On the mapping space homotopy groups and the free loop space homology groups

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Let X be a Poincaré duality space, Y a space and $f\colon X \to Y$ a based map. We show that the rational homotopy group of the connected component of the space of maps from X to Y containing f is contained in the rational homology group of a space $L_f Y$ which is the pullback of f and the evaluation map from the free loop space LY to the space Y. As an application of the result, when X is a closed oriented manifold, we give a condition of a noncommutativity for the rational loop homology algebra $\mathbf{H}_*(L_f Y;\mathbb{Q})$ defined by Gruher and Salvatore which is the extension of the Chas–Sullivan loop homology algebra.

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1 Introduction

We assume that all topological spaces in this paper have a base point. Let M be a simply connected d–dimensional closed oriented manifold and LM the free loop space of M. We denote by $\mathrm{aut}_1 M$ the path component of the monoid of the self-homotopy equivalences of M containing the identity map.

In [8], Félix and Thomas constructed the injective map from the rational homotopy group of $\mathrm{aut}_1 M$ to the rational homology group of LM:

$$\pi_*(\mathrm{aut}_1 M)\otimes\mathbb{Q} \longrightarrow H_{*-1+d}(LM;\mathbb{Q}). \tag{1-1}$$

Now recall that Jones [13] proved that $H^*(LM;\mathbf{k})$ is isomorphic as a vector space to the Hochschild homology of the singular cochain algebra $S^*(M;\mathbf{k})$ of M over a field $\mathbf{k}$:

$$H^*(LM;\mathbf{k}) \cong \mathrm{HH}_*(S^*(M;\mathbf{k});S^*(M;\mathbf{k})).$$

and the dual of the above isomorphism and the Poincaré duality of M yield an isomorphism of graded vector spaces $H_{*+d}(LM;\mathbf{k}) \cong \mathrm{HH}^{-*}(S^*(M;\mathbf{k});S^*(M;\mathbf{k}))$. We now note that the cochain algebra $S^*(M;\mathbb{Q})$ over $\mathbb{Q}$ is weakly equivalent to a free commutative differential graded algebra over $\mathbb{Q}$, $(\Lambda V, d)$, called a Sullivan model for M; see the end of Section 5, and so $H_{*+d}(LM;\mathbb{Q}) \cong \mathrm{HH}^{-*}(\Lambda V;\Lambda V)$.

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On the other hand, Block and Lazarev [1] and Lupton and Smith [14] constructed an isomorphism from the n–th rational homotopy groups of $\mathrm{aut}_1 M$ to the $(-n)$–th homology of the differential graded module of derivations of ΛV:

$$\pi_n(\mathrm{aut}_1 M) \otimes \mathbb{Q} \xrightarrow{\cong} H^{-n}(\mathrm{Der}^*(\Lambda V, \Lambda V)).$$

Also, we see that there is a map $J_1^*\colon H^*(\mathrm{Der}^*(\Lambda V, \Lambda V)) \to \mathrm{HH}^{*+1}(\Lambda V; \Lambda V)$; see Section 5 for a proper definition. The result of Félix and Thomas [8] also shows that a topological meaning of the map J_1^* is the map (1-1). That is, we get the following commutative square:

$$\begin{array}{ccc} H_{n-1+d}(LM;\mathbb{Q}) & \xrightarrow{\cong} & \mathrm{HH}^{-n+1}(\Lambda V; \Lambda V) \\ {\scriptstyle (1\text{-}1)}\uparrow & & \uparrow {\scriptstyle J_1^*} \\ \pi_n(\mathrm{aut}_1 M) \otimes \mathbb{Q} & \xrightarrow[\cong]{} & H^{-n}(\mathrm{Der}^*(\Lambda V, \Lambda V)). \end{array} \tag{1-2}$$

The objective of this paper is to give a generalization of their works such as that mentioned below.

Let X and Y be simply connected spaces with homologies over $\mathbf{k}$ of finite type and $f_1, f_2\colon X \to Y$ based maps. Here, the complex $S^*(X;\mathbf{k})$ is regarded as a $S^*(Y;\mathbf{k})$–bimodule; that is a right and left $S^*(Y;\mathbf{k})$–structure is via f_1^* and f_2^*, respectively. Denote by $P(Y; f_1, f_2)$ a pullback of the diagram

$$\begin{array}{ccc} P(Y; f_1, f_2) & \longrightarrow & \mathrm{map}([0,1], Y) \\ {\scriptstyle \chi}\downarrow & & \downarrow {\scriptstyle (p_0,p_1)} \\ X & \xrightarrow[(f_1,f_2)]{} & Y \times Y, \end{array}$$

where (p_0, p_1) is the map defined by $(p_0, p_1)(\varphi) = (\varphi(0), \varphi(1))$. Our first result is described as follows.

Theorem 1.1 *There is an isomorphism of* $\mathbf{k}$*–vector spaces*

$$\Theta_X\colon \mathrm{HH}_*(S^*(Y;\mathbf{k}); S^*(X;\mathbf{k})) \xrightarrow{\cong} H^*(P(Y; f_1, f_2); \mathbf{k}).$$

In the proof, we use a cubical singular cochain complex instead of singular cochain algebra. In [4], Chen proved Theorem 1.1 in the case in which $\mathbf{k} = \mathbb{R}$. Our proof of the theorem is using ideas of Chen. As the relevant result of Theorem 1.1, we refer to the paper of Hess, Parent and Scott [12, Theorem 3.1]. They proved an integral version of the theorem, which also takes into account comultiplicative structure, that is, Theorem 1.1 is a weaker assertion than their results. However, the important thing is that the isomorphism of Theorem 1.1 is given by the map Θ_X described in Section 4.

Assume that X is a $\mathbf{k}$–Poincaré duality space of formal dimension d; see Section 5. Let $\mathrm{map}(X,Y;f)$ be the path component of the space of free maps from X to Y containing the based map $f\colon X \to Y$ and denote by L_fY the space $P(Y;f,f)$, especially. We consider the natural map

$$g\colon \Omega\,\mathrm{map}(X,Y;f)\times X \longrightarrow L_fY, \quad g(\gamma,x)(t)=\gamma(t)(x)$$

and the composite map for $n \geq 2$

$$\Gamma_1\colon \pi_n(\mathrm{map}(X,Y;f))\otimes\mathbf{k} \xrightarrow{\cong} \pi_{n-1}(\Omega\,\mathrm{map}(X,Y;f))\otimes\mathbf{k} \xrightarrow{h} H_{n-1}(\Omega\,\mathrm{map}(X,Y;f);\mathbf{k}) \xrightarrow{\Gamma} H_{n+d-1}(L_fY;\mathbf{k}).$$

Here ΩZ is the based loop space of Z, h is the Hurewicz map, Γ is the map defined by $\Gamma(a)=H(g)(a\otimes[X])$ and $[X]\in S_d(X;\mathbf{k})$ the fundamental class of X.

