Multiple genus 2 Heegaard splittings: a missed case

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A gap in a paper by Rubinstein and the second author [5] is explored: new examples are found of closed orientable 3–manifolds with possibly multiple genus 2 Heegaard splittings. Properties common to all the examples in that paper are not universally shared by the new examples: some of the new examples have Hempel distance 3, and it is not clear that a single stabilization always makes the multiple splittings isotopic.

1 Introduction

In 1998, Rubinstein and the second author [5] studied the question of when there could be more than one distinct genus 2 Heegaard splitting of the same closed orientable 3–manifold. The goal of the project was modest: to provide a complete list of ways in which such multiple splittings could be constructed, but with no claim that each example on the list did in fact have multiple non-isotopic splittings (there could be isotopies from one splitting to another that are not apparent). Nor was there a claim that the list had no redundancies; a 3–manifold and its multiple splittings might appear more than once on the list. Such a list would still be useful, for if every example on the list could be shown to have a certain property, then that property would be true for any closed orientable 3–manifold M that has multiple genus 2 splittings. Two examples were given in [5]:

- If M is atoroidal then the hyperelliptic involutions determined by the two genus 2 Heegaard splittings commute.

- Any two genus 2 Heegaard splittings of M become isotopic after a single stabilization.

Despite this modest goal, the argument in [5] contains a gap[1] . In 2008, the first author discovered a class of examples that do not appear on the list and which, moreover, have

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[1]The error is on page 533: The last sentence of the first paragraph of Case 2 should have read, "The *same curves* cannot then be twisted in X since M is hyperbolike." This leaves open an additional possibility for P_X, P_Y, see the paper by the second author [7].

mathematical properties that distinguish them in important ways from the examples that do appear in [5]. It is true that, even for the new examples, the hyperelliptic involutions commute. But we know of no argument showing that the new examples all share the second property above; that is, we cannot show that the newly discovered multiple splittings necessarily become isotopic after a single stabilization (though they do after two stabilizations).

A third property, shared by all examples in [5] but not by some of the new examples, is not listed above because the notion of Hempel distance of Heegaard splittings (see Hempel [3]) did not exist at the time [5] was written. But a retrospective look (see Section 6 below) will verify that all the splittings described in [5] have Hempel distance no greater than 2, whereas results of the first author [2] illustrate that at least some of the new examples have Hempel distance 3. (This also verifies that the gap in the argument in [5] actually led to missed examples.)

Here is an outline: In Section 2 we describe a general method for constructing closed orientable 3–manifolds that appear to have multiple genus 2 Heegaard splittings; these examples (called *Dehn-derived*) are based around Dehn surgery on a pair of strategically placed curves. It is shown in [7] that these examples do fill the gap in [5]. It follows from the construction that the hyperelliptic involutions of the alternate splittings always coincide.

It is not immediately obvious that curves supporting Dehn-derived examples can be found, but in Sections 3 and 4 we give three specific classes of examples. The classes are denoted M_H (Section 3), $M_{\times I}$ and M_{hybrid} (Section 4). (M_H can be viewed as a third variation of [5, Example 4.2].) For the examples M_H and M_{hybrid} a single stabilization suffices to make the alternate splittings equivalent, but this property is at least not apparent in most cases of $M_{\times I}$.

In Section 5 it is shown, using new results of the first author [1], that *any* Dehn-derived example is in fact of type M_H, $M_{\times I}$ or M_{hybrid}. Finally, in Section 6 we verify that all of the old examples that are listed in [5] are of Hempel distance 2, whereas at least some Dehn-derived examples are of distance 3. (It is easy to see that all Dehn-derived examples are of distance no more than 3.)

