Polynomial $6j$–symbols and states sums

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For a given $2r$–th root of unity ξ, we give explicit formulas of a family of 3–variable Laurent polynomials $\mathsf{J}_{i,j,k}$ with coefficients in $\mathbb{Z}[\xi]$ that encode the $6j$–symbols associated with nilpotent representations of $U_\xi(\mathfrak{sl}(2))$. For a given abelian group G, we use them to produce a state sum invariant $\tau^r(M,L,h_1,h_2)$ of a quadruplet (compact 3–manifold M, link L inside M, homology class $h_1 \in H_1(M,\mathbb{Z})$, homology class $h_2 \in H_2(M,G)$) with values in a ring R related to G. The formulas are established by a "skein" calculus as an application of the theory of modified dimensions introduced by the authors and Turaev in [4]. For an oriented 3–manifold M, the invariants are related to $\tau(M,L,\varphi \in H^1(M,\mathbb{C}^*))$ defined by the authors and Turaev in [3] from the category of nilpotent representations of $U_\xi(\mathfrak{sl}(2))$. They refine them as $\tau(M,L,\varphi)=\sum_{h_1}\tau^r(M,L,h_1,\widetilde{\varphi})$ where $\widetilde{\varphi}$ correspond to φ with the isomorphism $H_2(M,\mathbb{C}^*)\simeq H^1(M,\mathbb{C}^*)$.

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Introduction

The $6j$–symbols are tensors describing the associativity of the tensor product in a tensor category. Formulas exist for the classical and quantum $6j$–symbols associated to the defining representations of $\mathfrak{sl}(2)$ and its powers (see Kirillov and Reshetikhin [6] and Masbaum and Vogel [7]). At a root of unity ξ, new representations appear for the quantum group $U_\xi(\mathfrak{sl}(2))$. There are essentially two new families: the nilpotent and the cyclic representations. Unlike the cyclic family, the nilpotent representations can be enriched to form a ribbon category. For ξ a fourth root of unity, this was already observed by O Viro in [11] who used these representations to construct a ribbon graph invariant related to the multivariable Alexander polynomial.

The theory of modified dimensions developed with V Turaev by the authors in [4; 3] produces a family of modified $6j$–symbols that share properties similar with usual $6j$–symbol. Nevertheless this family has a very different nature than previously defined $6j$–symbols. Indeed, the whole family of nilpotent representation can be thought as a unique module with parameters. This module is then a nontrivial one parameter

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deformation of the so called Kashaev module. For this reason, the $6j$–symbols can be described by a finite set of parameterized functions and more precisely by a family of 3–variable Laurent polynomials $\mathsf{J}_{i,j,k}$ with coefficients in $\mathbb{Z}[\xi]$.

These Laurent polynomials have wonderful properties. They have some symmetries (see (3)) and satisfy a Biedenharn–Elliott type identity (see (5)) and an orthonormality relation (see (6)). These three identities, imply that from a triangulation of a 3–manifold, one can compute a state sum, that is a weighted sum of product of these J polynomials associated with the tetrahedra of the triangulation, which is a topological invariant of M. Furthermore, F Costantino and J Murakami [2] show that the asymptotical behavior of these polynomial $6j$–symbols is related to the volume of truncated tetrahedra.

The main substance of this paper is the careful computation of the $6j$–symbols associated with nilpotent representations of $U_\xi(\mathfrak{sl}(2))$. This is done in Section 3. But once the $6j$–symbols are identified with certain values of the J polynomials, all the machinery of tensor category can be forgotten. This is what we want to highlight by the structure of this document. Hence the first part only defines the J polynomials and announces their properties. Here the tensor categories does not appear except in the fact that we do not have, without them, a direct proof of the identities. The second part is a short exposition of how the polynomials can be used to construct a Turaev–Viro [10] type invariant, following and refining their ideas and those of the authors with Turaev [3].

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1 Polynomial $6j$–symbols

Fix a nonzero positive integer r'. In this section, we define a set of formal $6j$–symbols $\mathsf{J}^{r'}_{i_1,i_2,i_3}$ for $i_1, i_2, i_3 \in \mathbb{Z}$ and give some of their properties. Since r' is fixed, we write J_{i_1,i_2,i_3} for $\mathsf{J}^{r'}_{i_1,i_2,i_3}$.

Let $\mathbb{N}$ be the set of positive integers including zero. Let $r = 2r'+1$ and $\xi = e^{im\pi/r}$ for m coprime with $2r$. Let $\mathfrak{L} = \mathbb{Z}[\xi][q_1^{\pm 1}, q_2^{\pm 1}, q_3^{\pm 1}]$ be the ring of Laurent polynomials in three variables, with coefficients in $\mathbb{Z}[\xi]$. We denote with a bar the involutive ring automorphism of $\mathfrak{L}$ defined by $\overline{\xi} = \xi^{-1}, \overline{q}_1 = q_1^{-1}, \overline{q}_2 = q_2^{-1}$ and $\overline{q}_3 = q_3^{-1}$. For any invertible element X of a ring and $N \in \mathbb{N}$ let $\langle X \rangle$ and $\mathsf{F}_N(X)$ be analogues of quantum integer and quantum factorial, given by

$$\langle X \rangle = X - X^{-1}, \quad \mathsf{F}_N(X) = \prod_{i=0}^{N-1} \langle \xi^i X \rangle = \langle X \rangle \langle \xi X \rangle \cdots \langle \xi^{N-1} X \rangle.$$

Also, for $n, N \in \mathbb{N}$ and $i_1, i_2, i_3 \in \{-r', -r'+1, \dots, r'\}$ such that $n \le N$ we set

$$\{n\} = \langle \xi^n \rangle = \xi^n - \xi^{-n}, \qquad \{N\}! = \{1\}\{2\} \cdots \{N\},$$
$$\begin{bmatrix} N \\ n \end{bmatrix} = \frac{\{N\}!}{\{n\}!\{N-n\}!}, \qquad \{i_1, i_2, i_3\} = \frac{\{2r'\}!}{\{r'-i_1\}!\{r'-i_2\}!\{r'-i_3\}!}.$$

Remark that $\mathsf{F}_k(X^{-1}) = (-1)^k \mathsf{F}_k(X)$ and $\{2r'\}! = (-1)^{r'} r$. For $N = r$ notice that

$$\mathsf{F}_r(X) = \prod_{i=0}^{r-1} (\xi^i X - \xi^{-i} X^{-1}) = \prod_{i=0}^{r-1} \xi^i X^{-1} (X^2 - \xi^{-2i}).$$

After multiplying the right hand side of this equation by X^r we see that this polynomial has roots $\pm\xi^{-i}$ for $i = 0, \dots, r-1$, and so up to the sign $(-1)^{r'} = \xi^{r(r-1)/2}$ is equal to $X^{2r} - 1$. Thus, we have shown that $\mathsf{F}_r(X) = \prod_{i=0}^{r-1} \xi^i X^{-1}(X^2 - \xi^{-2i}) = (-1)^{r'} \langle X^r \rangle$.

Let us consider the finite set

$$\mathcal{H}_{r'} = \big\{(i_1, i_2, i_3) \in \mathbb{N} : -r' \le i_1,\ i_2,\ i_3,\ i_1 + i_2 + i_3 \le r'\big\}.$$

One can easily show that $\mathrm{card}(\mathcal{H}_{r'}) = \frac{1}{3} r(2r^2 + 1)$. It can be useful to have in mind the action of the tetrahedral group $\mathfrak{S}_4$ on $\mathcal{H}_{r'}$ by permuting i_1, i_2, i_3 and $i_4 = -(i_1 + i_2 + i_3)$.

For all $(i_1, i_2, i_3) \in \mathcal{H}_{r'}$, we define a Laurent polynomial

$$\mathsf{J}_{i_1,i_2,i_3}(q_1, q_2, q_3) \in \mathfrak{L}$$

as follows.

- If $i_1, i_3 \le i_1 + i_2 + i_3$ then let $N = r' - i_1 - i_2 - i_3$ and define

$$\begin{aligned} \mathsf{J}_{i_1,i_2,i_3}(q_1, q_2, q_3) = {} & \{i_1, i_2, i_3\} \mathsf{F}_{i_2+i_3}(q_1 \xi^{-i_3-r'}) \mathsf{F}_{i_1+i_2}(q_3 \xi^{-i_2-r'}) \\ & \times \Bigg(\sum_{n=0}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \mathsf{F}_{N-n}(q_2 \overline{q}_1 \xi^{i_3+r'+1}) \mathsf{F}_{N-n}(q_2 \overline{q}_3 \xi^{i_3+i_2-i_1-r'}) \\ & \qquad \times \mathsf{F}_n(q_1 \overline{q}_2 \xi^{-2i_3-N}) \mathsf{F}_n(q_3 \overline{q}_2 \xi^{i_1+r'+1}) \mathsf{F}_{r'-i_2}(q_2 \xi^{-i_1-r'-n}) \Bigg). \end{aligned} \tag{1}$$

- If $i_2, i_3 \geq i_1 + i_2 + i_3$ then let $N = r' + i_1 + i_2 + i_3$ and define

$$(2)\quad \mathsf{J}_{i_1,i_2,i_3}(q_1,q_2,q_3) = \frac{\mathsf{F}_{r'+i_2-N}(q_3\overline{q}_1\xi^{N-2i_2+1})}{\{N\}!}\left(\sum_{n=0}^{N}\begin{bmatrix}N\\n\end{bmatrix}\mathsf{F}_n(q_2\xi^{-i_1+r'+1})\right.$$
$$\left.\times\,\mathsf{F}_{N-n}(q_2\xi^{-i_1-i_2+n+1})\mathsf{F}_{r'+i_3}(q_1\overline{q}_2\xi^{N-n-2i_3+1})\mathsf{F}_{r'+i_1}(q_2\overline{q}_3\xi^{n-2i_1+1})\right).$$

- For other $(i_1,i_2,i_3) \in \mathcal{H}_{r'}$, the polynomial J_{i_1,i_2,i_3} is obtained from Equation (3), below.

The definition of these symbols come from the $6j$–symbols associated to nilpotent representations of $U_\xi(\mathfrak{sl}(2))$ (see Definition 20 and Theorem 29). Theorem 29 shows that Equations (1) and (2) agree when both conditions are satisfied. We also extend the definition for $(i_1,i_2,i_3) \in \mathbb{Z}^3 \setminus \mathcal{H}_{r'}$ by $\mathsf{J}_{i_1,i_2,i_3} = 0$. Theorem 19 implies that $\mathsf{J}_{i_1,i_2,i_3}(q_1,q_2,q_3)$ is an element of $\mathfrak{L}$.

It is well known that the $6j$–symbols satisfy certain relations. We use Equation (27) to show the family of polynomials defined above satisfy equivalent relations. Indeed the theory of modified $6j$–symbols developed in [3] shows that these identities are satisfied as functions over some open dense subset of $\mathbb{C}^n$. Therefore, since the elements $\mathsf{J}_{*,*,*}$ are Laurent polynomials they satisfy the identities formally.

Let us now discuss these relations. Since the $6j$–symbols have tetrahedral symmetry we have

$$(3)\quad \mathsf{J}_{i_1,i_2,i_3}(q_1,q_2,q_3) = \mathsf{J}_{i_2,i_1,i_3}(\overline{q}_2,\overline{q}_1,\overline{q}_3) = \mathsf{J}_{i_2,i_3,i_4}(q_1\overline{q}_2\xi^{-2i_3}, q_1\overline{q}_3\xi^{2i_2}, q_1)$$

where $i_4 = -i_1 - i_2 - i_3$. These two equalities generate the 24 symmetries of the tetrahedral group. In particular, if σ is permutation of the set $\{1,2,3\}$ then $\mathsf{J}_{i_1,i_2,i_3}(q_1,q_2,q_3) = \mathsf{J}_{i_{\sigma(1)},i_{\sigma(2)},i_{\sigma(3)}}(q^{\varepsilon}_{\sigma(1)}, q^{\varepsilon}_{\sigma(2)}, q^{\varepsilon}_{\sigma(3)})$ where $\varepsilon = \varepsilon(\sigma)$ is the signature of σ.

The other relations involve a function called a modified dimension (see [3]). We introduce the following polynomial in q_1 which is a formal analog of the inverse of this function:

$$(4)\quad \mathsf{D}(q_1) = \mathsf{F}_{2r'}(q_1\xi) = (-1)^{r'}\frac{q_1^r - q_1^{-r}}{q_1 - q_1^{-1}}.$$

The J polynomials satisfy the Biedenharn–Elliott identity: For $x \in \mathbb{Z}$ let $\overline{x}$ be the element of $\{-r', -r'+1, \dots, r'\}$ congruent to x modulo r. For any $i_1, i_2, i_3, i_4, i_5, i_6 \in \mathbb{Z}$,

$$\mathsf{J}_{i_1,i_2,i_3}(q_1,q_2,q_3)\,\mathsf{J}_{i_1',i_2',i_3'}(q_0\xi^{2i_4}/q_1, q_0\xi^{2i_5}/q_2, q_0\xi^{2i_6}/q_3) \\ = \sum_{n=-r'}^{r'} \frac{\mathsf{J}_{i_1,i_6',-i_5'}(q_0\xi^{2n},q_2,q_3)\,\mathsf{J}_{i_2,i_4',-i_6'}(q_0\xi^{2n},q_3,q_1)\,\mathsf{J}_{i_3,i_5',-i_4'}(q_0\xi^{2n},q_1,q_2)}{\mathsf{D}(q_0\xi^{2n})} \tag{5}$$

where
$$\begin{cases} i_1' = -i_1 + \overline{i_5 - i_6}, \\ i_2' = -i_2 + \overline{i_6 - i_4}, \\ i_3' = -i_3 + \overline{i_4 - i_5}, \end{cases} \qquad \begin{cases} i_4' = \overline{i_4 - n}, \\ i_5' = \overline{i_5 - n}, \\ i_6' = \overline{i_6 - n}, \end{cases}$$

and q_0 is an independent variable.

For any $(i_1, i_2, i_3) \in \mathcal{H}_{r'}$ and any $i_1' \in \mathbb{Z}$ the orthonormality relation is expressed as

$$\sum_{n=-r'}^{r'} \frac{\mathsf{J}_{i_1,\overline{i_2-n},\overline{i_3+n}}(q_1\xi^{2n},q_2,q_3)\,\mathsf{J}_{-i_1',\overline{n-i_2},\overline{-i_3-n}}(\overline{q}_1\xi^{-2n},\overline{q}_2,\overline{q}_3)}{\mathsf{D}(q_2\overline{q}_3\xi^{-2i_1})\,\mathsf{D}(q_1\xi^{2n})} = \delta_{i_1,i_1'} \tag{6}$$

where $\delta_{i_1,i_1'}$ is the Kronecker symbol.

2 3–Manifold invariant

In this section we derive a set of topological invariant of links in a closed 3–manifold M from the family $\mathsf{J}_{***}(q_1,q_2,q_3)$. These invariants are indexed by element of $H_1(M,\mathbb{Z})$ and they refine the invariant constructed in [3, Section 10.4].

Let M be a closed 3–manifold and L a link in M. Here we follow the exposition of [3] inspired by Baseilhac and Benedetti [1]. A *quasiregular triangulation of* M is a decomposition of M as a union of embedded tetrahedra such that the intersection of any two tetrahedra is a union (possibly, empty) of several of their vertices, edges, and (2–dimensional) faces. Quasiregular triangulations differ from usual triangulations in that they may have tetrahedra meeting along several vertices, edges, and faces. Nevertheless, the edges of a quasiregular triangulation have distinct ends. A *Hamiltonian link* in a quasiregular triangulation $\mathcal{T}$ is a set $\mathcal{L}$ of unoriented edges of $\mathcal{T}$ such that every vertex of $\mathcal{T}$ belongs to exactly two edges of $\mathcal{L}$. Then the union of the edges of $\mathcal{T}$ belonging to $\mathcal{L}$ is a link L in M. We call the pair $(\mathcal{T},\mathcal{L})$ an *H–triangulation* of (M,L).