Let $\rho\colon (TV,d)\to S^*(Y;\mathbf{k})$ be a minimal free associative model for $S^*(Y;\mathbf{k})$ (see Halperin and Lemaire [11]) and $\mathrm{Der}^*(TV,S^*(X;\mathbf{k});f^*\circ\rho)$ the complex of $(f^*\circ\rho)$–derivations; see Section 5 for a proper definition. The next theorem is our main result of this paper.

Theorem 1.2 *If X is a $\mathbf{k}$–Poincaré duality space of formal dimension d, then, for any $n\geq 2$, there exists an isomorphism of $\mathbf{k}$–vector spaces Θ_X^* from $H_{*+d}(L_fY;\mathbf{k})$ to $\mathrm{HH}^*(TV;S^*(X;\mathbf{k}))$ and a $\mathbf{k}$–linear map Θ_1 from $\pi_n(\mathrm{map}(X,Y;f))\otimes\mathbf{k}$ to $H^{-n}(\mathrm{Der}^*(TV,S^*(X;\mathbf{k});f^*\circ\rho))$ such that the following square is commutative:*

$$\begin{array}{ccc} H_{n-1+d}(L_fY;\mathbf{k}) & \xrightarrow[\cong]{\Theta_X^*} & \mathrm{HH}^{-n+1}(TV;S^*(X;\mathbf{k})) \\ \uparrow{\scriptstyle \Gamma_1} & & \uparrow{\scriptstyle J_1^*} \\ \pi_n(\mathrm{map}(X,Y;f))\otimes\mathbf{k} & \xrightarrow[\Theta_1]{} & H^{-n}(\mathrm{Der}^*(TV,S^*(X;\mathbf{k});f^*\circ\rho)). \end{array}$$

If $\mathbf{k}=\mathbb{Q}$, $X=Y$ and f is the identity map, then the diagram in Theorem 1.2 coincides with the diagram (1-2); see the proof of Corollary 1.3. Thus Theorem 1.2 is regarded as a generalization of [8]. We here note that, in general, the map Θ_1 in Theorem 1.2 is not isomorphism. In the last paragraph, we use Theorem 1.2 to deduce the following corollary.

Corollary 1.3 *If $\mathbf{k}$ is $\mathbb{Q}$, then the map Γ_1 is injective.*

In [3], Chas and Sullivan constructed a product on $\mathbf{H}_*(LM) := H_{*+d}(LM)$ called the *loop product* and $\mathbf{H}_*(LM)$ is a commutative graded algebra. By Gruher and Salvatore [10], when X is a simply connected d–dimensional closed oriented manifold, we see that $\mathbf{H}_*(L_f Y)$ also has a graded algebra structure similar to the construction of loop products. As an application of the main result, we give a condition of a noncommutativity for $\mathbf{H}_*(L_f Y;\mathbb{Q})$ in rational cases. For details see Section 6.

The organization of this paper is as follows. In Section 2, we recall the Hochschild homology and cohomology. Section 3 gives a fundamental definition and facts on cubical singular chain complexes. Section 4 concentrates on the proof of Theorem 1.1. In Section 5, we prove the main result. Moreover, fundamental facts on rational homotopy theory and a proof of Corollary 1.3 are presented. Noncommutativity for $\mathbf{H}_*(L_f Y;\mathbb{Q})$ is described in Section 6.

2 Hochschild homology and cohomology

We begin with the definition of the Hochschild chain complex. Let (A,d) be a differential graded algebra over a field $\mathbf{k}$ with augmentation $\varepsilon\colon A\to \mathbf{k}$ and $\overline{A} = \operatorname{Ker}\varepsilon$ an augmentation ideal of A. Denote by $s\overline{A}$ the suspension of $\overline{A}$, that is $(s\overline{A})^n = \overline{A}^{n+1}$ and $T(s\overline{A})$ the tensor algebra on $s\overline{A}$. The *two-sided normalized bar construction* is the complex

$$\overline{\mathbf{B}}(A;A;A) = A\otimes T(s\overline{A})\otimes A$$

with the differential $d_{\overline{\mathbf{B}}} = d_1 + d_2$ defined by

$$d_1(a[a_1|a_2|\cdots|a_k]b) = d(a)[a_1|a_2|\cdots|a_k]b - \sum_{i=1}^{k}(-1)^{\varepsilon_i}a[a_1|a_2|\cdots|d(a_i)|\cdots|a_k]b + (-1)^{\varepsilon_{k+1}}a[a_1|a_2|\cdots|a_k]d(b),$$

$$d_2(a[a_1|a_2|\cdots|a_k]b) = (-1)^{|a|}aa_1[a_2|\cdots|a_k]b + \sum_{i=2}^{k}(-1)^{\varepsilon_i}a[a_1|\cdots|a_{i-1}a_i|\cdots|a_k]b - (-1)^{\varepsilon_k}a[a_1|a_2|\cdots|a_{k-1}]a_k b.$$

Here $\varepsilon_i = |a| + \sum_{j<i}|sa_j|$ and an element $a\otimes(sa_1\otimes sa_2\otimes\cdots\otimes sa_k)\otimes b$ in $\overline{\mathbf{B}}(A;A;A)$ is denoted by $a[a_1|a_2|\cdots|a_k]b$. We denote $\overline{\mathbf{B}}_n(A;A;A)$ by $A\otimes(s\overline{A})^{\otimes n}\otimes A$ for $n\geq 0$.

Let A^{op} be the opposite graded algebra of A and $A^e = A\otimes A^{\mathrm{op}}$. Recall that any A–bimodule can be considered as a left (or right) A^e–module.

Lemma 2.1 [5, Lemma 4.3] *The left A^e–module map*

$$\varepsilon_A\colon \overline{\mathbf{B}}(A;A;A) \to A$$

defined by $\varepsilon_A([\,]) = 1$ and $\varepsilon_A([a_1|a_2|\cdots|a_k]) = 0$ for $k > 0$ is a semifree resolution of A as a left A^e–module.

Let (M, d_M) be a differential graded A–bimodule, that is also a right A^e–module. The *Hochschild chain complex of A with coefficient in M* is the complex

$$C_*(A;M) = (M \otimes_{A^e} \overline{\mathbf{B}}(A;A;A),\ D = d_M \otimes 1 + 1 \otimes d_{\overline{\mathbf{B}}}).$$

The homology of $C_*(A;M)$ is denoted by $\mathrm{HH}_*(A;M)$ called the *Hochschild homology*. Similarly, the *Hochschild cochain complex of A with coefficient in M* is the complex

$$C^*(A;M) = (\mathrm{Hom}_{A^e}(\overline{\mathbf{B}}(A;A;A), M), D'),$$

where $D'(\varphi) = d_M \circ \varphi - (-1)^{|\varphi|}\varphi \circ d_{\overline{\mathbf{B}}}$ for $\varphi \in \mathrm{Hom}_{A^e}(\overline{\mathbf{B}}(A;A;A), M)$ and the *Hochschild cohomology* is the homology of $C^*(A;M)$, written by $\mathrm{HH}^*(A;M)$.