2 Dehn derived multiple splittings

A *primitive k–tuple* of curves in the boundary of a genus g handlebody H is a collection $\lambda_1, \ldots, \lambda_k \subset \partial H$ of $k \leq g$ disjoint simple closed curves so that, for some

properly embedded collection $D_1, \ldots, D_k$ of disks in H, $|\lambda_i \cap D_j| = \delta_{ij}$, $1 \leq i, j \leq k$. It is easy to see that the closed complement in H of such a collection of meridian disks is a genus $g - k$ handlebody. In particular, if $k = g$ then $\lambda_1, \ldots, \lambda_g$ is called a *complete set of primitive curves* and the corresponding collection of disks $D_1, \ldots, D_g$ is called a *complete set of meridian disks*. The closed complement of a complete set of meridian disks in H is a 3–ball.

Suppose $\Lambda = \lambda_1, \ldots, \lambda_k \subset \partial H$ is a primitive k–tuple of curves in H and let $\alpha_1, \ldots, \alpha_k$ be the properly embedded collection of curves in H obtained by pushing Λ slightly into the interior of H. We can view H as the boundary connect sum of a genus $g - k$ handlebody H' and k solid tori $W_1, \ldots, W_k$, with λ_i a longitude of W_i and so α_i a core curve of W_i. Then Dehn surgery on $\alpha_i \subset W_i$ still gives a solid torus. Hence any Dehn surgery on the family of curves $\alpha_1, \ldots, \alpha_k$ leaves H still a handlebody.

Definition 2.1 *Suppose $M_0 = H_a \cup_S H_b$ is a Heegaard splitting of a closed 3–manifold M_0. A simple closed curve $\lambda \subset S$ is doubly primitive if λ is a primitive curve in both handlebodies H_a and H_b.*

Suppose M_0 is a closed orientable 3–manifold and that $M_0 = H_a \cup_S H_b$ is a genus 2 Heegaard splitting of M_0. Suppose further that $\lambda_1, \lambda_2 \subset S$ are two disjoint doubly primitive curves in S.

Proposition 2.2 *Suppose M is a manifold obtained by some specified Dehn surgeries on λ_1 and λ_2. For $i = 1, 2$, let A_i (resp. B_i) be the manifold obtained from the handlebody H_a (resp. H_b) by pushing the curve λ_i into $\mathrm{int}(H_a)$ (resp. $\mathrm{int}(H_b)$) and performing the specified Dehn surgery on the curve.*

Then $A_1 \cup_S B_2$ and $A_2 \cup_S B_1$ are two (possibly different) genus 2 Heegaard splittings of M.

Proof A_i (resp. B_i) is obtained from H_a (resp. H_b) by Dehn surgery on a pushed in copy α_i of a single primitive curve in S. It was just observed that this makes each A_i (resp. B_i) a handlebody.   □

Definition 2.3 *Two genus 2 Heegaard splittings $X \cup_Q Y$ and $A \cup_P B$ of a closed 3–manifold M are called Dehn derived (from the splitting $M_0 = H_a \cup_S H_b$ via $\lambda_1 \cup \lambda_2 \subset S$) if the two splittings are created as in Proposition 2.2.*

Corollary 2.4 *Suppose $M = A \cup_P B = X \cup_Q Y$ are a Dehn-derived pair of Heegaard splittings. Then the two hyperelliptic involutions of M, one determined by the Heegaard splitting $A \cup_P B$ and the other by the Heegaard splitting $X \cup_Q Y$, coincide.*

Proof Let $M_0 = H_a \cup_S H_b$ be the Heegaard split 3–manifold from which the two splittings of M are Dehn derived, via $\lambda_1 \cup \lambda_2 \subset S$. The hyperelliptic involution preserves the isotopy class (though perhaps reversing the orientation) of any simple closed curve in S. We may then position λ_i so that the curves are preserved (perhaps reversing orientation) by the hyperelliptic involution on $M_0 = H_a \cup_S H_b$. Then the hyperelliptic involution on M_0 naturally induces a single hyperelliptic involution on M. □

3 A simple set of examples

It is not immediately obvious how to create examples of a Dehn-derived pair of splittings or, very naively, whether examples even exist. In this section we present and briefly discuss an important concrete class of examples.

