On R L Cohen's ζ–element

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Let p be a prime greater than three. In the p–local stable homotopy groups of spheres, R L Cohen constructed the infinite ζ–element $\zeta_{n-1} \in \pi_{2p^{n+1}-2p^n+2p-5}(S)$ of order p. In the stable homotopy group $\pi_{2p^{n+1}-2p^n+2p^2-3}(V(1))$ of the Smith–Toda spectrum $V(1)$, X Liu constructed an essential element ϖ_k for $k \geq 3$. Let $\beta_s^* = j_0 j_1 \beta^s \in [V(1), S]_{2sp^2-2s-2p}$ denote the Spanier–Whitehead dual of the generator $\beta_s'' = \beta^s i_1 i_0 \in \pi_{2sp^2-2s}(V(1))$, which defines the β–element β_s. Let $\xi_{s,k} = \beta_{s-1}^* \varpi_k$. In this paper, we show that the composite of R L Cohen's ζ–element ζ_{n-1} with $\xi_{s,n}$ is nontrivial, where $n > 4$ and $1 < s < p-1$. As a corollary, $\xi_{s,n}$ is also nontrivial for $1 < s < p-1$.

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1 Introduction and statements of results

Let A be the mod p Steenrod algebra and S the sphere spectrum localized at a prime p greater than three. To determine the stable homotopy groups of spheres $\pi_*(S)$ is one of the central problems in homotopy theory. One of the main tools to reach it is the Adams spectral sequence (ASS, for short) [1]

$$E_2^{s,t} = \mathrm{Ext}_A^{s,t}(\mathbb{Z}_p, \mathbb{Z}_p) \Rightarrow \pi_{t-s}(S),$$

where the E_2–term is the cohomology of A. If a family of generators x_i in $E_2^{s,*}$ converges nontrivially in the ASS, then we get a family of homotopy elements f_i in $\pi_*(S)$ and we say that f_i is represented by $x_i \in E_2^{s,*}$ and has filtration s in the ASS. The main purpose of this paper is to detect a new family of homotopy elements in $\pi_*(S)$ which has filtration $s+4$ in the ASS.

In this paper, we need the following spectra which are all related to the sphere spectrum. Let M be the mod p Moore spectrum given by the cofibration

$$S \xrightarrow{p} S \xrightarrow{i_0} M \xrightarrow{j_0} \Sigma S. \tag{1-1}$$

Let $\alpha\colon \Sigma^q M \longrightarrow M$ be the Adams self-map and K be its cofibre given by the cofibration

$$\Sigma^q M \xrightarrow{\alpha} M \xrightarrow{i_1} K \xrightarrow{j_1} \Sigma^{q+1} M. \tag{1-2}$$

This spectrum which we briefly write as K is known to be the Smith–Toda spectrum $V(1)$. Here $q=2(p-1)$ as usual. Let

$$\beta\colon \Sigma^{(p+1)q}V(1)\longrightarrow V(1)$$

denote the v_2–map.

Definition 1.1 For $t\geq 1$, the known β–element β_t is defined to be the composite map

$$\beta_t=j_1j_0\beta^t i_1i_0\in\pi_{q(tp+(t-1))-2}(S),$$

where the maps i_0, i_1, j_0, j_1 are given in (1-1) and (1-2).

Theorem 1.2 (Smith [12]) *With notation as above, for $p\geq 5$ and $t\geq 1$, β_t is a nontrivial element of order p in $\pi_{q(tp+(t-1))-2}(S)$.*

The known results on $\mathrm{Ext}_A^{*,*}(\mathbb{Z}_p,\mathbb{Z}_p)$ are as follows. By definition, $\mathrm{Ext}_A^{0,*}(\mathbb{Z}_p,\mathbb{Z}_p)=\mathbb{Z}_p$. From Liulevicius [7], $\mathrm{Ext}_A^{1,*}(\mathbb{Z}_p,\mathbb{Z}_p)$ has $\mathbb{Z}_p$–bases consisting of $a_0\in\mathrm{Ext}_A^{1,1}(\mathbb{Z}_p,\mathbb{Z}_p)$, $h_i\in\mathrm{Ext}_A^{1,p^iq}(\mathbb{Z}_p,\mathbb{Z}_p)$ for all $i\geq 0$ and $\mathrm{Ext}_A^{2,*}(\mathbb{Z}_p,\mathbb{Z}_p)$ has $\mathbb{Z}_p$–bases consisting of α_2, a_0^2, a_0h_i $(i>0)$, g_i $(i\geq 0)$, k_i $(i\geq 0)$, b_i $(i\geq 0)$, and h_ih_j $(j\geq i+2, i\geq 0)$ whose internal degrees are $2q+1$, 2, p^iq+1, $p^{i+1}q+2p^iq$, $2p^{i+1}q+p^iq$, $p^{i+1}q$ and p^iq+p^jq, respectively. In 1980, Aikawa [2] determined $\mathrm{Ext}_A^{3,*}(\mathbb{Z}_p,\mathbb{Z}_p)$ by the lambda algebra.

The problem of understanding the stable homotopy ring has long been one of the touchstones of algebraic topology. Low dimensional computation has proceeded slowly and has given little insight into the general structure of $\pi_*(S)$. So far, not so many families of homotopy elements in $\pi_*(S)$ have been detected.

In [10], Douglas C Ravenel obtained the following:

Theorem 1.3 [10] *For $p>3$ and $i\geq 1$, b_i is not a permanent cycle. (At $p=3$, b_1 is not permanent but b_2 is; b_0 is permanent for all odd primes.)*

In [3], R L Cohen detected a new element $\zeta_n\in\pi_{q(p^{n+1}+1)-3}(S)$ which is called the ζ–element.

Theorem 1.4 [3, Theorem IV.b] *For every $n\geq 1$,*

$$h_0b_n\in\mathrm{Ext}_A^{3,p^{n+1}q+q}(\mathbb{Z}_p,\mathbb{Z}_p)$$

survives to E_∞ in the ASS and represents a nontrivial element

$$\zeta_n\in\pi_{q(p^{n+1}+1)-3}(S)$$

of order p.

In [6], X Liu detected a new element in the stable homotopy groups of spheres and obtained the following:

Theorem 1.5 [6, Theorem 1.4] *Let $p \geq 5$, $n \geq 3$. Then*

$$k_0 h_n \neq 0 \in \mathrm{Ext}_A^{3, p^n q + 2pq + q}(\mathbb{Z}_p, \mathbb{Z}_p)$$

is a permanent cycle in the ASS and converges nontrivially to an element of order p in $\pi_{p^n q + 2pq + q - 3}(S)$.

On the way to proving the above theorem, X Liu detected a new element in the stable homotopy groups of $V(1)$ and gave the following:

Theorem 1.6 [6, Theorem 1.5] *Let $p \geq 5$, $n \geq 3$ and $h_n \in \mathrm{Ext}_A^{1, p^n q}(\mathbb{Z}_p, \mathbb{Z}_p)$ be the known generator in [7]. Then*

$$(\beta i_1 i_0)_*(h_n) \in \mathrm{Ext}_A^{2, p^n q + (p+1)q + 1}(H^* V(1), \mathbb{Z}_p)$$

is a permanent cycle in the ASS and converges to a nontrivial element

$$\varpi_n \in \pi_{p^n q + (p+1)q - 1}(V(1))$$

of order p.

For convenience, we let

$$\beta_s^* = j_0 j_1 \beta^s \in [V(1), S]_{spq + (s-1)q - 2} \tag{1-3}$$

denote the Spanier–Whitehead dual of the generator

$$\beta_s'' = \beta^s i_1 i_0 \in \pi_{s(p+1)q}(V(1)),$$

which defines the β–element β_s. Let

$$\xi_{s,n} = \beta_{s-1}^* \varpi_n \in \pi_{p^n q + spq + (s-1)q - 3}(S). \tag{1-4}$$

In this paper, we also prove that a composite map involving R L Cohen's ζ–element is nontrivial. Our main result can be stated as follows:

Theorem 1.7 *Let $p \geq 5$, $n > 4$ and $2 \leq s < p-1$. Then the composite map $\zeta_{n-1}\xi_{s,n}$ is nontrivial in $\pi_{2p^n q + spq + sq - 6}(S)$.*

From Theorem 1.7, the following consequence is immediate.

Corollary 1.8 *Let $p \geq 5, n > 4$ and $2 \leq s < p-1$. Then the map $\xi_{s,n}$ is nontrivial in $\pi_{p^n q + spq + (s-1)q - 3}(S)$.*

In the paper, we make use of the ASS and the May spectral sequence (MSS, for short) to show our main results.

The paper is arranged as follows. In Section 2, we recall some useful results on the MSS and, importantly, give a method for determining the May E_1–term $E_1^{\widetilde{s},\widetilde{t},*}$ for specialized integers $\widetilde{s}$ and $\widetilde{t}$. By the method given in Section 2, we first determine some May E_r–terms in Section 3. At the end of Section 3, two important theorems (Theorem 3.9 and Theorem 3.10) are given. Section 4 is devoted to showing our main results.

2 The MSS and a method on the MSS

In this section, we first recall some knowledge on the MSS. Then we will give an important method for detecting generators of the May E_1–term $E_1^{s,t,*}$ for some particular integers s and t.

