \times S^2$, one obtains $\Sigma_2 \times \Sigma_2$, the product of two surfaces of genus 2.*

Proof It follows from Proposition 2 that after the standard surgeries on the four pairs of Bing tori, each copy of A is replaced by $T_0 \times T_0$. This pinwheel gives $\Sigma_2 \times \Sigma_2$. □

If we fix one of the copies of B^4 in Figure 12 then the union of the other three is a regular neighborhood of a section and a fiber in $\mathbb{F}_0 \cong S^2 \times S^2$. After the handle trading process, this neighborhood becomes the manifold M of Figure 13(b). It is the complement of one copy of A in Figure 12.



(a) N (b) M

Figure 13

Thus $S^2 \times S^2 = A \cup M$. The manifold N of Figure 13(a) is the neighborhood of a pair of transversely intersecting tori of self-intersection 0. Lemma 1 points out that $T^2 \times S^2 = A \cup N$, a fact which can also be discerned from the Kirby calculus pictures.

Corollary 1 *One can pass from $S^2 \times S^2$ to $T^2 \times S^2$ by trading a Borromean pair of 2–handles in M for a Borromean pair of 1–handles.* □

Note also that the standard surgery on A gives $T_0 \times T_0$, and $T^4 = (T_0 \times T_0) \cup N$.

We next formalize the process that we used to obtain the new pinwheel structure for $S^2 \times S^2$. We first need a lemma.

Lemma 2 *Consider the framed solid torus $V = D^2 \times S^1$ and let $B^{(k)}$ be the k–twisted Bing double of the core circle $\{0\} \times S^1$. There are unique surgeries on the components of $B^{(k)}$ so that the resulting manifold has $b_1 = 3$, and the result W of these surgeries is $W = T_0 \times S^1$. Let $h \in H_1(\partial(T_0 \times S^1))$ be the class of $\{\mathrm{pt}\} \times S^1$. Then under the identification $\partial(T_0 \times S^1) = \partial V$, $h = k[\partial D^2] + [S^1]$.*

Proof That there are unique surgeries on $B^{(k)}$ giving a manifold with $b_1 = 3$ is clear. The union of $V \times [0,1]$ with a 2–handle attached to $\{0\} \times S^1 \times \{1\}$ with respect to the k–twisted framing is a 4–ball $D^2 \times D$ where D is the core disk of the 2–handle. The correspondingly framed Bing double of $\{0\} \times S^1 \times \{0\}$ bounds a pair of disjoint properly embedded 2–disks in $D^2 \times D$, and their complement is $T_0 \times D^2$. The surgeries on $B^{(k)}$ are achieved by this process, ie $\partial(T_0 \times D^2) = (V \times 0) \cup W$; so $W = T_0 \times S^1$. Because we are using the k–framing to attach the 2–handle to $\{\mathrm{pt}\} \times S^1$, it follows that $h = k[\partial D^2] + [S^1]$. □

Now suppose that we have a manifold X with a pinwheel structure with components C_i, $i = 1, \dots, n$, and suppose also that each S_i (and therefore each T_i) is a 2–sphere. We have $\partial C_i = (T_i' \times S^1) \cup (T^2 \times [-1,1]) \cup (S_i' \times S^1)$ as in Section 4, where $S_i' \cong T_i' \cong D^2$. We suppose that the components are ordered so that $T_i' \times S^1$ is identified with $S_{i+1}' \times S^1$ (via φ_i – see Equation (1)).

Consider $\{\text{point}\} \times S^1$ which is a fiber of the normal circle bundle $\partial N(T_i)$. The *preferred framing* of this circle is the one it receives from $N(T_i)$. That is, the preferred framing of $\{\text{point}\} \times S^1 \subset T_i' \times S^1$ is the one which extends across the normal disk in $N(T_i)|_{T_i'} \cong D^2 \times D^2$. Of course, the interior of $N(T_i)$ is not contained in C_i. We similarly define the preferred framing for $S_{i+1}' \times S^1$. The condition that we shall need is:

Handle Trading Condition Suppose that for each i, the circle $\{\mathrm{pt}\} \times S^1 \subset T_i' \times S^1$ with its preferred framing bounds a proper framed disk D in C_i.

If this condition holds, then the disk D has a neighborhood $D^2 \times D$ in C_i extending the preferred framing on $D^2 \times \partial D = T_i' \times S^1$. (For example, consider the case where C_i is obtained from the ruled surface $\mathbb{F}_n$ by removing tubular neighborhoods of the negative section, T_i, and the fiber S_i. Then the framed disk D is obtained from that part of another fiber which lies outside of $N(T_i)$.) As in the proof of Lemma 2, the Bing doubles of $\{\text{pt}\}\times S^1$ with respect to this framing bound disjoint disks in C_i. We may then use these 2–disks to attach 2–handles to C_{i+1} and at the same time a cancelling pair of 1–handles to C_i. We shall refer to this process as *handle trading*.

Lemma 3 *If a manifold X admits a pinwheel structure with all interface surfaces S_i, T_i of genus 0, and if this pinwheel structure satisfies the Handle Trading Condition at each interface, then handle trading produces another pinwheel structure on X where all of the interface surfaces are tori.*

Proof We need to see that the new interface surfaces are tori. Lemma 2 implies that the handle trading process turns $T_i'\times S^1$ and $S_{i+1}'\times S^1$ into copies of $T_0\times S^1$ such that the fiber $\{\text{pt}\}\times S^1$ is identified with $\{\text{pt}\}\times S^1$ in both $T_i'\times S^1$ and $S_{i+1}'\times S^1$. (Note that φ_i preserves the preferred framing.) Hence the new interface surfaces have neighborhoods

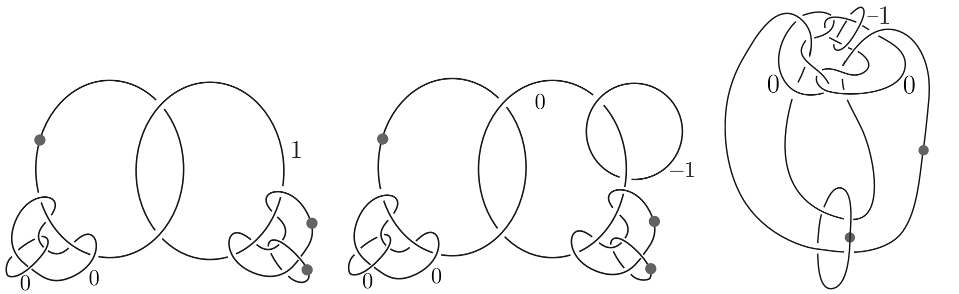
$$S_i''\times D^2\cup_{\alpha_i} T_0\times D^2 \quad \text{and} \quad T_i''\times D^2\cup_{\beta_i} T_0\times D^2$$

so we indeed get a pinwheel structure with tori as interface surfaces. □

For our next example, consider the pinwheel structure on $\mathbb{CP}^2$ which is illustrated in Figure 9. Each pinwheel component is given by a pair of cancelling handles as in Figure 8 with $n=1$. The Handle Trading Condition is satisfied at each interface because the meridian of the "dot" in Figure 8 is identified with the meridian of a "$+1$" in the next component, and this clearly bounds a disk and the framing extends. We may again trade handles to obtain the manifold $A_{(1)}$ of Figure 14(a). If we blow up and slide the $+1$–framed handle in Figure 14(a) over the exceptional curve, we obtain the manifold $\widehat{A}$ of Figure 14(b) (which is diffeomorphic to $A_{(1)}\,\#\,\overline{\mathbb{CP}}^2$), and we see $A\subset\widehat{A}$. From Figure 14(c) it is clear that at least one of the Bing tori in $A\subset\widehat{A}$ must intersect the exceptional curve.

This means that after blowing up in each component of the pinwheel structure of $\mathbb{CP}^2$, we obtain a 3–fold pinwheel structure for $\mathbb{CP}^2\,\#\,3\,\overline{\mathbb{CP}}^2$ on which we can perform our handle trading, and so that each component of the new pinwheel structure is a copy of $\widehat{A}$. Note that the Euler characteristic of $\widehat{A}$ is $e=2$; its betti numbers are $b_1(\widehat{A})=2$, $b_2(\widehat{A})=3$.

For $g\geq 1$, let $\mathbb{F}_n(g)$ denote the irrational ruled surface whose base has genus g and with $c_1=n$. Then $A_{(1)}$ is the complement in $\mathbb{F}_1(1)$ of the negative section and another torus representing the homology class of the fiber, but (nonminimal) genus one, similar



(a) $A_{(1)}$ (b) $\widehat{A}$ (c) $\widehat{A}$

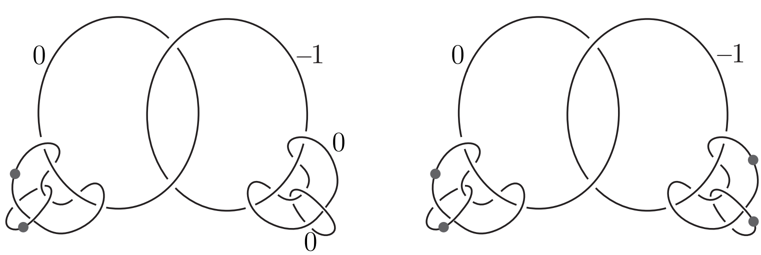
Figure 14

to Figure 5, except now rather than two copies of $T^2 \times D^2$ we replace (a) by a tubular neighborhood of the negative section, a D^2–bundle over T^2 with $c_1 = -1$, and (b) with the bundle with $c_1 = +1$, a tubular neighborhood of a (positive) section.

The blowup $\mathbb{F}_1(1)\#\overline{\mathbb{CP}}^2$ is diffeomorphic to $\mathbb{F}_0(1)\#\overline{\mathbb{CP}}^2 \cong (T^2\times S^2)\#\overline{\mathbb{CP}}^2$. One way to see this is to blow up, separating the fiber F and the negative section S_-, and then blow down the new exceptional curve $E' = F - E$ to get $\mathbb{F}_0(1)$. This correspondence takes S_- to $S' - E'$ where S' is a section of $\mathbb{F}_0(1)$, and it takes fiber to fiber. We have

$$(T^2 \times S^2)\,\#\,\overline{\mathbb{CP}}^2 = ((T^2 \times D^2)\,\#\,\overline{\mathbb{CP}}^2) \cup (T^2 \times D^2)$$

and to get $\widehat{A}$, we remove the proper transform torus $(T^2 \times \{0\}) - E'$ and the disk $\{\text{pt}\} \times D^2$ from the first summand together with the punctured torus of Figure 5(b). We can also see this by an argument similar to that of Corollary 1: Start with $\mathbb{F}_1 \cong \mathbb{CP}^2\,\#\,\overline{\mathbb{CP}}^2$ decomposed as the union of a 4–ball and a regular neighborhood of the union of S_- and F. We can also express $\mathbb{CP}^2\,\#\,\overline{\mathbb{CP}}^2$ as $A_{(1)} \cup M_1$ where M_1 is the manifold of Figure 15(a).



(a) M_1 (b) N_1

Figure 15

We pass to $A_{(1)} \cup N_1 \cong \mathbb{F}_1(1)$ and $(T^2 \times S^2) \# \overline{\mathbb{CP}}^2 \cong \mathbb{F}_1(1) \# \overline{\mathbb{CP}}^2 = \hat{A} \cup N_1$. The manifold N_1 of Figure 15(b) is a regular neighborhood of a pair of transversely intersecting tori of self-intersections -1 and 0 as described above. The standard surgeries on $B_T \subset A \subset \hat{A}$ replace A with $T_0 \times T_0$ and so $(T^2 \times S^2) \# \overline{\mathbb{CP}}^2$ is replaced with $T^4 \# \overline{\mathbb{CP}}^2$, and these surgeries replace $\hat{A}$ with the complement of transversely intersecting tori $\hat{T}_T$ and T_S where $\hat{T}_T$ represents the homology class $[T_T]-[E]$ in $T^4 \# \overline{\mathbb{CP}}^2$, where E is the exceptional curve.

Proposition 7 *The standard surgeries on the three Bing double links B_T in the three copies of $A \subset \hat{A}$ in $\mathbb{CP}^2 \# 3\,\overline{\mathbb{CP}}^2$ result in a symplectic 4–manifold Q_3, and the core tori of the surgeries are Lagrangian in Q_3.*

Proof It follows from our discussion that the under the standard surgeries, the Bing tori in $B_T \subset A \subset \hat{A}$ become Lagrangian tori in $T^4 \# \overline{\mathbb{CP}}^2 \smallsetminus (\hat{T}_T \cup T_S)$. This process carries the pinwheel structure on $\mathbb{CP}^2 \# 3\,\overline{\mathbb{CP}}^2$ to a pinwheel structure on a manifold Q_3 whose components are copies of the complements of the symplectic tori $\hat{T}_T$ and T_S in $T^4 \# \overline{\mathbb{CP}}^2$. It now follows from [14] that Q_3 is a symplectic manifold with a symplectic structure extending those of the pinwheel components. □

Presumably $Q_3 = \mathrm{Sym}^2(\Sigma_3)$; however this has not yet been shown.