Consider a genus 2 handlebody H, constructed from two 0–handles by connecting them with three 1–handles. With this structure H has a natural $\mathbb{Z}_3$ symmetry, shown as $\frac{2\pi}{3}$ rotation about the green axis in Figure 1. Let $\lambda_1 \subset \partial H$ be the red curve shown in the figure and λ_2, λ_3 be the other two simple closed curves to which λ_1 is carried by the $\mathbb{Z}_3$ symmetry. Then each λ_i is a primitive curve on ∂H and, indeed, any two of the curves, say λ_1, λ_2 constitute a complete set of primitive curves (that is, a primitive pair). In this case the corresponding pair of meridian disks are the meridian disks of the two 1–handles through which λ_3 passes.

Let $\overline{H}$ be the genus 3 handlebody obtained by removing from H a neighborhood of the arc in which the axis of symmetry intersects one of the 0–handles. It is easy to see that in $\overline{H}$ the collection $\lambda_1, \lambda_2, \lambda_3 \subset \partial \overline{H}$ is a complete set of primitive curves, that is a primitive 3–tuple.

To construct some Dehn-derived pairs of Heegaard splittings, begin with two genus 2 handlebodies A and B, on each of whose boundaries lie three disjoint simple closed curves corresponding to $\lambda_1, \lambda_2, \lambda_3 \subset \partial H$. Let $\lambda_{ia} \subset \partial A$ (resp. $\lambda_{ib} \subset \partial B$) be the curve corresponding to λ_i in A (resp. B), for each $1 \le i \le 3$. Adopting (for comparison purposes) notation from [5, Section 4.2], let $\alpha_{na}, \alpha_{sa}, \rho_a \subset A$ be the triple of curves obtained by pushing $\lambda_{1a}, \lambda_{2a}, \lambda_{3a}$ into the interior of A and let $\alpha_{nb}, \alpha_{sb}, \rho_b \subset B$ be the triple of curves obtained by pushing $\lambda_{1b}, \lambda_{2b}, \lambda_{3b}$ into the interior of B. Let N be a manifold constructed by identifying an annular neighborhood of λ_{1a} in ∂A with an annular neighborhood of λ_{1b} in ∂B and an annular neighborhood of λ_{2a} in ∂A with an annular neighborhood of λ_{2b} in ∂B. (After the identification, call the annuli $\mathcal{A}_n$ and $\mathcal{A}_s$ with core curves λ_1, λ_2 respectively.) Then identify the two 4–punctured spheres $\partial A - (\mathcal{A}_n \cup \mathcal{A}_s)$ and $\partial B - (\mathcal{A}_n \cup \mathcal{A}_s)$ by any homeomorphism.
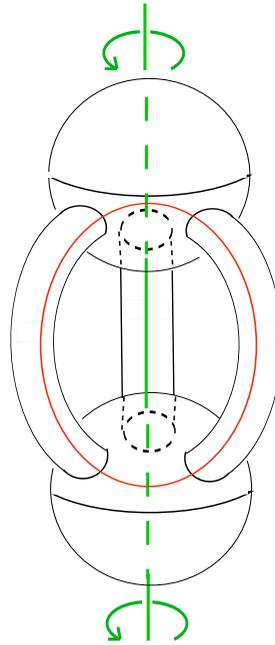
Figure 1