Ever since the introduction of the Adams spectral sequence in the late 1950's, the study of the stable homotopy groups of spheres has been split into essentially two parts. First, there is the purely algebraic problem of computing $\mathrm{Ext}_A^{*,*}(\mathbb{Z}_p, \mathbb{Z}_p)$. The second, and more geometric problem is to determine which elements of $\mathrm{Ext}_A^{*,*}(\mathbb{Z}_p, \mathbb{Z}_p)$ survive to E_∞ in this spectral sequence and represent maps between spheres. For the first part above, we know that the most successful method for computing $\mathrm{Ext}_A^{*,*}(\mathbb{Z}_p, \mathbb{Z}_p)$ through a range of dimensions is the MSS. From [11, Theorem 3.2.5], there is a May spectral sequence (MSS) $\{E_r^{s,t,*}, d_r\}$ which converges to $\mathrm{Ext}_A^{s,t}(\mathbb{Z}_p, \mathbb{Z}_p)$ with E_1–term

(2-1) $$E_1^{*,*,*} = E(h_{m,i} \mid m > 0, i \geq 0) \otimes P(b_{m,i} \mid m > 0, i \geq 0) \otimes P(a_n \mid n \geq 0),$$

where E is the exterior algebra, P is the polynomial algebra and

$$h_{m,i} \in E_1^{1,2(p^m-1)p^i,2m-1}, \quad b_{m,i} \in E_1^{2,2(p^m-1)p^{i+1},p(2m-1)}, \quad a_n \in E_1^{1,2p^n-1,2n+1}.$$

One has

(2-2) $$d_r\colon E_r^{s,t,u} \to E_r^{s+1,t,u-r}$$

and if $x \in E_r^{s,t,*}$ and $y \in E_r^{s',t',*}$, then

(2-3) $$d_r(x \cdot y) = d_r(x) \cdot y + (-1)^s x \cdot d_r(y).$$

There exists a graded commutativity in the MSS:

(2-4) $$x \cdot y = (-1)^{ss'+tt'} y \cdot x$$

for $x, y = h_{m,i}, b_{m,i}$ or a_n. The first May differential d_1 is given by

$$(2\text{-}5)\qquad \begin{cases} d_1(h_{i,j}) = \sum_{0<k<i} h_{i-k,k+j} h_{k,j}, \\ d_1(a_i) = \sum_{0\le k<i} h_{i-k,k} a_k, \\ d_1(b_{i,j}) = 0. \end{cases}$$

For each element $x \in E_1^{s,t,u}$, we define $\dim x = s$, $\deg x = t$, $M(x) = u$. Then we have

$$(2\text{-}6)\qquad \begin{cases} \dim h_{i,j} = \dim a_i = 1, \\ \dim b_{i,j} = 2, \\ \deg h_{i,j} = 2(p^i - 1)p^j = 2(p-1)(p^{i+j-1} + \cdots + p^j), \\ \deg b_{i,j} = 2(p^i - 1)p^{j+1} = 2(p-1)(p^{i+j} + \cdots + p^{j+1}), \\ \deg a_i = 2p^i - 1 = 2(p-1)(p^{i-1} + \cdots + 1) + 1, \\ \deg a_0 = 1, \\ M(h_{i,j}) = M(a_{i-1}) = 2i - 1, \\ M(b_{i,j}) = (2i-1)p, \end{cases}$$

where $i \ge 1$, $j \ge 0$. For more details about the MSS, see May [8; 9] and Ravenel [11].

Now we give an important theorem on the MSS.

By the knowledge on the p–adic expression in number theory, we see that for each integer $t \ge 0$, it can be expressed uniquely as

$$t = q(c_n p^n + c_{n-1} p^{n-1} + \cdots + c_1 p + c_0) + e,$$

where $0 \le c_i < p$ $(0 \le i < n)$, $p > c_n > 0$, $0 \le e < q$.

Theorem 2.1 [4, Proposition 1.1] *With notation as above, let s_1 be a positive integer with $0 < s_1 < p$. If there exists some $0 \le j \le n$ such that $s_1 < c_j$, then in the MSS, we have*

$$E_1^{s_1,t,*} = 0.$$

Now we give a method for detecting generators of the E_1–term $E_1^{\widetilde{s},\widetilde{t},*}$ of the MSS for specialized $\widetilde{s}$ and $\widetilde{t}$.

In this paper, we mainly consider the May E_1–terms of the form $E_1^{s,t+b,*}$, where s, t, b are three integers with $s > 0$, $t > 0$ and $b \ge 0$ satisfying the following conditions:

$$(2\text{-}7)\qquad \begin{cases} (1)\quad t = (\overline{c}_0 + \overline{c}_1 p + \cdots + \overline{c}_n p^n)q \text{ with } 0 \le \overline{c}_i < p \ (0 \le i < n). \\ (2)\quad 0 \le b < q. \\ (3)\quad s < b + q. \end{cases}$$

We denote a_i, $h_{i,j}$ and $b_{i,j}$ by x, y and z respectively. By the graded commutativity of $E_1^{*,*,*}$, generators take the following form:

$$g = (x_1 \cdots x_u)(y_1 \cdots y_v)(z_1 \cdots z_l) \in E_1^{s,t+b,*}. \tag{2-8}$$

Under the above conditions, we claim that

$$u = b.$$

Otherwise, by the characteristics of $\deg a_i$, $\deg b_{i,j}$, $\deg h_{i,j}$ and $\deg g$, we would get $u = b + wq$ for some integer $w > 0$. It would follow that $\dim g \geq b + wq > s = \dim g$, which is a contradiction. The claim is proved.

From the discussion above, we have further that g is of the following form:

$$g = (x_1 \cdots x_b)(y_1 \cdots y_v)(z_1 \cdots z_l) \in E_1^{s,t+b,*}. \tag{2-9}$$

Since $\deg g = \Sigma \deg x_i + \Sigma \deg y_j + \Sigma \deg z_k$, then we have that $\max\{\deg x_i \mid 1 \leq i \leq b\} \leq t + b$, $\max\{\deg y_i \mid 1 \leq i \leq v\} \leq t + b$, $\max\{\deg z_i \mid 1 \leq i \leq l\} \leq t + b$.

By (2-6), the degrees of x_i, y_i and z_i can be expressed uniquely as:

$$\begin{cases} \deg x_i = (x_{i,0} + x_{i,1} p + \cdots + x_{i,n} p^n) q + 1, \\ \deg y_i = (y_{i,0} + y_{i,1} p + \cdots + y_{i,n} p^n) q, \\ \deg z_i = (0 + z_{i,1} p + \cdots + z_{i,n} p^n) q, \end{cases} \tag{2-10}$$

$$\begin{cases} \text{(a)} & (x_{i,0}, \ldots, x_{i,k}, x_{i,k+1}, \ldots, x_{i,n}) \text{ is of the form } (1, \ldots, 1, 0, \ldots, 0), \\ & (x_{i,0}, \ldots, x_{i,k}, x_{i,k+1}, \ldots, x_{i,n}) = (0, \ldots, 0, 0, \ldots, 0) \text{ if } x_i = a_0, \\ \text{(b)} & (y_{i,0}, \ldots, y_{i,j}, y_{i,j+1}, \ldots, y_{i,k}, y_{i,k+1}, \ldots, y_{i,n}) \text{ is of the form} \\ & (0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0), \\ \text{(c)} & (0, z_{i,1}, \ldots, z_{i,n}) \text{ is of the form } (0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0). \end{cases} \tag{2-11}$$

By the graded commutativity of $E_1^{*,*,*}$, the generator

$$g = (x_1 \cdots x_b)(y_1 \cdots y_v)(z_1 \cdots z_l) \in E_1^{s,t+b,*}$$

can be arranged in the following way:

$$\begin{cases} \text{(i)} & \text{If } i > j, \text{ we put } a_i \text{ on the left side of } a_j. \\ \text{(ii)} & \text{If } j < k, \text{ we put } h_{i,j} \text{ on the left side of } h_{w,k}. \\ \text{(iii)} & \text{If } i > w, \text{ we put } h_{i,j} \text{ on the left side of } h_{w,j}. \\ \text{(iv)} & \text{Apply the rules (ii) and (iii) to } b_{i,j}. \end{cases} \tag{2-12}$$

Then by (2-11) and (2-12), the factors $x_{i,j}$, $y_{i,j}$ and $z_{i,j}$ in g must satisfy the following conditions:

$$
(2\text{-}13)\qquad \begin{cases}
(1)\quad x_{1,j} \geq x_{2,j} \geq \cdots \geq x_{b,j}. \\
(2)\quad x_{i,0} \geq x_{i,1} \geq \cdots \geq x_{i,n}. \\
(3)\quad \text{If } y_{i,j-1} = 0 \text{ and } y_{i,j} = 1, \text{then for all } k < j \;\; y_{i,k} = 0. \\
(4)\quad \text{If } y_{i,j} = 1 \text{ and } y_{i,j+1} = 0, \text{ then for all } k > j \;\; y_{i,k} = 0. \\
(5)\quad y_{1,0} \geq y_{2,0} \geq \cdots \geq y_{v,0}. \\
(6)\quad \text{If } y_{i,0} = y_{i+1,0},\ y_{i,1} = y_{i+1,1}, \ldots, y_{i,j} = y_{i+1,j}, \text{ then} \\
\qquad\;\; y_{i,j+1} \geq y_{i+1,j+1}. \\
(7)\quad \text{Apply the similar rule (3)} \sim \text{(6) to } z_{i,j}.
\end{cases}
$$

From $\deg g = \sum_{i=1}^{b} \deg x_i + \sum_{i=1}^{v} \deg y_i + \sum_{i=1}^{l} \deg z_i$, by the properties of the p–adic expression in number theory we get the following group of equations:

$$
(2\text{-}14)\qquad \begin{cases}
x_{1,0}+\cdots+x_{b,0}+y_{1,0}+\cdots+y_{v,0}+0+\cdots+0 = \overline{c}_0+k_0 p, \\
x_{1,1}+\cdots+x_{b,1}+y_{1,1}+\cdots+y_{v,1}+z_{1,1}+\cdots+z_{l,1} = \overline{c}_1+k_1 p-k_0, \\
x_{1,2}+\cdots+x_{b,2}+y_{1,2}+\cdots+y_{v,2}+z_{1,2}+\cdots+z_{l,2} = \overline{c}_2+k_2 p-k_1, \\
\qquad\vdots \\
x_{1,j}+\cdots+x_{b,j}+y_{1,j}+\cdots+y_{v,j}+z_{1,j}+\cdots+z_{l,j} = \overline{c}_j+k_j p-k_{j-1}, \\
\qquad\vdots \\
x_{1,n}+\cdots+x_{b,n}+y_{1,n}+\cdots+y_{v,n}+z_{1,n}+\cdots+z_{l,n} = \overline{c}_n-k_{n-1},
\end{cases}
$$

where $k_i \geq 0$ for $0 \leq i \leq n-1$.

In the above group of equations, we get two integer sequences $K = (k_0, k_1, \ldots, k_{n-1})$ and $S = (\overline{c}_0 + k_0 p, \overline{c}_1 + k_1 p - k_0, \ldots, \overline{c}_n - k_{n-1})$ denoted by $(c_0, c_1, \ldots, c_n)$ which are determined by $(k_0, k_1, \ldots, k_{n-1})$ and $(\overline{c}_0, \overline{c}_1, \ldots, \overline{c}_n)$. Notice that the elements x_i, y_i and z_i are uniquely determined by their degrees.

Each solution of (2-14) which satisfies (2-13) determines a unique generator g of $E_1^{s,t+b,*}$, because among g all the elements x_i, y_i and z_i can be determined by their degrees, respectively. Note that the generator g obtained above may be trivial, because there are perhaps more than one $h_{i,j}$ among g.