Proposition 1 [1, Proposition 4.20] *Any pair (a closed connected* 3*–manifold* M*, a nonempty link* $L \subset M$*) admits an* H*–triangulation.*

The language of both triangulation and skeleton are useful here. In particular, it is convenient to use triangulation to give the notion of a Hamiltonian link and skeleton to define the state sum. A skeleton of M is a 2–dimensional polyhedron $\mathcal{P}$ in M such that $M \setminus \mathcal{P}$ is a disjoint union of open 3–balls and locally $\mathcal{P}$ looks like a plane, or a union of 3 half-planes with common boundary line in $\mathbb{R}^3$, or a cone over the 1–skeleton of a tetrahedron (see, for instance [1; 9]). A typical skeleton of M is constructed from a triangulation $\mathcal{T}$ of M by taking the union $\mathcal{P}_{\mathcal{T}}$ of the 2–cells dual to its edges. This construction establishes a bijective correspondence $\mathcal{T} \leftrightarrow \mathcal{P}_{\mathcal{T}}$ between the quasiregular triangulations $\mathcal{T}$ of M and the skeletons $\mathcal{P}$ of M such that every 2–face of $\mathcal{P}$ is a disk adjacent to two distinct components of $M \setminus \mathcal{P}$ and no connected component of the 1–dimensional strata of $\mathcal{P}$ is a circle. To specify a Hamiltonian link $\mathcal{L}$ in a triangulation $\mathcal{T}$, we provide some faces of $\mathcal{P}_{\mathcal{T}}$ with dots such that each component of $M \setminus \mathcal{P}_{\mathcal{T}}$ is adjacent to precisely two (distinct) dotted faces. These dots correspond to the intersections of $\mathcal{L}$ with the 2–faces.

Let R be a commutative ring with a morphism $\mathbb{Z}[\xi] \to R$. We still denote by ξ the image of ξ in R and we assume that the group of $2r$–th root of 1 in R is of order $2r$ generated by ξ. Let $R^\times$ be the group of units of R and consider any subgroup G of $\{x^r : x \in R^\times\}$, for example $(R, G) = (\mathbb{C}, \mathbb{C}^*)$. Clearly any element $x \in G$ has exactly r r–th roots in R. They form a set $\mathrm{Root}_r(x) = \{y\xi^{-2r'}, \dots, y, y\xi^2, \dots, y\xi^{2r'}\}$ for some $y \in R$ such that $y^r = x$. We call (R, G) a *coloring pair*.

Let $(\mathcal{T}, \mathcal{L})$ be a H–triangulation of (M, L). Let $\mathcal{P}_{\mathcal{T}}$ be a skeleton dual to $\mathcal{T}$. The skeleton $\mathcal{P}_{\mathcal{T}}$ gives M a cell decomposition $M_{\mathcal{P}}$. So a n–chain of cellular homology with coefficients in G can be represented by a map from the oriented n–cells of $M_{\mathcal{P}}$ to G.

By a *G–coloring* of $\mathcal{T}$ (or of $\mathcal{P}_{\mathcal{T}}$), we mean a G–valued 2–cycle Φ on $\mathcal{P}_{\mathcal{T}}$, that is a map from the set of oriented faces of $\mathcal{P}_{\mathcal{T}}$ to G such that

(1) the product of the values of Φ on the three oriented faces adjacent to any oriented edge of $\mathcal{P}_{\mathcal{T}}$ is 1,

(2) $\Phi(-f) = \Phi(f)^{-1}$ for any oriented face f of $\mathcal{P}_{\mathcal{T}}$, where $-f$ is f with opposite orientation.

Each G–coloring Φ of $\mathcal{T}$ represents a homology class $[\Phi] \in H_2(M, G)$. When M is oriented, a G–coloring of $\mathcal{T}$ can be seen as a 1–cocycle (a map on the set of oriented edges of $\mathcal{T}$; see [3]). In general, it can also be interpreted as a map on the set of co-oriented edges of $\mathcal{T}$ but we prefer to adopt the point of view of $\mathcal{P}_{\mathcal{T}}$.

A *state* φ of a G–coloring Φ is a map assigning to every oriented face f of $\mathcal{P}_{\mathcal{T}}$ an element $\varphi(f)$ of $\mathrm{Root}_r(\Phi(f))$ such that $\varphi(-f) = \varphi(f)^{-1}$ for all f. The set of all states of Φ is denoted $\mathrm{St}(\Phi)$. A state φ can also be seen as a 2–chain on $\mathcal{P}_{\mathcal{T}}$ with values in $R^{\times}$ but its boundary, the 1–chain $\delta\varphi$ might not be trivial. Nevertheless, as $\varphi^r = \Phi$, we have $(\delta\varphi)^r = 1$. We call the height of φ the unique map h_φ assigning to every oriented edge e of $\mathcal{P}_{\mathcal{T}}$ an element of $\{-r', -r'+1, \dots, r'\}$ such that $(\delta\varphi)(e) = \xi^{2h_\varphi(e)}$. It follows that modulo r, h_φ is a 1–cycle on $\mathcal{P}_{\mathcal{T}}$. In the case when h_φ is also a 1–cycle with integer coefficients, let us denote its homology class by $[h_\varphi] \in H_1(M, \mathbb{Z})$. For $h \in H_1(M, \mathbb{Z})$, set $\mathrm{St}_h(\Phi) = \{\varphi \in \mathrm{St}(\Phi) : \delta h_\varphi = 0 \text{ and } [h_\varphi] = h\}$.

Given a G–coloring Φ of $(\mathcal{T}, \mathcal{L})$, we define a certain partition function (state sum) as follows: For each vertex x of $\mathcal{P}_{\mathcal{T}}$, fix a little 3–ball B centered at x whose intersection with $\mathcal{P}_{\mathcal{T}}$ is homeomorphic to the cone on the 1–skeleton of a tetrahedron. The trace of $\mathcal{P}_{\mathcal{T}}$ on ∂B gives a triangulation of this sphere whose one skeleton is a tetrahedron with four vertices v_1, v_2, v_3, v_4. Let f_1, f_2, f_3 be the regions of $\mathcal{P}_{\mathcal{T}}$ contained in the triangles $xv_2v_3, xv_3v_1, xv_1v_2$, respectively (see Figure 1). Also, let e_1, e_2, e_3, e_4 be

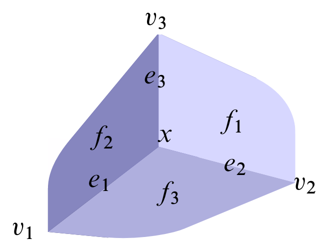


Figure 1: One side of $\mathcal{P}_{\mathcal{T}}$ near the vertex x

the segments of $\mathcal{P}_{\mathcal{T}}$ contained in xv_1, xv_2, xv_3, xv_4, respectively. The segment xv_i is oriented from x to v_i and induces an orientation on e_i. Similarly, the triangles above induce orientations on f_1, f_2, f_3. For each $\varphi \in \mathrm{St}(\Phi)$, if h_φ does not satisfy the cycle condition at x (ie if $\sum_i h_\varphi(e_i) \neq 0$) we set $\mathsf{J}(\varphi, x) = 0$, otherwise define

$$\mathsf{J}(\varphi, x) = \mathsf{J}_{h_\varphi(e_1), h_\varphi(e_2), h_\varphi(e_3)}\big(\varphi(f_1), \varphi(f_2), \varphi(f_3)\big) \in R.$$

Equation (3) implies that $\mathsf{J}(\varphi, x)$ does not depend of the choice of ordering of the vertices v_1, v_2, v_3, v_4. For example, if one chooses the ordering v_2, v_1, v_3, v_4 then e_1 and e_2 are exchanged, xv_2v_3 becomes xv_1v_3 and so f_1 becomes $-f_2$, etc... and the first equality of (3) implies that the two expressions for $\mathsf{J}(\varphi, x)$ are equal.

We say that $g \in G$ is *admissible* if $\langle g \rangle = g - g^{-1}$ is invertible in R. We call a G–coloring Φ *admissible* if it takes admissible values. If φ is a state of an admissible coloring Φ, and f is an unoriented face of $\mathcal{P}_{\mathcal{T}}$, then we define

$$\mathsf{d}(\varphi, f) = \mathsf{D}(g)^{-1} = \frac{(-1)^{r'} \langle g \rangle}{\langle g^r \rangle} \in R$$

where g is $\varphi(\vec{f})$ for any orientation $\vec{f}$ of f. Note that $\mathsf{d}(\varphi, f)$ does not depend on the orientation of f, as $\mathsf{D}(g) = \mathsf{D}(g^{-1})$.

Lemma 2 *Let (R, G) be a coloring pair with the following property:*

$(*)$ *For all $g_1, \dots, g_n \in G$ there exists $x \in G$ such that $xg_1, \dots, xg_n$ are all admissible.*

Then for any H–triangulation $(\mathcal{T}, \mathcal{L})$ of (M, L) and for any homology class $h_2 \in H_2(M, G)$, there exists an admissible G–coloring Φ on $\mathcal{T}$ representing h_2.

Proof Take any G–coloring Φ of $\mathcal{T}$ representing h_2. For f an oriented face of $\mathcal{P}_\mathcal{T}$ and $-f$ the same face with opposite orientation, we have that $\langle\Phi\rangle(-f) = -\langle\Phi\rangle(f)$ and thus $\Phi(-f)$ is admissible if and only if $\Phi(f)$ is admissible. As mentioned above $M \setminus \mathcal{P}_\mathcal{T}$ is the disjoint union of open 3–balls. We say that a such a 3–ball b is *bad* for Φ if there is a oriented face f in $\mathcal{T}$ incident to b such that $\Phi(f)$ is not admissible. It is clear that Φ is admissible if and only if Φ has no bad 3–balls. We show how to modify Φ in its homology class to reduce the number of bad 3–balls. Let b be a bad 3–ball for Φ and let E_b be the set of all oriented faces of $\mathcal{T}$ which are oriented away from b. From Property $(*)$ of the lemma, there exists $x \in G$ such that $x\Phi(f)$ is admissible for all $f \in E_b$. Let c be the G–valued 3–chain on $M_\mathcal{P}$ assigning x to b and 1 to all other 3–balls (recall $M_\mathcal{P}$ is the cell decomposition of M coming from $\mathcal{P}_\mathcal{T}$). Taking the boundary of this 3–chain we obtain a G–valued 2–chain δc on $M_\mathcal{P}$. The 2–cycle $(\delta c)\Phi$ on $\mathcal{P}_\mathcal{T}$ takes values in $\{x\Phi(f); (x\Phi(f))^{-1} : f \in E_b\}$ which are admissible on all faces of $\mathcal{P}_\mathcal{T}$ incident to b and takes the same values as Φ on all other faces of $\mathcal{T}$. Here we use the fact every 2–face of $\mathcal{P}_\mathcal{T}$ is a disk adjacent to two distinct components of $M \setminus \mathcal{P}_\mathcal{T}$. The transformation $\Phi \mapsto (\delta c)\Phi$ decreases the number of bad 3–balls. Repeating this argument, we find a 2–cycle without bad 3–balls. □

Let Φ be an admissible G–coloring of $\mathcal{T}$ and $h_1 \in H_1(M, \mathbb{Z})$. Then we define

$$\tau(\mathcal{T}, \mathcal{L}, h_1, \Phi) = r^{-2N} \sum_{\varphi \in \mathrm{St}_{h_1}(\Phi)} \prod_{f \in \mathcal{P}_2 \setminus \mathcal{L}} \mathsf{d}(\varphi, f) \prod_{x \in \mathcal{P}_0} \mathsf{J}(\varphi, x) \in R$$

where $\mathcal{P}_2 \setminus \mathcal{L}$ is the set of unoriented faces of $\mathcal{P}_\mathcal{T}$ without dots, $\mathcal{P}_0$ is the set of vertices of $\mathcal{P}_\mathcal{T}$ and N is the number of connected component of $M \setminus \mathcal{P}_\mathcal{T}$ (that is the number of vertices in $\mathcal{T}$).

When the coloring pair does not satisfy Property $(*)$ of Lemma 2, we explain how to perturb a nonadmissible G–coloring Φ: Consider the set S of element of $R[X^{\pm 1}]$ that are monic polynomials in X (ie Laurent polynomials whose leading coefficient is 1). Then S is a multiplicative set that does not contain zero divisor, hence we can form $R' = S^{-1}R[X^{\pm 1}] \supset R$. Let G' be the multiplicative group generated by G and X^r. Then (R', G') is a coloring pair with property $(*)$ as any $\langle X^{kr}h\rangle$ with $h \in G$ and $k \in \mathbb{Z}^*$ is invertible in R'. Using the inclusion above we can view Φ and $[\Phi]$ as taking values in G'. Then Lemma 2 implies there exists a 2–boundary x with values in G' such that $\Phi' = x\Phi$ is an admissible G'–coloring. We say that Φ' is a perturbation of Φ.

Theorem 3 *Let L be a link in a 3–manifold M, (R, G) be a coloring pair and $(h_1, h_2) \in H_1(M, \mathbb{Z}) \times H_2(M, G)$. Choose any H–triangulation $(\mathcal{T}, \mathcal{L})$ of (M, L) and let Φ be any admissible (or perturbation of a) G–coloring representing h_2. Then $\tau_R(M, L, h_1, h_2) = \tau(\mathcal{T}, \mathcal{L}, h_1, \Phi)$ belongs to R and it is an invariant of the diffeomorphism class of the four-tuple*

$$\big(M,\ L,\ h_1 \in H_1(M, \mathbb{Z}),\ h_2 \in H_2(M, G)\big).$$

Proof First, let us assume that (R, G) satisfy $(*)$ of Lemma 2, so there exists an admissible Φ representing h_2. In this case the proof is essentially the same as the proof of Theorem 22 in [3]. Here we sketch the main steps:

(I) In [1] it is shown that any two H–triangulation are related by a finite sequence of so called elementary H–moves. One can then colors this sequence and makes it a sequence of "colored H–moves."

(II) One shows that the state sum $\tau(\mathcal{T}, \mathcal{L}, h_1, \Phi)$ is invariant under an elementary "admissible colored H–move," ie an elementary H–move where the colors of the H–triangulation on both sides of the move are admissible. The main point here is that Equation (5) implies that if one performs a so called Pachner $2-3$ move (which consists in replacing in $\mathcal{T}$ two tetrahedra glued along a face with 3 tetrahedra having a common edge) the state sum is unchanged. Similarly, Equation (6) imply the invariance of the state sum under the lune move which consists in removing two tetrahedra which have 2 common faces and then gluing by pairs the orphan faces.
Here the following observation makes the refinement with h_1 possible: if two states φ, φ' of Φ differ only on a set of faces then $h_\varphi, h_{\varphi'}$ differ only on the set E of edges adjacent to these faces. Assume that the set E is included in a simply connected part of $\mathcal{P}_\mathcal{T}$. Then if φ and φ' have nontrivial contributions in the state sum (which implies $\delta h_\varphi = \delta h_{\varphi'} = 0$), we have $[h_\varphi] = [h_{\varphi'}]$ since h_φ and $h_{\varphi'}$ are equal outside the simply connected space containing E. Hence the colored H–moves do not modify the partial state sum associated to any homology class h_1.