This construction defines a genus 2 Heegaard structure on N, of course, but it also defines a genus 2 Heegaard splitting on M_0, the manifold obtained from N by arbitrary Dehn surgery on just the two curves $\rho_a \subset A$ and $\rho_b \subset B$, for surgery on these pushed-in primitive curves leaves A and B still handlebodies, handlebodies which we denote respectively H_a and H_b. What's more, the curves λ_1, λ_2 are each primitive in both H_a and H_b (though they are not necessarily a primitive pair in either). Thus the Heegaard splitting $M_0 = H_a \cup H_b$ gives rise to two potentially different genus 2 Heegaard structures on any manifold M_H that is obtained by simultaneously doing further Dehn surgery on the two curves λ_1, λ_2. That is, a manifold M_H obtained by arbitrary Dehn surgery on all four curves $\lambda_1, \lambda_2, \rho_a, \rho_b \subset N$ has two possibly distinct genus 2 Heegaard splittings, Dehn derived from the Heegaard splitting $M_0 = H_a \cup H_b$. One Heegaard structure $M_H = A_1 \cup B_2$ is obtained by pushing λ_1 to $\alpha_{na} \subset \operatorname{int} A$ and λ_2 to $\alpha_{sb} \subset \operatorname{int} B$ before doing Dehn surgeries on the four curves; the other $M_H = A_2 \cup B_1$ is obtained by pushing λ_1 to $\alpha_{nb} \subset \operatorname{int} B$ and λ_2 to $\alpha_{sa} \subset \operatorname{int} A$ before doing the Dehn surgeries. In each case, exactly two of the four Dehn surgered curves lie in each handlebody A and B before the Dehn surgery, and in that handlebody are a pushed-in primitive pair.

Proposition 3.1 *The two Heegaard splittings $A_1 \cup B_2$ and $A_2 \cup B_1$ of M_H become isotopic after at most a single stabilization.*

Proof Let $\bar{A}$ and $\bar{B}$ be the genus 3 handlebodies derived from A and B respectively, just as $\bar{H}$ was derived from H. Here is a natural genus 3 Heegaard splitting of M_H: in contrast to the construction above, push *both* λ_1 to α_{na} and λ_2 to α_{sa}, so both curves (as well as ρ_a) lie in A before doing the Dehn surgeries. Although A may no longer be a handlebody after the Dehn surgeries, it follows from the discussion above that the result on $\bar{A}$ of the surgery on the three curves $\alpha_{na}, \alpha_{sa}, \rho_a \subset \bar{A} \subset A$ is still a genus three handlebody $\bar{A}'$. The complement of $\bar{A}'$ in M_H is also a handlebody B': a single 1–handle is added to B and surgery is done on the single curve $\rho_b \subset B$. Thus $M_H = \bar{A}' \cup B'$ is a genus 3 Heegaard splitting of M_H.

It's fairly easy to see that this Heegaard splitting is a stabilization of $A_1 \cup B_2$ (and so, symmetrically, $A_2 \cup B_1$). Indeed, an alternate way to construct $\bar{A}' \cup B'$ is to begin with $A_1 \cup B_2$ and add to A_1 (and so subtract from B_2) a regular neighborhood of the curve $\alpha_{sb} \subset \text{int}(B)$ and a straight arc from ∂B to α_{sb}. From this point of view, the inclusion $B' \subset B_2$ defines a genus 3 Heegaard splitting of the genus 2 handlebody B_2, and any such Heegaard splitting is necessarily stabilized (see the paper by the second author and Thompson [8, Lemma 2.7]). The pair of stabilizing disks are also a pair of stabilizing disks for $\bar{A}' \cup B'$ □

4 A second construction, and a hybrid

Here is another natural, but less naive, way to find disjoint pairs of primitive curves on the boundary of a genus 2 handlebody and so to create a Dehn-derived pair of Heegaard splittings. Let F denote a torus with the interior of a disk removed. Then $F \times I$ is a genus 2 handlebody. For γ any properly embedded essential simple closed curve in F, $\gamma \times \{0\}$ (or symmetrically $\gamma \times \{1\}$) is a primitive curve in the handlebody $F \times I$. Indeed, for δ a properly embedded arc in F intersecting γ once, $\delta \times I$ is a meridian disk in $F \times I$ that intersects $\gamma \times \{0\}$ exactly once.