3 Cubical singular chain complex

Let $I^n = [0,1]^n$ be the n times product of the closed unit interval, $[0,1]$. An *n–cube in a topological space Z* is a continuous map $I^n \to Z$. An n–cube $\sigma\colon I^n \to Z$ is *degenerate* if there exist a integer i, $1 \le i \le n$, and an $(n-1)$–cube $\sigma'\colon I^{n-1} \to Z$ such that $\sigma(t_1, t_2, \dots, t_n) = \sigma'(t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n)$ for any $(t_1, t_2, \dots, t_n) \in I^n$. Note that all 0–cube are nondegenerate. We denote by $C_n(Z;\mathbf{k})$ the free $\mathbf{k}$–module generated by the set of all nondegenerate n–cubes in Z. We define the map

$$\lambda_i^{\varepsilon}\colon I^{n-1} \longrightarrow I^n; (t_1, t_2, \dots, t_{n-1}) \longmapsto (t_1, \dots, t_{i-1}, \varepsilon, t_i, \dots, t_{n-1})$$

for $\varepsilon = 0, 1$ and $1 \le i \le n$. Let $\partial = \sum_{i=1}^n (\lambda_i^{0*} - \lambda_i^{1*})\colon C_n(Z;\mathbf{k}) \to C_{n-1}(Z;\mathbf{k})$. Then ∂ is a well-defined differential of $C_*(Z;\mathbf{k})$ (see Massey [15, page 13]) and the chain complex $(C_*(Z;\mathbf{k}), \partial)$ is called the *cubical singular chain complex of Z*. The *cubical singular cochain complex of Z over* $\mathbf{k}$ is the complex $C^n(Z;\mathbf{k}) = \mathrm{Hom}_{\mathbf{k}}^{-n}(C_*(Z), \mathbf{k})$. The differential $d\colon C^{n-1}(Z;\mathbf{k}) \to C^n(Z;\mathbf{k})$ is defined by $d(\varphi) = \varphi\partial$ for $\varphi \in C^{n-1}(Z;\mathbf{k})$.

Remark 3.1 We see that the cubical singular chain complex $C_*(Z;\mathbf{k})$ is quasi-isomorphic to the singular chain complex $S_*(Z;\mathbf{k})$ by the method of acyclic models of Selick [18, Theorem 5.2.3’].

The Alexander–Whitney map and the Eilenberg–Zilber map are also defined in cubical singular chain complexes [15, pages 133, 137]. The Eilenberg–Zilber map

$$\mathrm{EZ}\colon C_n(Z_1;\mathbf{k})\otimes C_m(Z_2;\mathbf{k})\longrightarrow C_{n+m}(Z_1\times Z_2;\mathbf{k})$$

is defined by $\mathrm{EZ}(\varphi\otimes\psi)=\varphi\times\psi$ where φ (resp. ψ) is an n (resp. m)–cube. The Alexander–Whitney map is defined as follows. Let J be any subset of $\{1,2,\dots,n+m\}$ and J^c be the complementary subset of J. If $J=\{j_1,j_2,\dots,j_l\}$, then denote $\lambda^{\varepsilon}_J=\lambda^{\varepsilon}_{j_1}\lambda^{\varepsilon}_{j_2}\cdots\lambda^{\varepsilon}_{j_l}$. For any $(n+m)$–cube $\sigma\colon I^{n+m}\to Z_1\times Z_2$, we define a map $\mathrm{AW}\colon C_{n+m}(Z_1\times Z_2;\mathbf{k})\to(C_*(Z_1;\mathbf{k})\otimes C_*(Z_2;\mathbf{k}))_{n+m}$ by

$$\mathrm{AW}(\sigma)=\sum_J(-1)^{\varepsilon(J)}(\mathrm{pr}_1\,\sigma\lambda^0_{J^c})\otimes(\mathrm{pr}_2\,\sigma\lambda^1_J)\in(C_*(Z_1;\mathbf{k})\otimes C_*(Z_2;\mathbf{k}))_{n+m}$$

where $\mathrm{pr}_i\colon Z_1\times Z_2\to Z_i$ is the projection and $\varepsilon(J)$ is the cardinal number of the set $\{(i,j)\in J\times J^c\mid j<i\}$. We can see that EZ and AW are chain maps; see [15, pages 133, 138].

In the rest of this section, we recall the map called the *integration map* or the *slant product*. Let $\sigma\in C_q(Z_1;\mathbf{k})$, then define a map $\int_\sigma\colon C^{n+q}(Z_1\times Z_2;\mathbf{k})\to C^n(Z_2;\mathbf{k})$ by $(\int_\sigma(x))(\varphi)=x(\sigma\times\varphi)$ for any $\varphi\in C_n(Z_2;\mathbf{k})$. The equality

$$d\Big(\int_\sigma x\Big)=(-1)^q\Big(\int_\sigma dx-\int_{\partial\sigma}x\Big)\tag{3-1}$$

is easily seen as follows:

$$\begin{aligned}\Big(\int_\sigma dx\Big)(\varphi)&=dx(\sigma\times\varphi)=x(\partial\sigma\times\varphi)+(-1)^q x(\sigma\times\partial\varphi)\\&=\Big(\int_{\partial\sigma}x\Big)(\varphi)+(-1)^q d\Big(\int_\sigma x\Big)(\varphi).\end{aligned}$$

We note that the Equation (3-1) is a particular version of Stokes' theorem.

4 Proof of Theorem 1.1

In this section, we denote $C^*(-;\mathbf{k})$ by $C^*(-)$ for convenience. We begin recalling the $C^*(Y)$–bimodule structure on $C^*(X)$ defined for $\nu\in C^*(X)$ and $\omega,\omega'\in C^*(Y)$ by

$$\omega'\cdot\nu\cdot\omega=f_2^*(\omega')\nu f_1^*(\omega).$$

Let $\Delta^n=\{(t_1,t_2,\dots,t_n)\in\mathbb{R}^n\mid 0\le t_1\le t_2\le\cdots\le t_n\le 1\}$ be the standard n–simplex and $\kappa_n\colon I^n\to\Delta^n$ be a nondegenerate cubical chain defined by

$$\kappa_n(t_1,t_2,\dots,t_n)=(x_1,x_2,\dots,x_n),\quad x_i=1-t_1t_2\cdots t_i.$$

We now consider a map $\alpha_k\colon \Delta^k \times P(Y; f_1, f_2) \longrightarrow X \times Y^{\times k}$ defined by

$$\alpha_k((t_1, t_2, \ldots, t_k), \gamma) = (\chi(\gamma), \gamma(t_1), \gamma(t_2), \ldots, \gamma(t_k)).$$

Then we obtain the following composition map Θ_X^n

$$\Theta_X^n\colon C^*(X) \otimes_{C^*(Y)^e} \bar{\mathbf{B}}_n(C^*(Y); C^*(Y); C^*(Y)) \xrightarrow{s_n} C^*(X) \otimes C^*(Y)^{\otimes n} \xrightarrow{\mathrm{AW}} C^*(X \times Y^{\times n}) \xrightarrow{\int_{\kappa_n} \alpha_n^*} C^*(P(Y; f_1, f_2)),$$

where $\quad s_n(\nu \otimes \omega[\omega_1|\omega_2|\cdots|\omega_n]\omega') = (-1)^{\varrho}\omega'\nu\omega \otimes \omega_1 \otimes \omega_2 \otimes \cdots \otimes \omega_n,$

$$\varrho = |\omega'|(|\nu| + |\omega| + \sum_{j=0}^{n} |s\omega_j|) + \sum_{j=0}^{n-1}\sum_{i=1}^{j}(|\nu| + |s\omega_i|),$$

and put $\Theta_X = \sum_{n\geq 0} \Theta_X^n\colon C_*(C^*(Y); C^*(X)) \to C^*(P(Y; f_1, f_2))$. Essentially, the map Θ_X is the map in Félix, Oprea and Tanré [7, Theorem 9.64] which is defined using iterated integrals.