Thus for the $E_1^{s,t+b,*}$–term where $t=(\overline{c}_0+\overline{c}_1p+\cdots+\overline{c}_np^n)q$ with $0\le\overline{c}_i<p$ $(0\le i<n)$, $0<\overline{c}_n<p$, $0\le b<q$, the method for determining $E_1^{s,t+b,*}$ is as follows:

(2-15)
(1) List up all the possible (b,v,l) such that $b+v+2l=s$.

(2) For each given (b,v,l), list all the sequences $K=(k_0,k_1,\dots,k_{n-1})$ and $S=(c_0,c_1,\dots,c_n)$ such that $c_i\le b+v+l$ for all $0\le i\le n$.

(3) For each given (b,v,l), the sequences $K=(k_0,k_1,\dots,k_{n-1})$ and $S=(c_0,\dots,c_n)$, solve the group of equations (2-14) by virtue of (2-13). Then determine all the generators of $E_1^{s,t+b,*}$ by setting the corresponding second degrees.

(4) By use of $h_{i,j}^k=0$ for $k>1$, we get all the nontrivial generators of the May E_1–term $E_1^{s,t+b,*}$.

3 The determination of two Adams E_2–terms

In this section, by the method for determining the May E_1–term $E_1^{\widetilde{s},\widetilde{t},*}$ we first determine some May E_r–terms for $r\ge1$ (cf Proposition 3.1). Then we give two important theorems about Adams E_2–terms which will be used in the proof of Theorem 1.7.

We first give the following important proposition about the May E_1–term.

Proposition 3.1 *Let $p\ge5$, $n>4$, $0\le s<p-3$ and $r\ge1$. Then the May E_1–term satisfies*

$$E_1^{s+6-r,t(s,n)+1-r,*}=\begin{cases}G & r=1 \text{ and } s=p-4,\\ 0 & r\ge2, \text{ or } r=1 \text{ and } s<p-4.\end{cases}$$

Here $t(s,n)=q[2p^n+(s+2)p+(s+2)]+s$ and G is the $\mathbb{Z}_p$–module generated by the following twenty-three elements:

$$\mathcal{G}_1=a_n^{p-5}a_1h_{n,0}h_{3,0}h_{n-1,1}h_{n-1,2}h_{n-2,2},$$
$$\mathcal{G}_2=a_n^{p-5}a_1h_{n+1,0}h_{n,0}h_{n-1,1}h_{n-2,2}h_{1,2},$$
$$\mathcal{G}_3=a_n^{p-5}a_3h_{n,0}h_{1,0}h_{n-1,1}h_{n-1,2}h_{n-2,2},$$
$$\mathcal{G}_4=a_n^{p-4}h_{3,0}h_{1,0}h_{n-1,1}h_{n-1,2}h_{n-2,2},$$

$$\begin{aligned}
\mathcal{G}_5 &= a_n^{p-4} h_{n,0} h_{1,0} h_{2,1} h_{n-1,2} h_{n-2,2},\\
\mathcal{G}_6 &= a_{n+1} a_n^{p-5} h_{n,0} h_{1,0} h_{n-1,1} h_{n-2,2} h_{1,2},\\
\mathcal{G}_7 &= a_n^{p-4} h_{n+1,0} h_{1,0} h_{n-1,1} h_{n-2,2} h_{1,2},\\
\mathcal{G}_8 &= a_n^{p-4} h_{n,0} h_{1,0} h_{n,1} h_{n-2,2} h_{1,2},\\
\mathcal{G}_9 &= a_n^{p-4} h_{n,0} h_{1,0} h_{n-1,1} h_{n-1,2} h_{1,2},\\
\mathcal{G}_{10} &= a_n^{p-4} h_{n,0} h_{2,0} h_{n-1,2} h_{n-2,2} h_{1,2},\\
\mathcal{G}_{11} &= a_n^{p-5} a_3 h_{n,0} h_{3,0} h_{n-1,2} h_{n-2,2} h_{n-3,3},\\
\mathcal{G}_{12} &= a_n^{p-5} a_3 h_{n+1,0} h_{n,0} h_{n-2,2} h_{1,2} h_{n-3,3},\\
\mathcal{G}_{13} &= a_{n+1} a_n^{p-5} h_{n,0} h_{3,0} h_{n-2,2} h_{1,2} h_{n-3,3},\\
\mathcal{G}_{14} &= a_n^{p-4} h_{n+1,0} h_{3,0} h_{n-2,2} h_{1,2} h_{n-3,3},\\
\mathcal{G}_{15} &= a_n^{p-4} h_{n,0} h_{3,0} h_{n-1,2} h_{1,2} h_{n-3,3},\\
\mathcal{G}_{16} &= a_n^{p-4} h_{n,0} h_{3,0} h_{n-2,2} h_{1,2} h_{n-2,3},\\
\mathcal{G}_{17} &= a_n^{p-5} a_3 h_{n+1,0} h_{n,0} h_{n-2,2} b_{n-2,1},\\
\mathcal{G}_{18} &= a_{n+1} a_n^{p-5} h_{n,0} h_{3,0} h_{n-2,2} b_{n-2,1},\\
\mathcal{G}_{19} &= a_n^{p-4} h_{n+1,0} h_{3,0} h_{n-2,2} b_{n-2,1},\\
\mathcal{G}_{20} &= a_n^{p-4} h_{n,0} h_{3,0} h_{n-1,2} b_{n-2,1},\\
\mathcal{G}_{21} &= a_n^{p-4} h_{n,0} h_{3,0} h_{n-2,2} b_{n-1,1},\\
\mathcal{G}_{22} &= a_n^{p-4} h_{n+1,0} h_{n,0} h_{1,2} b_{n-2,1},\\
\mathcal{G}_{23} &= a_n^{p-4} h_{n+1,0} h_{n,0} h_{n-2,2} b_{1,1}.
\end{aligned}$$

Proof By (2-1) and (2-6), it is easy to show that in the MSS

$$E_1^{s+6-r,t(s,n)+1-r,*} = 0$$

for $r \geq s+2$. Thus in the rest of the proof, we always assume $1 \leq r < s+2$.

Consider $g = w_1 w_2 \cdots w_l \in E_1^{s+6-r,t(s,n)-r+1,*}$ in the MSS, where w_i is one of a_k, $h_{r,j}$ or $b_{u,z}$, $0 \leq k \leq n+1$, $0 \leq r+j \leq n+1$, $0 \leq u+z \leq n$, $r > 0$, $j \geq 0$, $u > 0$, $z \geq 0$. Assume that

$$\deg w_i = q(c_{i,n} p^n + c_{i,n-1} p^{n-1} + \cdots + c_{i,1} p + c_{i,0}) + e_i,$$

where $c_{i,j} = 0$ or 1, $e_i = 1$ if $w_i = a_{k_i}$, or $e_i = 0$. It follows that

$$\dim g = \sum_{i=1}^{l} \dim w_i = s + 6 - r, \tag{3-1}$$

$$\begin{aligned}\deg g &= \sum_{i=1}^{l} \deg w_i \\ &= q\Bigg(\Big(\sum_{i=1}^{l} c_{i,n}\Big)p^n + \cdots + \Big(\sum_{i=1}^{l} c_{i,1}\Big)p + \Big(\sum_{i=1}^{l} c_{i,0}\Big)\Bigg) + \Big(\sum_{i=1}^{l} e_i\Big) \qquad (3\text{-}2) \\ &= q\big(2p^n + (s+2)p + (s+2)\big) + (s+1-r).\end{aligned}$$

Note that $\dim h_{i,j} = \dim a_i = 1$, $\dim b_{i,j} = 2$, $1 \le r < s+3$ and $0 \le s < p-3$. Then

$$l \le s + 6 - r < p + 3 - r \le p + 2$$

from
$$\dim g = \sum_{i=1}^{l} \dim w_i = s + 6 - r.$$

We claim
$$s + 1 - r \ge 0.$$

It is easy to get

$$\sum_{i=1}^{l} e_i \le l \le p + 1.$$

However, by $1 \le r < s+2$ and $p \ge 5$, we would also have

$$\sum_{i=1}^{l} e_i = q + (s - r + 1) > 2p - 2 - 1 \ge p + 2$$

which contradicts

$$\sum_{i=1}^{l} e_i \le l \le p + 1.$$

The claim is proved.

Using $0 \le s+2, s+1-r < p$ and the knowledge on p–adic expression in number theory, we have

$$
\text{(3-3)} \qquad \begin{cases}
\sum_{i=1}^{l} e_i = s+1-r+\lambda_{-1}q & \lambda_{-1} \ge 0, \\
\sum_{i=1}^{l} c_{i,0} + \lambda_{-1} = s+2+\lambda_0 p & \lambda_0 \ge 0, \\
\sum_{i=1}^{l} c_{i,1} + \lambda_0 = s+2+\lambda_1 p & \lambda_1 \ge 0, \\
\sum_{i=1}^{l} c_{i,2} + \lambda_1 = 0+\lambda_2 p & \lambda_2 \ge 0, \\
\sum_{i=1}^{l} c_{i,3} + \lambda_2 = 0+\lambda_3 p & \lambda_3 \ge 0, \\
\sum_{i=1}^{l} c_{i,4} + \lambda_3 = 0+\lambda_4 p & \lambda_4 \ge 0, \\
\vdots & \vdots \\
\sum_{i=1}^{l} c_{i,n-1} + \lambda_{n-2} = 0+\lambda_{n-1} p & \lambda_{n-1} \ge 0, \\
\sum_{i=1}^{l} c_{i,n} + \lambda_{n-1} = 2.
\end{cases}
$$

From $e_i = 0$ or 1, $c_{i,j} = 0$ or 1, and $l \le p+1$, we easily have

$$(\lambda_{-1}, \lambda_0, \lambda_1) = (0,0,0).$$

Consider the fourth equality of (3-3), $\sum_{i=1}^{l} c_{i,2} = 0 + \lambda_2 p$. By $c_{i,2} = 0$ or 1, and $l \le p+1$, we get that

$$\lambda_2 = 0 \text{ or } 1.$$

Thus we divide our proof into the following two cases.

Case 1 $\lambda_2 = 0$.

We claim

$$\lambda_3 = 0.$$

If $\lambda_3 = 1$, we would have

$$\sum_{i=1}^{l} c_{i,1} = s+2, \quad \sum_{i=1}^{l} c_{i,2} = 0, \quad \sum_{i=1}^{l} c_{i,3} = p.$$

From
$$\sum_{i=1}^{l} c_{i,1} = s+2$$

and (2-6), there would be $s+2$ factors among g where $\deg x_i = q(\text{higher terms on } p + p + \text{lower terms on } p) + \delta_i$, where δ_i may equal 0 or 1. Similarly, from

$$\sum_{i=1}^{l} c_{i,3} = p,$$

there would be p factors among g such that $\deg w_i = q(\text{higher terms on } p + p^3 + \text{lower terms on } p) + \delta_i$. Thus, by $l \le p+1$ and (2-6), there would be at least $p+s+2-(p+1) = s+1$ factors in g such that $\deg w_i = q(\text{higher terms on } p + p^3 + p^2 + p^1 + \text{lower terms on } p) + \delta_i$. Thus we would have

$$\sum_{i=1}^{l} c_{i,2} \ge s+1$$

which contradicts
$$\sum_{i=1}^{l} c_{i,2} = 0.$$

The claim is proved.