Following this observation, and the example of the previous section, here is a recipe for constructing candidate 3–manifolds. Begin with two copies A and B of the surface F and choose two essential (not necessarily disjoint) simple closed curves $\alpha_0, \alpha_1 \subset A$ and two essential (not necessarily disjoint) simple closed curves $\beta_0, \beta_1 \subset B$. Let $\lambda_{0a} = \alpha_0 \times \{0\} \subset \partial(A \times I), \lambda_{1a} = \alpha_1 \times \{1\} \subset \partial(A \times I), \lambda_{0b} = \beta_0 \times \{0\} \subset \partial(B \times I), \lambda_{1b} = \beta_1 \times \{1\} \subset \partial(B \times I)$. Identify an annular neighborhood of λ_{0a} in $A \times \{0\}$ with an annular neighborhood of λ_{0b} in $B \times \{0\}$ and call the core curve of the resulting

annulus λ_0. Similarly identify an annular neighborhood of λ_{1a} in $A \times \{1\}$ with an annular neighborhood of λ_{1b} in $B \times \{1\}$ and call the core curve of the resulting annulus λ_1. Complete the identification of $\partial(A \times I)$ with $\partial(B \times I)$ along the remaining 4–punctured sphere arbitrarily. Call the resulting closed 3–manifold M_0, with Heegaard splitting $M_0 = (A \times I) \cup (B \times I)$.

The 3–manifold $M_{\times I}$ obtained from M_0 by doing arbitrary Dehn surgeries to the simple closed curves λ_0 and λ_1 has a Dehn-derived pair of Heegaard splittings: one comes from first pushing λ_0 into $A \times I$ and λ_1 into $B \times I$ before the Dehn surgery, the other comes from first pushing λ_1 into $A \times I$ and λ_0 into $B \times I$ before the Dehn surgery.

Remarks on stabilization It is not apparent to us that a single stabilization will make the two Dehn-derived splittings of $M_{\times I}$ equivalent. The argument of Proposition 3.1 does not immediately carry over: if both curves λ_{0a} and λ_{1a} are pushed into $A \times I$ there is no apparent arc so that the complement $\overline{A \times I}$ of a neighborhood of the arc in $A \times I$ is a genus 3 handlebody after an arbitrary Dehn surgery on the pushed in λ_{0a} and λ_{1a}. If there is a proper arc γ in A that intersects both curves $\alpha_0 \subset A$ and $\alpha_1 \subset A$ in a single point, then the complement $\overline{A \times I}$ after pushing the interior of γ into $A \times I$ is a genus 3 handlebody, and so a single stabilization suffices, but having such an arc γ is not the general situation. (What is required for such an arc γ to exist is that the slopes of α_0 α_1 in A are a distance at most two apart in the Farey graph (see Minsky [4, Figure 1]). In that case γ has the slope that is incident to the slopes of both α_0 and α_1 in the Farey graph.)

On the other hand, it is relatively easy to show that two stabilizations suffice to make the two splittings equivalent. To see this, push both λ_0 and λ_1 into $A \times I$ and connect them to respectively $A \times \{0\}$ and $A \times \{1\}$ by straight arcs. Then add a regular neighborhood of the arcs and of the pushed in curves λ_0 and λ_1 to $B \times I$ to create a genus 4 handlebody $\overline{B \times I}$ and simultaneously subtract the regular neighborhood from $A \times I$ to get the genus 4 handlebody $\overline{A \times I}$. The resulting genus 4 Heegaard splitting $M_0 = \overline{A \times I} \cup \overline{B \times I}$ becomes a Heegaard splitting $H_a^+ \cup H_b^+$ of $M_{\times I}$ after the prescribed Dehn surgery on λ_0 and λ_1. Using the argument of Proposition 3.1 it is easy to see that the Heegaard splitting $H_a^+ \cup H_b^+$ destablizes to the genus 3 splitting obtained by instead pushing λ_0 into $B \times I$ and then adding to $B \times I$ a regular neighborhood of $\lambda_1 \subset (A \times I)$ and a straight arc attaching it to $A \times \{1\}$. The argument of Proposition 3.1 applied again shows that this Heegaard splitting destabilizes to the genus 2 splitting in which λ_0 is pushed into $B \times I$ and λ_1 into $A \times I$, one of the Dehn-derived splittings. But this destabilization process is clearly symmetric: we could equally well have destabilized to the other genus 2 splitting, in which λ_0 is pushed into $A \times I$ and λ_1 into $B \times I$, and this is the other Dehn-derived splitting.