We next consider $\mathbb{CP}^2 \# 2\,\overline{\mathbb{CP}}^2$. As above, we write T^4 as the union $(T_0 \times T_0) \cup N$, where N is a regular neighborhood of the plumbing of two disk bundles with $c_1 = 0$ over tori. In this case, the tori are the standard symplectic tori T_T and T_S where $T^4 = T_T \times T_S$. Let $T_{0,T}$ and $T_{0,S}$ be the copies of T_0 in T_T and T_S where we are removing disks containing the intersection point of T_T and T_S. Now suppose that, instead of N, we wish to remove from T^4 the complement of T_T and two parallel copies of T_S. This can be achieved by adjoining to $\partial(T_{0,T} \times T_{0,S}) = \partial N$ the manifold $P \times T_{0,S}$, where P is a pair of pants (Figure 16). See also Figure 27 below. We have $T^4 \smallsetminus (T_T \cup (T_S \amalg T_S)) = (T_{0,T} \times T_{0,S}) \cup (P \times T_{0,S})$, and the intersection identifies $\partial T_{0,T} \times T_{0,S}$ with $\zeta \times T_{0,S}$, where ζ is a boundary circle of P as shown in Figure 16.

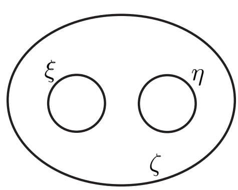


Figure 16: P

Instead of two parallel copies of T_S, we actually wish to remove a torus $2T_S$ representing the homology class $2[T_S]$. Then, instead of $P \times T_{0,S}$ we obtain a nontrivial

P–bundle, Ω, over $T_{0,S}$. This bundle is the restriction over $T_{0,S}$ of the P–bundle over $T_S = S^1 \times S^1$ whose monodromy over the first S^1 is the identity, and whose monodromy over the second S^1 is a diffeomorphism $P \to P$ which keeps ζ pointwise fixed and which interchanges the other two boundary circles, ξ and η. In other words, Ω restricted over the second S^1–factor is S^1 times the complement in a solid torus of a loop going twice around the core. Then

$$T^4 = (T_{0,T} \times T_{0,S}) \cup N = ((T_{0,T} \times T_{0,S}) \cup_{\partial T_{0,T} \times T_{0,S} = \zeta \times T_{0,S}} \Omega) \cup W$$

where W is obtained by plumbing two trivial disk bundles over T^2 twice, with both intersection points positive.

In $T_{0,T} \times T_{0,S}$ perform the standard surgeries on the tori in $\widetilde{B}_T$ whose result is A. We obtain $A \cup (\Omega \cup W) = A \cup N \cong T^2 \times S^2$, and if we replace N by M we obtain $A \cup M = S^2 \times S^2$.

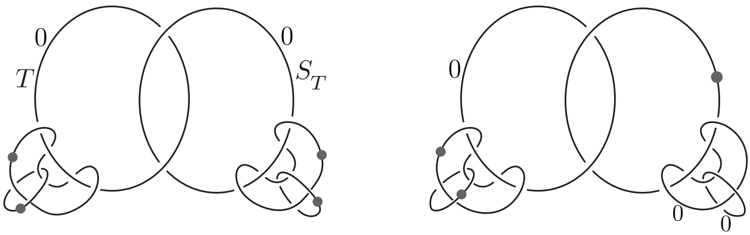
Back in T^4, the surfaces T_T and $2T_S$ intersect at two points. Blow up to separate the surfaces at one of these intersection points. In $N \# \overline{\mathbb{CP}}^2$ we have a regular neighborhood L of the union of tori $T_{I,T}$ representing the homology class $[T_T]-[E]$ and $T_{I,S}$ representing $2[T_S]-[E]$ where E is the exceptional curve. The neighborhood L is the plumbing of two disk bundles over T^2 with $c_1 = -1$. Write $N \# \overline{\mathbb{CP}}^2 = R \cup L$, where $R \cong N \# \overline{\mathbb{CP}}^2 \smallsetminus L$. Then the standard surgeries on the tori in $\widetilde{B}_T \subset T^4 \# \overline{\mathbb{CP}}^2$ give

$$T^2 \times S^2 \# \overline{\mathbb{CP}}^2 = A \cup (N \# \overline{\mathbb{CP}}^2) = A \cup (R \cup L) = I_0 \cup L$$

where $I_0 = A \cup R$ is the complement of a copy of L in $T^2 \times S^2 \# \overline{\mathbb{CP}}^2$. Thus I_0 is the complement in $T^2 \times S^2 \# \overline{\mathbb{CP}}^2$ of tori $T'_{I,T}$ and $T'_{I,S}$ representing the homology classes $[T]-[E]$ and $2[S_T]-[E]$.

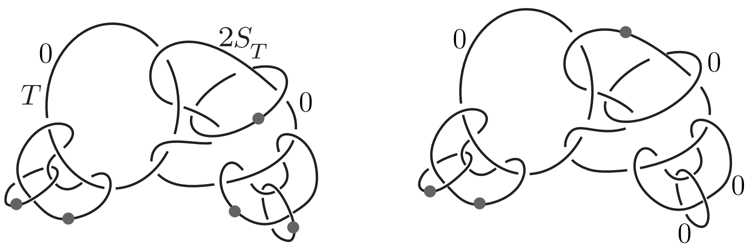
Since $\widetilde{B}_T \subset T_{0,T} \times T_{0,S}$ is disjoint from N, the tori $T'_{I,T}$ and $T'_{I,S}$ in $T^2 \times S^2 \# \overline{\mathbb{CP}}^2$ are constructed analogously to $T_{I,T}$ and $T_{I,S}$: The regular neighborhood of $T \cup S_T$ in $T^2 \times S^2$ may be identified with N. We have $T^2 \times S^2 = A \cup N = A \cup (\Omega \cup W)$, where Ω is now viewed as a P–bundle over $S_{T,0} = S_T \smallsetminus D^2$.

We next wish to see that this situation arises by attaching and subtracting 2–handles starting with the complement of a pair of surfaces in $S^2 \times S^2 \# \overline{\mathbb{CP}}^2$. To see this, recall the Kirby calculus depiction of $T^2 \times S^2 = N \cup A$ as shown in Figure 17. Similarly, we have the decomposition of $T^2 \times S^2$ into $W \cup (A \cup \Omega)$ in Figure 18. Next blow up W. This is shown in Figure 19. Then Figure 20 shows $T^2 \times S^2 \# \overline{\mathbb{CP}}^2$ as the union of L and I_0. To accomplish this, we move the exceptional curve and the 1–handle over to the other side of the picture. We need to make two points here. First, only part of the exceptional curve is seen in Figure 20(b) because E intersects both $T'_{I,T}$



(a) N (b) A

Figure 17: $T^2 \times S^2 = N \cup A$



(a) W (b) $A \cup \Omega$

Figure 18: $T^2 \times S^2 = W \cup (A \cup \Omega)$

and $T'_{I,S}$. Second, when the 1–handle from Figure 19 is moved to the other side, it becomes a 3–handle.

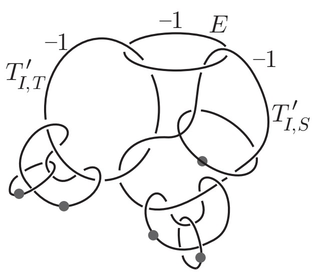
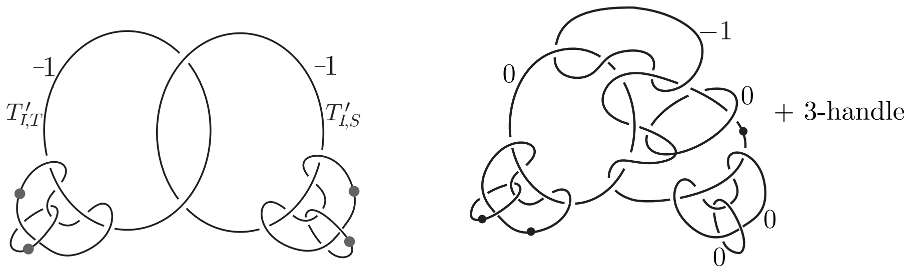


Figure 19: W blown up

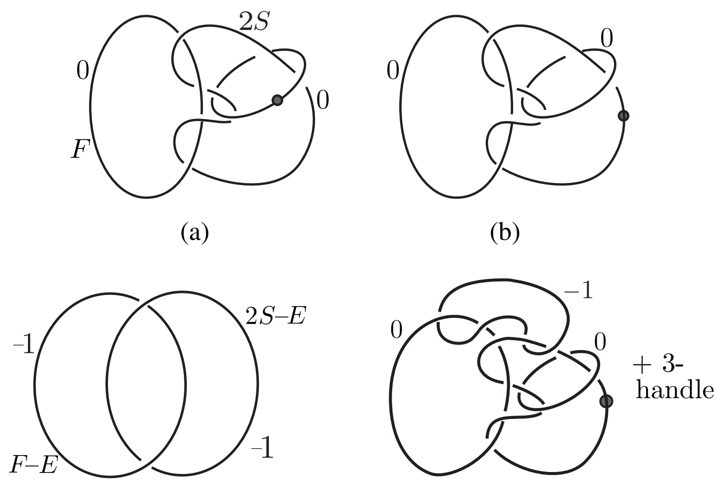
By removing 1– and 2–handle pairs from Figure 18 and Figure 20, we obtain Figure 21. The union of 21(a) and 21(b) is $S^2 \times S^2$, and the union of 21(c) and 21(d) is $S^2 \times S^2 \# \overline{\mathbb{CP}}^2$.

The complement of $F - E$ and $2S - E$ in $S^2 \times S^2 \# \overline{\mathbb{CP}}^2$ is given in Figure 21(d). In that figure, slide the left-hand 0 over the -1. This is shown in Figure 22. (After this handle slide, the 0–framed 2–handle cancels with the 3–handle.) Figure 22 also



(a) L (b) I_0

Figure 20: $T^2 \times S^2 \# \overline{\mathbb{CP}}^2 = L \cup I_0$



(a) (b) (c) (d)

Figure 21: $(a) \cup (b) = S^2 \times S^2$ and $(c) \cup (d) = S^2 \times S^2 \# \overline{\mathbb{CP}}^2$

illustrates meridians μ and ν to the 1– and 2–handle. As indicated in the figure, we call this manifold I'_0.

Lemma 4 *The 3–fold pinwheel described by Figure 23(a) is a pinwheel structure for $S^2 \times S^2 \# \overline{\mathbb{CP}}^2 \cong \mathbb{CP}^2 \# 2\,\overline{\mathbb{CP}}^2$.*

Proof This is an exercise in Kirby calculus. The union of the three manifolds I'_0, $B_1 \# \overline{\mathbb{CP}}^2$, and B_0 glued in a cyclical fashion as indicated in Figure 23(a) is shown in Figure 24. What must be seen is that the union of this with $T^2 \times D^2$ is $\mathbb{CP}^2 \# 2\,\overline{\mathbb{CP}}^2$. Handle slides, cancellations, and two blowdowns plus one anti-blowdown applied to Figure 24 reduce it to the Borromean link with two "0"s and a "dot". A handle

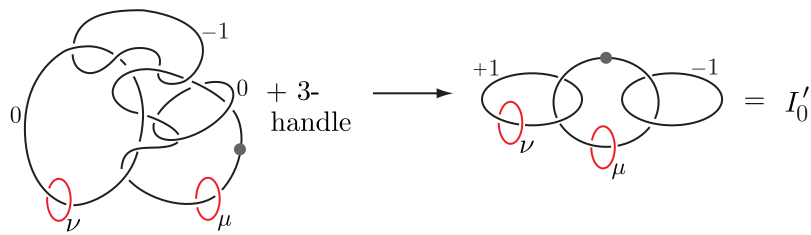


Figure 22: I_0'

picture for $T^2 \times D^2$ is the Borromean link with two "dots" and a "0". In the union, all these handles cancel and we obtain S^4. Taking into account the blowdowns and anti-blowdown, we see that we get $\mathbb{CP}^2 \# 2\,\overline{\mathbb{CP}}^2$. □

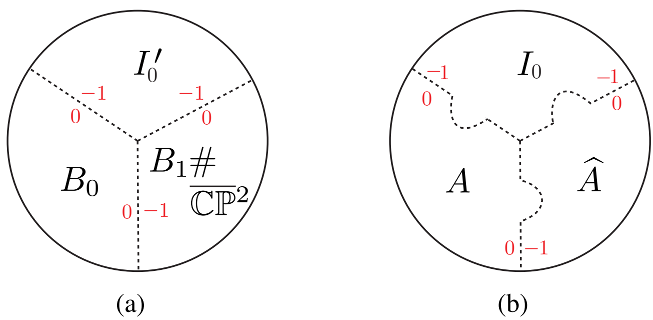


Figure 23: Pinwheel structures for $\mathbb{CP}^2 \# 2\,\overline{\mathbb{CP}}^2$

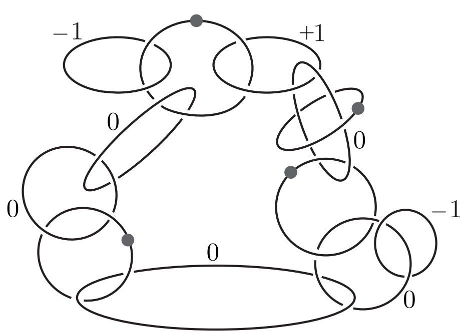


Figure 24

As is indicated by the new 0–framed 2–handles in Figure 24, the Handle Trading Condition is satisfied for the pinwheel structure of Lemma 4(a), and so Lemma 3 gives us the pinwheel structure of Figure 23(b) on $\mathbb{CP}^2 \# 2\,\overline{\mathbb{CP}}^2$.