(III) The 2–cycles representing h_2 in the sequence of colored H–moves of Step (I) are not necessarily always admissible G–colorings. However, using $(*)$ one can prove that $\tau(\mathcal{T},\mathcal{L},h_1,\Phi)$ depends only on the cohomology class of the admissible G–coloring Φ (see Lemmas 27 and 28 of [3]). This then allows us to modify a sequence of colored H–moves to a sequence of admissible colored H–moves such that the state sum is the same at each step, thus showing the theorem when (R,G) satisfy $(*)$ of Lemma 2.

Let us now consider the case where the coloring pair does not satisfy property $(*)$. We will prove that the perturbed state sum belongs to R. Let $\Phi' = x\Phi$ be a perturbation of any G–coloring Φ representing h_2. The idea is that the only component of Φ' which depends on $X \in R'$ is the boundary δ and as the state sum depend of the coloring only up to a boundary, it does not depend on X. To be more precise, let ρ: $R' \to R'$ be the ring morphisms which is the identity on R and sends X to X^2. Then $\Phi'' = \rho(\Phi')$ is also an admissible R'–coloring of $\mathcal{T}$. Moreover, $\rho(x)$ is a boundary and $\rho(\Phi)=\Phi$, hence Φ'/Φ'' is a boundary. But from above we know that two admissible colorings representing the same homology class give equal state sums. Thus, $\tau(\mathcal{T},\mathcal{L},h_1,\Phi') = \tau(\mathcal{T},\mathcal{L},h_1,\Phi'') = \rho(\tau(\mathcal{T},\mathcal{L},h_1,\Phi'))$ which implies that $\tau(\mathcal{T},\mathcal{L},h_1,\Phi) \in R$. □

We also define

$$\tau_R(M,L,h_2) = \sum_{h_1 \in H_1(M,\mathbb{Z})} \tau_R(M,L,h_1,h_2) \in R$$

This sum is finite because $\tau_R(M,L,h_1,h_2)=0$ for all but finitely many h_1. Indeed, if an admissible coloring Φ represent h_2 then we have $\mathrm{St}(\Phi) = \bigcup_{h_1} \mathrm{St}_{h_1}(\Phi)$ is finite and thus $\mathrm{St}_{h_1}(\Phi) = \varnothing$ for all but finitely many $h_1 \in H_1(M,\mathbb{Z})$.

The first fundamental example is obtained when $(R,G) = (\mathbb{C},\mathbb{C}^*)$. It is easy to see that this coloring pair satisfies Property $(*)$ of Lemma 2 since $1,-1 \in \mathbb{C}^*$ are the only nonadmissible elements. In this case, if M is oriented we denote the Poincaré dual of $h_2 \in H_2(M,\mathbb{C}^*)$ by $h_2^* \in H^1(M,\mathbb{C}^*)$. Then

$$\tau_{\mathbb{C}}(M,L,h_2) = \tau(M,L,h_2^*)$$

where $\tau(M,L,h_2^*)$ is the invariant defined in [3, Section 10.4].

We now consider a universal example: Let $H = H_1(M,\mathbb{Z})$, and assume that M is oriented so that for any abelian group G we have $H_2(M,G) \simeq H^1(M,G) \simeq \mathrm{Hom}(H,G)$. Then $H_2(M,H)$ has a particular universal element η whose image in $\mathrm{Hom}(H,H)$ is the identity. We will assume that the order of the torsion of H is coprime with r. Then multiplication by r is an injective morphism $m_r\colon H \to H$. Denote the image of m_r by rH, then $(\mathbb{Z}[\xi][H], rH)$ is a coloring pair and we consider $\tau(M, L, r\eta) \in \mathbb{Z}[\xi][H]$.

Proposition 4 *The invariant $\tau(M, L, r\eta)$ takes values in $\mathbb{Z}[\xi][rH]$ making it possible to define*

$$\tau(M, L) = m_r^*(\tau(M, L, r\eta)) \in \mathbb{Z}[\xi][H].$$

Then for any pair (R, G) as above and any $\psi \in \mathrm{Hom}(H, G)$ we have

$$\tau(M, L, \bar{\psi}) = \psi_*(\tau(M, L))$$

where $\bar{\psi}$ is the image of ψ in $H_2(M, G)$.

Proof First, let us show that for each $h_1 \in H_1(M, \mathbb{Z})$ we have $\tau(M, L, h_1, r\eta) \in \mathbb{Z}[\xi][rH]$. We choose a base of the free part of H; that is we write $H = \mathrm{Tor}(H) \oplus \mathbb{Z}x_1 \oplus \cdots \oplus \mathbb{Z}x_k$. Then define the ring morphism $\rho_i\colon \mathbb{Z}[\xi][H] \to \mathbb{Z}[\xi][H]$ as the identity on this basis except that $\rho_i(\mathrm{e}^{x_i}) = \xi\, \mathrm{e}^{x_i}$. Clearly, the set of states and thus the state sum is invariant by ρ_i for any i and thus belongs to $\mathbb{Z}[\xi][rH]$. The last point follows from the fact that $\psi_*(\eta) = \psi$. □

Remark 5 Suppose that $\mathcal{T}$ is not a quasiregular triangulation but a generalized triangulation where some edges might be loops. Then not all homology classes of $H_2(M, G)$ can be represented by admissible colorings on $\mathcal{T}$.

Nevertheless, suppose that an admissible coloring Φ is given on $\mathcal{T}$. Then one can prove that the state sum $\tau(\mathcal{T}, \mathcal{L}, h_1, \Phi)$ as above is still equal to the invariant $\tau(M, L, h_1, [\Phi])$. This might be useful for effective computations. This can be proven using the fact that up to perturbing the coloring, the triangulation $\mathcal{T}$ can be transformed into a quasiregular one by a sequence of elementary moves such that at each step, the locally modified coloring is admissible.

3 Skein calculus

3.1 The category $\mathcal{C}^H$ of $U_\xi^H(\mathfrak{sl}(2))$ weight modules

For $x \in \mathbb{C}$ we extend the notation ξ^x by setting $\xi^x = \mathrm{e}^{im\pi x/r}$. Also, if $(\alpha, k) \in \mathbb{C} \times \mathbb{N}$,

$$\{\alpha\} = \xi^\alpha - \xi^{-\alpha} \quad \text{and} \quad \{\alpha; k\}! = \mathsf{F}_k\left(\xi^\alpha\right) = \{\alpha\}\{\alpha+1\}\cdots\{\alpha+k-1\}.$$

Many computations in this section use the identity

$$\{x+z\}\{y+z\} - \{x\}\{y\} = \{x+y+z\}\{z\}.$$

Let $U_\xi^H(\mathfrak{sl}(2))$ be the "unrolled" quantization of $\mathfrak{sl}(2)$, ie the $\mathbb{C}$–algebra with generators E, F, K, K^{-1}, H and the following defining relations:

$$KK^{-1} = K^{-1}K = 1, \quad KEK^{-1} = \xi^2 E, \quad KFK^{-1} = \xi^{-2} F,$$
$$HK = KH, \quad [H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = \frac{K - K^{-1}}{\xi - \xi^{-1}}.$$

This algebra is a Hopf algebra with coproduct Δ, counit ε, and antipode S defined by the formulas

$$\begin{aligned}
&\Delta(E) = 1 \otimes E + E \otimes K, && \varepsilon(E) = 0, && S(E) = -EK^{-1},\\
&\Delta(F) = K^{-1} \otimes F + F \otimes 1, && \varepsilon(F) = 0, && S(F) = -KF,\\
&\Delta(K) = K \otimes K, && \varepsilon(K) = 1, && S(K) = K^{-1},\\
&\Delta(H) = H \otimes 1 + 1 \otimes H, && \varepsilon(H) = 0, && S(H) = -H.
\end{aligned}$$

Following [4], we define $\bar{U}_\xi^H(\mathfrak{sl}(2))$ to be the quotient of $U_\xi^H(\mathfrak{sl}(2))$ by the relations $E^r = F^r = 0$. It is easy to check that the operations above turn $\bar{U}_\xi^H(\mathfrak{sl}(2))$ into a Hopf algebra.

Let V be a $\bar{U}_\xi^H(\mathfrak{sl}(2))$–module. An eigenvalue $\lambda \in \mathbb{C}$ of the operator $H\colon V \to V$ is called a *weight* of V and the associated eigenspace $E_\lambda(V)$ is called a *weight space*. We call V a *weight module* if V is finite-dimensional, splits as a direct sum of weight spaces, and $\xi^H = K$ as operators on V.

Let $\mathcal{C}^H$ be the tensor category of weight $\bar{U}_\xi^H(\mathfrak{sl}(2))$–modules. By Section 6.2 of [4], $\mathcal{C}^H$ is a ribbon Ab-category with ground ring $\mathbb{C}$. The braiding $c_{V,W}\colon V \otimes W \to W \otimes V$ on $\mathcal{C}^H$ is defined by $v \otimes w \mapsto \tau(R(v \otimes w))$ where τ is the permutation $x \otimes y \mapsto y \otimes x$ and R is the operator of $V \otimes W$ defined by

$$R = \xi^{H \otimes H/2} \sum_{n=0}^{r-1} \frac{\{1\}^{2n}}{\{n\}!} \xi^{n(n-1)/2} E^n \otimes F^n. \tag{7}$$

The inverse of the twist on a weight module V is given by the operator

$$\theta_V^{-1} = K^{r-1} \xi^{-H^2/2} \sum_{n=0}^{r-1} (-1)^n \frac{\{1\}^{2n}}{\{n\}!} \xi^{3n(n-1)/2} F^n K^{-n} E^n \tag{8}$$

(also see Ohtsuki [8, Chapter 4.5] where this formula is given with $\zeta = \xi^2$ instead of ξ).

For an isomorphism classification of simple weight modules over the usual quantum $\mathfrak{sl}(2)$, see for example Kassel [5, Chapter VI]. This classification implies that simple weight $\bar{U}_\xi^H(\mathfrak{sl}(2))$–modules are classified up to isomorphism by highest weights. For $\alpha \in \mathbb{C}$, we denote by V_α the simple weight $\bar{U}_\xi^H(\mathfrak{sl}(2))$–module of highest weight $\alpha + r - 1$. This notation differs from the standard labeling of highest weight modules. Note that $V_{-r+1} = \mathbb{C}$ is the trivial module and V_0 is the so called Kashaev module.

The well-known Reshetikhin–Turaev construction defines a $\mathbb{C}$–linear functor F from the category of $\mathcal{C}^H$–colored ribbon graphs with coupons to $\mathcal{C}^H$. Let $B = (\mathbb{C} \setminus \mathbb{Z}) \cup r\mathbb{Z}$. The modules $\{V_\alpha\}_{\alpha \in B}$ are called typical and all have dimension $r = 2r' + 1$. Note that F is trivial on all closed $\mathcal{C}^H$–colored ribbon graph that have at least one color in B. In [4], the definition of F is extended to a nontrivial map F' defined on closed $\mathcal{C}^H$–colored ribbon graphs with at least one edge colored by a typical module. Let us recall how one can compute F'. If $T \subset \mathbb{R} \times [0,1]$ is a $\mathcal{C}^H$–colored (1-1)-tangle with the two ends colored by the same typical module V_α, we can form its "braid closure" $\hat{T}$. Then we say that T is a *cutting presentation* of the closed $\mathcal{C}^H$–colored ribbon graph $\hat{T}$. In this situation, $F(T)$ is an endomorphism of V_α that is a scalar. Then $F'(\hat{T})$ is this scalar multiplied by the modified dimension of V_α which is given by

$$\mathsf{d}(V_\alpha) = (-1)^{r'} \frac{\{\alpha\}}{\{r\alpha\}} = \prod_{k=1}^{2r'} \frac{1}{\{\alpha + k\}}.$$

It can be shown that $F'(\hat{T})$ does not depend on the cutting presentation T of $\hat{T}$ (see [4]).

For $\alpha \in B$ let us consider the basis of V_α given by $(v_i = F^i v_0)_{i=0..2r'}$ where v_0 is a highest weight vector of V_α. Then the $U_\xi^H(\mathfrak{sl}(2))$–module structure of V_α is given by

$$H.v_i = (\alpha + 2(r'-i))v_i, \quad E.v_i = \frac{\{i\}\{i-\alpha\}}{\{1\}^2} v_{i-1}, \quad F.v_i = v_{i+1}.$$

Remark 6 The family of module indexed by B can be seen as a vector bundle $\mathcal{E} \twoheadrightarrow B$ on which elements of $U_\xi^H(\mathfrak{sl}(2))$ act by continuous linear transformations. Then the v_i are sections of this vector bundle that form a trivialization $\mathcal{E} \simeq B \times \mathbb{C}^r$. In fact one can extend $\mathcal{E}$ to a unique vector bundle $\mathcal{E}'$ over $\mathbb{C} \supset B$ with an action of $U_\xi^H(\mathfrak{sl}(2))$ but the fiber over $k \in \mathbb{Z} \setminus r\mathbb{Z}$ is not an irreducible module.

Let v be the 2–dimensional simple weight $U_\xi^H(\mathfrak{sl}(2))$–module of highest weight 1 and basis (v_0, v_1) with $E.v_1 = v_0$ and $v_1 = F.v_0$. The categorical dimension of V_α is zero, while that of v is equal to $\mathrm{qdim}(\mathrm{v}) = \{-2\}/\{1\} = -\xi - \xi^{-1}$.

3.2 Duality in $\mathcal{C}^H$

As in [3], the ribbon structure of $\mathcal{C}^H$ induce the existence of functorial left and right duality given by $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ and the morphisms

$$\begin{aligned}
b_V &: \mathbb{C} \to V \otimes V^* && \text{is given by} && 1 \mapsto \sum v_j \otimes v_j^*, \\
d_V &: V^* \otimes V \to \mathbb{C} && \text{is given by} && f \otimes w \mapsto f(w), \\
d'_V &: V \otimes V^* \to \mathbb{C} && \text{is given by} && v \otimes f \mapsto f(K^{1-r} v), \\
b'_V &: \mathbb{C} \to V^* \otimes V && \text{is given by} && 1 \mapsto \sum v_j^* \otimes K^{r-1} v_j.
\end{aligned}$$

For $\alpha \in B$, the classification of simple modules implies that $V_{-\alpha}^*$ is isomorphic to V_α. We consider the isomorphism $w_\alpha \colon V_\alpha \to V_{-\alpha}^*$ given by

$$v_i \mapsto -\xi^{i^2-1-i\alpha} v_{2r'-i}^*.$$

The isomorphism w_α is the unique map up to a scalar that sends v_0 to $-\frac{1}{\xi} v_{2r'}^*$ and $v_i = F^i v_0$ to

$$\begin{aligned}
-\frac{1}{\xi} v_{2r'}^* \circ (-KF)^i &= -\frac{1}{\xi} v_{2r'}^* \circ ((-K)^i \xi^{i(i-1)} F^i) \\
&= -(-1)^i \xi^{i^2-i-1+i(-\alpha-2r')} v_{2r'-i}^* = -\xi^{i^2-1-i\alpha} v_{2r'-i}^*.
\end{aligned}$$

Let $w_{\mathsf{v}} = w_{1-2r'} \colon \mathsf{v} \overset{\sim}{\to} \to \mathsf{v}^*$ be the isomorphism given by $v_0 \mapsto -\xi v_1^*$ and $v_1 \mapsto v_0^*$.

Lemma 7 *For $\alpha \in B$, one has*

$$d_{V_\alpha}(w_{-\alpha} \otimes \mathrm{Id}_{V_\alpha}) = d'_{V_{-\alpha}}(\mathrm{Id}_{V_{-\alpha}} \otimes w_\alpha) \tag{9}$$

and similarly $d_{\mathsf{v}}(w_{1-2r'} \otimes \mathrm{Id}_{\mathsf{v}}) = d'_{\mathsf{v}}(\mathrm{Id}_{\mathsf{v}} \otimes w_{1-2r'})$.