A further, call it a *hybrid* example of a Dehn-derived pair of splittings comes by combining the two constructions above: Identify annular neighborhoods of $\lambda_1, \lambda_2 \subset \partial H$ from Section 3 with annular neighborhoods of $\lambda_{0b}, \lambda_{1b} \subset \partial(B \times I)$ and identify the rest of ∂H with the rest of $\partial(B \times I)$ in any way. This gives a closed 3–manifold N with a Heegaard splitting $H \cup (B \times I)$. Let M_0 be a 3–manifold obtained by doing an arbitrary Dehn surgery on $\lambda_3 \subset \partial H$, after pushing it into $\text{int}(H)$. Then M_0 has the genus 2 Heegaard splitting (exploiting the notation used above) $M_0 = H_a \cup (B \times I)$. Let M_{hybrid} be a closed 3–manifold obtained from M_0 by arbitrary Dehn surgeries on the two remaining curves $\lambda_1, \lambda_2 \subset \partial H_a \subset M_0$. The Dehn-derived pair of Heegaard splittings for M_{hybrid} is obtained by alternatively pushing λ_1 into H_a and λ_2 into $B \times I$ or vice versa. A single stabilization suffices to make the two splittings equivalent, essentially by the same argument as for M_H, in Proposition 3.1.

5 A taxonomy of Dehn-derived splittings

Sections 3 and 4 give concrete examples of pairs of Dehn-derived fillings. In this section we show that these examples in fact constitute all pairs of Dehn-derived splittings. The argument exploits Berge's classification of pairs of primitive curves on genus 2 handlebodies [1], though the classification here is slightly different.

Let H be a genus 2 handlebody, with $\lambda_1, \lambda_2, \lambda_3 \subset \partial H$ the disjoint simple closed curves described in Section 3. Denote by ρ the curve in the interior of H obtained by pushing λ_3 into H and let H_{surg} denote the handlebody obtained from H by a specified Dehn surgery on $\rho \subset \text{int}(H)$. As in Section 4, let F denote a torus with the interior of a disk removed.

Proposition 5.1 (Berge) *Suppose α and β are disjoint non-parallel primitive curves on the boundary of a genus 2 handlebody H. Then either*

(A) *there is a Dehn surgery on $\rho \subset H$ and a homeomorphism $h \colon H \to H_{\text{surg}}$ so that $h(\alpha) = \lambda_1 \subset \partial H_{\text{surg}}$ and $h(\beta) = \lambda_2 \subset \partial H_{\text{surg}}$ or*

(B) *there is a homeomorphism $h \colon H \to F \times I$ so that $h(\alpha) \subset F \times \{0\}$ and $h(\beta) \subset F \times \{1\}$.*

Proof This classification is a variant of that described in [1]. The Type II pair there, as well as some pairs of Type I, are exactly as described in alternative (B). The interest is in the third example of a Type I pair, in [1, Lemma 3.8 (3) via Figure 3]. In that example, H is viewed as divided into two solid tori by a separating disk D; let λ_a and λ_b be longitudes of the two solid tori into which D divides H. Then β is parallel to

λ_b, and α is the band sum, via a band that crosses D once, of λ_b with a torus knot on the solid torus containing λ_a. This picture is equivalent to letting α be the band sum $\lambda_a \# \lambda_b$ (through D) of λ_b with λ_a, and then performing a Dehn surgery on a disjoint copy of λ_a that has been pushed into H, to become a core of the solid torus on which λ_a lies. Now translate: relabel $\lambda_b \subset \partial H$ as λ_2 and $\lambda_a \subset \partial H$ as λ_3. Then $\lambda_a \# \lambda_b$ corresponds to λ_1. The construction just described is then to push λ_3 into the interior of H and perform some surgery to get H_{surg}. Afterwards α corresponds to $\lambda_1 \subset \partial H_{\text{surg}}$ and β corresponds to $\lambda_2 \subset \partial H_{\text{surg}}$. This is exactly alternative (A). □