Proposition 8 *The standard surgeries on the three Bing double links $B_T \subset A \subset \mathbb{CP}^2 \# 2\,\overline{\mathbb{CP}}^2$ in the pinwheel structure given by Figure 23(b) result in a symplectic 4–manifold Q_2 and the core tori of the surgeries are Lagrangian in Q_2.*

Proof As in Proposition 7, this follows from [14]. The manifold Q_2 has a symplectic pinwheel structure as the 3–fold sum of $(\widetilde{X}_i; \widetilde{S}_i, \widetilde{T}_i)$ where

$$(\widetilde{X}_0; \widetilde{S}_0, \widetilde{T}_0) = (T^4 \# \overline{\mathbb{CP}}^2; T_{I,T}, T_{I,S})$$
$$(\widetilde{X}_1; \widetilde{S}_1, \widetilde{T}_1) = (T^4 \# \overline{\mathbb{CP}}^2; \hat{T}_T, T_S)$$
$$(\widetilde{X}_2; \widetilde{S}_2, \widetilde{T}_2) = (T^4; T_T, T_S). \qquad \square$$

6 Component pieces and their fundamental groups

In this section we shall examine the fundamental groups of the complements of our pair B_T of Bing tori in the pinwheel components A, $\widehat{A}$, and I_0 of Section 5. One important goal is to see that when we choose our basepoint to lie on the central torus T_c, appropriate elements of these fundamental groups are represented by loops lying on the boundaries of these components. This will assist in the computations in Section 7 where we piece together the fundamental group of a manifold from the components of a pinwheel structure on it.

We are helped greatly by the analysis of Baldridge and Kirk in [5], where they study the fundamental group of the complement of the Lagrangian tori in $T_0 \times T_0$ which are exactly the core tori Λ_1 and Λ_2 of the result of surgeries on $B_T \subset A$. Following the notation in that paper (with only minor changes) and referring to Figure 25, the disjoint Lagrangian tori are $\Lambda_1 = a' \times x'$ and $\Lambda_2 = a'' \times y'$. The relevant classes in π_1 are $x = \{p\} \times x$, $y = \{p\} \times y$, $a = a \times \{q\}$, and $b = b \times \{q\}$. Since Λ_1 and Λ_2 are Lagrangian, we refer to pushoffs of loops in Λ_i to parallel loops in Λ_i as "Lagrangian pushoffs".

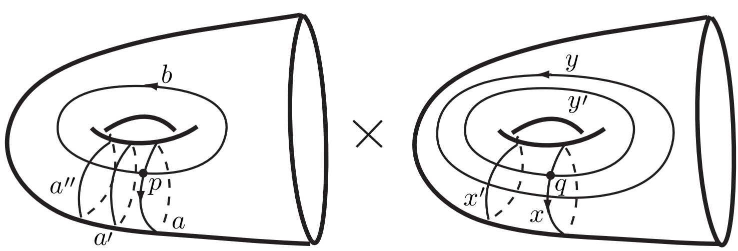


Figure 25: $T_{0,T} \times T_{0,S}$

Lemma 5 [5] *There are basepaths in $T_{0,T} \times T_{0,S} \smallsetminus (\Lambda_1 \cup \Lambda_2) = T_{0,T} \times T_{0,S} \smallsetminus \widetilde{B}_T$ to the boundaries of tubular neighborhoods of Λ_1 and Λ_2 so that the generators of the fundamental group $\pi_1(T_{0,T} \times T_{0,S} \smallsetminus \widetilde{B}_T, (p,q))$ that are represented by Lagrangian pushoffs of x' and a' on Λ_1 are x and a and the meridian to Λ_1 represents the commutator $[b^{-1}, y^{-1}]$. Similarly, Lagrangian pushoffs of the loops y' and a'' on Λ_2 represent the elements y and bab^{-1}, and the meridian to Λ_2 represents $[x^{-1}, b]$. Furthermore, $\pi_1(T_{0,T} \times T_{0,S} \smallsetminus \widetilde{B}_T, (p,q))$ is generated by the elements x, y, a and b, and the relations $[x,a] = [y,a] = [y, bab^{-1}] = 1$ (among others) hold.*

It may help in visualizing this lemma to consider (as done in [6]) Figure 26 where the punctured torus T_0 is viewed as a punctured disk with identifications. The loop $\{p\} \times x'$ is based at (p,q) via a path that travels backwards along y from q to q' (ie $\{p\}$ times this) and after traversing x' travels back to q along the same path. We base the other paths similarly. Then the lemma holds as stated. We wish to move the basepoint to the point $z_c = (p_c, q_c)$ which lies on the torus $\partial T_{0,T} \times \partial T_{0,S}$. This is done by using the path $\gamma \times \delta$ to join (p_c, q_c) to (p,q). Changing the base point in this way does not affect Lemma 5. Since the loop $\gamma * \gamma^{-1}$ is nullhomotopic, we may view the loop representing x based at z_c as $\{p_c\} \times \xi$ where $\xi = \delta * x * \delta^{-1}$, and similarly for y, a, and b. Thus the generators of $\pi_1(T_{0,T} \times T_{0,S} \smallsetminus \widetilde{B}_T, z_c)$ are all represented by loops in $\partial(T_{0,T} \times T_{0,S})$.

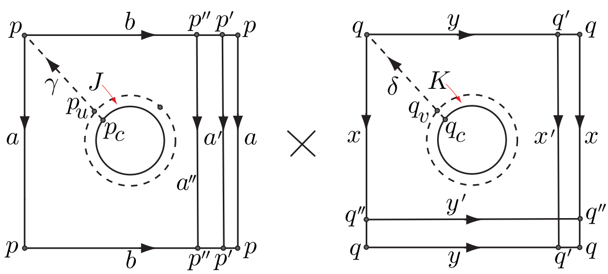


Figure 26: $T_{0,T} \times T_{0,S}$

One obtains A from $T_0 \times T_0$ by surgeries on Λ_1 (killing x) and Λ_2 (killing y); so the lemma implies that $\pi_1(A, z_c)$ is generated by a and b. In fact, it quickly follows from Figure 4(a) that $\pi_1(A, z_c)$ is the free group on two generators (a and b) which are represented by the meridians to the 1–handles. (By "meridian" of a 1–handle, we mean a loop parallel to its core. Thinking of a 1–handle as a "scooped out" 2–handle gives more intuitive meaning to this terminology.) Notice that the classes of a and b generate $H_1(T_{0,T})$ and those of x, y generate $H_1(T_{0,S})$.

Lemma 6 *The fundamental group $\pi_1(A, z_c)$ is the free group generated by a and b. Furthermore, $\pi_1(A \smallsetminus B_T, z_c) = \pi_1(T_{0,T} \times T_{0,S} \smallsetminus \widetilde{B}_T, z_c)$, and its generators x, y, a, and b are represented by loops on $\partial A = \partial(T_{0,T} \times T_{0,S})$.* □

Recalling the terminology of Section 4 (and thinking of A or $T_{0,T} \times T_{0,S}$ as the i–th component of an appropriate pinwheel)

$$\begin{aligned}\partial A &= \partial(T_{0,T} \times T_{0,S})\\ &= (T_{0,T} \times \partial T_{0,S}) \cup (\partial T_{0,T} \times T_{0,S})\\ &= ((T_i \smallsetminus D^2) \times S^1) \cup T^2 \times I \cup ((S_i \smallsetminus D^2) \times S^1)\\ &= (T_i' \times S^1) \cup (T^2 \times I) \cup (S_i' \times S^1).\end{aligned}$$

In order to make the following discussion a bit easier, consider in Figure 26 the annuli $J = \partial T_{0,T} \times [u, c]$ and $K = \partial T_{0,S} \times [v, c]$. These are collars on the boundaries of the two copies of T_0. Let J_c and K_c be the boundary circles of J and K containing the points p_c and q_c, and let J_u and K_v be the boundary circles containing p_u and q_v. Then $S_i' \times S^1$, the normal circle bundle over S_i', is represented in this figure by $J_u \times (T_{0,S} \smallsetminus K)$ and $T_i' \times S^1$ is represented by $(T_{0,T} \smallsetminus J) \times K_v$. In the above formula for ∂A, the $T^2 \times I$ summand is $(\partial T_i' \times S^1 \times [v_i, c]) \cup (\partial S_i' \times S^1 \times [c, u_i])$. In the figure, this is $(J_c \times K) \cup (J \times K_c)$. In $\pi_1(T_{0,T} \times T_{0,S} \smallsetminus \widetilde{B}_T, z_c)$, the loop $J_c \times \{q_c\} = \partial T_i' \times \{q_c\}$ (in the given trivialization) represents the commutator $[a, b]$, and similarly $\{p_c\} \times K_c = \{p_c\} \times \partial S_i'$ represents $[x, y]$.

We see from Figures 14(b) and 14(c) that $\widehat{A}$ is obtained from A by attaching a 2–handle with framing -1, and Figure 14(b) shows that the attaching circle for this 2–handle is the boundary of a normal disk to the torus T in $T^2 \times S^2$. This is the boundary of the punctured torus $S_{T,0}$. Equivalently, after the standard surgeries on B_T, the attaching circle of the 2–handle is the boundary of $T_{0,S}$; so in terms of our generators for $\pi_1(A \smallsetminus B_T, z_c) = \pi_1(T_{0,S} \times T_{0,T} \smallsetminus \widetilde{B}_T, z_c)$, attaching the 2–handle given by the exceptional curve adds the relation $[x, y] = 1$.

Lemma 7 *The fundamental group $\pi_1(\widehat{A}, z_c) = \pi_1(A, z_c)$ and $\pi_1(\widehat{A} \smallsetminus B_T, z_c) = \pi_1((T_{0,T} \times T_{0,S}) \# \overline{\mathbb{CP}}^2 \smallsetminus \widetilde{B}_T, z_c)$ is obtained from $\pi_1(A \smallsetminus B_T, z_c)$ by adding the relation $[x, y] = 1$. Furthermore, the generators x, y, a, and b of the fundamental group $\pi_1(\widehat{A} \smallsetminus B_T, z_c) = \pi_1((T_{0,T} \times T_{0,S}) \# \overline{\mathbb{CP}}^2 \smallsetminus \widetilde{B}_T, z_c)$ are represented by loops on $\partial \widehat{A} = \partial((T_{0,T} \times T_{0,S}) \# \overline{\mathbb{CP}}^2)$.* □

We next discuss the fundamental group of $I_0 \smallsetminus B_T$. Recall that one can construct I_0 by doing surgery on $\widetilde{B}_T$ in $T_{0,T} \times T_{0,S} \subset T^4 \# \overline{\mathbb{CP}}^2 \smallsetminus (T_{I,T} \cup T_{I,S})$. Using the notation of the last section, $T^4 \# \overline{\mathbb{CP}}^2 \smallsetminus L = (T_{0,T} \times T_{0,S}) \cup R$ where $R = N \# \overline{\mathbb{CP}}^2 \smallsetminus L$

(and the neighborhood L of $T_{I,T} \cup T_{I,S}$ is the plumbing of two disk bundles over T^2 with $c_1 = -1$). So $\pi_1(I_0 \smallsetminus B_T) = \pi_1((T_{0,T} \times T_{0,S} \smallsetminus \widetilde{B}_T) \cup R)$. We will discuss basepoints momentarily; we first need to describe R. To construct R, we begin with the manifold Ω, the nontrivial P (pair of pants)-bundle over $T_{0,S}$ which was constructed in the last section. Referring to Figure 16 and below,

$$T^4 = ((T_{0,T} \times T_{0,S}) \cup_{\partial T_{0,T} \times T_{0,S} = \zeta \times T_{0,S}} \Omega) \cup W.$$

(Recall that W is a regular neighborhood of $T_T \cup 2T_S$.)

A schematic is shown in Figure 27, where we see $T^4 = N \cup (T_{0,T} \times T_{0,S})$. In N, the manifold Ω is the complement of a neighborhood of $T_T \cup 2T_S$. The indicated points on $\partial(T_{0,T} \times T_{0,S})$ designate $\partial T_{0,T} \times \partial T_{0,S} = \zeta \times \partial T_{0,S}$, the torus corresponding to the intersection point of T_T and T_S. There are also tori $T_\xi = \xi \times \partial T_{0,S}$ and $T_\eta = \eta \times \partial T_{0,S}$ corresponding to the two intersections of the tori T_T and $2T_S$.