Lemma 4.1 *The map Θ_X is a chain map.*

Proof The Equation (3-1) enables us to give

$$\begin{aligned} d\Theta_X^n &= (-1)^n \int_{\kappa_n} d\alpha_n^* \,\mathrm{AW}\, s_n - (-1)^n \int_{\partial\kappa_n} \alpha_n^* \,\mathrm{AW}\, s_n \\ &= (-1)^n \int_{\kappa_n} \alpha_n^* \,\mathrm{AW}\, ds_n + (-1)^{n+1} \int_{\kappa_{n-1}} \alpha_{n-1}^* \,\mathrm{AW}\, \delta s_n, \end{aligned}$$

where the map $\delta\colon C^*(X) \otimes C^*(Y)^{\otimes n} \to C^*(X) \otimes C^*(Y)^{\otimes(n-1)}$ is defined by

$$\begin{aligned} \delta(\nu \otimes \omega_1 \otimes \cdots \otimes \omega_n) &= \nu f_1^*(\omega_1) \otimes \omega_2 \otimes \cdots \otimes \omega_n \\ &\quad + \sum_{i=2}^{n} (-1)^{i-1} \nu \otimes \omega_1 \otimes \cdots \otimes \omega_{i-1}\omega_i \otimes \cdots \otimes \omega_n \\ &\quad + (-1)^{|\omega_n|(|\nu| + \sum_{i=1}^{n-1}|\omega_i|)+n} f_2^*(\omega_n)\nu \otimes \omega_1 \otimes \cdots \otimes \omega_{n-1}. \end{aligned}$$

A straightforward calculation shows $(-1)^n ds_n = s_n(d \otimes 1 + 1 \otimes d_1)$ and $(-1)^{n-1}\delta s_n = s_{n-1}(1 \otimes d_2)$. We hence have $d\Theta_X = \Theta_X D$. □

Let $\rho\colon (TV, d) \to C^*(Y)$ be a minimal free associative model for $C^*(Y)$ [11], that is, TV is a tensor algebra over $\mathbf{k}$, ρ is a quasi-isomorphism of differential graded algebras,

$V = \{V^p\}_{p\geq 2}$, each V^p is finite dimensional and d is decomposable; $d(V) \subset T^{\geq 2}V$. Since the map

$$C_*(\rho; C^*(X))\colon C_*(TV; C^*(X)) \to C_*(C^*(Y); C^*(X))$$

is a quasi-isomorphism by [5, Proposition 2.4], it is only necessary to show that the composition map $\Theta_X \circ C_*(\rho; C^*(X))$ is a quasi-isomorphism to prove Theorem 1.1. We put $\bar{\Theta}_X = \Theta_X \circ C_*(\rho; C^*(X))$.

We begin with the Hochschild chain complex $C_*(TV; C^*(X))$ and filter it by

$$F^p = C^{\geq p}(X) \otimes_{(TV)^e} \bar{\mathbf{B}}(TV; TV; TV).$$

Lemma 4.2 *The spectral sequence associated the above filtration, denote by (E_r, d_r), satisfy $E_2^{p,q} \cong H^p(X) \otimes \mathrm{HH}_q(TV; \mathbf{k})$ as $\mathbf{k}$–vector spaces.*

Proof Recall that there is an isomorphism as follows:

$$\begin{aligned} E_0^{p,q} = \frac{(F^p)^{p+q}}{(F^{p+1})^{p+q}} &\cong C^p(X) \otimes (\mathbf{k} \otimes_{(TV)^e} \bar{\mathbf{B}}(TV, TV, TV)^q) \\ &= C^p(X) \otimes C_q(TV; \mathbf{k}), \\ \nu \otimes \omega[\omega_1|\cdots|\omega_n]\omega' &\longmapsto (-1)^{|\omega'|(|\nu|+|\omega|+\sum_i |s\omega_i|)} \omega'\nu\omega \otimes (1 \otimes [\omega_1|\cdots|\omega_n]). \end{aligned}$$

If the degrees of ω or ω' are not zero, then $\nu \otimes \omega[\omega_1|\cdots|\omega_n]\omega'$ is zero in $E_0^{p,q}$. It follows that the above correspondence is one-to-one. The differential $d_0\colon E_0^{p,q} \to E_0^{p,q+1}$ is the induced map of the differential of the Hochschild chain complex $C_*(TV; C^*(X))$. Since $\overline{TV}^0 = 0$, we have $d_0 = 1 \otimes D$ where D is the differential of $C_*(TV; \mathbf{k})$ and so $E_1^{p,q} \cong C^p(X) \otimes \mathrm{HH}_q(TV; \mathbf{k})$. The differential d_1 is defined by

$$\begin{aligned} d_1\colon E_1^{p,q} = H^{p+q}(F^p/F^{p+1}) &\xrightarrow{\partial^*} H^{p+q+1}(F^{p+1}) \\ &\xrightarrow{\pi_*} H^{p+q+1}(F^{p+1}/F^{p+2}) = E_1^{p+1,q} \end{aligned}$$

where ∂^* is the connecting homomorphism and $\pi\colon F^{p+1} \to F^{p+1}/F^{p+2}$ is the quotient map. For any $\sum \sigma \otimes [\omega_1|\omega_2|\cdots|\omega_k] \in H^{p+q}(F^p/F^{p+1})$, we have

$$d_1\Big(\sum \sigma \otimes [\omega_1|\omega_2|\cdots|\omega_k]\Big) = \sum d(\sigma) \otimes [\omega_1|\omega_2|\cdots|\omega_k]$$

since $\overline{TV}^0 = 0$ and $\overline{TV}^1 = 0$. Therefore, we conclude that $d_1 = \partial \otimes 1$ and it means that $E_2^{p,q} \cong H^p(X) \otimes \mathrm{HH}_q(TV; \mathbf{k})$. □

We next recall the Serre spectral sequence associated to the fibration $\chi\colon P(Y;f_1,f_2)\to X$. For any nondegenerate p–cube $\sigma\colon I^p\to X$, a $(q+p)$–cube $\bar\sigma\colon I^q\times I^p\to P(Y;f_1,f_2)$ is a *fibered q–cube over σ* if the diagram

$$\begin{array}{ccc} I^q\times I^p & \xrightarrow{\bar\sigma} & P(Y;f_1,f_2) \\ {\scriptstyle \mathrm{pr}_2}\downarrow & & \downarrow{\scriptstyle \chi} \\ I^p & \xrightarrow[\sigma]{} & X \end{array}$$

is commutative. Denote by $\widetilde{F}_p$ the subcomplex of $C_*(P(Y;f_1,f_2))$ generated by nondegenerate cubes fibered by some $\sigma\in C_{\le p}(X)$ and put

$$\widetilde{F}^p=\{\varphi\in C^*(P(Y;f_1,f_2))\mid \varphi|_{\widetilde{F}_{p-1}}=0\}. \tag{4-1}$$

Then, we get a spectral sequence, written by $(\widetilde{E}_r,\widetilde{d}_r)$, associated to the filtration which is called the Serre spectral sequence.