By induction on j, we have that

$$\lambda_j = 0$$

for $3 \le j \le n-1$. Then we have the following cases.

Case 1.1 If there are two factors $h_{1,n}$ and $b_{1,n-1}$ in g, then up to sign $g = h_{1,n} b_{1,n-1} \tilde{g}$ with $\tilde{g} \in E_1^{s+3-r, q((s+2)p+(s+2))+(s+1-r),*}$.

When $r = 1$, by Method (2-15), we easily get that

$$E_1^{s+2, q((s+2)p+(s+2))+s,*} = \mathbb{Z}_p\{a_2^s h_{2,0}^2\} = 0$$

by $h_{i,j}^k = 0$ for $k > 1$.

When $r \ge 2$, by Theorem 2.1 we have that

$$E_1^{s+3-r, q((s+2)p+(s+2))+(s+1-r),*} = 0.$$

From the above discussion, there cannot exist two factors $h_{1,n}$ and $b_{1,n-1}$ in g.

Case 1.2 If there are two factors $b_{1,n-1}$ and $b_{1,n-1}$ in g, then up to sign $g = b_{1,n-1}b_{1,n-1}\widetilde{g}$ with $\widetilde{g} \in E_1^{s+2-r,q((s+2)p+(s+2))+(s+1-r),*}$. By Theorem 2.1,

$$E_1^{s+2-r,q((s+2)p+(s+2))+(s+1-r),*} = 0,$$

showing that there cannot exist two factors $b_{1,n-1}$ and $b_{1,n-1}$ in g.

Case 2 $\lambda_2 = 1$.

If $r \geq 3$, then we would have $l \leq s+6-r < p+3-r \leq p$. It is easy to see that λ_2 is impossible to equal 1 when $r \geq 3$. Thus in the rest of this case, we always assume $r \leq 2$.

From the fifth equality of (3-3), $\sum_{i=1}^{l} c_{i,3} + 1 = \lambda_3 p$, and $0 \leq \sum_{i=1}^{l} c_{i,3} \leq l \leq p+1$, we can deduce

$$\lambda_3 = 1.$$

By induction on j, we get that

$$\lambda_j = 1$$

for $3 \leq j \leq n-1$. Thus (3-3) turns into

$$\text{(3-4)} \qquad \begin{cases} \sum_{i=1}^{l} e_i = s+1-r, \\ \sum_{i=1}^{l} c_{i,0} = s+2, \\ \sum_{i=1}^{l} c_{i,1} = s+2, \\ \sum_{i=1}^{l} c_{i,2} = p, \\ \sum_{i=1}^{l} c_{i,3} = p-1, \\ \qquad \vdots \\ \sum_{i=1}^{l} c_{i,n-1} = p-1, \\ \sum_{i=1}^{l} c_{i,n} = 1. \end{cases}$$

From the fourth equality of (3-4), $\sum_{i=1}^{l} c_{i,2} = p$, using $c_{i,2} = 0$ or 1, we have

$$l \geq p.$$

Note $l \leq s+5$. So $s \geq p-5$. By $0 \leq s < p-3$, we see s may equal $p-5$ or $p-4$.

Case 2.1 When $s = p-4$, $g = w_1 w_2 \cdots w_l \in E_1^{p+2-r,t(p-4,n)+1-r,*}$.

Case 2.1.1 $r = 2$. Then we have

$$g = w_1 w_2 \cdots w_l \in E_1^{p,t(p-4,n)+1-r,*} = E_1^{p,2p^n q+(p-2)pq+(p-2)q+(p-5),*}.$$

Note that $l \geq p$. From $\dim g = p$, we have $\dim w_i = 1$ for each i. Then up to sign the generator g must be of the form $(x_1 x_2 \cdots x_{p-5})(y_1 y_2 y_3 y_4 y_5)$ as in (2-8). By Method (2-15) and (3-4), we have that up to sign g can be one of the following:

$$\begin{aligned}
&a_{n+1} a_n^{p-7} a_3 h_{n,0}^3 h_{n-2,2}^2, && a_n^{p-6} a_3 h_{n+1,0} h_{n,0}^2 h_{n-2,2}^2,\\
&a_n^{p-6} a_3 h_{n,0}^3 h_{n-1,2} h_{n-2,2}, && a_{n+1} a_n^{p-6} h_{n,0}^2 h_{3,0} h_{n-2,2}^2,\\
&a_n^{p-5} h_{n+1,0} h_{n,0} h_{3,0} h_{n-2,2}^2, && a_n^{p-5} h_{n,0}^2 h_{3,0} h_{n-1,2} h_{n-2,2},\\
&a_{n+1} a_n^{p-6} h_{n,0}^3 h_{n-2,2} h_{1,2}, && a_n^{p-5} h_{n+1,0} h_{n,0}^2 h_{n-2,2} h_{1,2},\\
&a_n^{p-5} h_{n,0}^3 h_{n-1,2} h_{1,2}.
\end{aligned}$$

By (2-1), $h_{i,j}^k = 0$ for $k > 1$. Thus all the generators above are trivial.

Case 2.1.2 $r = 1$. Then we have

$$g = w_1 w_2 \cdots w_l \in E_1^{p+1,2p^n q+(p-2)pq+(p-2)q+(p-4),*}.$$

In this case, we see that l may equal p or $p+1$.

If $l = p$, then up to sign the generator g has the form $(x_1 x_2 \cdots x_{p-4})(y_1 y_2 y_3)(z_1)$ as in (2-8). By Method (2-15) and (3-4), we have that up to sign g may be one of the following:

$$\begin{aligned}
&a_{n+1} a_n^{p-6} a_3 h_{n,0}^2 h_{n-2,2} b_{n-2,1}, && a_n^{p-5} a_3 h_{n+1,0} h_{n,0} h_{n-2,2} b_{n-2,1},\\
&a_n^{p-5} a_3 h_{n,0}^2 h_{n-1,2} b_{n-2,1}, && a_n^{p-5} a_3 h_{n,0}^2 h_{n-2,2} b_{n-1,1},\\
&a_{n+1} a_n^{p-5} h_{n,0} h_{3,0} h_{n-2,2} b_{n-2,1}, && a_n^{p-4} h_{n+1,0} h_{3,0} h_{n-2,2} b_{n-2,1},\\
&a_n^{p-4} h_{n,0} h_{3,0} h_{n-1,2} b_{n-2,1}, && a_n^{p-4} h_{n,0} h_{3,0} h_{n-2,2} b_{n-1,1},\\
&a_{n+1} a_n^{p-5} h_{n,0}^2 h_{1,2} b_{n-2,1}, && a_n^{p-4} h_{n+1,0} h_{n,0} h_{1,2} b_{n-2,1},\\
&a_n^{p-4} h_{n,0}^2 h_{1,2} b_{n-1,1}, && a_{n+1} a_n^{p-5} h_{n,0}^2 h_{n-2,2} b_{1,1},\\
&a_n^{p-4} h_{n+1,0} h_{n,0} h_{n-2,2} b_{1,1}, && a_n^{p-4} h_{n,0}^2 h_{n-1,2} b_{1,1}.
\end{aligned}$$

By (2-1), $h_{i,j}^k = 0$ for $k > 1$. Thus among the generators above, only the following seven generators are nontrivial:

$$a_n^{p-5} a_3 h_{n+1,0} h_{n,0} h_{n-2,2} b_{n-2,1},$$
$$a_{n+1} a_n^{p-5} h_{n,0} h_{3,0} h_{n-2,2} b_{n-2,1},$$
$$a_n^{p-4} h_{n+1,0} h_{3,0} h_{n-2,2} b_{n-2,1},$$
$$a_n^{p-4} h_{n,0} h_{3,0} h_{n-1,2} b_{n-2,1},$$
$$a_n^{p-4} h_{n,0} h_{3,0} h_{n-2,2} b_{n-1,1},$$
$$a_n^{p-4} h_{n+1,0} h_{n,0} h_{1,2} b_{n-2,1},$$
$$a_n^{p-4} h_{n+1,0} h_{n,0} h_{n-2,2} b_{1,1},$$

denoted by $\mathcal{G}_{17}, \mathcal{G}_{18}, \cdots, \mathcal{G}_{23}$, respectively.

If $l = p+1$, then we have that dim $w_i = 1$ for $1 \le i \le p+1$. Then up to sign the generator g must be of the form $(x_1 x_2 \cdots x_{p-4})(y_1 y_2 y_3 y_4 y_5)$ as in (2-8). By Method (2-15) and (3-4), we have that up to sign g may be one of the following:

$$a_{n+1} a_n^{p-7} a_3 a_0 h_{n,0}^3 h_{n-2,2}^2, \quad a_n^{p-6} a_3 a_0 h_{n+1,0} h_{n,0}^2 h_{n-2,2}^2,$$
$$a_n^{p-6} a_3 a_0 h_{n,0}^3 h_{n-1,2} h_{n-2,2}, \quad a_{n+1} a_n^{p-6} a_0 h_{n,0}^2 h_{3,0} h_{n-2,2}^2,$$
$$a_n^{p-5} a_0 h_{n+1,0} h_{n,0} h_{3,0} h_{n-2,2}^2, \quad a_n^{p-5} a_0 h_{n,0}^2 h_{3,0} h_{n-1,2} h_{n-2,2},$$
$$a_{n+1} a_n^{p-6} a_0 h_{n,0}^3 h_{n-2,2} h_{1,2}, \quad a_n^{p-5} a_0 h_{n+1,0} h_{n,0}^2 h_{n-2,2} h_{1,2},$$
$$a_n^{p-5} a_0 h_{n,0}^3 h_{n-1,2} h_{1,2}, \quad a_{n+1} a_n^{p-7} a_3 a_1 h_{n,0}^2 h_{n-1,1} h_{n-2,2}^2,$$
$$a_n^{p-6} a_3 a_1 h_{n+1,0} h_{n,0} h_{n-1,1} h_{n-2,2}^2, \quad a_n^{p-6} a_3 a_1 h_{n,0}^2 h_{n,1} h_{n-2,2}^2,$$
$$a_n^{p-6} a_3 a_1 h_{n,0}^2 h_{n-1,1} h_{n-1,2} h_{n-2,2}, \quad a_{n+1} a_n^{p-6} a_1 h_{n,0} h_{3,0} h_{n-1,1} h_{n-2,2}^2,$$
$$a_n^{p-5} a_1 h_{n+1,0} h_{3,0} h_{n-1,1} h_{n-2,2}^2, \quad a_n^{p-5} a_1 h_{n,0} h_{3,0} h_{n,1} h_{n-2,2}^2,$$
$$a_n^{p-5} a_1 h_{n,0} h_{3,0} h_{n-1,1} h_{n-1,2} h_{n-2,2}, \quad a_{n+1} a_n^{p-6} a_1 h_{n,0}^2 h_{2,1} h_{n-2,2}^2,$$
$$a_n^{p-5} a_1 h_{n+1,0} h_{n,0} h_{2,1} h_{n-2,2}^2, \quad a_n^{p-5} a_1 h_{n+1,0} h_{n,0} h_{2,1} h_{n-2,2}^2,$$
$$a_n^{p-5} a_1 h_{n,0}^2 h_{2,1} h_{n-1,2} h_{n-2,2}, \quad a_{n+1} a_n^{p-6} a_1 h_{n,0}^2 h_{n-1,1} h_{n-2,2} h_{1,2},$$
$$a_n^{p-5} a_1 h_{n+1,0} h_{n,0} h_{n-1,1} h_{n-2,2} h_{1,2}, \quad a_n^{p-5} a_1 h_{n,0}^2 h_{n,1} h_{n-2,2} h_{1,2},$$
$$a_n^{p-5} a_1 h_{n,0}^2 h_{n-1,1} h_{n-1,2} h_{1,2}, \quad a_{n+1} a_n^{p-6} a_3 h_{n,0} h_{1,0} h_{n-1,1} h_{n-2,2}^2,$$
$$a_n^{p-5} a_3 h_{n+1,0} h_{1,0} h_{n-1,1} h_{n-2,2}^2, \quad a_n^{p-5} a_3 h_{n,0} h_{1,0} h_{n,1} h_{n-2,2}^2,$$
$$a_n^{p-5} a_3 h_{n,0} h_{1,0} h_{n-1,1} h_{n-1,2} h_{n-2,2}, \quad a_{n+1} a_n^{p-5} h_{3,0} h_{1,0} h_{n-1,1} h_{n-2,2}^2,$$

$$a_n^{p-4}h_{3,0}h_{1,0}h_{n,1}h_{n-2,2}^2, \quad a_n^{p-4}h_{3,0}h_{1,0}h_{n-1,1}h_{n-1,2}h_{n-2,2},$$
$$a_{n+1}a_n^{p-5}h_{n,0}h_{1,0}h_{2,1}h_{n-2,2}^2, \quad a_n^{p-4}h_{n+1,0}h_{1,0}h_{2,1}h_{n-2,2}^2,$$
$$a_n^{p-4}h_{n,0}h_{1,0}h_{2,1}h_{n-1,2}h_{n-2,2}, \quad a_{n+1}a_n^{p-5}h_{n,0}h_{1,0}h_{n-1,1}h_{n-2,2}h_{1,2},$$
$$a_n^{p-4}h_{n+1,0}h_{1,0}h_{n-1,1}h_{n-2,2}h_{1,2}, \quad a_n^{p-4}h_{n,0}h_{1,0}h_{n,1}h_{n-2,2}h_{1,2},$$
$$a_n^{p-4}h_{n,0}h_{1,0}h_{n-1,1}h_{n-1,2}h_{1,2}, \quad a_{n+1}a_n^{p-7}a_3a_2h_{n,0}^2h_{n-2,2}^3,$$
$$a_n^{p-6}a_3a_2h_{n+1,0}h_{n,0}h_{n-2,2}^3, \quad a_n^{p-6}a_3a_2h_{n,0}^2h_{n-1,2}h_{n-2,2}^2,$$
$$a_{n+1}a_n^{p-6}a_3h_{n,0}h_{2,0}h_{n-2,2}^3, \quad a_n^{p-5}a_3h_{n+1,0}h_{2,0}h_{n-2,2}^3,$$
$$a_n^{p-5}a_3h_{n,0}h_{2,0}h_{n-1,2}h_{n-2,2}^2, \quad a_{n+1}a_n^{p-6}h_{3,0}h_{2,0}h_{n-2,2}^3,$$
$$a_n^{p-4}h_{3,0}h_{2,0}h_{n-1,2}h_{n-2,2}^2, \quad a_{n+1}a_n^{p-5}h_{n,0}h_{2,0}h_{n-2,2}^2h_{1,2},$$
$$a_n^{p-4}h_{n+1,0}h_{2,0}h_{n-2,2}^2h_{1,2}, \quad a_n^{p-4}h_{n,0}h_{2,0}h_{n-1,2}h_{n-2,2}h_{1,2},$$
$$a_{n+1}a_n^{p-7}a_3^2h_{n,0}^2h_{n-2,2}^2h_{n-3,3}, \quad a_n^{p-6}a_3^2h_{n+1,0}h_{n,0}h_{n-2,2}^2h_{n-3,3},$$
$$a_n^{p-6}a_3^2h_{n,0}^2h_{n-1,2}h_{n-2,2}h_{n-3,3}, \quad a_n^{p-6}a_3^2h_{n,0}^2h_{n-2,2}^2h_{n-2,3},$$
$$a_{n+1}a_n^{p-6}a_3h_{n,0}h_{3,0}h_{n-2,2}^2h_{n-3,3}, \quad a_n^{p-5}a_3h_{n+1,0}h_{3,0}h_{n-2,2}^2h_{n-3,3},$$
$$a_n^{p-5}a_3h_{n,0}h_{3,0}h_{n-1,2}h_{n-2,2}h_{n-3,3}, \quad a_n^{p-5}a_3h_{n,0}h_{3,0}h_{n-2,2}^2h_{n-2,3},$$
$$a_{n+1}a_n^{p-6}a_3h_{n,0}^2h_{n-2,2}h_{1,2}h_{n-3,3}, \quad a_n^{p-5}a_3h_{n+1,0}h_{n,0}h_{n-2,2}h_{1,2}h_{n-3,3},$$
$$a_n^{p-5}a_3h_{n,0}^2h_{n-1,2}h_{1,2}h_{n-3,3}, \quad a_n^{p-5}a_3h_{n,0}^2h_{n-2,2}h_{1,2}h_{n-2,3},$$
$$a_n^{p-5}a_3h_{n,0}^2h_{n-2,2}^2h_{1,n}, \quad a_{n+1}a_n^{p-5}h_{n,0}^2h_{n-2,2}^2h_{n-3,3},$$
$$a_n^{p-4}h_{n,0}^2h_{n-1,2}h_{n-2,2}h_{n-3,3}, \quad a_n^{p-4}h_{n,0}^2h_{n-2,2}^2h_{n-2,3},$$
$$a_{n+1}a_n^{p-5}h_{n,0}h_{3,0}h_{n-2,2}h_{1,2}h_{n-3,3}, \quad a_n^{p-4}h_{n+1,0}h_{3,0}h_{n-2,2}h_{1,2}h_{n-3,3},$$
$$a_n^{p-4}h_{n,0}h_{3,0}h_{n-1,2}h_{1,2}h_{n-3,3}, \quad a_n^{p-4}h_{n,0}h_{3,0}h_{n-2,2}h_{1,2}h_{n-2,3},$$
$$a_n^{p-4}h_{n,0}h_{3,0}h_{n-2,2}^2h_{1,n+1}, \quad a_{n+1}a_n^{p-5}h_{n,0}^2h_{1,2}^2h_{n-3,3},$$
$$a_n^{p-4}h_{n+1,0}h_{n,0}h_{1,2}^2h_{n-3,3}, \quad a_n^{p-4}h_{n,0}^2h_{1,2}^2h_{n-2,3},$$
$$a_n^{p-4}h_{n,0}^2h_{n-2,2}h_{1,2}h_{1,n}.$$

Note that $h_{i,j}^k = 0$ for $k > 1$. Among the generators above, the nontrivial generators are as follows:

$$a_n^{p-5}a_1h_{n,0}h_{3,0}h_{n-1,1}h_{n-1,2}h_{n-2,2}, \quad a_n^{p-5}a_1h_{n+1,0}h_{n,0}h_{n-1,1}h_{n-2,2}h_{1,2},$$
$$a_n^{p-5}a_3h_{n,0}h_{1,0}h_{n-1,1}h_{n-1,2}h_{n-2,2}, \quad a_n^{p-4}h_{3,0}h_{1,0}h_{n-1,1}h_{n-1,2}h_{n-2,2},$$
$$a_n^{p-4}h_{n,0}h_{1,0}h_{2,1}h_{n-1,2}h_{n-2,2}, \quad a_{n+1}a_n^{p-5}h_{n,0}h_{1,0}h_{n-1,1}h_{n-2,2}h_{1,2},$$

$$a_n^{p-4}h_{n+1,0}h_{1,0}h_{n-1,1}h_{n-2,2}h_{1,2},\quad a_n^{p-4}h_{n,0}h_{1,0}h_{n,1}h_{n-2,2}h_{1,2},$$
$$a_n^{p-4}h_{n,0}h_{1,0}h_{n-1,1}h_{n-1,2}h_{1,2},\quad a_n^{p-4}h_{n,0}h_{2,0}h_{n-1,2}h_{n-2,2}h_{1,2},$$
$$a_n^{p-5}a_3h_{n,0}h_{3,0}h_{n-1,2}h_{n-2,2}h_{n-3,3},\quad a_n^{p-5}a_3h_{n+1,0}h_{n,0}h_{n-2,2}h_{1,2}h_{n-3,3},$$
$$a_{n+1}a_n^{p-5}h_{n,0}h_{3,0}h_{n-2,2}h_{1,2}h_{n-3,3},\quad a_n^{p-4}h_{n+1,0}h_{3,0}h_{n-2,2}h_{1,2}h_{n-3,3},$$
$$a_n^{p-4}h_{n,0}h_{3,0}h_{n-1,2}h_{1,2}h_{n-3,3},\quad a_n^{p-4}h_{n,0}h_{3,0}h_{n-2,2}h_{1,2}h_{n-2,3},$$

denoted by $\mathcal{G}_1$, $\mathcal{G}_2$, $\cdots$, $\mathcal{G}_{16}$, respectively.

Case 2.2 When $s=p-5$, $g=w_1w_2\cdots w_l\in E_1^{p+1-r,t(p-5,n)+1-r,*}$.