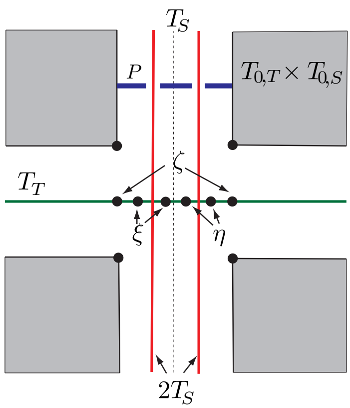


Figure 27

The basepoint $z_c = (p_c, q_c)$ for $T_{0,T} \times T_{0,S}$ lies on $\zeta \times \partial T_{0,S}$. (See Figure 26.) Use an arc from p_c to p_ξ inside $P \subset T_T$ to move this basepoint to the point $z_\xi = (p_\xi, q_c)$ on T_ξ for the purpose of calculating $\pi_1(T^4 \smallsetminus W)$ and $\pi_1(T^4 \smallsetminus (W \cup \widetilde{B}_T))$. See Figure 28(a).

Let x be the generator of $\pi_1(T_S, q_c)$ over which the monodromy of Ω is trivial and y the generator over which the monodromy exchanges ξ and η. Then since we have $T^4 \smallsetminus W = (T_{0,T} \times T_{0,S}) \cup \Omega$ and $z_c \in \zeta \times T_{0,S} = (T_{0,T} \times T_{0,S}) \cap \Omega$, the groups $\pi_1(T^4 \smallsetminus W, z_c)$ and $\pi_1(T^4 \smallsetminus (W \cup \widetilde{B}_T), z_c)$ are obtained from $\pi_1(T_{0,T} \times T_{0,S}, z_c)$ and $\pi_1((T_{0,T} \times T_{0,S}) \smallsetminus \widetilde{B}_T, z_c)$ by adding generators ξ and η, and they satisfy relations $[x,\xi] = [x,\eta] = 1$, $y\xi y^{-1} = \eta$, $y\eta y^{-1} = \xi$, and $[a,b] = \xi^{-1}\eta^{-1}$. This last relation holds because $\xi^{-1}\eta^{-1} = \zeta = \partial(T_{0,T} \times \{\mathrm{pt}\})$. Now we can move the basepoint from z_c to z_ξ using the basepath shown in Figure 28(a); so these relations also hold in $\pi_1(T^4 \smallsetminus W, z_\xi)$ and $\pi_1(T^4 \smallsetminus (W \cup \widetilde{B}_T), z_\xi)$

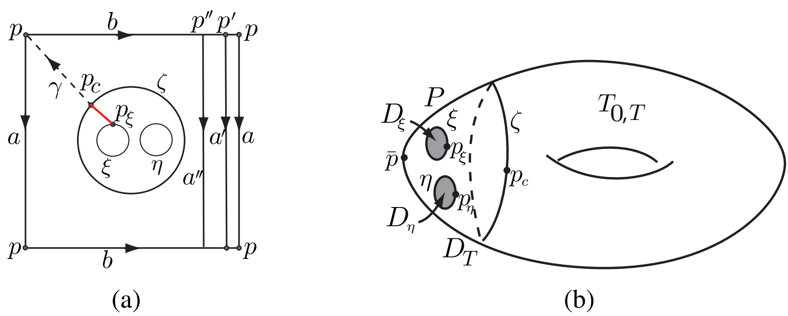


Figure 28

Let $(\overline{p}, \overline{q})$ denote the intersection point $T_T \cap T_S$. Then $T_{0,S}$ is obtained from T_S by removing a disk D_S containing $\overline{q}$ and similarly for $T_{0,T}$ and a disk D_T containing $\overline{p}$. We may assume that $T_T \cap 2T_S = \{(\overline{p}_\xi, \overline{q}), (\overline{p}_\eta, \overline{q})\}$ and that P is D_T minus disk neighborhoods D_ξ of $\overline{p}_\xi$ and D_η of $\overline{p}_\eta$.

We next need to blow up to remove the intersection point $(\overline{p}_\eta, \overline{q})$ of T_T and $2T_S$. For any transverse intersection of surfaces in a 4–manifold, the result of blowing up at the intersection point is to replace a neighborhood of the intersection point, a 4–ball containing a pair of transverse 2–disks, with the disk bundle over S^2 with $c_1 = -1$ containing a pair of disk fibers. In terms of the complements of the surfaces, this adds a copy of $S^1 \times I \times D^2$ (the disk bundle minus two fibers) to the previous complement.

In our situation, we have $N \# \overline{\mathbb{CP}}^2 = R \cup L$ where L is a neighborhood of the tori $T_{I,T}$ (representing the homology class $[T_T] - [E]$) and $T_{I,S}$ (representing $[2T_S] - [E]$). The discussion in the above paragraph shows that $R = \Omega \cup (S^1 \times I \times D^2)$. Let $\eta \times I$ denote a collar in P of the boundary component η. The gluing of $S^1 \times I \times D^2$ to Ω is given by

$$S^1 \times I \times \partial D^2 \to \eta \times I \times \partial T_{0,S}, \quad (t, r, s) \mapsto (ts^{-1}, r, s),$$

that is, ∂D^2 is identified with a fiber of the Hopf S^1–bundle, and S^1 is sent to $\eta \times \{q_c\}$.

We need to apply Van Kampen's Theorem to $T^4 \# \overline{\mathbb{CP}}^2 \smallsetminus L = (T^4 \smallsetminus W) \cup (S^1 \times I \times D^2)$. Formally, the basepoint that we use needs to lie on the intersection $\eta \times I \times \partial T_{0,S}$; so for the purposes of this calculation we move the basepoint to (p_η, q_c) by means of a path in $P \times \{q_c\}$. When we are done, we move the basepoint back to z_ξ along the same path. So, in effect, we are calculating $\pi_1(T^4 \# \overline{\mathbb{CP}}^2 \smallsetminus L, z_\xi)$. Thus the gluing formula above implies that in $\pi_1(T^4 \# \overline{\mathbb{CP}}^2 \smallsetminus L, z_\xi)$ and $\pi_1(T^4 \# \overline{\mathbb{CP}}^2 \smallsetminus (L \cup \widetilde{B}_T), z_\xi)$ we have

$[x,y]\eta^{-1}=1$ and $[S^1]=\eta$. Thus η lies in the image of $\pi_1(T_{0,T}\times T_{0,S}\smallsetminus\widetilde{B}_T,z_\xi)$, and since $\xi=y\eta y^{-1}$, ξ also lies in the image of $\pi_1(T_{0,T}\times T_{0,S}\smallsetminus\widetilde{B}_T,z_\xi)$. (Here we use the homotopy equivalence of $T_{0,T}\times T_{0,S}$ with $T_{0,T}\times T_{0,S}\cup(\kappa\times\{q_c\})$ where κ is the arc from p_c to p_ξ shown in Figure 28.)

Lemma 8 *The inclusion-induced map*

$$\pi_1(T_{0,T}\times T_{0,S}\smallsetminus\widetilde{B}_T,z_\xi)\to\pi_1(T^4\#\overline{\mathbb{CP}}^2\smallsetminus(L\cup\widetilde{B}_T),z_\xi)$$

(equivalently, $\pi_1(A\smallsetminus B_T,z_\xi)\to\pi_1(I_0\smallsetminus B_T,z_\xi)$) is surjective. □

Our analysis of the blowup also shows that it changes $\partial(T^4\smallsetminus W)$ to $\partial(T^4\#\overline{\mathbb{CP}}^2\smallsetminus L)$ by removing $\eta\times I\times\partial T_{0,S}$ and adding $(D_\eta\times\partial T_{0,S})\cup(S^1\times D_S)$. In fact,

$$\partial I_0=\partial(T^4\#\overline{\mathbb{CP}}^2\smallsetminus L)=(T_{I,T,0}\times S^1)\cup(T_{\xi,c}\times I)\cup(S^1\times T_{I,S,0})$$

where we can identify the punctured torus $T_{I,T,0}$ with $T_{0,T}\cup P\cup D_\eta$ and $S^1\times T_{I,S,0}$ with $\partial_{\xi,\eta}\Omega\cup(S^1\times D_S)$. Here, $\partial_{\xi,\eta}\Omega=\partial\Omega\smallsetminus(\zeta\times T_{0,S})$ is the inside boundary of Ω. The torus $T_{\xi,c}$ lies on ∂I_0 and contains the basepoint z_ξ. Thus the interface regions of I_0 are $S'\cong S^1\times T_{I,S,0}$ and $T'\cong T_{I,T,0}\times S^1$. The punctured torus $T_{I,T,0}$ has fundamental group generated by a and b, and the punctured torus $T_{I,S,0}$ has fundamental group generated by x and y^2, and the S^1 factor in $S^1\times T_{I,S,0}$ is generated by ξ.

Lemma 9 *The relation $[\xi,a]=1$ holds in the group*

$$\pi_1(I_0\smallsetminus B_T,z_\xi)=\pi_1(T^4\#\overline{\mathbb{CP}}^2\smallsetminus(L\cup\widetilde{B}_T),z_\xi).$$

Proof Since $A\smallsetminus B_T\subset I_0\smallsetminus B_T$, Lemma 5 implies that $[a,x]=[a,y]=1$. We have just seen that the blowup introduces the relation $\eta=[x,y]$; so $\xi=y\eta y^{-1}=y[x,y]y^{-1}$. Thus a also commutes with ξ. □

We still need to check our claim above that appropriate elements of

$$\pi_1(T^4\#\overline{\mathbb{CP}}^2\smallsetminus(L\cup\widetilde{B}_T),z_\xi)=\pi_1(I_0\smallsetminus B_T,z_\xi)$$

have representatives on $\partial(T^4\#\overline{\mathbb{CP}}^2\smallsetminus L)=\partial I_0$. These elements are the generators a, b, x, and also $y^2\xi^k$ for $k\in\mathbb{Z}$. Before blowing up, the boundary in question is

$$\partial(T^4\smallsetminus W)=((T_{0,T}\cup P)\times\partial T_{0,S})\cup\partial_{\xi,\eta}\Omega.$$

Consider first a and b. They are represented by loops in $T_{0,T}\times\{q_c\}$, and they become based at $z_\xi=(p_\xi,q_c)$ by means of a path in $P\times\{q_c\}$ as shown in Figure 28.

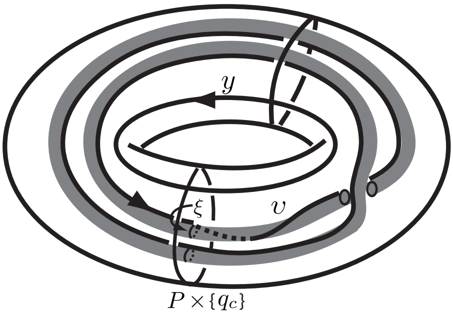


Figure 29: Ω restricted over y

Thus a and b (based at z_ξ) have representative loops lying in $(T_{0,T} \cup P) \times \{q_c\} \subset (T_{0,T} \cup P) \times \partial T_{0,S}$. These loops are unaffected by the blowup at the point $(\overline{p}_\eta, \overline{q})$.

Next consider x. It was originally represented by a loop with basepoint $z_c = (p_c, q_c)$ on $\zeta \times T_{0,S}$, and we used the basepath $\gamma \times \{q_c\}$ to move its basepoint to z_ξ. The bundle Ω is trivial over x; so one can construct an annulus whose intersection with each fiber $P\times$(point in x) is the path $\gamma\times$(point in x). This gives an isotopy starting at x and ending at a representative $\widetilde{x}$ of x based at (p_ξ, q_c) which lies on $\partial_{\xi,\eta}\Omega$. Also, because the monodromy of the bundle Ω over y has order two, there is a loop $\upsilon \subset \partial_{\xi,\eta}\Omega$, depicted in Figure 29, which represents y^2. Other loops on $\partial_{\xi,\eta}\Omega$ which are lifts of y^2 are of the form $y^2\xi^k$ and are represented by $\upsilon\,\xi^k$. These loops lie on $\partial(T^4 \smallsetminus W)$, and our description of how blowing up changes the boundary shows that $\widetilde{x}$ and $\upsilon\,\xi^k$ lie on the boundary of $T^4 \# \overline{\mathbb{CP}}^2 \smallsetminus L$ after blowing up. They lie on the branch of the blown up surfaces corresponding to $\partial_{\xi,\eta}\Omega$; ie on $\partial_{\xi,\eta}\Omega \cup (S^1 \times D_S)$.