Proposition 4.3 [19, Chapter II 8, Proposition 6] *There is an isomorphism of* $\mathbf{k}$–*vector space*

$$\widetilde{E}_2^{p,q}\cong H^p(X)\otimes H^q(\Omega Y).$$

Lemma 4.4 *The map $\overline{\Theta}_X$ is filtration preserving. Moreover, the morphism of spectral sequences induced by $\overline{\Theta}_X$ is of the form*

$$1\otimes H(\overline{\Theta}_{\mathrm{pt}})^{\pm}\colon E_2^{p,n-p}\longrightarrow \widetilde{E}_2^{p,n-p}$$

at the 2–terms. Here, pt *is the one point space and the map $H(\overline{\Theta}_{\mathrm{pt}})^{\pm}$ from $\mathrm{HH}_*(TV;\mathbf{k})$ to $H^*(\Omega Y)$ is defined by*

$$H(\overline{\Theta}_{\mathrm{pt}})^{\pm}([\omega_1|\omega_2|\cdots|\omega_k])=(-1)^{p(k+n-p)}H(\overline{\Theta}_{\mathrm{pt}})([\omega_1|\omega_2|\cdots|\omega_k]).$$

Proof Given $\nu\otimes[\omega_1|\omega_2|\cdots|\omega_k]\in F^p$ and n–cube $\bar\sigma\colon I^n\to P(Y;f_1,f_2)$ in $\widetilde{F}_{p-1}$ where $n=|\nu|+\sum_i|\omega_i|-k$. By the definition of $\widetilde{F}_{p-1}$, there exists a nondegenerate m–cube σ $(m<n)$ such that the following square commutes:

$$\begin{array}{ccc} I^{n-m}\times I^m & \xrightarrow{\bar\sigma} & P(Y;f_1,f_2) \\ {\scriptstyle \mathrm{pr}_2}\downarrow & & \downarrow{\scriptstyle \chi} \\ I^m & \xrightarrow[\sigma]{} & X. \end{array}$$

Then, we have

$$\overline{\Theta}_X(\nu\otimes[\omega_1|\omega_2|\cdots|\omega_k])(\bar\sigma)=(\nu\otimes\rho(\omega_1)\otimes\rho(\omega_2)\otimes\cdots\otimes\rho(\omega_k))\,\mathrm{AW}(\alpha_k(\kappa_k\times\bar\sigma)).$$

It is only necessary to show $\mathrm{AW}(\alpha_k(\kappa_k \times \overline{\sigma})) = 0$ in $C_{|\nu|}(X) \otimes \bigotimes_i C_{|\omega_i|}(Y)$. We may write

$$\mathrm{AW}(\alpha_k(\kappa_k \times \overline{\sigma})) = \sum \pm \psi \otimes \psi_1 \otimes \psi_2 \otimes \cdots \otimes \psi_k$$

where ψ is a $|\nu|$–cube in X and ψ_i is a $|\omega_i|$–cube in Y. By the definition of the Alexander–Whitney map, there is a subset $J = \{j_1 < j_2 < \cdots < j_{|\nu|}\}$ of $\{1, 2, \ldots, n+k\}$ such that the diagram is commutative:

$$\begin{array}{ccccc} I^k \times I^n & \xrightarrow{\kappa_k \times \overline{\sigma}} & \Delta^k \times P(Y; f_1, f_2) & \xrightarrow{\alpha_k} & X \times Y^{\times k} \\ \uparrow{\scriptstyle \lambda^0_{J^c}} & & & & \downarrow{\scriptstyle \mathrm{pr}_1} \\ I^{|\nu|} & & \xrightarrow[\psi]{\qquad\qquad\qquad} & & X. \end{array}$$

We put $\lambda^0_{J^c}(t) = ((u_1, u_2, \ldots, u_k), (u_{k+1}, u_{k+2}, u_{k+n})) \in I^k \times I^n$ for any $t \in I^{|\nu|}$. If $(u_1, u_2, \ldots, u_k) \neq 0$ for some t, we see that ψ is a degenerate cube by the commutativity of the above diagram. Hence, $\psi = 0$ in $C_{|\nu|}(X)$. If $(u_1, u_2, \ldots, u_k) = 0$ for any t, that is the composition map

$$j \colon I^{|\nu|} \xrightarrow{\lambda^0_{J^c}} I^k \times I^n \xrightarrow{\mathrm{pr}_2} I^n$$

is the inclusion, then we see the commutativity of the diagram

$$I^{|\nu|} \xrightarrow[j]{} I^{n-m} \times I^m \xrightarrow[\mathrm{pr}_2]{} I^m \xrightarrow[\sigma]{} X, \qquad \psi = \sigma \circ \mathrm{pr}_2 \circ j \colon I^{|\nu|} \to X. \tag{4-2}$$

Since $m \leq p-1 < p \leq |\nu|$, ψ is a degenerate cube. Therefore, we conclude that $\mathrm{AW}(\alpha_k(\kappa_k \times \overline{\sigma})) = 0$. This finishes a proof of the first assertion.

Recall that $E_0^{p,q} \cong C^p(X) \otimes C_q(TV, \mathbf{k})$ and $\widetilde{E}_0^{p,n-p} \cong \mathrm{Hom}_{\mathbf{k}}((\widetilde{F}_p)^n/(\widetilde{F}_{p-1})^n, \mathbf{k})$. We consider the case $|\nu| = p$ and $\overline{\sigma} \in (\widetilde{F}_p)^n/(\widetilde{F}_{p-1})^n$, that means the σ is a p–cube. Then, the diagram (4-2) shows that $\psi = \sigma$. Therefore we have

$$\mathrm{AW}(\alpha_k(\kappa_k \times \overline{\sigma})) = \sigma \otimes (-1)^{p(k+n-p)} \mathrm{AW}(\alpha_k(\kappa_k \times \overline{\sigma}|_{I^{n-p}})).$$

in $C_{|\nu|}(X) \otimes \bigotimes_i C_{|\omega_i|}(Y)$ where the sign $(-1)^{p(k+n-p)}$ is appeared by the Alexander–Whitney map and so

$$\overline{\Theta}_X(\nu \otimes [\omega_1|\omega_2|\cdots|\omega_k])(\overline{\sigma}) = \nu(\sigma) \otimes (-1)^{p(k+n-p)} \overline{\Theta}_{\mathrm{pt}}([\omega_1|\omega_2|\cdots|\omega_k])(\overline{\sigma}|_{I^{n-p}}).$$

The equality shows the second assertion. □

Before proving Theorem 1.1, we recall the following theorem.