Following Proposition 5.1 there is a fairly clear description of the cases of multiple Heegaard splittings that are missing from [5]. According to [7] the only missing cases are pairs of splittings that are Dehn-derived from an initial splitting $H_{AX} \cup H_{BY}$ of a manifold M_0. First determine which of alternatives (A) and (B) apply to the pairs of surgery curves as they lie on the boundaries of the respective handlebodies: $\{a_1, c_2\} \subset H_{AX}$ or $\{a_2, c_1\} \subset H_{BY}$. If both are of type (A) then the pair of splittings is Dehn-derived as in the construction of M_H in Section 3. If both are of type (B) then the pair of splittings is Dehn-derived as in the construction of $M_{\times I}$ in Section 4. If one is of type (A) and one of type (B) then the pair of splittings is Dehn-derived as in the construction of M_{hybrid} in Section 4.

It is worth mentioning that there is another view of a pair of primitive curves lying on a handlebody as in (A) of Proposition 5.1, a view that more closely resembles that in (B): Let α, β, γ be simple closed curves in F so that each pair of curves intersects in exactly one point. (For example, choose curves in F of slopes $0, 1, \infty$.) Then it is fairly easy to see that the three curves $\alpha \times \{0\}$, $\beta \times \{1\}$, $\gamma \times \{\frac{1}{2}\}$ lie in the handlebody $F \times I$ just as $\lambda_1, \lambda_2, \rho$ lie in H in the description preceding Proposition 5.1. So the primitive curves in description (A) can be made to look like a special case of those in description (B), but with the cost that an extra Dehn surgery has to be performed on a specific curve in the interior of $F \times I$. This is the twisted product view of [1, 3.2].

6 Distance

It would seem possible that the Dehn-derived pairs of Heegaard splittings exhibited above could coincidentally all be contained among the examples already listed in [5], for there is no claim that the types of examples of multiple Heegaard splittings we have offered here and in [5] do not overlap. But in fact there is an invariant which does show that at least some Dehn-derived pairs of Heegaard splittings described above did not already occur in a different guise in [5]. This invariant had not yet been introduced when [5] was written and is called the *(Hempel) distance* of the Heegaard splitting (see Hempel [3]). We briefly review:

Definition 6.1 *A Heegaard splitting $H_1 \cup_S H_2$ has Hempel distance at most n if there is a sequence $c_0, \ldots, c_n$ of essential simple closed curves in the splitting surface S so that*

- *for each $i = 1, \ldots, n$, $c_i \cap c_{i-1} = \varnothing$*
- *c_0 bounds a disk in H_1*
- *c_n bounds a disk in H_2*

If the splitting has distance $\leq n$ but not $\leq n - 1$, then the distance $d(S) = n$.

A Heegaard splitting of distance 0 is called *reducible*; one of distance ≤ 1 is called *weakly reducible*. Any Heegaard splitting of a reducible manifold is reducible. A Heegaard splitting of distance ≤ 2 is said to have the *disjoint curve property* (see Thompson [9]); any Heegaard splitting of a toroidal 3–manifold has the disjoint curve property [3; 9]. A weakly reducible genus 2 Heegaard splitting is also reducible, so an irreducible Heegaard splitting of genus 2 has distance at least 2 [9].

In the other direction we have:

Proposition 6.2 *Suppose the manifold M has a Dehn-derived pair of Heegaard splittings. Then each of these Heegaard splittings has Hempel distance at most 3.*

Proof Suppose the splittings are Dehn-derived from a splitting $M_0 = H_a \cup_S H_b$ via the disjoint pair of simple closed curves $\lambda_1, \lambda_2 \subset S$. With no loss of generality, consider the splitting $M = A \cup_S B$ obtained by pushing λ_1 into $\text{int}(H_a)$ and λ_2 into $\text{int}(H_b)$ before doing Dehn surgery on the λ_i. Since λ_1 is primitive in H_a there is a properly embedded essential disk $D_a \subset H_a$ that is disjoint from λ_1. (For example D_a can be obtained from a meridian disk $D_1 \subset H_a$ that intersects λ_1 in a single point by band-summing together two copies of D_1 along a subarc of $\lambda_1 - D_1$.) D_a is then also disjoint from the curve $\alpha_1 \subset H_a$ obtained by pushing λ_1 into $\text{int}(H_a)$, so D_a remains intact as a meridian of A after surgery on α_1. Hence ∂D_a and λ_1 are disjoint curves in ∂A.