Lemma 10 *The elements a, b, x and $y^2\xi^k$, $k \in \mathbb{Z}$, of $\pi_1(T^4 \# \overline{\mathbb{CP}}^2 \smallsetminus (L \cup \widetilde{B}_T), z_\xi) = \pi_1(I_0 \smallsetminus B_T, z_\xi)$ are represented by loops on $\partial(T^4 \# \overline{\mathbb{CP}}^2 \smallsetminus L) = \partial I_0$. The representatives of a and b are contained in $T_{I,T,0} \times S^1$, and the representatives of x and $y^2\xi^k$ are contained in $S^1 \times T_{I,S,0}$.* □

7 Exotic 4–manifolds with $b^+ = 1$

We now begin to exhibit examples which illustrate how pinwheel surgeries can be utilized to produce families of exotic 4–manifolds. There are two advantages that this technique holds over previous methods. The first is that one is able to identify the nullhomologous tori upon which surgery is done. These are copies of $B_T \subset A$ embedded in standard manifolds. The second is that there is a torus common to all the components of a pinwheel structure. Taking a basepoint in this torus can greatly simplify calculations of fundamental groups.

7.1 $\mathbb{CP}^2 \# 3\,\overline{\mathbb{CP}}^2$

As a first example, we show how the construction of exotic smooth manifolds homeomorphic to $\mathbb{CP}^2 \# 3\,\overline{\mathbb{CP}}^2$ (cf [2; 5; 6]) fits into our framework. We have seen in Proposition 7 that $\mathbb{CP}^2 \# 3\,\overline{\mathbb{CP}}^2$ has a 3–fold pinwheel structure where each component is a copy of $\widehat{A}$, and that after performing the standard surgeries on the tori in the three copies of B_T, we obtain a symplectic manifold Q_3. Since each of these surgeries increases b_1 by one and adds a hyperbolic pair to H_2, we see that Q_3 has $b_1 = 6$, $b^+ = 7$, and $b^- = 9$. We have also identified three pairs of Lagrangian tori $\widetilde{B}_T$ in three copies of $T_0 \times T_0 \subset (T^4 \# \overline{\mathbb{CP}}^2) \smallsetminus (\hat{T}_T \cup T_S)$. Each pair consists of tori Λ_1 and Λ_2. Recall that we obtain $\mathbb{CP}^2 \# 3\,\overline{\mathbb{CP}}^2$ by surgery killing the class "x" in each copy of Λ_1 and "y" in each copy of Λ_2. We have used the quotes here, because we really need to index the copies of $\widehat{A}$, Λ_i, etc.

We now proceed by labelling the components of the pinwheel. The components of the pinwheel structure, which are copies of $\widehat{A}$, will be denoted $\widehat{A}_i$, the index i viewed as an integer mod 3. The boundary of $\widehat{A}$ is the plumbing of two S^1–bundles with torus base and Euler numbers 0 and 1. Hence $\partial\widehat{A} = T_0 \times S^1 \cup_\varphi S^1 \times T_0$ for an appropriate gluing map $\varphi : T^2 \to T^2$. Using our previous notation (see above the Handle Trading Condition) we get for the pinwheel components that $\partial\widehat{A}_i = (S_i' \times S^1) \cup (T^2 \times [-1,1]) \cup (T_i' \times S^1)$.

Before proceeding further, we establish notation for our use of basepoints: The representative loops in $\widehat{A}_i$ for x_i, y_i, a_i, and b_i are based at $z_{c_i} = (p_{c_i}, q_{c_i}) \in T_{c_i} = T^2 \times 0 \subset T^2 \times [-1,1]$. The pinwheel components are glued together so that $T_i' \times S^1$ is identified with $S_{i+1}' \times S^1$; so we slide the basepoint for a_i and b_i down to $(p_{T_i}, q_{T_i}) \in \partial T_i' \times S^1$ by isotoping the loops off $T^2 \times (-1,1)$ using the $[-1,1]$–factor. Similarly we push the basepoint for x_i, y_i up to $(p_{S_i}, q_{S_i}) \in \partial S_i' \times S^1$.

Each component $\widehat{A}_i$ is glued to $\widehat{A}_{i+1}$ identifying $T_i' \times S^1$ with $S_{i+1}' \times S^1$ so that a_i is identified with x_{i+1}, b_i is identified with y_{i+1}, (p_{T_i}, q_{T_i}) is identified with $(p_{S_{i+1}}, q_{S_{i+1}})$, and the corresponding basepaths are identified. This identifies the loops in π_1 that represent a_i and b_i with those representing x_{i+1} and y_{i+1}.

The union of the three copies of $T^2 \times [-1,1]$ is T^3 and gets filled in with the manifold $T^2 \times D^2$. We may suppose that all three of the points $(p_{T_i}, q_{T_i}) = (p_{S_{i+1}}, q_{S_{i+1}})$ lie in $\{\mathrm{pt}\} \times \partial D^2 \subset T^2 \times D^2$, and we finally choose for our basepoint $z_0 = \{\mathrm{pt}\} \times \{0\}$ via straight lines to the center z_0 of $\{\mathrm{pt}\} \times D^2$.

In each $\widehat{A}_i$, perform surgery on the Lagrangian tori $\Lambda_{1,i}$ and $\Lambda_{2,i}$ as follows. On $\Lambda_{1,i}$ perform $+1$ surgery on x_i with respect to the Lagrangian framing, and on $\Lambda_{2,i}$ perform $+1$ surgery on y_i with respect to the Lagrangian framing. It follows from [4] that the resultant manifold will admit a symplectic structure extending the one on the complement of the tori. Furthermore, according to Lemma 5, these surgeries add the relations $x_i\,[b_i^{-1}, y_i^{-1}] = 1$ and $y_i\,[x_i^{-1}, b_i] = 1$.

Theorem 2 *By surgeries on all six Lagrangian tori in $\widetilde{B}_{T,i}$, $i = 0, 1, 2$ in Q_3, one can obtain an infinite family of mutually nondiffeomorphic smooth minimal 4–manifolds all homeomorphic to $\mathbb{CP}^2 \# 3\,\overline{\mathbb{CP}}^2$. Furthermore, all these manifolds can be obtained by surgeries on six nullhomologous tori which comprise three copies of $B_T \subset \mathbb{CP}^2 \# 3\,\overline{\mathbb{CP}}^2$.*

Presumably, these are the manifolds of [2; 5; 6]. What is new here is the construction technique and the fact that all these manifolds are obtained by surgeries on the standard manifold $\mathbb{CP}^2 \# 3\,\overline{\mathbb{CP}}^2$.

Proof Let X_1 be the symplectic manifold which is the result of these six surgeries on Q_3. The fundamental group $\pi_1(X_1, z_0)$ is generated by a_i, b_i, x_i, y_i, $i = 0, 1, 2$, and the relations $[x_i, y_i] = 1$ (by Lemma 7), and $[x_i, a_i] = [y_i, a_i] = 1$ (by Lemma 5) hold. Furthermore, $a_i = x_{i+1}$ and $b_i = y_{i+1}$ for $i = 0, 1, 2$. Since $\pi_1(\widehat{A}_i \smallsetminus B_{T,i})$ is generated by a_i, b_i, x_i, and y_i, we see that $\pi_1(X_1, z_0)$ has generators a_i, b_i for $i = 0, 1, 2$, and $[a_i, b_i] = 1$. There is no further generator coming from gluing the components of the pinwheel because the circle $I_0 \cup I_1 \cup I_2$ bounds a disk in the central $T^2 \times D^2$. (Each $I_i \cong [-1, 1]$.)

The relations arising from the surgeries are $x_i\,[b_i^{-1}, y_i^{-1}] = 1$ and $y_i\,[x_i^{-1}, b_i] = 1$. Translating these and the other relations, we see that $\pi_1(X_1, z_0)$ is generated by the a_i and b_i which satisfy

$$[a_i, b_i] = 1, \quad [a_{i-1}, a_i] = [b_{i-1}, a_i] = 1, \quad a_{i-1} = [b_{i-1}^{-1}, b_i^{-1}], \quad b_{i-1} = [b_i, a_{i-1}^{-1}].$$

So $b_1 = [b_2, a_1^{-1}] = [[b_0, a_2^{-1}], a_1^{-1}] = 1$, using the commutativity relations $[a_1, a_2] = 1$ and $[b_0, a_1] = 1$. Now it follows from the other relations that $\pi_1(X_1, z_0)$ is trivial.

The infinite family of manifolds referred to in the statement of the theorem is constructed by changing the surgery on the last torus $\Lambda_{2,2}$ so that it now kills y_2 times the n–th power of the meridian $[x_2^{-1}, b_2]$ to $\Lambda_{2,2}$. Call this manifold X_n. This notation agrees with that of Theorem 1. Changing the surgery in this way has the effect of replacing the relation $b_1 = [b_2, a_1^{-1}]$ with $b_1 = [b_2, a_1^{-1}]^n$ in the calculation, which goes through just as before to show that $\pi_1(X_n)$ is trivial. Theorem 1 now implies that the X_n satisfy the conclusion of the theorem.

We have seen that there are surgeries on the three copies of $\widetilde{B}_T$ in Q_3 that give $\mathbb{CP}^2 \# 3\,\overline{\mathbb{CP}}^2$ and the core tori are the three copies of B_T. The composition of the inverse of these surgeries together with the surgeries on Q_3 that give X_n, give a description of surgeries on the six tori in the copies of B_T in $\mathbb{CP}^2 \# 3\,\overline{\mathbb{CP}}^2$ that give rise to X_n. In other words, all of the exotic manifolds X_n can be obtained by surgeries on these explicitly given nullhomologous tori in $\mathbb{CP}^2 \# 3\,\overline{\mathbb{CP}}^2$.

We still need to see that the manifolds X_n are minimal, ie that they contain no sphere of self-intersection -1. We first show that they have just two basic classes, $\pm k$. Since X_n is homeomorphic to $\mathbb{CP}^2 \# 3\,\overline{\mathbb{CP}}^2$, it makes sense to talk about homology generators h and e_i, $i = 1, 2, 3$, where $h^2 = 1$ and $e_i^2 = -1$. We can see from the pinwheel construction that $h - e_i$ is represented by a genus 2 surface in X_n. This surface is obtained as follows: The attaching circle γ_i of the 0–framed 2–handle in the i–th pinwheel component for $\mathbb{CP}^2 \# 3\,\overline{\mathbb{CP}}^2$ is the meridian to the dotted circle giving the 1–handle. Thus γ_i is identified with the meridian to γ_{i+1} in an adjacent component. Thus γ_i bounds a disk in that component, and the union of this disk with the core of the 2–handle attached to γ_i is a sphere representing the homology class $h - e_i$. After the surgeries on the three copies of B_T, neither of these two 2–disks remains. Instead, γ_i bounds a punctured torus in the surgered (and handle-traded) pinwheel component C_i. Similarly, the meridian to γ_{i+1} bounds a punctured torus in the surgered C_{i+1}. Thus $h - e_i$ is represented by a surface of genus 2 in X_n. Similarly, each class e_i is represented by a torus in X_n. It follows that $h = (h - e_i) + e_i$ is represented by a surface of genus 3.

If k is a basic class for (the small perturbation Seiberg–Witten invariant) on X_n, write $k = ah - \sum_0^2 b_i e_i$. The adjunction inequality holds for the small perturbation invariant, and using the surfaces described in the paragraph above and the fact that k^2 must equal $(3\,\mathrm{sign} + 2\,e)(X_n) = 6$, we see that the only basic classes of X_n are $\pm(3h - e_0 - e_1 - e_2)$. The difference of these two classes has square 24, but if one of these manifolds failed to be minimal, it would have to have a pair of basic classes, $\kappa \pm e$, whose difference has square -4. Thus the X_n are minimal. □

7.2 $\mathbb{CP}^2 \# 2\,\overline{\mathbb{CP}}^2$

We shall next consider the construction of exotic manifolds which are homeomorphic to $\mathbb{CP}^2 \# 2\,\overline{\mathbb{CP}}^2$. As in Section 5, we can start with a pinwheel structure on $\mathbb{CP}^2 \# 2\,\overline{\mathbb{CP}}^2$ whose components are I_0', $B_1 \# \overline{\mathbb{CP}}^2$ and B_0 as in Figure 23(a) and after handle trading obtain the pinwheel structure shown in Figure 23(b). The pinwheel components are glued together as follows: $C_0 = I_0$ is glued to $C_1 = \widehat{A}$, identifying the loop representing a_0 with that representing x_1, and b_0 with y_1. Also $C_1 = \widehat{A}$ is glued to $C_2 = A$ identifying the loop representing a_1 with that representing x_2 and b_1 with y_2, and A is glued to I_0, identifying a_2 with x_0 and b_2 with $y_0^2 \xi^k$ for some k.

This k is determined by our construction in Section 5, but its precise value will not be important to us. The basepoints in the copies of T_c (recall that for I_0 this means the point $(p_\xi, q_c) \in T_{\xi,c}$) are all identified in the central $T^2 \times D^2$ of $\mathbb{CP}^2 \# 2\,\overline{\mathbb{CP}}^2$ in a fashion completely analogous to the $\mathbb{CP}^2 \# 3\,\overline{\mathbb{CP}}^2$ case, and the basepoint for our calculation again becomes $z_0 = \{\mathrm{pt}\} \times \{0\} \in T^2 \times D^2$.

We follow the pattern established above to construct examples. After performing the standard surgeries on the tori in the three copies of B_T in our pinwheel structure, we obtain the symplectic manifold Q_2 of Proposition 8 which has $b_1 = 6$ and the same Euler number and signature as $\mathbb{CP}^2 \# 2\,\overline{\mathbb{CP}}^2$. Each of the pinwheel components has fundamental group generated by the elements a_i, b_i, x_i and y_i.