Proof Let us denote by f the left hand side of (9) and by g the right hand side of (9). By a direct computation on $v_i \otimes v_{2r'-i} \in V_{-\alpha} \otimes V_\alpha$,

$$f(v_i \otimes v_{2r'-i}) = d_{V_\alpha}(-\xi^{i^2-1+i\alpha} v_{2r'-i}^* \otimes v_{2r'-i}) = -\xi^{i^2-1+i\alpha}$$

and

$$\begin{aligned}
g(v_i \otimes v_{2r'-i}) &= d'_{V_{-\alpha}}(-\xi^{(2r'-i)^2-1-(2r'-i)\alpha} v_i \otimes v_i^*) \\
&= -\xi^{4r'^2-4r'i+i^2-1-(2r'-i)\alpha} v_i^*(K^{-2r'} v_i) \\
&= -\xi^{4r'^2-4r'i+i^2-1-(2r'-i)\alpha} \xi^{(-2r')(-\alpha+2(r'-i))} \\
&= -\xi^{i^2-1+i\alpha}.
\end{aligned}$$

The analogous equation for v follows similarly

$$g(v_0 \otimes v_1) = -q v_1^*(v_1) = -\xi = \xi^{-2r'} = v_0^*(K^{-2r'} v_0) = f(v_0 \otimes v_1). \qquad \square$$

For $\alpha \in B$, we denote d^α and b^α as the following morphisms

$$d^\alpha = d_{V_\alpha} \circ (w_{-\alpha} \otimes \mathrm{Id}_{V_\alpha})\colon V_{-\alpha} \otimes V_\alpha \to \mathbb{C} \tag{10}$$

$$b^\alpha = (\mathrm{Id}_{V_\alpha} \otimes (w_{-\alpha})^{-1}) \circ b_{V_\alpha}\colon \mathbb{C} \to V_\alpha \otimes V_{-\alpha} \tag{11}$$

Similarly,

$$d^{\mathrm{v}} = d_{\mathrm{v}} \circ (w_{\mathrm{v}} \otimes \mathrm{Id})\colon \mathrm{v} \otimes \mathrm{v} \to \mathbb{C} \quad b^{\mathrm{v}} = (\mathrm{Id} \otimes w_{\mathrm{v}}) \circ b_{\mathrm{v}}\colon \mathbb{C} \to \mathrm{v} \otimes \mathrm{v}.$$

We use the isomorphism $w_{-\alpha}$ to identify V_α^* with $V_{-\alpha}$. Under this identification, we get $d_{V_\alpha} \cong d'_{V_{-\alpha}} \cong d^\alpha$ and $b_{V_\alpha} \cong b'_{V_{-\alpha}} \cong b^\alpha$. Similarly, $d_{\mathrm{v}} \cong d'_{\mathrm{v}^*} \cong d^{\mathrm{v}}$ and $b_{\mathrm{v}} \cong b'_{\mathrm{v}^*} \cong b^{\mathrm{v}}$. Graphically, for a $\mathcal{C}^H$–colored ribbon graph Γ, this means that one can reverse the orientation of an edge colored by V_α and simultaneously replace its coloring by $V_{-\alpha}$. Also, if Γ has an oriented edge colored by v, one can forgot its orientation. We will represent edges colored by v with dashed unoriented edges (see for example Figure 2).

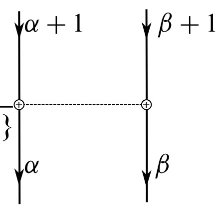
$X = \frac{1}{\{1\}}$ 

Figure 2: The family of maps X

3.3 Multiplicity modules in $V_\alpha \otimes V_{-\alpha\pm1} \otimes \mathrm{v}$

We consider the following spaces of morphisms of $\mathcal{C}^H$ using the notation

$$H^W_{U,V} = \mathrm{Hom}_{\mathcal{C}^H}(U \otimes V, W), \qquad H^{U,V}_W = \mathrm{Hom}_{\mathcal{C}^H}(W, U \otimes V),$$
$$H_{U,V,W} = \mathrm{Hom}_{\mathcal{C}^H}(U \otimes V \otimes W, \mathbb{I}), \qquad H^{U,V,W} = \mathrm{Hom}_{\mathcal{C}^H}(\mathbb{I}, U \otimes V \otimes W),$$

where U, V, W are weight modules. If there is no ambiguity, for $\alpha \in B$ we replace V_α with α in this notation, eg $H^{V_\alpha}_{V_\beta,V_\gamma} = H^\alpha_{\beta,\gamma}$. Also, since V_α^* and $V_{-\alpha}$ are identified we can replace V_α^* with $-\alpha$, eg $H^{V_\alpha^*}_{V_\beta,V_\gamma} = H^{-\alpha}_{\beta,\gamma}$.

We define the symmetric multiplicity module of U, V, W to be the space $H(U, V, W)$ obtained by identifying the 12 following isomorphic spaces

$$\begin{aligned} &H^{U,V,W} \simeq H^{W,U,V} \simeq H^{V,W,U} \simeq H^U_{W^*,V^*} \simeq H^W_{V^*,U^*} \simeq H^V_{U^*,W^*} \simeq \\ &H^{U,V}_{W^*} \simeq H^{W,U}_{V^*} \simeq H^{V,W}_{U^*} \simeq H_{W^*,V^*,U^*} \simeq H_{V^*,U^*,W^*} \simeq H_{U^*,W^*,V^*} \end{aligned} \tag{12}$$

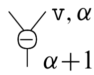
where each of these isomorphisms come from certain duality morphisms (see [9]).

For $\alpha \in \mathbb{C} \setminus \mathbb{Z}$ the character formula implies

$$\mathrm{v} \otimes V_\alpha \simeq V_{\alpha-1} \oplus V_{\alpha+1}. \tag{13}$$

Therefore, for $\alpha \in \mathbb{C} \setminus \mathbb{Z}$, the space $H_\beta^{\mathrm{v},\alpha}$ is the zero space if $\beta \neq \alpha \pm 1$ and $H_\beta^{\mathrm{v},\alpha}$ has dimension 1 if $\beta = \alpha \pm 1$.

Consider the morphism



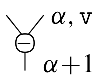
$: V_{\alpha+1} \to \mathrm{v} \otimes V_\alpha$

given by $\quad v_0 \mapsto v_0 \otimes v_0 \quad$ and $\quad v_i \mapsto \xi^{-i} v_0 \otimes v_i + \dfrac{\{i\}}{\{1\}} v_1 \otimes v_{i-1}.$

This morphism forms a basis of $H_{\alpha+1}^{\mathrm{v},\alpha}$. Thus, this morphism and the cyclic isomorphisms

$$H_{\alpha+1}^{\alpha,\mathrm{v}} \simeq H_{-\alpha}^{\mathrm{v},-\alpha-1}, \quad H_{\alpha+1,\mathrm{v}}^{\alpha} \simeq H_{-\alpha}^{\mathrm{v},-\alpha-1}, \quad H_{\mathrm{v},\alpha+1}^{\alpha} \simeq H_{\alpha+1}^{\mathrm{v},\alpha}$$

induce a basis on $H_{\alpha+1}^{\alpha,\mathrm{v}}$, $H_{\alpha+1,\mathrm{v}}^{\alpha}$, and $H_{\mathrm{v},\alpha+1}^{\alpha}$. Each of these basis consists of the single morphism which we denote by



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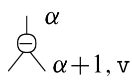

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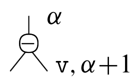
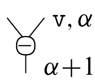


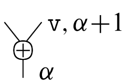
,

respectively. Moreover, the morphism



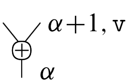
and isomorphisms represented in Equation (12) define a basis vector $\omega^-(\alpha)$ for the symmetric module $H(\mathrm{v}, \alpha, -\alpha-1)$.

Similarly, consider the basis of $H_\alpha^{\mathrm{v},\alpha+1}$ given by the morphism



$$: \quad v_{2r'} \mapsto \xi^{-1}\{\alpha - 2r'\} v_1 \otimes v_{2r'},$$
$$v_i \mapsto -\xi^{\alpha-i-1}\{1\} v_0 \otimes v_{i+1} + \xi^{-1}\{\alpha - i\} v_1 \otimes v_i.$$

As above this morphism and the isomorphisms in (12) induce basis of $H_\alpha^{\alpha+1,\mathrm{v}}$, $H_{\alpha,\mathrm{v}}^{\alpha+1}$, $H_{\mathrm{v},\alpha}^{\alpha+1}$ and $H(\mathrm{v}, \alpha+1, -\alpha)$ which each consist of one morphism which we denote by



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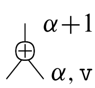

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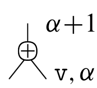


, $\quad \omega^+(\alpha),$

respectively.

The next proposition is illustrated by Figure 3. It computes the pairing of some of the families of morphisms defined above.

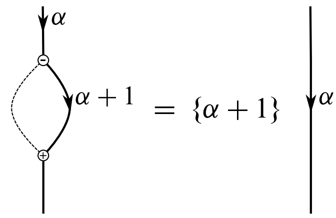
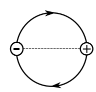


Figure 3: The duality for $H(V_\alpha, V_{\alpha\pm1}, \mathrm{v})$

Proposition 8 *We have the relation*

$$\ominus^{\alpha}_{\mathrm{v},\alpha+1} \circ \oplus^{\mathrm{v},\alpha+1}_{\alpha} = \{\alpha+1\}\,\mathrm{Id}_{V_\alpha}\,. \tag{14}$$

The evaluation of F' on the colored Θ–graph



induce the pairing

$$H(\mathrm{v}, \alpha, -\alpha-1) \otimes H(\mathrm{v}, \alpha+1, -\alpha) \to \mathbb{C}$$

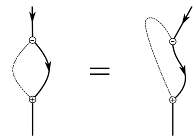
determined by

$$\big\langle \omega^-(\alpha), \omega^+(\alpha) \big\rangle = (-1)^{r'} \frac{\{\alpha\}\{\alpha+1\}}{\{l\alpha\}} = \prod_{k=2}^{2r'} \frac{1}{\{\alpha+k\}}.$$

Proof The duality follows from the value of $\mathrm{d}(V_\alpha)$ and from the first statement which is the result of the following computation:

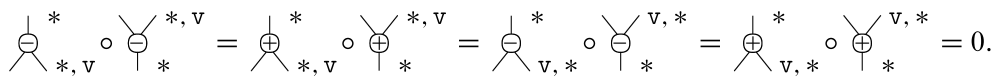
$$\begin{aligned}
&\ominus^{\alpha}_{\mathrm{v},\alpha+1} \circ \oplus^{\mathrm{v},\alpha+1}_{\alpha} (v_0)\\
&= (d_{\mathrm{v}} \otimes \mathrm{Id}_{V_\alpha}) \circ (w_{\mathrm{v}} \otimes \mathrm{Id}_{\mathrm{v}} \otimes \mathrm{Id}_{V_\alpha}) \circ \left(\mathrm{Id}_{\mathrm{v}} \otimes \ominus^{\mathrm{v},\alpha}_{\alpha+1}\right) \circ \oplus^{\mathrm{v},\alpha+1}_{\alpha} (v_0)\\
&= (d_{\mathrm{v}} \otimes \mathrm{Id}) \circ (w_{\mathrm{v}} \otimes \mathrm{Id} \otimes \mathrm{Id}) \circ \left(\mathrm{Id} \otimes \ominus^{\mathrm{v},\alpha}_{\alpha+1}\right)(-\xi^{\alpha-1}\{1\}v_0 \otimes v_1 + \xi^{-1}\{\alpha\}v_1 \otimes v_0)\\
&= (d_{\mathrm{v}} \otimes \mathrm{Id}) \circ (w_{\mathrm{v}} \otimes \mathrm{Id} \otimes \mathrm{Id})(-\xi^{\alpha-1}\{1\}v_0 \otimes v_1 \otimes v_0 + \xi^{-1}\{\alpha\}v_1 \otimes v_0 \otimes v_0)\\
&= (d_{\mathrm{v}} \otimes \mathrm{Id})(\xi^{\alpha}\{1\}v_1^* \otimes v_1 \otimes v_0 + \xi^{-1}\{\alpha\}v_0^* \otimes v_0 \otimes v_0)\\
&= \xi^{\alpha}\{1\}v_0 + \xi^{-1}\{\alpha\}v_0 = \{\alpha+1\}v_0.
\end{aligned}$$

Here the first equality corresponds to the following isotopy:



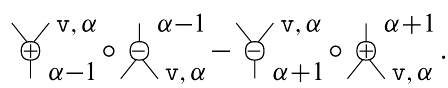
This completes the proof. □

Remark 9 If $\alpha \neq \beta$ are in $\mathbb{C} \setminus \mathbb{Z}$ then V_α and V_β are nonisomorphic simple modules and we have $\mathrm{Hom}(V_\alpha, V_\beta) = 0$. Thus,



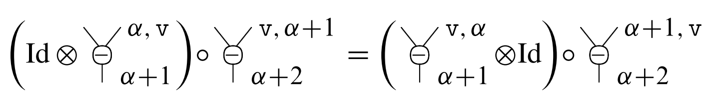
Here and after, the stars $*$ shall be replaced by any element in $\mathbb{C} \setminus \mathbb{Z}$ such that the morphisms are defined.

Corollary 10 (Fusion rule) *For any* $\alpha \in \mathbb{C} \setminus \mathbb{Z}$,

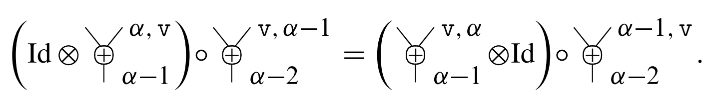
(15) $\{\alpha\}\, \mathrm{Id}_{\mathrm{v}\otimes V_\alpha} =$ 

Proof This is a direct consequence of Proposition 8 and the fact that $\mathrm{v} \otimes V_\alpha$ split into a direct sum of simple modules as in (13). (Also see Remark 9.) □

Lemma 11 *For all* $\alpha \in \mathbb{C} \setminus \mathbb{Z}$, *one has*



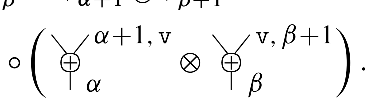
and similarly



Proof The first equality is true because both side are maps $V_{\alpha+2} \to \mathrm{v} \otimes V_\alpha \otimes \mathrm{v}$ determined by $v_0 \mapsto v_0 \otimes v_0 \mapsto v_0 \otimes v_0 \otimes v_0$. Similarly, an easy computation gives that the other two maps $V_{\alpha-2} \to \mathrm{v} \otimes V_\alpha \otimes \mathrm{v}$ are determined by $v_{2r'} \mapsto \xi^{-2}\{\alpha\}\{\alpha-1\} v_1 \otimes v_{2r'} \otimes v_1$. □

If $\alpha, \beta \in \mathbb{C} \setminus \mathbb{Z}$, we will use the following family of operators

$$X \colon V_\alpha \otimes V_\beta \to V_{\alpha+1} \otimes V_{\beta+1}$$

given by $X = \frac{1}{\{1\}}(\mathrm{Id} \otimes d^{\mathrm{v}} \otimes \mathrm{Id}) \circ$ 

The following lemma shows that the denominator of X disappears.

Lemma 12 *For $\alpha, \beta \in \mathbb{C} \setminus \mathbb{Z}$ the map $X\colon V_\alpha \otimes V_\beta \to V_{\alpha+1} \otimes V_{\beta+1}$ is given by*

$$X\colon v_i \otimes v_j \mapsto \xi^{\beta+i-j-1}\{\alpha - i\} v_i \otimes v_{j+1} + \xi^{-1}\{\beta - j\} v_{i+1} \otimes v_j$$

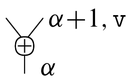
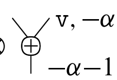
where $v_{2r'+1}$ should be understood as 0.