Symmetrically, there is a meridian $D_b \subset B$ so that ∂D_b and λ_2 are disjoint curves in ∂B. Then the sequence $\partial D_a, \lambda_1, \lambda_2, \partial D_b$ shows that the splitting $A \cup_S B$ has distance at most 3. $\square$

Proposition 6.3 *All examples of multiple Heegaard splittings appearing in [5, Section 4] have Hempel distance ≤ 2.*

Proof Following the comments above we can restrict attention to irreducible, atoroidal manifolds. We briefly run through the examples as they appear in [5, Section 4]. Typically the description of an example $H_1 \cup_S H_2$ in [5] consists of two parts: A collection of annuli $\mathcal{A} \subset S$ along which ∂H_1 and ∂H_2 are identified, followed by an arbitrary identification of $\partial H_1 - \mathcal{A}$ with $\partial H_2 - \mathcal{A}$. From this point of view the simple closed curves $\partial \mathcal{A} \subset S$ that separate one sort of region from the other will be called the *seams* of the Heegaard splitting. We will observe that in [5] some seam is always disjoint from an essential disk in H_1 and an essential disk in H_2. This demonstrates that the splitting has the disjoint curve property and so has distance ≤ 2.

To be specific: In [5, Subsection 4.1], [5, Subsection 4.2, Variation 1] and [5, Subsection 4.4, Variations 1 and 2], the meridians of the 1–handles e_a and e_b are disjoint from the seams. [5, Subsection 4.2, Variation 2] is slightly more complicated. It is a bit like the construction in Section 3 above: Handlebodies A and B are identified along neighborhoods of all three curves $\lambda_i, i = 1, 2, 3$, Dehn surgery is done to all three, with λ_1, λ_2 pushed into A and λ_3 into B (then vice versa). But there is a meridian of A disjoint from λ_1 and λ_2 and a meridian of B disjoint from λ_1 and λ_3, so a seam parallel to λ_1 demonstrates that the splitting of [5, Subsection 4.2, Variation 2] has the disjoint curve property.

The manifolds in [5, Subsection 4.3] and [5, Subsection 4.4, Variations 3, 4, and 7] are all toroidal, so they are of distance ≤ 2. What remains are [5, Subsection 4.4, Variations 5 and 6] and we adopt the terminology there. In Variation 5, with, say, $\rho_a \subset A_-$, the seams that are the boundary of the 4–punctured sphere $\partial A_- \cap \partial \Gamma$ are all disjoint from the meridian of the 1–handle $e_b \subset B$ and, in A_-, any one of these seams together with ρ_a lie in A_- as two of the λ_i's of Section 3 above lie in H. In particular, there is a meridian of A_- disjoint from both the seam and from ρ_a. Thus that seam again illustrates that the splitting has the disjoint curve property.

The argument for Variation 6 is much the same. First note that if, in that Variation, Dehn surgeries are done on two curves parallel to σ, then the resulting manifold has a Seifert piece and so has distance ≤ 2. So the only change we need to consider from Variation 5 is Dehn surgery on a single curve parallel to σ. If that curve lies in B the argument for Variation 5 suffices; if it is in A_- this merely forces us to pick a specific seam in the argument for Variation 5, a seam parallel to the new surgery curve. □

In contrast, some of the examples constructed in this paper can be shown to have distance 3, so they cannot have appeared in any case considered in [5]. See [2] (also [6]) for details.

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