On each $\Lambda_{1,i}$ perform $+1$ surgery on x_i with respect to the Lagrangian framing, and on $\Lambda_{2,i}$ perform $+1$ surgery on y_i with respect to the Lagrangian framing. Again it follows from [4] that the resultant manifold will admit a symplectic structure extending the one on the complement of the tori, and these surgeries add the relations $x_i\,[b_i^{-1}, y_i^{-1}] = 1$ and $y_i\,[x_i^{-1}, b_i] = 1$. We obtain a theorem analogous to Theorem 2.

Theorem 3 *By surgeries on all six of the Lagrangian tori in $\widetilde{B}_{T,i}$, $i = 0, 1, 2$ in Q_2, one can obtain an infinite family of mutually nondiffeomorphic smooth minimal 4–manifolds all homeomorphic to $\mathbb{CP}^2 \# 2\,\overline{\mathbb{CP}}^2$. Furthermore, all these manifolds can be obtained by surgeries on six nullhomologous tori which comprise three copies of $B_T \subset \mathbb{CP}^2 \# 2\,\overline{\mathbb{CP}}^2$.*

The proof is also the same as that of Theorem 2 with some minor modifications.

Proof Let X_1 be the symplectic manifold which is the result of these six surgeries on Q_2. The fundamental group $\pi_1(X_1, z_0)$ is generated by a_i, b_i, x_i, y_i, $i = 0, 1, 2$. After applying the identifications coming from the pinwheel structure, we see that $\pi_1(X_1, z_0)$ is generated by a_i, b_i, $i = 0, 1, 2$, and y_0, and it satisfies the relations coming from those of A, $\widehat{A}$ and I_0 with their surgeries:

- A: $a_1[b_2^{-1}, b_1^{-1}] = 1$, $b_1[a_1^{-1}, b_2] = 1$, $[a_1, a_2] = 1$, $[b_1, a_2] = 1$.
- $\widehat{A}$: $a_0[b_1^{-1}, b_0^{-1}] = 1$, $b_0[a_0^{-1}, b_1] = 1$, $[a_0, a_1] = 1$, $[b_0, a_1] = 1$, $[a_0, b_0] = 1$.
  (The last equality comes from the blowup relation $[x_1, y_1] = 1$.)
- I_0: $a_2[b_0^{-1}, y_0^{-1}] = 1$, $y_0[a_2^{-1}, b_0] = 1$, $[a_2, a_0] = 1$, $[y_0, a_0] = 1$, $[b_2, a_0] = 1$.
  (The last equality holds because of Lemma 9 and $b_2 = y_0^2 \xi^k$.)

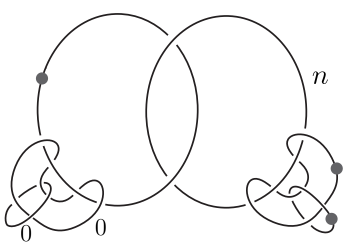
Thus $b_0 = [b_1, a_0^{-1}] = [[b_2, a_1^{-1}], a_0^{-1}] = 1$, using the fact that a_0 commutes with b_2 and a_1. It then follows from the other relations that $\pi_1(X_1, z_0) = 1$. (Recall that $\xi = y_0[x_0, y_0]y_0^{-1}$.)

The infinite family of manifolds referred to in the statement of the theorem is constructed by changing the surgery on the torus $\Lambda_{2,2}$ so that it now kills y_2 times the n–th power of the meridian $[x_2^{-1}, b_2]$ to $\Lambda_{2,2}$. Call this manifold X_n. This notation agrees with that of Theorem 1. Changing the surgery in this way has the effect of replacing the relation $b_1 = [b_2, a_1^{-1}]$ with $b_1 = [b_2, a_1^{-1}]^n$ in the calculation, which goes through just as before to show that $\pi_1(X_n) = 1$. Theorem 1 now implies that the X_n satisfy the conclusion of the theorem.

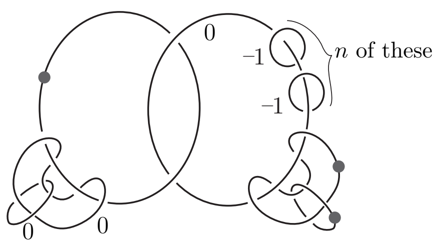
The proof that the X_n are minimal follows an argument similar to that in the proof of Theorem 2. Once again we see from the pinwheel structure that X_n contains genus 2 surfaces representing the classes $h-e_1$ and $h-e_2$ and a genus one surface representing e_1, say; so there is again a genus 3 surface representing $h=(h-e_1)+e_1$. We see that the only possible basic classes are $\pm(3h-e_1-e_2)$, and the proof proceeds as before. □

8 More exotic rational surfaces

In the last section we showed how to use pinwheel surgery to obtain infinite families of manifolds homeomorphic but not diffeomorphic to $\mathbb{CP}^2 \# k\,\overline{\mathbb{CP}}^2$ for $k = 2, 3$. In this section, we outline how to use similar techniques to construct families for $k = 4, \dots, 7$ and $k = 9$, and we explain why we are having difficulty seeing a similar construction for $k = 8$. We begin by generalizing the construction of the manifold $\widehat{A}$. Recall the notation B_n from Figure 8. For $n > 0$, B_n is naturally the complement of the negative section and a fiber in the ruled surface $\mathbb{F}_n$. In accordance we also consider the manifolds $A_{(n)}$ and $\widehat{A}_{(n)}$ of Figure 30. In terms of this notation, our manifold $\widehat{A} = \widehat{A}_{(1)}$. Notice that $A \subset \widehat{A}_{(n)}$.



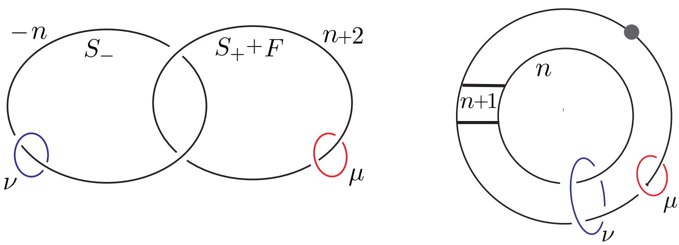
(a) $A_{(n)}$



(b) $\widehat{A}_{(n)}$

Figure 30

We introduce next another family of manifolds which we shall need. Let K_n denote the complement in $\mathbb{F}_n$ of S_-, the negative section, and $S_+ + F$, a positive section plus a fiber. These are embedded complex curves in $\mathbb{F}_n$ which meet transversely in a single point and have self-intersections $-n$ and $n+2$. It follows that K_n is the rational ball of [7] whose boundary is the lens space $L((n+1)^2, -n)$, and the fundamental group $\pi_1(K_n) = \mathbb{Z}_{n+1}$. (The manifold $K_0 \cong B^4$. See Figure 31(b).)



(a) Neighborhood of $S_- \cup (S_+ + F)$ (b) K_n

Figure 31: $\mathbb{F}_n$

Figure 31 displays $\mathbb{F}_n$ as the union of a neighborhood of the intersecting spheres S_- and $S_+ + F_n$ and K_n. (The "$n+1$" indicates $n+1$ full right hand twists.) The loops μ and ν are the boundaries of normal disks to the two surfaces, and they are shown in Figure 31(b) lying in ∂K_n. It is important to note that μ and ν are isotopic in K_n. We can see this in Figure 31(b) since there is an isotopy of ν across a belt disk of the 2–handle. Alternatively, a fiber of $\mathbb{F}_n$ intersects both S_- and $S_+ + F$ transversely in single points, and the complementary annulus in the fiber gives the isotopy.

We next describe an operation that will lead to a mild generalization of handle trading. Consider K_n and abstractly attach a pair of 2–handles to the Bing double of μ with both framings equal to 0. Call this new manifold K_n^+. Since ν is isotopic to μ in K_n, the Bing double of ν is isotopic to the Bing double of μ in K_n, and using this isotopy and the cores of the new 2–handles in K_n^+, we get disks in K_n^+ with boundary the Bing double of ν, and we can remove neighborhoods of these disks from K_n^+ to obtain the manifold K_n^0.

This procedure will be used as follows: Visualize K_n as a pinwheel component, and consider the adjacent pinwheel component C_i with interface surface T_i whose meridian m_{T_i} is identified with the meridian μ of $S_+ + F$. Suppose that the meridian m_{T_i} satisfies the Handle Trading Condition; so in the pinwheel, we can add the pair of 2–handles to K_n (removing them from C_i) in order to construct K_n^+. Next consider the pinwheel component C_j on the other side of K_n with interface surface S_j whose meridian m_{S_j} is identified with the meridian ν of S_-. We can view the construction of K_n^0 from K_n^+ as handle trading. In other words, the Handle Trading Condition is not satisfied at the S_- interface of K_n, but after trading handles at the $S_+ + F$

interface, it becomes possible to trade handles in K_n^+. See Figure 32. Note that the interface surfaces of the new pinwheel component K_n^0 are tori (as usual after handle trading). We call this operation *pushing through*.

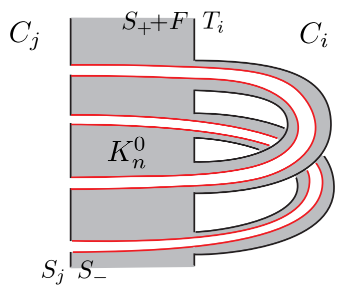


Figure 32: Pushing through

Lemma 11 *The manifold K_n^0 is the complement in the irrational ruled surface $\mathbb{F}_n(1)$ of symplectic tori representing the negative section, S'_-, and a positive section plus a fiber, $S'_+ + F'$.*

Proof To change $\mathbb{F}_n$ into $\mathbb{F}_n(1)$, we need to remove a neighborhood of a fiber, $F \times D^2 \cong S^2 \times D^2$, and replace it with $F \times T_0$. We are working with the smaller manifold, K_n; so the change we would need to make is to replace $S^1 \times [0,1] \times D^2$ with $S^1 \times [0,1] \times T_0$. Thus our discussion is local to this $S^1 \times [0,1] \times D^2$ in K_n.

The isotopy in K_n used for pushing through sits inside of a fiber of $\mathbb{F}_n$. Its trace is the annulus $S^1 \times [0,1]$. The pushing through procedure adds a pair of 2–handles to the link $L_\mu = L \times 1$ where L is the Bing double of $S^1 \times 0$ in $S^1 \times D^2$. Let D_1, D_2 be the core disks of these 2–handles.

We then bore out a neighborhood of $(L \times [0,1]) \cup D_1 \cup D_2$. The claim is that the resultant manifold is $S^1 \times [0,1] \times T_0$. Notice that each $S^1 \times \{t\} \times D^2$ has been replaced by the 3–manifold obtained by adding a pair of 2–handles to $L \times \{t\}$, ie by performing 0–surgery on both components of this link. This is $S^1 \times T_0$ as required. □

As we have mentioned above, the fundamental group of K_n is $\mathbb{Z}_{n+1}$, and it is generated by either of the meridians μ, ν. To construct K_n^0 we attach a pair of 2–handles and then subtract a pair of 2–handles. The handle additions add no generators to π_1 and the handle subtractions add a pair of generators a, b, which are identified with corresponding generators of the form x_i, y_i in the adjacent pinwheel component C_i.

8.1 $\mathbb{CP}^2 \# 4\,\overline{\mathbb{CP}}^2$

The pinwheel structure we shall use to construct families for $\mathbb{CP}^2 \# 4\,\overline{\mathbb{CP}}^2$ is given in Figure 33.

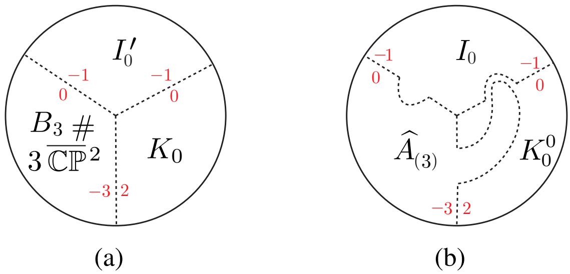


Figure 33: Pinwheel structures for $\mathbb{CP}^2 \# 4\,\overline{\mathbb{CP}}^2$

The interface surface between the pinwheel components I_0' and $B_3 \# 3\,\overline{\mathbb{CP}}^2$ satisfies the Handle Trading Condition because the corresponding interface surface of $B_3 \# 3\,\overline{\mathbb{CP}}^2$ is a fiber of $\mathbb{F}_3$ (the other interface surface of this component is a negative section), and the disk used for handle trading is provided by the positive section blown up three times. It is the core of the handle labelled n $(= 3)$ in Figure 8 after the blowups. The interface surface between the components K_0 and I_0' also satisfies the Handle Trading Condition using the normal disk to the 2–handle with framing $+1$ in I_0'. To see that this pinwheel structure actually gives $\mathbb{CP}^2 \# 4\,\overline{\mathbb{CP}}^2$, one argues as in Lemma 4. The starting Kirby calculus diagram is shown in Figure 34.