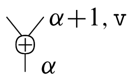
Proof First, a direct computations shows that

$$d^\alpha(v_i \otimes v_{2r'-j}) = -\delta_j^i \xi^{i\alpha+i^2-1}, \tag{16}$$

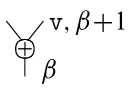
$$b^\alpha(1) = \sum_{i=0}^{2r'} -\xi^{-i\alpha+1-i^2} v_{2r'-i} \otimes v_i. \tag{17}$$

Then using

 $= (\mathrm{Id} \otimes d^\alpha) \circ (\mathrm{Id} \otimes$  $\otimes \mathrm{Id}) \circ (b^{\alpha+1} \otimes \mathrm{Id})$,

we have  $(v_i) = -\xi^i \{\alpha - i\} v_i \otimes v_1 - \xi^{-1}\{1\} v_{i+1} \otimes v_0.$

On the other hand, by definition we have

 $(v_j) = -\xi^{\beta-j-1}\{1\} v_0 \otimes v_{j+1} + \xi^{-1}\{\beta - j\} v_1 \otimes v_j.$

Combining these equalities with $d^{\mathrm{v}}(v_0 \otimes v_1) = -\xi$ and $d^{\mathrm{v}}(v_1 \otimes v_0) = 1$ the result follows. □

3.4 Multiplicity modules in $V_\alpha \otimes V_\beta \otimes V_\gamma$

It is well-known that, in the quantum plane $\mathbb{Z}\langle x, y\rangle_{/yx=\xi^2 xy}$, one has

$$(x+y)^i = \sum_{k=0}^{i} \xi^{k(i-k)} \begin{bmatrix} i \\ k \end{bmatrix} x^k y^{i-k}$$

for all $i \in \mathbb{N}$. Applying this to $y = K^{-1} \otimes F$ and $x = F \otimes 1$, we get

$$(\Delta F)^i = (x+y)^i = \sum_{k=0}^{i} \xi^{k(i-k)} \begin{bmatrix} i \\ k \end{bmatrix} F^k K^{k-i} \otimes F^{i-k}. \tag{18}$$

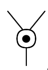
The character formula for typical modules (see [4]) also implies that for all $\alpha, \beta \in B$ with $\alpha+\beta \notin \mathbb{Z}$,

$$V_\alpha \otimes V_\beta = \sum_{k=-r'}^{r'} V_{\alpha+\beta+2k}. \tag{19}$$

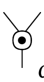
Hence, for $\alpha, \beta, \gamma \in \mathbb{C} \setminus \mathbb{Z}$,

$$\dim(H_\alpha^{\beta\gamma}) = \begin{cases} 1 & \text{if } \beta+\gamma-\alpha \in \{-2r', -2r'+2, \dots, 2r'\}, \\ 0 & \text{else.} \end{cases}$$

Now for $\alpha, \beta, \gamma \in \mathbb{C} \setminus \mathbb{Z}$ with $\beta+\gamma-\alpha = 2k \in \{-2r', -2r'+2, \dots, 2r'\}$ we define a map

 $\overset{\beta,\gamma}{\underset{\alpha}{2k}}$

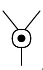
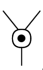
which will form a basis for the 1–dimensional space $H_\alpha^{\beta\gamma}$. First, suppose $\beta+\gamma-\alpha = -2r'$ then

 $\overset{\beta,\gamma}{\underset{\alpha}{-2r'}}\colon V_\alpha \to V_\beta \otimes V_\gamma$

$$v_0 \mapsto v_0 \otimes v_0$$

$$v_n \mapsto (\Delta F)^n v_0 \otimes v_0 = \sum_{k=0}^{n} \xi^{(n-k)(k-\beta-2r')} \begin{bmatrix} n \\ k \end{bmatrix} v_k \otimes v_{n-k}$$

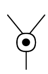
where the last equality follows from Equation (18). Now, let $n = r'+k$ and define

 $\overset{\beta,\gamma}{\underset{\alpha}{2k}} = X^{\circ n} \circ$  $\overset{\beta-n,\gamma-n}{\underset{\alpha}{-2r'}} \colon V_\alpha \to V_\beta \otimes V_\gamma.$

We now show that these bases are compatible with the cyclic isomorphisms defining the symmetric multiplicity modules. Let $\mathcal{R}$ be the cyclic isomorphism

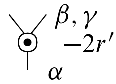
$$\begin{aligned} \mathcal{R}\colon H_\alpha^{\beta,\gamma} &\to H_{-\beta}^{\gamma,-\alpha} \\ f &\mapsto (d^\beta \otimes \mathrm{Id} \otimes \mathrm{Id}) \circ f \circ (\mathrm{Id} \otimes \mathrm{Id} \otimes b^\alpha). \end{aligned} \tag{20}$$

Remark 13 The family of maps

 $\overset{*,*}{\underset{*}{-2r'}}$

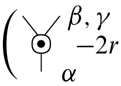
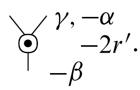
can be seen as a section of the vector bundle $\mathcal{E}_{-2r'}$ which is a restriction of $\mathcal{E} \otimes \mathcal{E} \otimes \mathcal{E}^*$ to the subset of B^3 defined by the equation $\beta+\gamma-\alpha = -2r'$. The cyclic isomorphism $\mathcal{R}$

is a lift to this vector bundle of the permutation on the basis: $(\alpha,\beta,\gamma)\mapsto(-\beta,\gamma,-\alpha)$. The following proposition means that the section

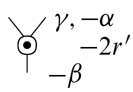
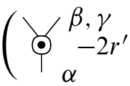


is a fixed point of the cyclic isomorphism $\mathcal{R}\colon \mathcal{E}_{-2r'}\to\mathcal{E}_{-2r'}$.

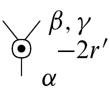
Proposition 14 *For all $(\alpha,\beta,\gamma)\in B^3$ with $\beta+\gamma-\alpha=-2r'$, we have*

$\mathcal{R}\Big($  $\Big) =$  .

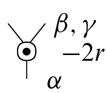
Proof Let

$f_1 =$  and $f_2 = \mathcal{R}\Big($  $\Big)$.

Since $f_1(v_0)=v_0\otimes v_0$, then f_2 is determined by its value on $v_0\in V_{-\beta}$ which must be a multiple of the unique weight vector $v_0\otimes v_0\in V_\gamma\otimes V_{-\alpha}$. Because of this, we don't need to compute all the terms to see that $f_2(v_0)=v_0\otimes v_0$. In particular, from the facts

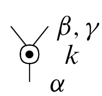
- $b_{V_\alpha}\colon 1\mapsto v_{2r'}\otimes v^*_{2r'}+\cdots$
- $w^{-1}_{-\alpha}(v^*_{2r'})=-\xi v_0$
- $w_{-\beta}(v_0)=-\xi^{-1}v^*_{2r'}$
-  $(v_{2r'})=(\Delta F)^{2r'}(v_0\otimes v_0)=v_{2r'}\otimes v_0+\cdots$ (because $(\Delta F)^{2r'}=F^{2r'}\otimes 1+\cdots$)
- $d_{V_\beta}(v^*_{2r'}\otimes v_{2r'})=1$

one can see that

$f_2(v_0)=(d_{V_\beta}\otimes\mathrm{Id}\otimes\mathrm{Id})\circ\Big(w_{-\beta}\otimes$  $\otimes\, w^{-1}_{-\alpha}\Big)\circ(\mathrm{Id}\otimes b_{V_\alpha})(v_0)$

is equal to $v_0\otimes v_0$. Thus, $f_1=f_2$. □

To establish the same statement for the maps



we will need the two following lemmas.

Lemma 15 *Let $\alpha,\beta,\gamma\in\mathbb{C}\setminus\mathbb{Z}$ such that $\alpha+\beta+\gamma=2-2r'$ then*

$$(X\otimes\mathrm{Id})\circ\left(\mathrm{Id}\otimes \underset{1-\alpha}{\overset{\beta-1,\gamma}{\odot}}{}_{-2r'}\right)\circ b^{\alpha-1}=(\mathrm{Id}\otimes X)\circ\left(\mathrm{Id}\otimes \underset{-\alpha}{\overset{\beta-1,\gamma-1}{\odot}}{}_{-2r'}\right)\circ b^{\alpha}.$$

Proof Both side of this equality are invariant maps $\mathbb{C}\to V_\alpha\otimes V_\beta\otimes V_\gamma$. Let Z_l and Z_r be the maps on the right and left hand sides, respectively. The space $H^{\alpha,\beta,\gamma}$ has dimension 1 so the maps Z_l and Z_r are proportional. Thus, to show they are equal it is enough to show the functions $(d^\alpha\otimes\mathrm{Id}\otimes d^\gamma)(v_0\otimes Z_i(1)\otimes v_{2r'})$, for $i=l,r$, are equal.

First, let us work on the left hand side. By considering the formulas for d^α and X the only terms of $b^{\alpha-1}(1)$ that contribute nontrivially to the function are $-\xi v_{2r'}\otimes v_0$ and $-\xi^{1-\alpha}v_{2r'-1}\otimes v_1$. Therefore, we only need to consider

$$\underset{*}{\overset{*,*}{\odot}}{}_{-2r'}(v_0)=v_0\otimes v_0\quad\text{and}\quad \underset{*}{\overset{*,*}{\odot}}{}_{-2r'}(v_1)=v_1\otimes v_0+\cdots$$

where the other term(s) contained in the $\cdots$ can be disregarded since $d^\gamma(v_i\otimes v_{2r'})$ is nonzero if and only if $i=0$. So $Z_l(1)$ is equal to

$$\begin{aligned}-\xi(X\otimes\mathrm{Id})(v_{2r'}\otimes v_0\otimes v_0)-\xi^{1-\alpha}(X\otimes\mathrm{Id})(v_{2r'-1}\otimes v_1\otimes v_0)+\cdots\\ =-\xi^{\beta-2}\{\alpha\}(v_{2r'}\otimes v_1\otimes v_0)-\xi^{-\alpha}\{\beta-2\}(v_{2r'}\otimes v_1\otimes v_0)+\cdots\end{aligned}$$

where as above the term(s) contained in the $\cdots$ can be disregarded since they do not contribute nontrivially to the function. So, we have

$$\begin{aligned}(d^\alpha\otimes\mathrm{Id}\otimes d^\gamma)(v_0\otimes Z_l(1)\otimes v_{2r'})&=-(\xi^{\beta-4}\{\alpha\}+\xi^{-\alpha-2}\{\beta-2\})v_1\\&=-\xi^{-2}\{\alpha+\beta-2\}v_1\end{aligned}$$

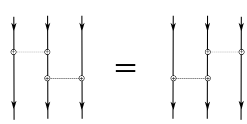
since $d^x(v_0\otimes v_{2r'})=-\xi^{-1}$. Similarly,

$$(d^\alpha\otimes\mathrm{Id}\otimes d^\gamma)(v_0\otimes Z_r(1)\otimes v_{2r'})=-\xi^{-2}\{\gamma-1\}v_1=-\xi^{-2}\{\alpha+\beta-2\}v_1.\quad\square$$

Lemma 16 *Let $\alpha,\beta,\gamma\in\mathbb{C}\setminus\mathbb{Z}$, then $X\otimes\mathrm{Id}$ and $\mathrm{Id}\otimes X$ commute:*

$$(X\otimes\mathrm{Id})\circ(\mathrm{Id}\otimes X)=(\mathrm{Id}\otimes X)\circ(X\otimes\mathrm{Id})\colon V_{\beta-1}\otimes V_{\alpha-2}\otimes V_{\gamma-1}\to V_\beta\otimes V_\alpha\otimes V_\gamma.$$

Graphically, this is illustrated by the following:



Proof Composing both side of the second equality of Lemma 11 with

$$\oplus\begin{smallmatrix}\beta\\ \beta-1,\,\mathrm{v}\end{smallmatrix} \otimes \mathrm{Id}_{V_\alpha} \otimes \oplus\begin{smallmatrix}\gamma\\ \mathrm{v},\,\gamma-1\end{smallmatrix}$$

gives the result. □

Proposition 17 *For all* $(\alpha,\beta,\gamma)\in(\mathbb{C}\setminus\mathbb{Z})^3$ *with* $\beta+\gamma-\alpha=k\in\{-2r',-2r'+2,\dots,2r'\}$ *we have*

$$\mathcal{R}\left(\odot\begin{smallmatrix}\beta,\gamma\\ k\\ \alpha\end{smallmatrix}\right)=\odot\begin{smallmatrix}\gamma,-\alpha\\ k\\ -\beta\end{smallmatrix}$$

where $\mathcal{R}\colon H_\alpha^{\beta,\gamma}\to H_{-\beta}^{\gamma,-\alpha}$ *is given in Equation* (20).

Proof We first give a reformulation of Proposition 14: tensoring the equality with Id_{V_β} on the left and composing on the right with b^β, we have

$$\left(\mathrm{Id}\otimes\odot\begin{smallmatrix}\gamma,-\alpha\\ -2r'\\ -\beta\end{smallmatrix}\right)\circ b^\beta=\left(\odot\begin{smallmatrix}\beta,\gamma\\ -2r'\\ \alpha\end{smallmatrix}\otimes\mathrm{Id}\right)\circ b^\alpha$$

for all $\alpha,\beta,\gamma\in(\mathbb{C}\setminus\mathbb{Z})^3$ with $\beta+\gamma-\alpha=-2r'$. Let $\alpha,\beta,\gamma\in\mathbb{C}\setminus\mathbb{Z}$ such that $\alpha+\beta+\gamma=2-2r'$ then from Lemmas 15 and 16 we have that for $p,q\in\mathbb{N}$,

$$(X^{\circ p+1}\otimes\mathrm{Id})\circ(\mathrm{Id}\otimes X^{\circ q})\circ\left(\mathrm{Id}\otimes\odot\begin{smallmatrix}\beta-1,\gamma\\ -2r'\\ 1-\alpha\end{smallmatrix}\right)\circ b^{\alpha-1}$$

$$=(X^{\circ p}\otimes\mathrm{Id})\circ(\mathrm{Id}\otimes X^{\circ q+1})\circ\left(\mathrm{Id}\otimes\odot\begin{smallmatrix}\beta-1,\gamma-1\\ -2r'\\ -\alpha\end{smallmatrix}\right)\circ b^\alpha.$$

Then by induction, for any $n\in\mathbb{N}$ and for any $\alpha,\beta,\gamma\in\mathbb{C}\setminus\mathbb{Z}$ such that $\alpha+\beta+\gamma=2n-2r'$, one has

$$(X^{\circ n}\otimes\mathrm{Id})\circ\left(\mathrm{Id}\otimes\odot\begin{smallmatrix}\beta-n,\gamma\\ -2r'\\ n-\alpha\end{smallmatrix}\right)\circ b^{\alpha-n}=(\mathrm{Id}\otimes X^{\circ n})\circ\left(\mathrm{Id}\otimes\odot\begin{smallmatrix}\beta-n,\gamma-n\\ -2r'\\ -\alpha\end{smallmatrix}\right)\circ b^\alpha.$$

Therefore, for $n=k/2+r'\in\mathbb{N}$ we have

$$\mathcal{R}\left(\odot\begin{smallmatrix}\beta,\gamma\\ k\\ \alpha\end{smallmatrix}\right)=(d^\beta\otimes\mathrm{Id}\otimes\mathrm{Id})\circ(\mathrm{Id}\otimes\left(X^{\circ n}\circ\odot\begin{smallmatrix}\beta-n,\gamma-n\\ -2r'\\ \alpha\end{smallmatrix}\right)\otimes\mathrm{Id})\circ(\mathrm{Id}\otimes b^\alpha)$$

$$=(d^\beta\otimes\mathrm{Id}\otimes\mathrm{Id})\circ(\mathrm{Id}\otimes\mathrm{Id}\otimes X^{\circ n})\circ\left(\mathrm{Id}\otimes\odot\begin{smallmatrix}\beta,\gamma-n\\ -2r'\\ \alpha-n\end{smallmatrix}\otimes\mathrm{Id}\right)\circ(\mathrm{Id}\otimes b^{\alpha-n})$$

$$
\begin{aligned}
&= X^{\circ n} \circ (d^{\beta} \otimes \mathrm{Id} \otimes \mathrm{Id}) \circ \left(\mathrm{Id} \otimes \odot\begin{smallmatrix}\beta,\gamma-n\\ -2r'\\ \alpha-n\end{smallmatrix} \otimes \mathrm{Id} \right) \circ (\mathrm{Id} \otimes b^{\alpha-n}) \\
&= X^{\circ n} \circ \odot\begin{smallmatrix}\gamma-n,n-\alpha\\ -2r'\\ -\beta\end{smallmatrix} \quad = \odot\begin{smallmatrix}\gamma,-\alpha\\ k\\ -\beta\end{smallmatrix}
\end{aligned}
$$

where the second to last equality is given by Proposition 14. □

The cyclic isomorphisms allow us to define the basis

$$\odot\begin{smallmatrix}\alpha\\ k\\ \beta,\gamma\end{smallmatrix}$$

of $H^{\alpha}_{\beta,\gamma}$ in two equivalent way: if $\alpha-\beta-\gamma=k$, let

$$\odot\begin{smallmatrix}\alpha\\ k\\ \beta,\gamma\end{smallmatrix} = (\mathrm{Id} \otimes d^{\gamma}) \circ \left(\odot\begin{smallmatrix}\alpha,-\gamma\\ k\\ \beta\end{smallmatrix} \otimes \mathrm{Id} \right) = (d^{-\beta} \otimes \mathrm{Id}) \circ \left(\mathrm{Id} \otimes \odot\begin{smallmatrix}-\beta,\alpha\\ k\\ \gamma\end{smallmatrix} \right).$$

Similarly, if $\alpha+\beta+\gamma=k$, we get a vector $\omega^k(\alpha,\beta,\gamma)$, which forms a canonical basis of the symmetric multiplicity module $H(\alpha,\beta,\gamma)$.