Case 2.2.1 $r=2$. Then we have

$$g=w_1w_2\cdots w_l\in E_1^{p-1,2p^nq+(p-3)pq+(p-3)q+(p-6),*}.$$

Recall that $l\geq p$. Thus in this case, the generator g is impossible to exist.

Case 2.2.2 $r=1$. Then we have

$$g=w_1w_2\cdots w_l\in E_1^{p,2p^nq+(p-3)pq+(p-3)q+(p-5),*}.$$

Note that $l\geq p$. From $\dim g=p$, we have that l must equal p. Then $\dim w_i=1$ for each i. So up to sign the generator g must be of the form $(x_1x_2\cdots x_{p-5})(y_1y_2y_3y_4y_5)$ as in (2-8). By Method (2-15) and (3-4), we have that up to sign g can be one of the following:

$$a_{n+1}a_n^{p-7}a_3h_{n,0}^2h_{n-2,2}^3,\quad a_n^{p-6}a_3h_{n+1,0}h_{n,0}h_{n-2,2}^3,$$
$$a_n^{p-6}a_3h_{n,0}^2h_{n-1,2}h_{n-2,2},\quad a_{n+1}a_n^{p-6}h_{n,0}h_{3,0}h_{n-2,2}^3,$$
$$a_n^{p-5}h_{n+1,0}h_{3,0}h_{n-2,2}^3,\quad a_n^{p-5}h_{n,0}h_{3,0}h_{n-1,2}h_{n-2,2}^2,$$
$$a_{n+1}a_n^{p-6}h_{n,0}^2h_{n-2,2}^2h_{1,2},\quad a_n^{p-5}h_{n+1,0}h_{n,0}h_{n-2,2}^2h_{1,2},$$
$$a_n^{p-5}h_{n,0}^2h_{n-1,2}h_{n-2,2}h_{1,2}.$$

By (2-1), $h_{i,j}^k=0$ for $k>1$. Thus none of the generators above is nontrivial.

Combining Cases 1 and 2, we complete the proof of Proposition 3.1. □

Now we begin to consider the representative of the composite map $\zeta_{n-1}\xi_{s+2,n}$ in the ASS. We need the following theorem about the β–element.

Theorem 3.2 [5, Theorem 2.2] *Let $p\geq 5$, $0\leq s<p-2$. Then the permanent cocycle*

$$a_2^sh_{2,0}h_{1,1}\in E_r^{s+2,t,*}$$

converges to the second Greek letter element

$$\widetilde{\beta}_{s+2} \in \mathrm{Ext}_A^{s+2,t}(\mathbb{Z}_p, \mathbb{Z}_p)$$

in the MSS, where $r \geq 1$, $t = (s+2)pq + (s+1)q + s$ *and* $\widetilde{\beta}_{s+2}$ *converges to the* β*–element*

$$\beta_{s+2} \in \pi_{(s+2)pq+(s+1)q-2}(S)$$

in the ASS, where β_{s+2} *is described in Definition 1.1.*

Now we consider some results on the product $h_0 b_{n-1} h_n \widetilde{\beta}_{s+2}$.

Lemma 3.3 (1) *The product* $h_0 b_{n-1} h_n \widetilde{\beta}_{s+2} \in \mathrm{Ext}_A^{s+6,t(s,n)}(\mathbb{Z}_p, \mathbb{Z}_p)$ *is represented by* $h_{1,0} b_{1,n-1} h_{1,n} a_2^s h_{2,0} h_{1,1} \in E_1^{s+6,t(s,n),*}$ *in the MSS, where the degree* $t(s,n) = q\big(2p^n + (s+2)p + (s+2)\big) + s$.

(2) *For the twenty-three generators of* $E_1^{p+1,t(p-4,n),*}$, *we have*

$$\begin{aligned} M(\mathcal{G}_i) &= (2n+1)p - 2n - 9 \qquad (1 \leq i \leq 16), \\ M(\mathcal{G}_j) &= (4n-4)p - 4n - 3 \qquad (j = 17, 18, 19, 20, 22), \\ M(\mathcal{G}_{21}) &= (4n-2)p - 4n - 5, \\ M(\mathcal{G}_{23}) &= (2n+2)p - 2n - 9. \end{aligned}$$

In particular,

$$M(h_{1,0} b_{1,n-1} h_{1,n} a_2^s h_{2,0} h_{1,1}) = p + 5s + 6.$$

Proof (1) Note that $h_{1,i}$, $b_{1,i}$ and $a_2^s h_{2,0} h_{1,1} \in E_1^{*,*,*}$ are all permanent cocycles in the MSS and converge nontrivially to h_i, b_i, $\widetilde{\beta}_{s+2} \in \mathrm{Ext}_A^{*,*}(\mathbb{Z}_p, \mathbb{Z}_p)$ for $0 \leq s < p$ and $i \geq 0$, respectively (cf Theorem 3.2). Then

$$h_{1,0} b_{1,n-1} h_{1,n} a_2^s h_{2,0} h_{1,1} \in E_1^{s+6,t(s,n),*}$$

is a permanent cocycle in the MSS and converges to

$$h_0 b_{n-1} h_n \widetilde{\beta}_{s+2} \in \mathrm{Ext}_A^{s+6,t(s,n)}(\mathbb{Z}_p, \mathbb{Z}_p).$$

(2) From (2-6), we have that

$$\begin{aligned} M(\mathcal{G}_1) &= M(a_n^{p-5} a_1 h_{n,0} h_{3,0} h_{n-1,1} h_{n-1,2} h_{n-2,2}) \\ &= M(a_n^{p-5}) + M(a_1) + M(h_{n,0}) + M(h_{3,0}) + M(h_{n-1,1}) \\ &\qquad + M(h_{n-1,2}) + M(h_{n-2,2}) \\ &= (p-5)(2n+1) + 3 + 2n - 1 + 5 + 2(n-1) - 1 \\ &\qquad + 2(n-1) - 1 + 2(n-2) - 1 \\ &= (2n+1)p - 2n - 9. \end{aligned}$$

The other results can be obtained similarly. □

By Proposition 3.1 and Lemma 3.3, we have the following:

Corollary 3.4 *For the May E_1–module G in Proposition 3.1,*

$$G = E_1^{p+1,t(p-4,n),(2n+1)p-2n-9} \oplus E_1^{p+1,t(p-4,n),(4n-4)p-4n-3} \oplus E_1^{p+1,t(p-4,n),(4n-2)p-4n-5} \oplus E_1^{p+1,t(p-4,n),(2n+2)p-2n-9},$$

where
$$\begin{aligned} E_1^{p+1,t(p-4,n),(2n+1)p-2n-9} &= \mathbb{Z}_p\{\mathcal{G}_i \mid 1 \le i \le 16\},\\ E_1^{p+1,t(p-4,n),(4n-4)p-4n-3} &= \mathbb{Z}_p\{\mathcal{G}_j \mid j = 17, 18, 19, 20, 22\},\\ E_1^{p+1,t(p-4,n),(4n-2)p-4n-5} &= \mathbb{Z}_p\{\mathcal{G}_{21}\},\\ E_1^{p+1,t(p-4,n),(2n+2)p-2n-9} &= \mathbb{Z}_p\{\mathcal{G}_{23}\}. \end{aligned}$$

To show the nontriviality of the product $h_0 b_{n-1} h_n \widetilde{\beta}_{s+2}$, we need to show the following four lemmas.

Lemma 3.5 *The May E_r–module $E_r^{p+1,t(p-4,n),(2n+2)p-2n-9} = 0$ for $r \ge 2$.*

Proof From Corollary 3.4, we have

$$E_1^{p+1,t(p-4,n),(2n+2)p-2n-9} = \mathbb{Z}_p\{\mathcal{G}_{23}\}.$$

By (2-3), (2-4) and (2-5), one has the first May differential of $\mathcal{G}_{23}$ as follows:

$$\begin{aligned} d_1(\mathcal{G}_{23}) &= d_1(a_n^{p-4} h_{n+1,0} h_{n,0} h_{n-2,2} b_{1,1})\\ &= (-1)^{p-2} a_n^{p-4} h_{n+1,0} h_{n,0} d_1(h_{n-2,2} b_{1,1}) + \cdots\\ &= -a_n^{p-4} h_{n+1,0} h_{n,0} h_{1,n-1} h_{n-3,2} b_{1,1} + \cdots\\ &= a_n^{p-4} h_{n+1,0} h_{n,0} h_{n-3,2} h_{1,n-1} b_{1,1} + \cdots\\ &\ne 0, \end{aligned}$$

showing that

$$E_2^{p+1,t(p-4,n),(2n+2)p-2n-9} = 0.$$

Thus it follows that

$$E_r^{p+1,t(p-4,n),(2n+2)p-2n-9} = 0$$

for $r \ge 2$. The proof of Lemma 3.5 is finished. □

Similarly, one has the following lemmas.

Lemma 3.6 *The May E_r–module $E_r^{p+1,t(p-4,n),(4n-2)p-4n-5} = 0$ for $r \ge 2$.*

Lemma 3.7 *The May E_r–module $E_r^{p+1,t(p-4,n),(4n-4)p-4n-3} = 0$ for $r \ge 2$.*

Proof From Corollary 3.4,

$$E_1^{p+1,t(p-4,n),(4n-4)p-4n-3} = \mathbb{Z}_p\{\mathcal{G}_j \mid j = 17, 18, 19, 20, 22\}.$$

By use of (2-3), (2-4) and (2-5), one has the first May differentials of the generators as follows:

$$\begin{aligned}
d_1(\mathcal{G}_{17}) &= d_1(a_n^{p-5}a_3h_{n+1,0}h_{n,0}h_{n-2,2}b_{n-2,1}) \\
&= \underline{a_n^{p-5}a_3h_{n+1,0}h_{n,0}h_{n-3,2}h_{1,n-1}b_{n-2,1}}_{17} + \cdots \\
&\neq 0, \\
d_1(\mathcal{G}_{18}) &= d_1(a_{n+1}a_n^{p-5}h_{n,0}h_{3,0}h_{n-2,2}b_{n-2,1}) \\
&= \underline{a_{n+1}a_n^{p-5}h_{n,0}h_{3,0}h_{n-3,2}h_{1,n-1}b_{n-2,1}}_{18} + \cdots \\
&\neq 0, \\
d_1(\mathcal{G}_{19}) &= d_1(a_n^{p-4}h_{n+1,0}h_{3,0}h_{n-2,2}b_{n-2,1}) \\
&= \underline{a_n^{p-4}h_{n+1,0}h_{3,0}h_{n-3,2}h_{1,n-1}b_{n-2,1}}_{19} + \cdots \\
&\neq 0, \\
d_1(\mathcal{G}_{20}) &= d_1(a_n^{p-4}h_{n,0}h_{3,0}h_{n-1,2}b_{n-2,1}) \\
&= \underline{a_n^{p-4}h_{n,0}h_{3,0}h_{2,2}h_{n-3,4}b_{n-2,1}}_{20} + \cdots \\
&\neq 0, \\
d_1(\mathcal{G}_{22}) &= d_1(a_n^{p-4}h_{n+1,0}h_{n,0}h_{1,2}b_{n-2,1}) \\
&= \underline{a_n^{p-4}h_{n+1,0}h_{3,0}h_{1,2}h_{n-3,3}b_{n-2,1}}_{22} + \cdots \\
&\neq 0.
\end{aligned}$$