Theorem 4.5 (McCleary [16, Theorem 3.26]) *Let E_r and $\widetilde{E}_r$ be first quadrant spectral sequences of cohomological type over a field $\mathbf{k}$ and $\phi_r\colon E_r \to \widetilde{E}_r$ a morphism of spectral sequences such that $E_2^{p,q} = E_2^{p,0} \otimes E_2^{0,q}$, $\widetilde{E}^{p,q} = \widetilde{E}^{p,0} \otimes \widetilde{E}^{0,q}$ and $\phi_2^{p,q} = \phi_2^{p,0} \otimes \phi_2^{0,q}$. Then any two of the following conditions imply the third:*

(1) $\phi_2^{p,0}\colon E_2^{p,0} \longrightarrow \widetilde{E}_2^{p,0}$ *is an isomorphism for all p.*

(2) $\phi_2^{0,q}\colon E_2^{0,q} \longrightarrow \widetilde{E}_2^{0,q}$ *is an isomorphism for all q.*

(3) $\phi_\infty^{p,q}\colon E_\infty^{p,q} \longrightarrow \widetilde{E}_\infty^{p,q}$ *is an isomorphism for all p, q.*

Proof of Theorem 1.1 Since the both spectral sequences E_r and $\widetilde{E}_r$ are strong convergent, by [16, Theorem 3.9], it is only enough to show that $H(\overline{\Theta}_{\mathrm{pt}})^{\pm}$ is an isomorphism to prove the theorem. We consider the following pullback diagram

$$\begin{array}{ccc} P(Y;1_Y,c_*) & \longrightarrow & \mathrm{map}(I,Y) \\ {\scriptstyle\chi}\downarrow & & \downarrow{\scriptstyle(p_0,p_1)} \\ Y & \xrightarrow[(1_Y,c_*)]{} & Y\times Y, \end{array}$$

where $c_*\colon Y \to Y$ is the constant map to the base point. The space $P(Y;1_Y,c_*)$ is contractible, we see that $H^*(P(Y;1_Y,c_*)) \cong \mathbf{k}$. On the other hand, when the $C^*(Y)$–bimodule structure on $C^*(Y)$ is defined by $\omega' \cdot \nu \cdot \omega = c_*^*(\omega')\nu\omega$, the Hochschild homology $\mathrm{HH}_*(C^*(Y);C^*(Y))$ is $\mathbf{k}$. In effect, we now note that any element $\nu \otimes \omega[\omega_1|\omega_2|\cdots|\omega_k]\omega'$ in $C_*(C^*(Y);C^*(Y))$ is zero if $|\omega'|>0$ since $c_*^*(\omega')=0$ and so assume that $|\omega'|=0$, that is $c_*^*(\omega') \in \mathbf{k}$. Define a map

$$h\colon C_*(C^*(Y);C^*(Y)) \to C_*(C^*(Y);C^*(Y))$$

by $$h(\nu\otimes\omega[\omega_1|\omega_2|\cdots|\omega_k]\omega') = \begin{cases} 0 & |\nu|=|\omega|=0, \\ 1\otimes 1[c_*^*(\omega')\nu\omega|\omega_1|\omega_2|\cdots|\omega_k]1 & \text{otherwise.} \end{cases}$$

An easy calculation gives us the equation $Dh+hD=1$, where D is the differential of $C_*(C^*(Y);C^*(Y))$. Hence, we have $\mathrm{HH}_*(C^*(Y);C^*(Y)) \cong \mathbf{k}$ and, by Theorem 4.5, the map $H(\overline{\Theta}_{\mathrm{pt}})^{\pm}$ is an isomorphism. □

5 Main result

Let (A,d) and (M,d) be differential graded algebras and $\xi\colon A \to M$ a differential graded algebra map. We here recall the complex of ξ–derivations from A to M, $\mathrm{Der}^*(A,M;\xi)$. An element θ in $\mathrm{Der}^n(A,M;\xi)$ is a $\mathbf{k}$–linear map of degree n

with $\theta(xy) = \theta(x)\xi(y) + (-1)^{n|x|}\xi(x)\theta(y)$. The differentials $\delta\colon \mathrm{Der}^*(A, M; \xi) \to \mathrm{Der}^{*+1}(A, M; \xi)$ send θ to $d\theta - (-1)^{|\theta|}\theta d$. Then we have the natural map

$$J_1\colon \mathrm{Der}^n(A, M; \xi) \longrightarrow C^{n+1}(A; M)$$
$$J_1(\theta)([\omega_1|\omega_2|\cdots|\omega_k]) = \begin{cases} (-1)^{|\theta|}\theta(\omega_1) & k = 1, \\ 0 & k \geq 2,\ k = 0, \end{cases}$$

and it is readily seen that J_1 is a cochain map of degree 1, that is, $J_1 D = -\delta J_1$.

Suppose that X is a **k**-*Poincaré duality space* of formal dimension d; that is, the space X is equipped with a *fundamental class* $[X] \in H_d(X)$ such that the cap product

$$-\cap [X]\colon H^*(X) \longrightarrow H_{d-*}(X)$$

is an isomorphism. We also denote by $[X] \in C_d(X)$ the representative element of $[X] \in H_d(X)$. By dualizing Theorem 1.1, we obtain the isomorphism of **k**–vector space

$$\Phi_X\colon H_*(L_f Y) \xrightarrow{\cong} H^*(L_f Y)^\vee \xrightarrow{H(\bar{\Theta}_X)^\vee} \mathrm{HH}_*(TV; C^*(X))^\vee,$$

where $(-)^\vee = \mathrm{Hom}_{\mathbf{k}}(-, \mathbf{k})$ is the graded dual space. Let $\varepsilon\colon C_*(X) \longrightarrow C^*(X)^\vee$ be the evaluation map; $\varepsilon(\sigma)(\omega) = (-1)^{|\sigma|}\omega(\sigma)$ for $\sigma \in C_*(X)$ and $\omega \in C^*(X)$. We here remark that the evaluation map ε is not a chain map by the definition of the differentials of $C^*(X)$. However, ε induces the map $H(C_*(X)) \to H(C^*(X)^\vee)$ in homology and the induced map is an isomorphism.