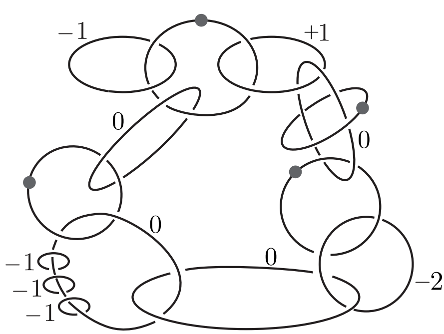


Figure 34

The model symplectic manifold Q_4 is built by performing standard surgeries on the copies of B_T in I_0 and $\widehat{A}_{(3)}$. No surgeries are performed in K_0^0. Thus (employing

Lemma 11) Q_4 has the symplectic pinwheel structure given by

$$(\widetilde{X}_0; \widetilde{S}_0, \widetilde{T}_0) = (T^4 \# \overline{\mathbb{CP}}^2; T_{I,T}, T_{I,S})$$
$$(\widetilde{X}_1; \widetilde{S}_1, \widetilde{T}_1) = (T^2 \times S^2; T_{S_+ + F}, T_{S_-})$$
$$(\widetilde{X}_2; \widetilde{S}_2, \widetilde{T}_2) = (T^4 \# 3\,\overline{\mathbb{CP}}^2; T_{T,3}, T_S)$$

where $T_{T,3}$ is a symplectic torus representing $T_T - E_1 - E_2 - E_3$ and in $T^2 \times S^2 = \mathbb{F}_0(1)$, T_{S_++F} and T_{S_-} are tori representing the obvious classes.

To construct the infinite family of mutually nondiffeomorphic manifolds all homeomorphic to $\mathbb{CP}^2 \# 4\,\overline{\mathbb{CP}}^2$, we follow the proof of Theorem 3, except that here we only surger the tori $\Lambda_{0,i}$ and $\Lambda_{2,i}$ coming from the pinwheel components I_0 and $\widehat{A}_{(3)}$. The component K_0^0 is diffeomorphic to $T^2 \times S^2 \setminus (T_{S_++F} \cup T_{S_-})$ where $\pi_1(T^2 \times S^2, z_c)$ is generated by a_1 and b_1 (where we identify K_0^0 with the component C_1). So $\pi_1(K_0^0, z_c)$ is normally generated by a_1, b_1, and a meridian μ to T_{S_++F}.

Theorem 4 *By surgeries on the four Lagrangian tori comprising the copies of $\widetilde{B}_{T,i}$, $i = 0, 2$ in Q_4 which live in $\widetilde{X}_0$ and $\widetilde{X}_2$ in Q_4, one can obtain an infinite family of mutually nondiffeomorphic minimal smooth 4–manifolds all homeomorphic to $\mathbb{CP}^2 \# 4\,\overline{\mathbb{CP}}^2$. Furthermore, all these manifolds can be obtained by surgeries on four nullhomologous tori which comprise two copies of $B_T \subset I_0 \cup \widehat{A}_{(3)} \subset \mathbb{CP}^2 \# 4\,\overline{\mathbb{CP}}^2$.*

Proof We first show that the manifold X obtained by performing $+1$ surgeries on the Lagrangian tori $\Lambda_{1,i}$ and $\Lambda_{2,i}$ in $\widetilde{X}_i$, $i = 0, 2$, is simply connected. Our pinwheel components are $C_0 = I_0$, $C_1 = K_0^0$, and $C_2 = \widehat{A}_{(3)}$. The fundamental group of $\widehat{A}_{(3)} \setminus B_T$ is the same as that of $\widehat{A} \setminus B_T$ since the extra blowups do not change π_1. The pinwheel construction identifies a_2 with x_0 and b_2 with $y_0^2\,\xi^k$. Furthermore, the pushing through process identifies a_0 and b_0 with the belt circles of the 2–handles first attached to K_0 to construct K_0^+, and these are in turn identified with $a_1 = x_2$ and $b_1 = y_2$ after pushing through. Thus $a_0 = a_1$ and $b_0 = b_1$.

Analogous to the proof of Theorem 3, we see that $\pi_1(X, z_0)$ is generated by a_i, b_i, $i = 0, 2$, y_0, and the meridian μ from K_0^0, and it satisfies the relations coming from those of $\widehat{A}_{(3)}$, and I_0 with their surgeries:

$$\widehat{A}_{(3)}\colon\quad a_0[b_2^{-1}, b_0^{-1}] = 1,\; b_0[a_0^{-1}, b_2] = 1,\; [a_0, a_2] = 1,\; [b_0, a_2] = 1,\; [a_0, b_0] = 1.$$
$$I_0\colon\quad a_2[b_0^{-1}, y_0^{-1}] = 1,\; y_0[a_2^{-1}, b_0] = 1,\; [a_2, a_0] = 1,\; [y_0, a_0] = 1,\; [b_2, a_0] = 1.$$

(Recall that this last equality holds because of Lemma 9 and $b_2 = y_0^2\,\xi^k$.)

Thus $b_0 = [b_2, a_0^{-1}] = 1$, and then it follows that a_0, a_2, and y_0 vanish, and so we also have $\xi = y_0[x_0, y_0]y_0^{-1} = y_0[a_2, y_0]y_0^{-1} = 1$; hence $b_2 = 1$. Finally, $\mu = \nu$ is identified with the boundary of the meridian to the 1–handle of $\widehat{A}_{(3)}$ (see Figure 30(b)). Thus $\mu = [a_2, b_2] = 1$; so $\pi_1(X, z_0) = 1$.

It follows as usual that the manifolds X_n, obtained by performing $1/n$–surgery rather than $+1$–surgery on the last torus $\Lambda_{2,2}$, are also simply connected. To see that these manifolds are minimal, we again use an argument based on the adjunction inequality is in the proof of Theorem 2. This is easy once we see that these manifolds have embedded genus 2 surfaces representing the classes $h - e_i$, $i = 1, \dots, 4$ and an embedded genus 3 surface representing h.

We shall describe these surfaces referring to Figures 33 and 34. The class e_4 is represented as follows. The 2–handle of I_0' which is labelled "-1" (on the upper left in Figure 34) gives a punctured sphere, punctured by the 1–handle of I_0'. The meridian of that 1–handle is isotopic to (identified with) the meridian of the 1–handle in $B_3 \# 3\,\overline{\mathbb{CP}}^2$. This meridian is, in turn, isotopic to the attaching circle of the 2–handle labelled "0" in $B_3 \# 3\,\overline{\mathbb{CP}}^2$. Thus we see a sphere of self-intersection -1 in $\mathbb{CP}^2 \# 4\,\overline{\mathbb{CP}}^2$. After the surgeries on the tori of the copy of B_T in $\widehat{A}_{(3)}$ this sphere is no longer extant, but a torus representing e_4 is, since normal disks to the 1–handle become punctured tori.

Instead of starting with the 2–handle of I_0' which is labelled "-1", we could begin with the handle labelled "$+1$". Then this construction produces a 2–sphere in $\mathbb{CP}^2 \# 4\,\overline{\mathbb{CP}}^2$ representing h. Notice that both classes h and e_4 intersect the spheres of self-intersection -1 which come from the three 2–handles labelled "-1" in $B_3 \# 3\,\overline{\mathbb{CP}}^2$. It follows easily that these three spheres represent the classes $h - e_i - e_4$, $i = 1, 2, 3$. Since the surgeries on $B_T \subset \widehat{A}_{(3)}$ turn normal disks to the "0" into punctured tori, the classes $h - e_i - e_4$, $i = 1, 2, 3$ are represented by tori in the surgered manifolds X_n. It follows that $h - e_i = (h - e_i - e_4) + e_4$, $i = 1, 2, 3$, are represented by genus 2 surfaces in X_n.

Returning to $\mathbb{CP}^2 \# 4\,\overline{\mathbb{CP}}^2$, there is a sphere representing a class of self-intersection 0 arising from the 2–handle labelled 0 at the bottom of Figure 34. (Alternatively, this sphere is formed from the cocores of the 2–handle labelled 0 in $B_3 \# 3\,\overline{\mathbb{CP}}^2$ and the 2–handle labelled -2 in K_0, since the pinwheel construction identifies their boundaries.) Since this sphere intersects both h and e_4 once and is orthogonal to $h - e_i - e_4$, $i = 1, 2, 3$, it represents $h - e_4$. Surgery plus the pushing through operation make this surface genus 2. Finally, $h = (h - e_4) + e_4$ is now seen to be represented by a genus 3 surface in X_n, and we may carry out the adjunction inequality argument as planned to see that X_n is minimal. □

8.2 $\mathbb{CP}^2 \# 5\,\overline{\mathbb{CP}}^2$

The construction of families for $\mathbb{CP}^2 \# 5\,\overline{\mathbb{CP}}^2$ follows our established pattern, using the pinwheel structure of Figure 35. One sees that these pinwheels give $\mathbb{CP}^2 \# 5\,\overline{\mathbb{CP}}^2$ via our usual Kirby calculus argument.

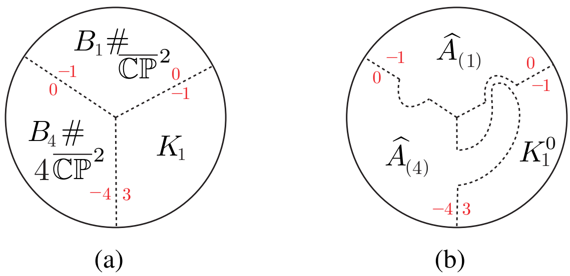


Figure 35: Pinwheel structures for $\mathbb{CP}^2 \# 5\,\overline{\mathbb{CP}}^2$

We get our infinite family of manifolds as before. The argument is parallel to, but easier than, that of Theorem 4.

8.3 $\mathbb{CP}^2 \# 6\,\overline{\mathbb{CP}}^2$

This construction is similar to that for $\mathbb{CP}^2 \# 4\,\overline{\mathbb{CP}}^2$. In that construction (Figure 33), replace the pinwheel component $B_3 \# 3\,\overline{\mathbb{CP}}^2$ with the manifold $L_{(0,-3)}$ which is obtained by removing the spheres $S_+ + F - E_1 - E_2 - E_3$ and $S_- - E_4 - E_5$ from $\mathbb{F}_1 \# 5\,\overline{\mathbb{CP}}^2$. These spheres have self-intersections 0 and -3 as required. See Figure 36. Note that, as for the manifolds K_n, the meridians to the spheres $S_+ + F - E_1 - E_2 - E_3$ and $S_- - E_4 - E_5$ are isotopic in $L_{(0,-3)}$. (This isotopy is given by the fiber of $\mathbb{F}_1$.)

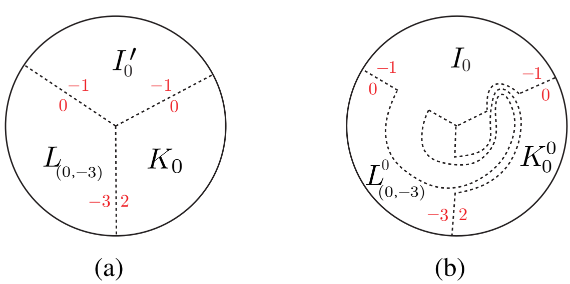


Figure 36: Pinwheel structures for $\mathbb{CP}^2 \# 6\,\overline{\mathbb{CP}}^2$

In this case we need to push through two pinwheel components. Surgery is then performed on the single pair of Bing tori in I_0. The notation $L^0_{(0,-3)}$ is meant to be analogous to K_n^0 above. The proof of Lemma 11 shows that $L^0_{(0,-3)}$ is the complement in the blown up irrational ruled surface $\mathbb{F}_n(1)\,\#\,5\,\overline{\mathbb{CP}}^2$ of symplectic tori representing $S'_+ + F - E_1 - E_2 - E_3$ and $S'_- - E_4 - E_5$. Standard surgeries on the tori in $B_T \subset I_0$ give rise to the symplectic manifold Q_6 which has the symplectic pinwheel structure given by

$$(\widetilde{X}_0; \widetilde{S}_0, \widetilde{T}_0) = (T^4\,\#\,\overline{\mathbb{CP}}^2; T_{I,T}, T_{I,S})$$
$$(\widetilde{X}_1; \widetilde{S}_1, \widetilde{T}_1) = (T^2 \times S^2; T_{S_+ + F}, T_{S_-})$$
$$(\widetilde{X}_2; \widetilde{S}_2, \widetilde{T}_2) = (\mathbb{F}_1(1)\,\#\,5\,\overline{\mathbb{CP}}^2; T_{S_+ + F - E_1 - E_2 - E_3}, T_{S_- - E_4 - E_5})$$

with notation as above. See the $\mathbb{CP}^2\,\#\,7\,\overline{\mathbb{CP}}^2$ case below for comments on a similar fundamental group calculation. Notice that our construction leaves none of the exceptional curves from the blowups, and our usual adjunction inequality argument shows that we get a family of minimal 4–manifolds.