In what follows we consider ribbons graphs with coupons colored by the elements $\omega^k(\alpha,\beta,\gamma)$. For such a coupon c, Proposition 17 implies that we do not need to know what edges are attached to the bottom of c and what edges are attached to its top. Only the information of the cyclic ordering of these edges is needed to compute F or F'.

The choice of an half twist θ' (a family of endomorphisms whose square are given by the twist) produces isomorphisms

$$H^{\alpha,\beta}_{\gamma} \to H^{\beta,\alpha}_{\gamma} \quad \text{given by} \quad f \mapsto \theta'_{\alpha}\theta'_{\beta}{\theta'_{\gamma}}^{-1} C_{V_\alpha,V_\beta} \circ f$$

(for details see [3]). These isomorphism produces isomorphisms $H(\alpha,\beta,\gamma) \to H(\beta,\alpha,\gamma)$. The following lemma shows that the bases we have defined above are compatible with these isomorphisms.

Lemma 18 *We can define an half twist on the set of typical modules $\{V_\alpha\}_{\alpha\in B}$ by the formula*

$$\theta'_{\alpha} = \xi^{(\alpha/2)^2-r'^2}\,\mathrm{Id}_{V_\alpha}.$$

Let $\alpha, \beta, \alpha+\beta \in \mathbb{C} \setminus \mathbb{Z}$. *Then*

$$C_{V_\alpha,V_\beta} \circ \begin{smallmatrix} & \alpha,\beta \\ \odot & -2r' \\ * & \end{smallmatrix} = \xi^{(1/2)(\alpha+2r')(\beta+2r')} \begin{smallmatrix} & \beta,\alpha \\ \odot & -2r' \\ * & \end{smallmatrix}, \tag{21}$$

$$C_{V_\alpha,\mathrm{v}} \circ \begin{smallmatrix} & \alpha,\mathrm{v} \\ \oplus & \\ & \alpha-1 \end{smallmatrix} = \xi^{-(1/2)\alpha}\xi^{r'} \begin{smallmatrix} & \mathrm{v},\alpha \\ \oplus & \\ & \alpha-1 \end{smallmatrix}, \tag{22}$$

$$C_{\mathrm{v},V_\alpha} \circ \begin{smallmatrix} & \mathrm{v},\alpha \\ \oplus & \\ & \alpha-1 \end{smallmatrix} = \xi^{-(1/2)\alpha}\xi^{r'} \begin{smallmatrix} & \alpha,\mathrm{v} \\ \oplus & \\ & \alpha-1 \end{smallmatrix}, \tag{23}$$

$$C_{V_\alpha,V_\beta} \circ X = \xi^{(-\alpha-\beta+1)/2} X \circ C_{V_{\alpha-1},V_{\beta-1}}, \tag{24}$$

and for $n = r'+k$,

$$C_{V_\alpha,V_\beta} \circ \begin{smallmatrix} & \alpha,\beta \\ \odot & 2k \\ & \alpha+\beta-2k \end{smallmatrix} = \frac{\theta'_{\alpha+\beta-2k}}{\theta'_\alpha \theta'_\beta} \begin{smallmatrix} & \beta,\alpha \\ \odot & 2k \\ & \alpha+\beta-2k \end{smallmatrix}. \tag{25}$$

Proof From Formula (8), we have that θ_{V_α} acts on the highest weight vector $v_0 \in V_\alpha$ as

$$K^{-2r'}\xi^{H^2/2}v_0 = \xi^{-2r'(\alpha+2r')+(\alpha+2r')^2/2}v_0 = \xi^{\alpha^2/2-2r'^2}v_0.$$

Hence θ' is an half twist.

Only the "Cartan" part $\xi^{H\otimes H}$ of the R-matrix (7) acts nontrivially on the tensor product of two highest weight vectors. Hence

$$C_{V_\alpha,V_\beta}(v_0\otimes v_0) = \xi^{(1/2)(\alpha+2r')(\beta+2r')} v_0\otimes v_0 \in V_\beta \otimes V_\alpha.$$

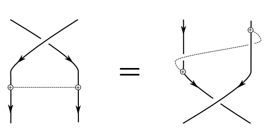
But $$v_0 \otimes v_0 = \begin{smallmatrix} & \alpha,\beta \\ \odot & -2r' \\ * & \end{smallmatrix}(v_0)$$

and this gives Equation (21).

Similarly, $$C_{V_\alpha,\mathrm{v}} \circ \begin{smallmatrix} & \alpha,\mathrm{v} \\ \ominus & \\ & \alpha+1 \end{smallmatrix} = \xi^{(1/2)\alpha}\xi^{r'} \begin{smallmatrix} & \mathrm{v},\alpha \\ \ominus & \\ & \alpha+1 \end{smallmatrix}$$

and Equation (22) follows from the duality of (14). Equation (22) is proved with analogous techniques.

To prove Equation (24), consider the isotopy



which illustrates the fact that $\{1\}C_{V_\alpha,V_\beta}\circ X$ is equal to

$$\text{(26)}\quad \left(\mathrm{Id}\otimes\left(\oplus^{\alpha}_{\alpha-1,\,\mathrm{v}}\circ C^{-1}_{V_{\alpha-1},\mathrm{v}}\right)\right)\circ\left(\left(C_{\mathrm{v},V_\beta}\circ\oplus^{\mathrm{v},\beta}_{\beta-1}\right)\otimes\mathrm{Id}\right)\circ C_{V_{\alpha-1},V_{\beta-1}}.$$

Here Equation (23) can be used to remove the braiding C_{v,V_β} in (26). Now Equation (22) implies that

$$\oplus^{\alpha}_{\alpha-1,\,\mathrm{v}}\circ C^{-1}_{V_{\alpha-1},\mathrm{v}}=\xi^{-\alpha/2}\xi^{r'}\oplus^{\alpha}_{\mathrm{v},\,\alpha-1}$$

and Equation (24) follows.

Finally, Equation (25) follows from

$$\begin{aligned}C_{V_\alpha,V_\beta}\circ X^n&=\xi^{-(1/2)\big((\alpha+\beta-1)+(\alpha+\beta-3)+\cdots+(\alpha+\beta-2n+1)\big)}X^n\circ C_{V_{\alpha-n},V_{\beta-n}}\\&=\xi^{-(1/2)n(\alpha+\beta-n)}X^n\circ C_{V_{\alpha-n},V_{\beta-n}}.\end{aligned}$$

Composing this equation with

$$\odot^{\alpha-n,\,\beta-n}_{\alpha+\beta-2k}\;\; -2r'$$

and applying Equation (21), the result follows. □

3.5 A Laurent polynomial invariant of planar trivalent graphs

In this section we discuss how to defined maps lead to invariant of planar graphs that are in some sense Laurent polynomial.

Let $\Gamma\subset\mathbb{R}\times[0,1]$ be a planar unitrivalent framed graph with trivalent vertices marked by heights, that are integers in $\{-2r',-2r'+2,\dots,2r'-2,2r'\}$ and whose set Γ_u of univalent vertices is included in $\mathbb{R}\times\{0,1\}$. The heights can be seen as a 0–chain h on the CW–complex Γ relative to Γ_u. A coloring of Γ is a complex 1–chain $c\in C_1(\Gamma,\Gamma_u;\mathbb{C})$ such that its boundary is $\delta c=h$. Let $\mathrm{Col}(\Gamma)$ be the affine space of coloring of Γ and $\mathrm{Col}_0(\Gamma)$ be the subset of coloring that have no values (no coefficients) in $\mathbb{Z}$.

Since a coloring is a realization of h as a boundary we have the set $\mathrm{Col}(\Gamma)$ is nonempty if and only if $[h]=0\in H_0(\Gamma,\Gamma_u;\mathbb{Z})$. This means that the sum of the heights of any connected component of Γ that does not meet Γ_u is zero. Let us assume that this is true and let $n=\dim H_1(\Gamma,\Gamma_u;\mathbb{C})$. Then $\mathrm{Col}(\Gamma)$ is an affine space over $H_1(\Gamma,\Gamma_u;\mathbb{C})$. We then choose a family of n edges $e_1,\dots,e_n$ of Γ. We assume that the union of the interior of these edges has a complement in Γ/Γ_u which is simply connected. Then the map

$$\mathrm{Col}(\Gamma)\to\mathbb{C}^n\quad\text{given by}\quad c\mapsto(c(e_1),\dots,c(e_n))$$

is bijective.

We will also suppose that every edge of Γ is in the support of a relative cycle. Hence, any coloring that takes an integer value on an edge can be infinitesimally modified to a coloring of $\mathrm{Col}_0(\Gamma)$. Then $\mathrm{Col}_0(\Gamma)$ is an open dense subset of $\mathrm{Col}(\Gamma)$.

If $c \in \mathrm{Col}_0(\Gamma)$, we can form a $\mathcal{C}$–colored ribbon graph $c(\Gamma)$ as follows. First, we choose an orientation of the edges of Γ. Color each oriented edge e of Γ with $V_{c(e)}$. Any trivalent vertex of Γ with height k is replaced with a trivalent coupon containing the morphism ω_k previously defined. Positioning the edges around the coupon involves some choice but the value under F (or F' if Γ is closed ie has no univalent vertices) of the resulting ribbon graph does not depend of these choices.

Theorem 19 *Let Γ be a planar unitrivalent framed graph with height h as above. Also, as above choose n edges $e_1, \dots, e_n$ of Γ. Supposing that Γ is not a circle, for any coloring $c \in \mathrm{Col}_0(\Gamma)$ define $x(c)$ as follows:*

(1) *If Γ has univalent vertices, then let $x(c)$ be any fixed coefficient of the matrix in the canonical bases of $F(c(\Gamma))$,*

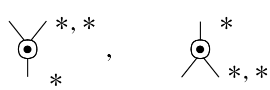
(2) *else, Γ is closed and let $x(c)$ be $F'(c(\Gamma))$.*

Then there exists a unique Laurent polynomial

$$P(q_1, \dots, q_n) \in \mathbb{Z}[\xi][q_1^{\pm 1}, \dots, q_n^{\pm 1}]$$

such that for any coloring $c \in \mathrm{Col}_0(\Gamma)$, $x(c) = P(\xi^{c(e_1)}, \dots, \xi^{c(e_n)})$.

Proof First consider the case $\Gamma_u \neq \varnothing$. For the existence of the Laurent polynomials, it is sufficient to remark that it is true for the elementary morphisms



and b^*, d^* from (10), (11). Now the uniqueness follows from the general fact that a Laurent polynomials in n variables with complex coefficients which vanishes on an open dense subset of $(\mathbb{C}^*)^n$ must be 0.

In the other case, Γ is a closed graph and $x(c) = F'(c(\Gamma))$. To compute $F'(c(\Gamma))$ we open $c(\Gamma)$ on an edge e to get a cutting presentation of $c(\Gamma)$. The invariant of this cutting presentation is then a scalar times the identity of $V_{c(e)}$. By the previous argument, this scalar is given by a Laurent polynomial P_e. $F'(c(\Gamma))$ is by definition this scalar times $\mathsf{d}(V_{c(e)}) = \mathsf{D}(\xi^{c(e)})^{-1} = \langle \xi^{c(e)} \rangle / \langle (\xi^{c(e)})^r \rangle$. This denominator seems to be a problem but in fact it must cancel. Indeed, as Γ is not a circle, we have $n \geq 2$. But $F'(c(\Gamma))$ does not depend on where we cut and open $c(\Gamma)$ (see [4, Theorem 3 and Section 6.2]). Hence cutting alternatively on the edges e_1, and then e_2, we get that there exits polynomials $P_1, P_2 \in \mathbb{Z}[\xi][q_1^{\pm1}, \dots, q_n^{\pm1}]$ such that

$$F'(c(\Gamma)) = \frac{P_i(\xi^{c(e_1)}, \dots, \xi^{c(e_n)})}{(\xi^{c(e_i)})^r - (\xi^{c(e_i)})^{-r}}$$

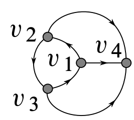
with $i \in \{1,2\}$ and thus

$$\frac{P_1}{q_1^r - q_1^{-r}} = \frac{P_2}{q_2^r - q_2^{-r}}.$$

Even if $\mathbb{Z}[\xi]$ is not a unique factorization domain, one easily see that this last equality implies that $(1/(q_1^r - q_1^{-r}))P_1 \in \mathbb{Z}[\xi][q_1^{\pm1}, \dots, q_n^{\pm1}]$. □

We now use this theorem applied to the tetrahedron graph to give an alternative definition of the polynomials J. This will prove in particular that their coefficients are in $\mathbb{Z}[\xi]$. As we will see Theorem 29 implies that this definition coincides with the formulas given in Section 1.

For $(i,j,k) \in \mathcal{H}_{r'}$ we consider the planar 1–skeleton Γ of the tetrahedron with heights as follows. Let v_1, v_2, v_3, v_4 be the vertices of Γ:

$\Gamma =$ 

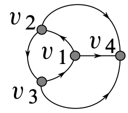
Assign v_1, v_2, v_3, v_4 the heights $2i, 2j, 2k, -2i-2j-2k$, respectively.