For simplicity we denote by $\bar{\mathbf{B}}(C^*(X))$ the two-sided normalized bar construction $\bar{\mathbf{B}}(C^*(X); C^*(X); C^*(X))$. In [9], Félix, Thomas and Vigué-Poirrier proved that the map of $C^*(X)$–bimodules with degree $-d$

$$\theta_{\varepsilon[X]}\colon \bar{\mathbf{B}}(C^*(X)) \longrightarrow C^*(X)^\vee$$

defined by $\theta_{\varepsilon[X]}(\omega[\,]\omega') = \varepsilon(\omega\omega' \cap [X])$ and $\theta_{\varepsilon[X]}(\omega[\omega_1|\omega_2|\cdots|\omega_k]\omega') = 0$ for $k > 0$ is a quasi-isomorphism [9, Theorem 12]. Here the $C^*(X)$–bimodule structure on $C^*(X)^\vee$ is defined by

$$(\omega_1 \cdot \varphi \cdot \omega_2)(\omega) = (-1)^{|\omega_1||\varphi|}\varphi(\omega_1\omega_2\omega)$$

for $\omega, \omega_i \in C^*(X)$ and $\varphi \in C^*(X)^\vee$. Therefore, by [5, Proposition 2.4], we have the isomorphism

$$\begin{aligned}\Psi_X\colon \mathrm{HH}^*(TV; C^*(X)) &\xrightarrow{\mathrm{HH}(TV;\varepsilon_{C^*(X)})^{-1}} \mathrm{HH}^*(TV; \bar{\mathbf{B}}(C^*(X))) \\ &\xrightarrow{\mathrm{HH}(TV;\theta_{\varepsilon[X]})} \mathrm{HH}^{*-d}(TV; C^*(X)^\vee) \\ &\xrightarrow{\iota_*} \mathrm{HH}_{-*+d}(TV; C^*(X))^\vee\end{aligned}$$

where ι_* the induced map of the isomorphism of complexes

$$\iota\colon \operatorname{Hom}_{(TV)^e}(\overline{\mathbf{B}}(TV), C^*(X)^\vee) \longrightarrow \operatorname{Hom}(C^*(X) \otimes_{(TV)^e} \overline{\mathbf{B}}(TV), \mathbf{k})$$

defined by $\iota(\varphi)(\omega \otimes \sigma) = (-1)^{|\sigma||\omega|}\varphi(\sigma)(\omega)$ for $\sigma \in \overline{\mathbf{B}}(TV)$ and $\omega \in C^*(X)$.

Now we define the map for any $n \geq 2$

$$\Theta_1\colon \pi_n(\operatorname{map}(X, Y; f)) \otimes \mathbf{k} \longrightarrow H^{-n}(\operatorname{Der}^*(TV, C^*(X); f^* \circ \rho))$$

by $\Theta_1(\alpha)(x) = (-1)^{n|x|} \int_{[S^n]}(C^*(\overline{\alpha})\rho(x))$ for any $\alpha \in \pi_n(\operatorname{map}(X, Y; f)) \otimes \mathbf{k}$ and $x \in TV$, where $\overline{\alpha}\colon S^n \times X \to Y$ is the adjoint of α and $[S^n] \in C_n(S^n)$ be the fundamental class defined by

$$[S^n]\colon I^n \to I^n/\partial I^n \cong S^n, \quad (t_1, t_2, \dots, t_n) \longmapsto [1-t_1, 1-t_2, \dots, 1-t_n].$$

A straightforward calculation shows that $\Theta_1(\alpha)$ is a $(f^* \circ \rho)$–derivation. If two maps $\overline{\alpha}$ and $\overline{\beta}\colon S^n \times X \to Y$ are homotopic, then we have $\int_{[S^n]} C^*(\overline{\alpha})\rho - \int_{[S^n]} C^*(\overline{\beta})\rho = \delta(\int_{[S^n]} \int_{\mathrm{id}_I} C^*(H)\rho)$ where $\mathrm{id}_I \in C_1(I)$ is the identity map and $H\colon I \times S^n \times X \to Y$ is a homotopy from $\overline{\alpha}$ to $\overline{\beta}$. Hence, Θ_1 is a well-defined map. In addition, the map Θ_1 is a homomorphism. Indeed, for any α and β in $\pi_n(\operatorname{map}(X, Y; f)) \otimes \mathbf{k}$, the adjoint of the sum $\alpha + \beta \in \pi_n(\operatorname{map}(X, Y; f)) \otimes \mathbf{k}$ is the composite map

$$S^n \times X \xrightarrow{\mu' \times 1} (S^n \vee S^n) \times X \xrightarrow{(\overline{\alpha}|\overline{\beta})} Y$$

where $\mu'\colon S^n \to S^n \vee S^n$ is the pinching map and $(\overline{\alpha}|\overline{\beta})$ is a map defined by $(\overline{\alpha}|\overline{\beta})((u, *), x) = \overline{\alpha}(u, x)$ and $(\overline{\alpha}|\overline{\beta})((*, u), x) = \overline{\beta}(u, x)$ for $u \in S^n$ and $x \in X$. Then, we see that the following diagram is commutative:

$$\begin{array}{ccc} (C_*(S^n) \otimes C_*(X)) & \xrightarrow{(C_*(i_1)+C_*(i_2))\otimes 1} & C_*(S^n \vee S^n) \otimes C_*(X) \\ \Big\downarrow{\scriptstyle C_*(\overline{\alpha})+C_*(\overline{\beta})} & & \Big\downarrow{\scriptstyle \mathrm{EZ}} \\ C_*(Y) & \xleftarrow[C_*(\overline{\alpha}|\overline{\beta})]{} & C_*((S^n \vee S^n) \times X), \end{array}$$

where i_1 and $i_2\colon S^n \to S^n \vee S^n$ are the inclusions on the first and second factors respectively. A commutativity of the diagram shows that $C^*(\overline{\alpha}|\overline{\beta}) = C^*(\overline{\alpha}) + C^*(\overline{\beta})$ and hence the map Θ_1 is a homomorphism.

Proof of Theorem 1.2 We consider the following diagram:

$$
\begin{array}{ccc}
H_{n-1+d}(L_f Y;\mathbf{k}) \overset{\Phi_X}{\to} \mathrm{HH}_{n-1+d}(TV;C^*(X))^\vee \overset{\Psi_X}{\leftarrow} & & \mathrm{HH}^{-n+1}(TV;C^*(X)) \\
\Gamma_1 \uparrow & & \uparrow J_1^* \\
\pi_n(\mathrm{map}(X,Y;f))\otimes\mathbf{k} & \xrightarrow[\Theta_1]{} & H^{-n}(\mathrm{Der}^*(TV,C^*(X);f^*\circ\rho)).
\end{array} \tag{5-1}
$$