8.4 $\mathbb{CP}^2\,\#\,7\,\overline{\mathbb{CP}}^2$

Use the pinwheel structure given in Figure 37. Again we need to push through two pinwheel components. Surgery is performed on a single pair of Bing tori.

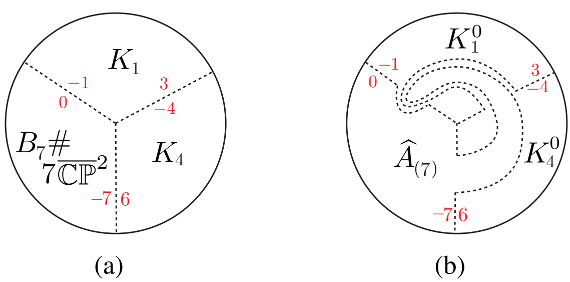


Figure 37: Pinwheel structures for $\mathbb{CP}^2\,\#\,7\,\overline{\mathbb{CP}}^2$

Standard surgeries on the tori in $B_T \subset \widehat{A}_{(7)}$ give rise to the symplectic manifold Q_7 which has the symplectic pinwheel structure given by

$$(\widetilde{X}_0; \widetilde{S}_0, \widetilde{T}_0) = (\mathbb{F}_1(1); T_{S_+ + F}, T_{S_-})$$
$$(\widetilde{X}_1; \widetilde{S}_1, \widetilde{T}_1) = (\mathbb{F}_4(1); T_{S_+ + F}, T_{S_-})$$
$$(\widetilde{X}_2; \widetilde{S}_2, \widetilde{T}_2) = (T^4\,\#\,7\,\overline{\mathbb{CP}}^2; T_{T,7}, T_S)$$

where $T_{T,7}$ is a symplectic torus representing $T_T - \sum_{i=1}^{7} E_i$, and the other notation is as above.

Theorem 5 *By surgeries on the two Lagrangian tori comprising the copy of $\widetilde{B}_T \subset \widetilde{X}_2$, in Q_7, one can obtain an infinite family of mutually nondiffeomorphic minimal smooth 4–manifolds all homeomorphic to $\mathbb{CP}^2 \# 7\,\overline{\mathbb{CP}}^2$. Furthermore, all these manifolds can be obtained by surgeries on two nullhomologous tori which comprise two copies of $B_T \subset \widehat{A}_{(7)} \subset \mathbb{CP}^2 \# 7\,\overline{\mathbb{CP}}^2$.*

Proof We first show that the manifold X obtained by performing $+1$ surgeries on the Lagrangian tori $\Lambda_{1,2}$ and $\Lambda_{2,2}$ in $\widetilde{X}_2$ is simply connected. The pushing through process identifies a_2 and b_2 with the belt circles of the 2–handles first attached to K_1 to construct K_1^+, and these are in turn identified with a_0 and b_0. The second pushing through operation identifies these classes with a_1 and b_1. Thus $a_0 = a_1 = a_2$ and $b_0 = b_1 = b_2$. Therefore, $\pi_1(X; z_0)$ is generated by $a = a_i$, $b = b_i$, and meridians $\mu_0 = \nu_0$ of $S_+ + F$ and S_- in $\mathbb{F}_1$ and $\mu_1 = \nu_1$ of $S_+ + F$ and S_- in $\mathbb{F}_4$.

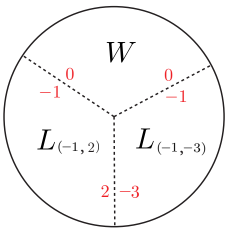
The usual relations coming from $\widehat{A}_{(7)}$ in this case reduce quickly to $a = 1$, $b = 1$. As in the proof of Theorem 4 we have $\mu_i = [a, b] = 1$, $i = 0, 1$; so $\pi_1(X, z_0) = 1$, and the rest of the proof follows analogously to the examples above. □

8.5 $\mathbb{CP}^2 \# 8\,\overline{\mathbb{CP}}^2$ and $\mathbb{CP}^2 \# 9\,\overline{\mathbb{CP}}^2$

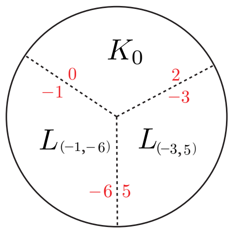
These constructions are slightly different from those above. We begin by explaining why the type of pinwheel surgeries that we have already described will not work for these manifolds. In each of our pinwheel surgery constructions, the manifold X_1 obtained by performing only $+1$ surgeries is symplectic and not diffeomorphic to $\mathbb{CP}^2 \# k\,\overline{\mathbb{CP}}^2$. The adjunction formula shows that its canonical class is represented by a symplectic surface of genus $10 - k$. The canonical class of the standard manifold $\mathbb{CP}^2 \# k\,\overline{\mathbb{CP}}^2$ has genus 1, and each surgery on a Bing torus pair B_T in a copy of A increases the genus of the disks which span meridians to the "0" and the "dot" by one. For example, letting $k = 3$, the canonical class of $\mathbb{CP}^2 \# 3\,\overline{\mathbb{CP}}^2$ is given by $-3h + e_1 + e_2 + e_3$, and in each pinwheel component $B_1 \# \overline{\mathbb{CP}}^2$, it is represented by a pair of transverse disks normal to the "0" and the "dot". Gluing these together along the interfaces, they add up to three 2–spheres representing the classes $-h + e_i$, and arranged in a cyclical fashion; so that smoothing their intersection points gives a torus representing $-3h + e_1 + e_2 + e_3$. The surgeries increase the genus of each of these surfaces to 2, and the result after smoothing double points is the canonical class of X_1 of genus $7 = 10 - k$. The other examples can be analyzed in a like fashion.

We notice that each pinwheel component that is surgered must increase the genus of a symplectic representative of the canonical class by at least 2; so the minimal genus which can be achieved by this technique is 3. However, the canonical class of a symplectic manifold homeomorphic but not diffeomorphic to $\mathbb{CP}^2 \# 8\,\overline{\mathbb{CP}}^2$ has genus 2; so it cannot be obtained by this particular type of pinwheel surgery.

However, one does have different pinwheel constructions for $\mathbb{CP}^2 \# 8\,\overline{\mathbb{CP}}^2$ and $E(1) = \mathbb{CP}^2 \# 9\,\overline{\mathbb{CP}}^2$ as in Figures 38(a) and 38(b). The manifolds $L_{(j,k)}$ are obtained by modifying the constructions of the K_n above. For example, $L_{(-1,-3)}$ is the complement of S_- and $S_+ + F - E_1 - \cdots - E_6$ in $\mathbb{F}_1 \# 6\,\overline{\mathbb{CP}}^2$. The component W is the complement of S and $2S + F - 2E$ in $\mathbb{F}_0 \# \overline{\mathbb{CP}}^2$.



(a) $\mathbb{CP}^2 \# 8\,\overline{\mathbb{CP}}^2$



(b) $\mathbb{CP}^2 \# 9\,\overline{\mathbb{CP}}^2$

Figure 38: Pinwheel structures for $\mathbb{CP}^2 \# 8\,\overline{\mathbb{CP}}^2$ and $\mathbb{CP}^2 \# 9\,\overline{\mathbb{CP}}^2$

We leave it as a simple exercise for the reader to see representatives for the canonical classes in these pinwheel structures, tori of self-intersections 1 and 0, respectively. Surgery on the Bing double of this last torus was the subject of the example in Section 3. By replacing K_0 with K_0^0 and each $L_{(j,k)}$ with $L^0_{(j,k)}$; ie by replacing the pinwheel components of $E(1) = \mathbb{CP}^2 \# 9\,\overline{\mathbb{CP}}^2$ with components obtained in like fashion from ruled surfaces over T^2 rather than over S^2, we obtain the model manifold Q_9, and it is easy to see that Q_9 is the elliptic surface over T^2 obtained by fiber summing $E(1)$ with the product elliptic surface $T^2 \times T^2$. Thus Q_9 is obtained from $E(1)$ by removing a neighborhood of a fiber, $T^2 \times D^2$ and replacing it with $T^2 \times T_0$. In other words, we have performed standard surgeries on the standard copy of B_T in the neighborhood of a fiber in $E(1)$ in order to obtain Q_9. The surgeries on the corresponding Lagrangian tori in $E(1)$ will give exactly the examples of Section 3.

We have not as yet been able to effect a construction of exotic manifolds homeomorphic to $\mathbb{CP}^2 \# 8\,\overline{\mathbb{CP}}^2$ utilizing pinwheels. However in [10] we constructed a manifold R_8 homeomorphic to $\mathbb{CP}^2 \# 8\,\overline{\mathbb{CP}}^2$ and with trivial Seiberg–Witten invariant (as for $\mathbb{CP}^2 \# 8\,\overline{\mathbb{CP}}^2$) such that R_8 contains nullhomologous tori upon which surgery yields an infinite family of exotic manifolds homeomorphic to $\mathbb{CP}^2 \# 8\,\overline{\mathbb{CP}}^2$.

8.6 The main theorem

The upshot of this section and the last is the following theorem:

Theorem 6 *For $k = 2, \dots, 7$ and $k = 9$, there are nullhomologous tori embedded in $\mathbb{CP}^2 \# k\, \overline{\mathbb{CP}}^2$ upon which surgery gives rise to an infinite family of mutually nondiffeomorphic minimal 4–manifolds homeomorphic to $\mathbb{CP}^2 \# k\, \overline{\mathbb{CP}}^2$.* □

9 Final remarks

The fact that surgery on a well-chosen collection of (nullhomologous) tori in a fixed smooth manifold can alter smooth structures gives rise to a target (and optimistic) classification scheme for simply connected smooth 4–manifolds.

Question Is it possible that every simply connected smooth 4–manifold is obtained by surgery on tori in a connected sum of copies of S^4, $\mathbb{CP}^2$, $\overline{\mathbb{CP}}^2$ and $S^2 \times S^2$?

Acknowledgements RF was partially supported by NSF grants DMS-0704091 and DMS-1006322 and RJS was supported by no federal research dollars.

References

[1] **A Akhmedov**, **R İ Baykur**, **B D Park**, *Constructing infinitely many smooth structures on small 4–manifolds*, J. Topol. 1 (2008) 409–428 MR2399137

[2] **A Akhmedov**, **B D Park**, *Exotic smooth structures on small 4–manifolds*, Invent. Math. 173 (2008) 209–223 MR2403396

[3] **A Akhmedov**, **B D Park**, *Exotic smooth structures on small 4–manifolds with odd signatures*, Invent. Math. 181 (2010) 577–603 MR2660453

[4] **D Auroux**, **S K Donaldson**, **L Katzarkov**, *Luttinger surgery along Lagrangian tori and non-isotopy for singular symplectic plane curves*, Math. Ann. 326 (2003) 185–203 MR1981618

[5] **S Baldridge**, **P Kirk**, *A symplectic manifold homeomorphic but not diffeomorphic to $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$*, Geom. Topol. 12 (2008) 919–940 MR2403801

[6] **R Fintushel**, **B D Park**, **R J Stern**, *Reverse engineering small 4–manifolds*, Algebr. Geom. Topol. 7 (2007) 2103–2116 MR2366188

[7] **R Fintushel**, **R J Stern**, *Rational blowdowns of smooth 4–manifolds*, J. Differential Geom. 46 (1997) 181–235 MR1484044

[8] **R Fintushel**, **R J Stern**, *Knots, links, and 4–manifolds*, Invent. Math. 134 (1998) 363–400 MR1650308

[9] **R Fintushel**, **R J Stern**, *Double node neighborhoods and families of simply connected 4–manifolds with $b^+ = 1$*, J. Amer. Math. Soc. 19 (2006) 171–180 MR2169045

[10] **R Fintushel**, **R J Stern**, *Surgery on nullhomologous tori and simply connected 4–manifolds with* $b^+ = 1$, J. Topol. 1 (2008) 1–15 MR2365649

[11] **D McDuff**, **M Symington**, *Associativity properties of the symplectic sum*, Math. Res. Lett. 3 (1996) 591–608 MR1418574

[12] **J W Morgan**, **T S Mrowka**, **Z Szabó**, *Product formulas along* T^3 *for Seiberg–Witten invariants*, Math. Res. Lett. 4 (1997) 915–929 MR1492130

[13] **P Orlik**, **F Raymond**, *Actions of the torus on 4–manifolds. I*, Trans. Amer. Math. Soc. 152 (1970) 531–559 MR0268911

[14] **M Symington**, *A new symplectic surgery: the 3–fold sum*, from: "Symplectic, contact and low-dimensional topology (Athens, GA, 1996)", Topology Appl. 88 (1998) 27–53 MR1634562

[15] **C H Taubes**, *More constraints on symplectic forms from Seiberg–Witten invariants*, Math. Res. Lett. 2 (1995) 9–13 MR1312973

Department of Mathematics, Michigan State University
East Lansing MI 48824, USA

Department of Mathematics, University of California, Irvine
Irvine CA 92697-3875, USA

ronfint@math.msu.edu, rstern@uci.edu

http://www.math.msu.edu/~ronfint/, http://math.uciedu/~rstern/

Received: 8 November 2010