Definition 20 Let $\mathrm{J}_{i,j,k} \in \mathbb{Z}[\xi][q_1^{\pm1}, q_2^{\pm1}, q_3^{\pm1}] = \mathfrak{L}$ be the Laurent polynomial of Theorem 19 associated to Γ and the edges $(e_1, e_2, e_3) = (v_2v_3, v_3v_1, v_1v_2)$. Thus, $\mathrm{J}_{i,j,k}$ is the unique Laurent polynomial such that for all $\alpha, \beta, \gamma \in \mathbb{C}$ with $\alpha, \beta, \gamma, \alpha-\beta, \beta-\gamma, \gamma-\alpha \notin \mathbb{Z}$,

$$\begin{bmatrix} 2i & \alpha \\ 2j & \beta \\ 2k & \gamma \end{bmatrix} = \mathrm{J}_{i,j,k}(\xi^\alpha, \xi^\beta, \xi^\gamma) \tag{27}$$

where
$$\begin{bmatrix} 2i & \alpha \\ 2j & \beta \\ 2k & \gamma \end{bmatrix} = F'(c(\Gamma))$$

is the invariant of the graph

$\Gamma =$ 

colored with

$$\begin{array}{lll} c(v_2v_3)=\alpha & c(v_3v_1)=\beta & c(v_1v_2)=\gamma \\ c(v_1v_4)=\beta-\gamma-2i & c(v_2v_4)=\gamma-\alpha-2j & c(v_3v_4)=\alpha-\beta-2k. \end{array}$$

If $|i|,|j|,|k|$ or $|i+j+k|$ is $>r'$, then by convention, set

$$\begin{bmatrix} 2i & \alpha \\ 2j & \beta \\ 2k & \gamma \end{bmatrix}=0.$$

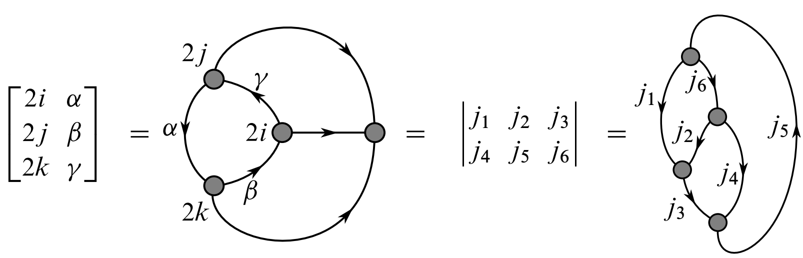
Here we change from the usual notation

$$\begin{vmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{vmatrix}$$

of [3] to the notation

$$\begin{bmatrix} 2i & \alpha \\ 2j & \beta \\ 2k & \gamma \end{bmatrix}.$$

We use the new notation because it is closely related to the polynomials J and easily adapts to the computations below. The correspondence between the two notation is given in Figure 4.



$$\begin{array}{llll} \text{with} & j_1=\alpha & j_2=-\beta & j_6=-\gamma \\ & j_4=\beta-\gamma-2i & j_5=\alpha-\gamma+2j & j_3=\alpha-\beta-2k \end{array}$$

Figure 4: The two notation for the $6j$–symbols $\mathrm{J}_{i,j,k}(\xi^\alpha,\xi^\beta,\xi^\gamma)$

3.6 Computations of the $6j$–symbols

The next proposition establishes the unexpected fact that the family of bases of the multiplicity modules constructed in Section 3.4 is self dual. Proposition 21 is illustrated by Figure 5 where the left hand side may be seen as a cutting presentation of the Θ–graph.

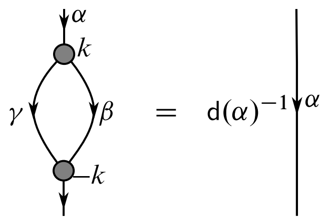


Figure 5: The duality for $H(V_\alpha, V_{-\beta}, V_{-\gamma})$

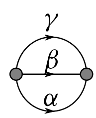
Proposition 21 *Let $\alpha = \beta + \gamma - 2r'$ then*

$$\underset{\beta,\gamma}{\overset{\alpha}{\odot}}{}_{-2r'} \circ X^{\circ 2r'} \circ \underset{\alpha}{\overset{\beta-2r',\,\gamma-2r'}{\odot}}{}_{-2r'} = \mathsf{d}(\alpha)^{-1}\,\mathrm{Id}_{V_\alpha}$$

and consequently, if $\alpha+\beta+\gamma = k$ then

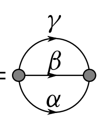
$$\left\langle \omega^k(\alpha,\beta,\gamma), \omega^{-k}(-\gamma,-\beta,-\alpha)\right\rangle = 1$$

where the duality $H(\alpha,\beta,\gamma)\otimes H(-\gamma,-\beta,-\alpha)\to\mathbb{C}$ is obtained by the evaluation of F' on the colored Θ–graph



.

Proof Let us denote by

$\Theta =$ 

the Θ–graph where the coupons are filled with the morphisms $\omega^k(\alpha,\beta,\gamma)$ and $\omega^{-k}(-\gamma,-\beta,-\alpha)$. We use properties of F' to compute $F'(\Theta)$ as follows. We have

$$F'(\Theta) = \mathsf{d}(V_\alpha)\left\langle \underset{\beta,\gamma}{\overset{\alpha}{\odot}}{}_{-2r'} \circ X^{\otimes 2r'} \circ \underset{\alpha}{\overset{\beta-2r',\,\gamma-2r'}{\odot}}{}_{-2r'} \right\rangle$$

$$= \mathsf{d}(V_{\gamma-2r'})\left\langle \underset{2r'-\beta,\,\alpha}{\overset{\gamma-2r'}{\odot}}{}_{-2r'} \circ \left(\mathrm{Id}\otimes \underset{\beta,\gamma}{\overset{\alpha}{\odot}}{}_{-2r'}\right) \circ \left(\mathrm{Id}\otimes X^{\circ 2r'}\right)\circ\left(b^{2r'-\beta}(1)\otimes \mathrm{Id}_{V_{\gamma-2r'}}\right)\right\rangle.$$

We compute the bracket of the right hand side of the last equality by evaluating the morphisms on the lowest weight vector $v_{2r'}\in V_{\gamma-2r'}$.

First remark that according to Lemma 12, X sends

$$V_{\delta+i}\otimes V_{\varepsilon+i}\ni v_i\otimes v_{2r'}\mapsto \xi^{-1}\{\varepsilon+i-2r'\}v_{i+1}\otimes v_{2r'}\in V_{\delta+i+1}\otimes V_{\varepsilon+i+1}.$$

Therefore, $X^{\circ 2r'}(v_i\otimes v_{2r'})=0$ if $i\geq 1$ and

$$X^{\circ 2r'}(v_0\otimes v_{2r'})=-\xi\left(\prod_{i=1}^{2r'}\{\varepsilon+i\}\right)v_{2r'}\otimes v_{2r'}$$

where here we use the equalities $\xi^{-2r'}=-\xi$ and $\{\varepsilon+i-2r'\}=-\{\varepsilon+i+1\}$. Applying this to $b^{2r'-\beta}(1)\otimes v_{2r'}\in V_{2r'-\beta}\otimes V_{\beta-2r'}\otimes V_{\gamma-2r'}$ we get

$$\begin{aligned}\mathrm{Id}\otimes X^{\circ 2r'}\big(b^{2r'-\beta}(1)\otimes v_{2r'}\big)&=\mathrm{Id}\otimes X^{\circ 2r'}\,(-\xi\, v_{2r'}\otimes v_0\otimes v_{2r'})\\&=\xi^2\left(\prod_{i=1}^{2r'}\{\gamma-2r'+i\}\right)v_{2r'}\otimes v_{2r'}\otimes v_{2r'}\in V_{2r'-\beta}\otimes V_\beta\otimes V_\gamma.\end{aligned}$$

Now using the fact that

$$Y^{*,*}_{*}{}_{-2r'}(v_{2r'})=v_{2r'}\otimes v_0+\cdots$$

and $d^*(v_0\otimes v_{2r'})=-\xi^{-1}$ we have

$$\lambda^{*}_{*,*}{}_{-2r'}(v_{2r'}\otimes v_{2r'})=(\mathrm{Id}\otimes d^*)\circ\left(Y^{*,*}_{*}{}_{-2r'}\otimes\mathrm{Id}\right)(v_{2r'}\otimes v_{2r'})=-\xi^{-1}v_{2r'}$$

and we see that the above bracket is equal to $\mathsf{d}(V_{\gamma-2r'})^{-1}$. $\square$

Remark that with Theorem 19, the previous result can be restated as saying: the Laurent polynomial associated to the Θ–graph with heights $k,-k$ is constant equal to 1.

Proposition 22

$$\left(\mathrm{Id}\otimes Y^{*,*}_{*}{}_{-2r'}\right)\circ Y^{*,*}_{*}{}_{-2r'}=\left(Y^{*,*}_{*}{}_{-2r'}\otimes\mathrm{Id}\right)\circ Y^{*,*}_{*}{}_{-2r'}\tag{28}$$

$$\left(\mathrm{Id}\otimes Y^{\mathrm{v},*}_{*}\right)\circ Y^{*,*}_{*}{}_{-2r'}=\left(Y^{*,\mathrm{v}}_{*}\otimes\mathrm{Id}\right)\circ Y^{*,*}_{*}{}_{-2r'}\tag{29}$$

Here the stars can be replaced by any colors in $\mathbb{C}\setminus\mathbb{Z}$ such that the compositions are matching, the source and target of the maps are the same in both side of the equalities and the colors meeting at a trivalent vertex satisfy the conditions given in Sections 3.3 and 3.4.

Proof All these maps send the highest weight vector of the bottom irreducible module v_0 to $v_0\otimes v_0\otimes v_0$. $\square$

Define the following operators:

- ${}^{-}X\colon V_\alpha \otimes V_\beta \to V_{\alpha-1} \otimes V_{\beta+1}$ by
$$ {}^{-}X = (\mathrm{Id} \otimes d^{\mathrm{v}} \otimes \mathrm{Id}) \circ \left(\underset{\alpha}{\ominus}^{\alpha-1,\,\mathrm{v}} \otimes \underset{\beta}{\oplus}^{\mathrm{v},\,\beta+1} \right), $$
- $X^{-}\colon V_\alpha \otimes V_\beta \to V_{\alpha+1} \otimes V_{\beta-1}$ by
$$ X^{-} = (\mathrm{Id} \otimes d^{\mathrm{v}} \otimes \mathrm{Id}) \circ \left(\underset{\alpha}{\oplus}^{\alpha+1,\,\mathrm{v}} \otimes \underset{\beta}{\ominus}^{\mathrm{v},\,\beta-1} \right), $$
- $\bar{X}\colon V_\alpha \otimes V_\beta \to V_{\alpha-1} \otimes V_{\beta-1}$ by
$$ \bar{X} = (\mathrm{Id} \otimes d^{\mathrm{v}} \otimes \mathrm{Id}) \circ \left(\underset{\alpha}{\ominus}^{\alpha-1,\,\mathrm{v}} \otimes \underset{\beta}{\ominus}^{\mathrm{v},\,\beta-1} \right). $$

From Corollary 10 we have a commutation rule for these operators:

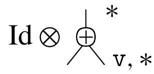
Lemma 23 *We have the following equalities of maps from $V_\alpha \otimes V_\beta$:*
$$ {}^{-}X \circ X = X \circ {}^{-}X \quad \text{and} \quad X^{-} \circ X = X \circ X^{-} $$
$$ {}^{-}X \circ \bar{X} = \bar{X} \circ {}^{-}X \quad \text{and} \quad X^{-} \circ \bar{X} = \bar{X} \circ X^{-} $$
$$ {}^{-}X \circ X^{-} - \{1\}\bar{X} \circ X = \{\alpha+1\}\{\beta\}\,\mathrm{Id}_{V_\alpha \otimes V_\beta} $$

Proof Consider the map $\mathrm{End}(\mathrm{v} \otimes V_\gamma) \to \mathrm{Hom}(V_* \otimes V_\gamma, V_* \otimes V_\gamma)$ given by
$$ y \mapsto \left(\overset{*}{\oplus}_{*,\,\mathrm{v}} \otimes \mathrm{Id} \right) \circ (\mathrm{Id} \otimes y) \circ \left(\underset{*}{\oplus}^{*,\,\mathrm{v}} \otimes \mathrm{Id} \right). $$
The identities of the lemma are obtained by composing this map with both sides of Equation (15). $\square$

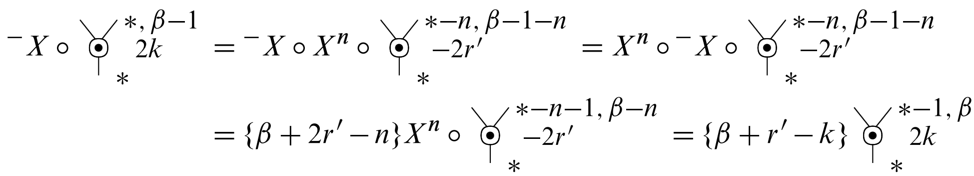
The following proposition describes how these operators act on multiplicity modules.

Proposition 24 *For any $\alpha, \beta, \gamma \in \mathbb{C} \setminus \mathbb{Z}$ and any $k \in \{-r', \dots, r'\}$, we have the following:*
$$ {}^{-}X \circ \underset{*}{\odot}_{2k}^{*,\,\beta-1} = \{\beta + r' - k\}\, \underset{*}{\odot}_{2k}^{*-1,\,\beta} $$
$$ X^{-} \circ \underset{*}{\odot}_{2k}^{\alpha-1,\,*} = \{\alpha + r' - k\}\, \underset{*}{\odot}_{2k}^{\alpha,\,*-1} $$
$$ \bar{X} \circ \underset{\gamma}{\odot}_{2k+2}^{*,\,*} = \frac{\{r'-k\}}{\{1\}} \{\gamma + k + r' + 1\}\, \underset{\gamma}{\odot}_{2k}^{*-1,\,*-1} $$

Proof Let us start with the first equality. If $k = -r'$ it is obtained by composing (29) with



where the factor $\{1-\beta\} = \{\beta + 2r'\}$ arises from the duality of (14). Now, for any $k = n - r' \in \{-r', \dots, r'\}$, we have

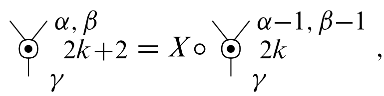


which proves the first equality. The proof of the second identity is similar.