One can check that the first May differential of each of the five generators contains at least a term which is not in the first May differential of the other generators. For example, $\underline{\quad}_{17}$ appears only in $d_1(\mathcal{G}_{17})$ and does not appear in $d_1(\mathcal{G}_j)$ $(j = 18,\ 19,\ 20,\ 22)$. Thus, the five first May differentials above are linearly independent, showing

$$E_2^{p+1,t(p-4,n),(4n-4)p-4n-3} = 0.$$

Then it follows that

$$E_r^{p+1,t(p-4,n),(4n-4)p-4n-3} = 0$$

for $r \geq 2$. The proof of Lemma 3.7 is completed. □

Lemma 3.8 *The May E_r–module $E_r^{p+1,t(p-4,n),(2n+1)p-2n-9} = 0$ for $r \geq 2$.*

Proof From Corollary 3.4,

$$E_1^{p+1,t(p-4,n),(2n+1)p-2n-9} = \mathbb{Z}_p\{\mathcal{G}_i \mid 1 \le i \le 16\}.$$

Using (2-3), (2-4) and (2-5), the first May differentials of the generators are as follows:

$$\begin{aligned}
d_1(\mathcal{G}_1) &= d_1(a_n^{p-5}a_1h_{n,0}h_{3,0}h_{n-1,1}h_{n-1,2}h_{n-2,2}) \\
&= (-1)^p a_n^{p-5}a_1h_{n,0}h_{3,0}h_{n-1,1}h_{n-1,2}d_1(h_{n-2,2}) + \cdots \\
&= -a_n^{p-5}a_1h_{n,0}h_{3,0}h_{n-1,1}h_{n-1,2}h_{1,n-1}h_{n-3,2} + \cdots \\
&= \underline{a_n^{p-5}a_1h_{n,0}h_{3,0}h_{n-1,1}h_{n-1,2}h_{n-3,2}h_{1,n-1}}_{\,1} + \cdots \\
&\neq 0, \\
d_1(\mathcal{G}_2) &= (-1)^{p-1}a_n^{p-5}a_1h_{n+1,0}h_{n,0}h_{n-1,1}d_1(h_{n-2,2}h_{1,2}) + \cdots \\
&= \underline{a_n^{p-5}a_1h_{n+1,0}h_{n,0}h_{n-1,1}h_{n-3,2}h_{1,2}h_{1,n-1}}_{\,2} + \cdots \\
&\neq 0, \\
d_1(\mathcal{G}_3) &= (-1)^p a_n^{p-5}a_3h_{n,0}h_{1,0}h_{n-1,1}h_{n-1,2}d_1(h_{n-2,2}) + \cdots \\
&= \underline{a_n^{p-5}a_3h_{n,0}h_{1,0}h_{n-1,1}h_{n-1,2}h_{n-3,2}h_{1,n-1}}_{\,3} + \cdots \\
&\neq 0, \\
d_1(\mathcal{G}_4) &= (-1)^p a_n^{p-4}h_{3,0}h_{1,0}h_{n-1,1}h_{n-1,2}d_1(h_{n-2,2}) + \cdots \\
&= \underline{a_n^{p-4}h_{3,0}h_{1,0}h_{n-1,1}h_{n-1,2}h_{n-3,2}h_{1,n-1}}_{\,4} + \cdots \\
&\neq 0, \\
d_1(\mathcal{G}_5) &= (-1)^p a_n^{p-4}h_{n,0}h_{1,0}h_{2,1}h_{n-1,2}d_1(h_{n-2,2}) + \cdots \\
&= \underline{a_n^{p-4}h_{n,0}h_{1,0}h_{2,1}h_{n-1,2}h_{n-3,2}h_{1,n-1}}_{\,5} + \cdots \\
&\neq 0, \\
d_1(\mathcal{G}_6) &= (-1)^{p-1}a_{n+1}a_n^{p-5}h_{n,0}h_{1,0}h_{n-1,1}d_1(h_{n-2,2}h_{1,2}) + \cdots \\
&= \underline{a_{n+1}a_n^{p-5}h_{n,0}h_{1,0}h_{n-1,1}h_{n-3,2}h_{1,2}h_{1,n-1}}_{\,6} + \cdots \\
&\neq 0, \\
d_1(\mathcal{G}_7) &= (-1)^{p-1}a_n^{p-4}h_{n+1,0}h_{1,0}h_{n-1,1}d_1(h_{n-2,2}h_{1,2}) + \cdots \\
&= \underline{a_n^{p-4}h_{n+1,0}h_{1,0}h_{n-1,1}h_{n-3,2}h_{1,2}h_{1,n-1}}_{\,7} + \cdots \\
&\neq 0, \\
d_1(\mathcal{G}_8) &= (-1)^{p-1}a_n^{p-4}h_{n,0}h_{1,0}h_{n,1}d_1(h_{n-2,2}h_{1,2}) + \cdots \\
&= \underline{a_n^{p-4}h_{n,0}h_{1,0}h_{n,1}h_{n-3,2}h_{1,2}h_{1,n-1}}_{\,8} + \cdots \\
&\neq 0,
\end{aligned}$$

$$
\begin{aligned}
d_1(\mathcal{G}_9) &= (-1)^{p-1} a_n^{p-4} h_{n,0} h_{1,0} h_{n-1,1} d_1(h_{n-1,2} h_{1,2}) + \cdots \\
&= \underline{a_n^{p-4} h_{n,0} h_{1,0} h_{n-1,1} h_{2,2} h_{1,2} h_{n-3,4}}_{\,9} + \cdots \\
&\neq 0, \\
d_1(\mathcal{G}_{10}) &= (-1)^{p-1} a_n^{p-4} h_{n,0} h_{2,0} h_{n-1,2} d_1(h_{n-2,2} h_{1,2}) + \cdots \\
&= \underline{a_n^{p-4} h_{n,0} h_{2,0} h_{n-1,2} h_{n-3,2} h_{1,2} h_{1,n-1}}_{\,10} + \cdots \\
&\neq 0, \\
d_1(\mathcal{G}_{11}) &= (-1)^{p} a_n^{p-5} a_3 h_{n,0} h_{3,0} h_{n-1,2} h_{n-2,2} d_1(h_{n-3,3}) + \cdots \\
&= \underline{a_n^{p-5} a_3 h_{n,0} h_{3,0} h_{n-1,2} h_{n-2,2} h_{n-4,3} h_{1,n-1}}_{\,11} + \cdots \\
&\neq 0, \\
d_1(\mathcal{G}_{12}) &= (-1)^{p} a_n^{p-5} a_3 h_{n+1,0} h_{n,0} h_{n-2,2} h_{1,2} d_1(h_{n-3,3}) + \cdots \\
&= \underline{a_n^{p-5} a_3 h_{n+1,0} h_{n,0} h_{n-2,2} h_{1,2} h_{n-4,3} h_{1,n-1}}_{\,12} + \cdots \\
&\neq 0, \\
d_1(\mathcal{G}_{13}) &= (-1)^{p} a_{n+1} a_n^{p-5} h_{n,0} h_{3,0} h_{n-2,2} h_{1,2} d_1(h_{n-3,3}) + \cdots \\
&= \underline{a_{n+1} a_n^{p-5} h_{n,0} h_{3,0} h_{n-2,2} h_{1,2} h_{n-4,3} h_{1,n-1}}_{\,13} + \cdots \\
&\neq 0, \\
d_1(\mathcal{G}_{14}) &= (-1)^{p} a_n^{p-4} h_{n+1,0} h_{3,0} h_{n-2,2} h_{1,2} d_1(h_{n-3,3}) + \cdots \\
&= \underline{a_n^{p-4} h_{n+1,0} h_{3,0} h_{n-2,2} h_{1,2} h_{n-4,3} h_{1,n-1}}_{\,14} + \cdots \\
&\neq 0, \\
d_1(\mathcal{G}_{15}) &= (-1)^{p} a_n^{p-4} h_{n,0} h_{3,0} h_{n-1,2} h_{1,2} d_1(h_{n-3,3}) + \cdots \\
&= \underline{a_n^{p-4} h_{n,0} h_{3,0} h_{n-1,2} h_{1,2} h_{n-4,3} h_{1,n-1}}_{\,15} + \cdots \\
&\neq 0, \\
d_1(\mathcal{G}_{16}) &= (-1)^{p} a_n^{p-4} h_{n,0} h_{3,0} h_{n-2,2} h_{1,2} d_1(h_{n-2,3}) + \cdots \\
&= \underline{a_n^{p-4} h_{n,0} h_{3,0} h_{n-2,2} h_{1,2} h_{1,3} h_{n-3,4}}_{\,16} + \cdots \\
&\neq 0.
\end{aligned}
$$

One can easily check that the first May differential of each of the sixteen generators contains at least a term which is not in the first May differential of the other generators. That is, $\underline{\quad}_k$ appears only in $d_1(\mathcal{G}_k)$ and does not appear in $d_1(\mathcal{G}_j)$ $(j \neq k)$. Thus, the sixteen May differentials above are linearly independent, showing that

$$E_2^{p+1, t(p-4,n), (2n+1)p-2n-9} = 0.$$

Then it follows that
$$E_r^{p+1,t(p-4,n),(2n+1)p-2n-9}=0$$
for $r\geq 2$. The proof of this lemma is completed. □

By use of Lemmas 3.5, 3.6, 3.7 and 3.8, we can prove the nontriviality of the product $h_0b_{n-1}h_n\widetilde{\beta}_{s+2}$ as follows:

Theorem 3.9 *Let $p\geq 5$, $n>4$, $0\leq s<p-3$. Then the product*
$$h_0b_{n-1}h_n\widetilde{\beta}_{s+2}\neq 0\in \mathrm{Ext}_A^{s+6,t(s,n)}(\mathbb{Z}_p,\mathbb{Z}_p).$$
Here $t(s,n)=2p^nq+(s+2)pq+(s+2)q+s$.