For the third, Lemma 23 implies

$$\{1\}\bar{X} \circ X = {}^{-}X \circ X^{-} - \{\alpha\}\{\beta - 1\}\,\mathrm{Id}_{V_{\alpha-1} \otimes V_{\beta-1}}\,.$$

Then since



the identity comes from the equality

$$\{\alpha + r' - k\}\{\beta + r' - k - 1\} - \{\alpha\}\{\beta - 1\} = \{\alpha + \beta - k + r' - 1\}\{r' - k\}$$

with $\gamma = \alpha + \beta - 2k - 2$. □

Proposition 25

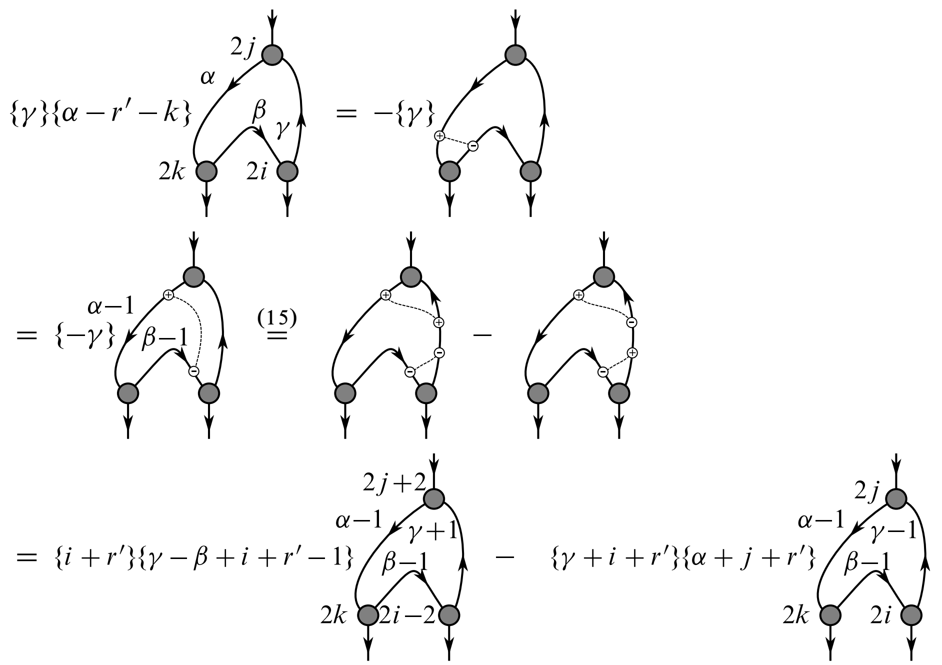
$$\begin{bmatrix} 2i & \alpha \\ 2j & \beta \\ 2k & \gamma \end{bmatrix} = \begin{bmatrix} 2j & \beta \\ 2k & \gamma \\ 2i & \alpha \end{bmatrix} = \begin{bmatrix} 2k & \gamma \\ 2i & \alpha \\ 2j & \beta \end{bmatrix} = \begin{bmatrix} 2k & -\gamma \\ 2j & -\beta \\ 2i & -\alpha \end{bmatrix}$$
$$= \begin{bmatrix} -2(i+j+k) & -\gamma \\ 2j & \beta - \gamma - 2i \\ 2i & \alpha - \gamma + 2j \end{bmatrix}$$
$$= \begin{bmatrix} -2(i+j+k) & -\beta \\ 2i & \alpha - \beta - 2k \\ 2k & \gamma - \beta + 2i \end{bmatrix} = \begin{bmatrix} -2(i+j+k) & -\alpha \\ 2k & \gamma - \alpha - 2j \\ 2j & \beta - \alpha + 2k \end{bmatrix}$$

Proof These identities are exactly the usual symmetries of $6j$–symbols. □

Lemma 26 *If $i \le r'$ and $j \ge -r'$ then*

$$\{i+r'\}\{\beta-\gamma-i+r'+2\}\begin{bmatrix}2i-2 & \alpha\\ 2j+2 & \beta\\ 2k & \gamma\end{bmatrix} = \{\gamma+i+r'-1\}\{\alpha+j+r'+1\}\begin{bmatrix}2i & \alpha\\ 2j & \beta\\ 2k & \gamma-2\end{bmatrix} + \{\gamma-1\}\{\alpha-k-r'\}\begin{bmatrix}2i & \alpha+1\\ 2j & \beta+1\\ 2k & \gamma-1\end{bmatrix}. \tag{30}$$

Proof We give a graphical proof:



then substitute γ with $\gamma-1$. □

Proposition 27 • *If $i+j+k=r'$ then*

$$\begin{bmatrix}2i & \alpha\\ 2j & \beta\\ 2k & \gamma\end{bmatrix} = \{i,j,k\}\{\alpha-r'-k;r'-i\}!\{\beta-r'-i;r'-j\}!\{\gamma-r'-j;r'-k\}!\,. \tag{31}$$

- *If* $i+j+k=-r'$ *then*

$$\begin{bmatrix}2i & \alpha\\ 2j & \beta\\ 2k & \gamma\end{bmatrix}=\{\delta+1;r'+i\}!\{\varepsilon+1;r'+j\}!\{\varphi+1;r'+k\}! \tag{32}$$

where we use the notation

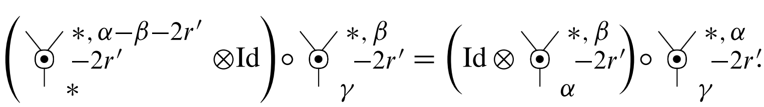
$$\begin{cases}\delta=\beta-\gamma-2i,\\ \varepsilon=\gamma-\alpha-2j,\\ \varphi=\alpha-\beta-2k.\end{cases}$$

Proof Remark first that these formulas are symmetric for the action of $\mathfrak{S}_3$ which permute simultaneously $\{i,j,k\}$ and $\{\alpha,\beta,\gamma\}$ and multiplies the last three variables by the signature of the permutation (see Proposition 25).

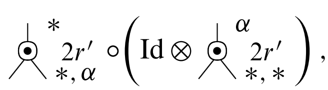
We will prove these identities by a recurrence on the natural number $n=2r'-\max(i,j,k)+\min(i,j,k)$. First, we prove Equation (31) when $n=0$. In this case up to a permutation we have $(i,j,k)=(-r',r',r')$. Hence, we must prove that

$$\begin{bmatrix}-2r' & \alpha\\ 2r' & \beta\\ 2r' & \gamma\end{bmatrix}=\{\alpha-2r';2r'\}!=\mathsf{d}(\alpha)^{-1}. \tag{33}$$

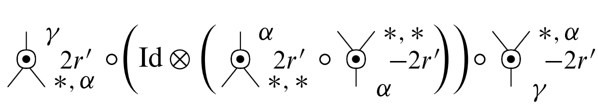
To do this, recall Equation (28):



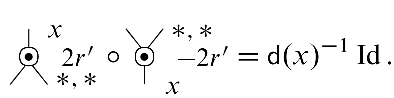
Composing this identity with



we get that

(34) $\mathsf{d}(\gamma)^{-1}\begin{bmatrix}-2r' & \alpha\\ 2r' & \beta\\ 2r' & \gamma\end{bmatrix}=$ 

Proposition 21 states that



Applying this identity twice in Equation (34) we arrive at (33). Remark that using the symmetries of the $6j$–symbols, Equation (33) also implies that

$$\begin{bmatrix} 2r' & \alpha \\ -2r' & \beta \\ -2r' & \gamma \end{bmatrix} = \{\beta-\gamma-2r';2r'\}! \tag{35}$$

which is the case $n=0$ for Equation (32).

Now let
$$\begin{bmatrix} 2i & \alpha \\ 2j & \beta \\ 2k & \gamma \end{bmatrix}'$$
be the right hand sides of Equation (31) (respectively Equation (32)). By induction, it is enough to show that
$$\begin{bmatrix} 2i & \alpha \\ 2j & \beta \\ 2k & \gamma \end{bmatrix}'$$
satisfy the relation of Lemma 26. Indeed, this relation applied to both sides of Equation (31) (respectively Equation (32)) after a well chosen permutation of (i,j,k) allows us to reduce n by 1 or 2.

- Let us start with Equation (31). Here $i+j+k=r'$ and direct computation shows

$$\{\gamma+i+r'-1\}\{\alpha+j+r'+1\}\begin{bmatrix} 2i & \alpha \\ 2j & \beta \\ 2k & \gamma-2 \end{bmatrix}' + \{\gamma-1\}\{\alpha-k-r'\}\begin{bmatrix} 2i & \alpha+1 \\ 2j & \beta+1 \\ 2k & \gamma-1 \end{bmatrix}'$$
$$= \{i,j,k\}\{\alpha-r'-k,r'-i+1\}!\{\beta-i-r'+1,r'-j-1\}!\{\gamma-j-r'-1,r'-k\}!$$
$$\times\big(-\{\beta-i-r'\}\{\gamma-j+r'-1\}+\{\beta+k-r'\}\{\gamma-1\}\big)$$

where we use that $\{x\pm(2r'+1)\}=-\{x\}$. Now this is equal to

$$\{i+r'\}\{\beta-\gamma-i+2+r'\}\begin{bmatrix} 2i-2 & \alpha \\ 2j+2 & \beta \\ 2k & \gamma \end{bmatrix}'$$

since

$$\{\beta+k-r'\}\{\gamma-1\}-\{\beta-i-r'\}\{\gamma-j+r'-1\}=\{r'-j\}\{\beta-\gamma-i+r'+2\}.$$

Thus,
$$\begin{bmatrix} 2i & \alpha \\ 2j & \beta \\ 2k & \gamma \end{bmatrix}'$$
satisfy the relation of Lemma 26 when $i+j+k=r'$.

- We now deal similarly with the case $i+j+k=-r'$.

$$\{\gamma+i+r'-1\}\{\alpha+j+r'+1\}\begin{bmatrix}2i & \alpha\\ 2j & \beta\\ 2k & \gamma-2\end{bmatrix}' + \{\gamma-1\}\{\alpha-k-r'\}\begin{bmatrix}2i & \alpha+1\\ 2j & \beta+1\\ 2k & \gamma-1\end{bmatrix}'$$
$$= \{\beta-\gamma-2i+3, i+r'\}!\{\gamma-\alpha-2j-1, j+r'\}!\{\alpha-\beta-2k+1, k+r'\}!$$
$$\times\big(\{\gamma+i+r'-1\}\{\alpha+j+r'+1\}+\{\gamma-1\}\{\alpha+i+j\}\big)$$

But now this is equal to

$$\{i+r'\}\{\beta-\gamma-i+2+r'\}\begin{bmatrix}2i-2 & \alpha\\ 2j+2 & \beta\\ 2k & \gamma\end{bmatrix}'$$

because

$$\big(\{\gamma+i+r'-1\}\{\alpha+j+r'+1\}+\{\gamma-1\}\{\alpha+i+j\}\big) = \{i+r'\}\{\gamma-\alpha-j-1+r'\}.$$

Thus,
$$\begin{bmatrix}2i & \alpha\\ 2j & \beta\\ 2k & \gamma\end{bmatrix}'$$
satisfy the relation of Lemma 26 when $i+j+k=-r'$ and this completes the proof. □

Let us now rewrite and generalize the relation of Lemma 26:

Lemma 28 *Let $(i,j,k)\in\mathcal{H}_{r'}$ and let $l=-i-j-k$. Here, we again use the "colors":*

$$\begin{cases}\delta=\beta-\gamma-2i,\\ \varepsilon=\gamma-\alpha-2j,\\ \varphi=\alpha-\beta-2k.\end{cases}$$

- *If $i+j+k=-l<r'$ and $k<r'$ then*

$$\begin{bmatrix}2i & \alpha\\ 2j & \beta\\ 2k & \gamma\end{bmatrix} = \frac{1}{\{k-r'\}\{-\alpha+k-r'\}}\left(\{-\varphi-k-r'\}\{\delta-l-r'\}\begin{bmatrix}2i & \alpha\\ 2j & \beta\\ 2k+2 & \gamma\end{bmatrix}\right.$$
$$\left.+\{\varphi-1\}\{-\delta-i-r'\}\begin{bmatrix}2i & \alpha\\ 2j & \beta-1\\ 2k+2 & \gamma\end{bmatrix}\right). \tag{36}$$

- *If* $N > 0$, $i + j + k = -l \leq r' - N$ *and* $k \leq r' - N$ *then*

$$\begin{bmatrix} 2i & \alpha \\ 2j & \beta \\ 2k & \gamma \end{bmatrix} = \frac{1}{\{k - r'; N\}!\{-\alpha + k - r'; N\}!} \left(\sum_{n=0}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \{-\varphi - k - r'; N - n\} \right.$$
$$\left. \times \{\delta - l - r'; N - n\}!\{\varphi - N; n\}!\{-\delta - i - r'; n\}! \begin{bmatrix} 2i & \alpha \\ 2j & \beta - n \\ 2k + 2N & \gamma \end{bmatrix} \right). \tag{37}$$

- *If* $N > 0$, $i + j + k = -l \geq N - r'$ *and* $k \geq N - r'$ *then*

$$\begin{bmatrix} 2i & \alpha \\ 2j & \beta \\ 2k & \gamma \end{bmatrix} = \frac{1}{\{l - r'; N\}!\{-\varepsilon + l - r'; N\}!} \tag{38}$$
$$\times \left(\sum_{n=0}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \{\varphi - l - r'; N - n\}!\{-\beta - k - r'; N - n\}! \right.$$
$$\left. \{-\varphi - N; n\}!\{\beta - i - r'; n\}! \begin{bmatrix} 2i & \alpha \\ 2j & \beta + n \\ 2k - 2N & \gamma \end{bmatrix} \right).$$

Proof The first relation (36) is obtained from Equation (30) by using the symmetry

$$\begin{bmatrix} 2i & \alpha \\ 2j & \beta \\ 2k & \gamma \end{bmatrix} = \begin{bmatrix} 2l & \beta \\ 2k & \beta - \gamma - 2i \\ 2i & \beta - \alpha + 2k \end{bmatrix}$$

and renaming the variables.

The second relation is shown by recurrence on N. For $N = 1$ it is just Equation (36). The induction step also follows from Equation (36). The meticulous reader who wants to check this relation carefully will have to use the identity:

$$\begin{bmatrix} N \\ n \end{bmatrix} \{\varphi + n - N - 1\} + \begin{bmatrix} N \\ n-1 \end{bmatrix} \{\varphi + n - 2N - 2\} = \begin{bmatrix} N+1 \\ n \end{bmatrix} \{\varphi - N - 1\}.$$

The third identity can be obtained from the second by using the symmetry

$$\begin{bmatrix} 2i & \alpha \\ 2j & \beta \\ 2k & \gamma \end{bmatrix} = \begin{bmatrix} 2i & \varepsilon \\ 2j & -\delta \\ 2l & \gamma \end{bmatrix}$$

and renaming the variables. $\square$

Theorem 29 • *For any $(i,j,k)\in\mathcal{H}_{r'}$ with $i,k\le i+j+k$, let $N=r'-i-j-k$. Then*

$$\begin{bmatrix}2i&\alpha\\2j&\beta\\2k&\gamma\end{bmatrix}=\{i,j,k\}\{\alpha-r'-k;j+k\}!\{\gamma-r'-j;i+j\}!\times\Bigg(\sum_{n=0}^{N}\begin{bmatrix}N\\n\end{bmatrix}\{\beta+k-\alpha+r'+1;N-n\}!\{\beta+k-\gamma-i+j-r';N-n\}!\{\alpha-\beta-2k-N;n\}!\{\gamma-\beta+i+r'+1;n\}!\{\beta-r'-i-n;r'-j\}!\Bigg).\tag{39}$$

• *For any $(i,j,k)\in\mathcal{H}_{r'}$ with $j,k\ge i+j+k$, let $N=r'+i+j+k$. Then*

$$\begin{bmatrix}2i&\alpha\\2j&\beta\\2k&\gamma\end{bmatrix}=\frac{\{\varepsilon+N+1;r'+j-N\}!}{\{N\}!}\Bigg(\sum_{n=0}^{N}\begin{bmatrix}N\\n\end{bmatrix}\{\beta-i-j+n+1;N-n\}\times\{\beta-i+r'+1;n\}!\{\varphi+N-n+1;r'+k\}!\{\delta+n+1;r'+i\}!\Bigg)\tag{40}$$

where we use the notation

$$\begin{cases}\delta=\beta-\gamma-2i,\\ \varepsilon=\gamma-\alpha-2j,\\ \varphi=\alpha-\beta-2k.\end{cases}$$

Proof The first formula is obtained by replacing

$$\begin{bmatrix}2i&\alpha\\2j&\beta-n\\2k+2N&\gamma\end{bmatrix}$$

with

$$\{i,j,k\}\{\alpha-r'-k+N;r'-i\}!\{\beta-r'-i-n;r'-j\}!\{\gamma-r'-j;r'-k-N\}!$$

in Equation (37) where $l=-i-j-k=N-r'$. The second formula is obtained from Equation (38) by replacing

$$\begin{bmatrix}2i&\alpha\\2j&\beta+n\\2k-2N&\gamma\end{bmatrix}$$

with

$$\{\delta+n+1;r'+i\}!\{\varepsilon+1;r'+j\}!\{\varphi+2N-n+1;r'+k-N\}!$$

where $l=-i-j-k=r'-N$. □

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