Proof From Lemma 3.3 (1), the product
$$h_0b_{n-1}h_n\widetilde{\beta}_{s+2}\in \mathrm{Ext}_A^{s+6,t(s,n)}(\mathbb{Z}_p,\mathbb{Z}_p)$$
is represented in the MSS by
$$h_{1,0}b_{1,n-1}h_{1,n}a_2^sh_{2,0}h_{1,1}\in E_1^{s+6,t(s,n),*}.$$
Now we show that nothing hits the permanent cocycle $h_{1,0}b_{1,n-1}h_{1,n}a_2^sh_{2,0}h_{1,1}$ under the May differential d_r for $r\geq 1$.

We divide the proof into the following two cases:

Case 1 When $0\leq s<p-4$, from Proposition 3.1 we know that in the May spectral sequence
$$E_1^{s+5,t(s,n),*}=0.$$
Then we have
$$E_r^{s+5,t(s,n),*}=0$$
for $r\geq 1$. From (2-2), one has that the permanent cocycle $h_{1,0}b_{1,n-1}h_{1,n}a_2^sh_{2,0}h_{1,1}\in E_1^{s+6,t(s,n),*}$ does not bound and converges nontrivially to $h_0b_{n-1}h_n\widetilde{\beta}_{s+2}$ in the May spectral sequence, then we have
$$h_0b_{n-1}h_n\widetilde{\beta}_{s+2}\neq 0\in \mathrm{Ext}_A^{s+6,t(s,n)}(\mathbb{Z}_p,\mathbb{Z}_p).$$

Case 2 When $s=p-4$, from Proposition 3.1 and Corollary 3.4, we have
$$\begin{aligned}E_1^{p+1,t(p-4,n),*}=E_1^{p+1,t(p-4,n),(2n+1)p-2n-9}&\oplus E_1^{p+1,t(p-4,n),(4n-4)p-4n-3}\\&\oplus E_1^{p+1,t(p-4,n),(4n-2)p-4n-5}\oplus E_1^{p+1,t(p-4,n),(2n+2)p-2n-9}.\end{aligned}$$
Note that in this case, by Corollary 3.4,
$$M(h_{1,0}b_{1,n-1}h_{1,n}a_2^{p-4}h_{2,0}h_{1,1})=6p-14.$$

By direct computations, we have that

$$\begin{aligned}
M(E_1^{p+1,t(p-4,n),(2n+1)p-2n-9})-(6p-13) &= (2n+1)p-2n-9-(6p-13)\\
&= (2n-5)(p-1)-1,\\
M(E_1^{p+1,t(p-4,n),(4n-4)p-4n-3})-(6p-13) &= (4n-4)p-4n-3-(6p-13)\\
&= (4n-10)(p-1),\\
M(E_1^{p+1,t(p-4,n),(4n-2)p-4n-5})-(6p-13) &= (4n-2)p-4n-5-(6p-13)\\
&= (4n-8)(p-1),\\
M(E_1^{p+1,t(p-4,n),(2n+2)p-2n-9})-(6p-13) &= (2n+2)p-2n-9-(6p-13)\\
&= (2n-4)(p-1).
\end{aligned}$$

Using $n>4$ and $p\geq 5$, we have that

$$\begin{aligned}
(2n-5)(p-1)-1&>0,\\
(4n-10)(p-1)&>0,\\
(4n-8)(p-1)&>0,\\
(2n-4)(p-1)&>0.
\end{aligned}$$

From (2-2), we see that the first May differential is given by

$$d_1\colon E_1^{s,t,u}\to E_r^{s+1,t,u-1}.$$

Thus by the reason of May filtration, we have

$$\begin{aligned}
h_{1,0}b_{1,n-1}h_{1,n}a_2^{p-4}h_{2,0}h_{1,1} &\notin d_1(E_1^{p+1,t(p-4,n),(2n+1)p-2n-9}),\\
h_{1,0}b_{1,n-1}h_{1,n}a_2^{p-4}h_{2,0}h_{1,1} &\notin d_1(E_1^{p+1,t(p-4,n),(4n-4)p-4n-3}),\\
h_{1,0}b_{1,n-1}h_{1,n}a_2^{p-4}h_{2,0}h_{1,1} &\notin d_1(E_1^{p+1,t(p-4,n),(4n-2)p-4n-5}),\\
h_{1,0}b_{1,n-1}h_{1,n}a_2^{p-4}h_{2,0}h_{1,1} &\notin d_1(E_1^{p+1,t(p-4,n),(2n+2)p-2n-9}).
\end{aligned}$$

Moreover, using Lemmas 3.5, 3.6, 3.7 and 3.8, one has

$$\begin{aligned}
E_r^{p+1,t(p-4,n),(2n+1)p-2n-9} &= 0\ (r\geq 2),\\
E_r^{p+1,t(p-4,n),(4n-4)p-4n-3} &= 0\ (r\geq 2),\\
E_1^{p+1,t(p-4,n),(4n-2)p-4n-5} &= 0\ (r\geq 2),\\
E_r^{p+1,t(p-4,n),(2n+2)p-2n-9} &= 0\ (r\geq 2).
\end{aligned}$$

From the discussion above, the permanent cocycle $h_{1,0}b_{1,n-1}h_{1,n}a_2^{p-4}h_{2,0}h_{1,1}$ cannot be hit by any differential in the MSS. Thus, $h_{1,0}b_{1,n-1}h_{1,n}a_2^{p-4}h_{2,0}h_{1,1}$ converges nontrivially to $h_0b_{n-1}h_n\widetilde{\beta}_{p-2} \in \mathrm{Ext}_A^{p+2,t(p-4,n)}(\mathbb{Z}_p,\mathbb{Z}_p)$ in the MSS. Consequently, we have

$$h_0b_{n-1}h_n\widetilde{\beta}_{p-2} \neq 0 \in \mathrm{Ext}_A^{p+2,t(p-4,n)}(\mathbb{Z}_p,\mathbb{Z}_p).$$

From Cases 1 and 2, the desired result follows. □

Theorem 3.10 *Let $p \geq 5$, $n > 4$, $0 \leq s < p-3$, $2 \leq r \leq s+6$. Then*

$$\mathrm{Ext}_A^{s+6-r,t(s,n)+1-r}(\mathbb{Z}_p,\mathbb{Z}_p) = 0,$$

where $t(s,n) = 2p^nq + (s+2)pq + (s+2)q + s$.

Proof From Proposition 3.1, in this case

$$E_1^{s+6-r,t(s,n)+1-r,*} = 0.$$

By the MSS, the desired result follows. □

4 Proof of Theorem 1.7

We are now in a position to prove Theorem 1.7. The method for proving the main theorem is by the classical ASS. To prove Theorem 1.7 is equivalent to proving the following theorem. The proof of Theorem 4.1 depends on Theorems 3.9 and 3.10 which are obtained at the end of Section 3.

Theorem 4.1 *Let $p \geq 5$, $n > 4$, $0 \leq s < p-3$. Then the product*

$$h_0b_{n-1}h_n\widetilde{\beta}_{s+2} \in \mathrm{Ext}_A^{s+6,t(s,n)}(\mathbb{Z}_p,\mathbb{Z}_p)$$

is a permanent cycle in the ASS, and converges nontrivially to the composite map

$$\xi_{s+2,n}\zeta_{n-1} \in \pi_{t(s,n)-s-6}(S)$$

of order p, where $t(s,n) = 2p^nq + (s+2)pq + (s+2)q + s$.

Proof From Theorem 1.4, the ζ–element ζ_{n-1} is represented by

$$h_0b_{n-1} \in \mathrm{Ext}_A^{3,p^nq+q}(\mathbb{Z}_p,\mathbb{Z}_p)$$

in the ASS. Meanwhile, from Theorem 1.6 we know that

$$(\beta i_1 i_0)_*(h_n) \in \mathrm{Ext}_A^{2,p^nq+(p+1)q+1}(H^*V(1),\mathbb{Z}_p)$$

is a permanent cycle in the ASS and converges to a nontrivial element

$$\varpi_n \in \pi_{p^nq+(p+1)q-1}(V(1))$$

of order p. Then we consider the composite map

$$\xi_{s+2,n}\zeta_{n-1} = \beta^*_{s+1}\varpi_n\zeta_{n-1}.$$

Since ϖ_n is represented by $(\beta i_1 i_0)_*(h_n) \in \mathrm{Ext}_A^{2,p^nq+(p+1)q+1}(H^*V(1),\mathbb{Z}_p)$ in the ASS, then the map

$$\xi_{s+2,n}\zeta_{n-1}$$

is represented by

$$(j_0 j_1 \beta^{s+2} i_1 i_0)_*(h_0 b_{n-1} h_n)$$

in the ASS.

From Theorem 3.2 and the knowledge of Yoneda products we know that the composite

$$(j_0 j_1 \beta^{s+2} i_1 i_0)_*\colon \mathrm{Ext}_A^{0,*}(\mathbb{Z}_p,\mathbb{Z}_p) \xrightarrow{(i_1 i_0)_*} \mathrm{Ext}_A^{0,*}(H^*V(1),\mathbb{Z}_p) \xrightarrow{(j_0 j_1 \beta^{s+2})_*} \mathrm{Ext}_A^{s+2,*+(s+2)pq+(s+1)q+s}(\mathbb{Z}_p,\mathbb{Z}_p)$$

is a multiplication up to nonzero scalar by $\widetilde{\beta}_{s+2} \in \mathrm{Ext}_A^{s+2,q[(s+2)p+(s+1)]+s}(\mathbb{Z}_p,\mathbb{Z}_p)$. It follows that the composite map $\xi_{s+2,n}\zeta_{n-1}$ is represented by

$$h_0 b_{n-1} h_n \widetilde{\beta}_{s+2} \in \mathrm{Ext}_A^{s+6,t(s,n)}(\mathbb{Z}_p,\mathbb{Z}_p)$$

in the ASS.

By Theorem 3.9,

$$h_0 b_{n-1} h_n \widetilde{\beta}_{s+2} \in \mathrm{Ext}_A^{s+6,t(s,n)}(\mathbb{Z}_p,\mathbb{Z}_p)$$

is nontrivial. Recall that the Adams differential is given by

$$\widetilde{d}_r : E_r^{s,t} \to E_r^{s+r,t+r-1}.$$

Then from Theorem 3.10, we see that $h_0 b_{n-1} h_n \widetilde{\beta}_{s+2}$ can not be hit by any differential in the ASS. Consequently, the corresponding homotopy element

$$\zeta_{n-1}\xi_{s+2,n}$$

is nontrivial and of order p. The proof of Theorem 4.1 is completed. □

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