Given $\alpha \in \pi_n(\mathrm{map}(X,Y;f))\otimes\mathbf{k}$. By the definition of Γ_1,

$$\Gamma_1(\alpha) = H_{n-1+d}(g(\alpha'\times 1_X))(EZ([S^{n-1}]\otimes[X]))$$

where $\alpha'\colon S^{n-1}\to\Omega\,\mathrm{map}(X,Y;f)$ is the adjoint map of α, and denote

$$\gamma_1(\alpha) = C_{n-1+d}(g(\alpha'\times 1_X))(EZ([S^{n-1}]\otimes[X])) \in C_{n-1+d}(L_f Y)$$

by the representative element of $\Gamma_1(\alpha)$. For any element $\nu\otimes[\omega_1|\omega_2|\cdots|\omega_k]$ in $C_{n-1+d}(TV;C^*(X))$, we have

$$
\begin{aligned}
&(\Phi_X\Gamma_1)(\alpha)(\nu\otimes[\omega_1|\omega_2|\cdots|\omega_k]) \\
&\qquad = \pm((\nu\otimes\rho\omega_1\otimes\rho\omega_2\otimes\cdots\otimes\rho\omega_k)\circ\mathrm{AW}\circ\alpha_{k*})(EZ(\kappa_k\otimes\gamma_1(\alpha))).
\end{aligned}
$$

We may write

$$(\mathrm{AW}\circ\alpha_{k*})(EZ(\kappa_k\otimes\gamma_1(\alpha))) = \sum \pm\psi\otimes\psi_1\otimes\psi_2\otimes\cdots\otimes\psi_k$$

where ψ is a $|\nu|$–cube in X and ψ_i is a $|\omega_i|$–cube in Y. If $k\geq 2$, it is readily seen that some ψ_i are degenerate, that is $(\Phi_X\Gamma_1)(\alpha)(\nu\otimes[\omega_1|\omega_2|\cdots|\omega_k]) = 0$. If $k=1$, we see that

$$\alpha_{1*}(EZ(\kappa_1\otimes\gamma_1(\alpha))) = \sum n_x\tau_x.$$

Here, we may write $[X] = \sum_{[X]} n_x x \in C_d(X)$ some $n_x\in\mathbf{k}$ and $x\colon I^d\to X$. Then, $(n+d)$–cube, τ_x is the compositions

$$\tau_x\colon I\times I^{n-1}\times I^d \xrightarrow{\kappa_1\times[S^{n-1}]\times x} \Delta^1\times S^{n-1}\times X \xrightarrow{\alpha_1(1\times g(\alpha'\times 1))} X\times Y.$$

Then, $$\mathrm{AW}(\tau_x) = \sum_{\substack{J\subset\{1,2,\dots,n+d\},\\ \#J=|\nu|}} (-1)^{\varepsilon(J)}\tau_{x1}\lambda^0_{J^c}\otimes\tau_{x2}\lambda^\varepsilon_J$$

where $\#J$ is the cardinal number of J and τ_{xi} is the composition of the projection pr_i and τ_x. If there is $i\in J$ such that $i<n$, then $\tau_{x1}\lambda^0_{J^c}$ is degenerate since

$\tau_{x1}\colon I \times I^{n-1} \times I^d \to X$ depends only on I^d. Hence,

$$\mathrm{AW}(\tau_x) = \sum_{\substack{J\subset\{1,2,\dots,n+d\},\\ \#J=|\nu|,\ \min J\geq n}} (-1)^{\varepsilon(J)} \tau_{x1}\lambda^0_{J^c} \otimes \tau_{x2}\lambda^{\varepsilon}_J$$

and so

$$\begin{aligned}(\Phi_X\Gamma_1)(\alpha)(\nu\otimes[\omega_1])&\\ &= (-1)^{|\nu|}(\nu\otimes\rho(\omega_1))\,\mathrm{AW}\Bigl(\sum_{[X]} n_x\tau_x\Bigr)\\ &= (-1)^{|\nu|+|\nu||\omega_1|}\sum_{[X]}\sum_{\substack{J\subset\{1,2,\dots,n+d\},\\ \#J=|\nu|,\ \min J\geq n}} (-1)^{\varepsilon(J)} n_x\bigl(\nu(\tau_{x1}\lambda^0_{J^c})\bigr)\bigl(\rho(\omega_1)(\tau_{x2}\lambda^{\varepsilon}_J)\bigr).\end{aligned}$$

On the other hand, $(\Psi_X J_1^*\Theta_1)(\alpha)(\nu\otimes[\omega_1|\omega_2|\cdots|\omega_k]) = 0$ for $k \geq 2$ and $k = 0$ by the definition of J_1, and

$$\begin{aligned}(\Psi_X J_1^*\Theta_1)(\alpha)(\nu\otimes[\omega_1])&\\ &= (-1)^{|\nu||s\omega_1|}\varepsilon(J_1^*\Theta_1(\alpha)([\omega_1])\cap[X])(\nu)\\ &= (-1)^{|\nu||s\omega_1|+|\nu|}\nu(J_1^*\Theta_1(\alpha)([\omega_1])\cap[X])\\ &= (-1)^{|\nu||s\omega_1|+|\nu|+n}\nu\bigl(\Theta_1(\alpha)(\omega_1)\cap[X]\bigr)\\ &= (-1)^{|\nu||s\omega_1|+|\nu|+n+n|\omega_1|}\nu\Bigl(\int_{[S^n]} C^*(\overline{\alpha})\rho(\omega_1)\cap[X]\Bigr)\\ &= (-1)^{|\nu||s\omega_1|+|\nu|+n+n|\omega_1|+|\nu|(d-|\nu|)}\\ &\qquad\times\sum_{[X]}\sum_{\substack{J\subset\{1,2,\dots,d\},\\ \#J=|\nu|}} (-1)^{\varepsilon(J)} n_x\bigl(\nu(x\lambda^0_{J^c})\bigr)\Bigl(\int_{[S^n]} C^*(\overline{\alpha})\rho(\omega_1)(x\lambda^1_J)\Bigr)\\ &= (-1)^{|\nu||s\omega_1|+nd+(n+d)|\nu|}\\ &\qquad\times\sum_{[X]}\sum_{\substack{J\subset\{1,2,\dots,n+d\},\\ J=|\nu|,\ \min J\geq n}} (-1)^{\varepsilon(J)+n|\nu|} n_x\bigl(\nu(\tau_{x1}\lambda^0_{J^c})\bigr)\bigl(\rho(\omega_1)(\overline{\alpha}([S^n]\times x)\lambda^1_J)\bigr).\end{aligned}$$

Since $\rho(\omega_1)(\tau_{x2}\lambda^1_J) = \rho(\omega_1)(\overline{\alpha}([S^n]\times x)\lambda^1_J)$, we have

$$(\Phi_X\Gamma_1)(\alpha)(\nu\otimes[\omega_1]) = (-1)^{nd+d|\nu|}(\Psi_X J_1^*\Theta_1)(\alpha)(\nu\otimes[\omega_1]).$$

If d is even, then the diagram (5-1) is commutative. We consider the case that d is odd. When we define Ψ_X, we replace $\theta_{\varepsilon[X]}$ with the map of degree $-d$, $\widetilde{\theta}_{\varepsilon[X]}\colon \overline{\mathbf{B}}(C^*(X)) \longrightarrow C^*(X)^\vee$ defined by $\widetilde{\theta}_{\varepsilon[X]}(\omega[\,]\omega') = (-1)^{|\omega\omega'|}\varepsilon(\omega\omega'\cap[X])$ and $\widetilde{\theta}_{\varepsilon[X]}(\omega[\omega_1|\omega_2|\cdots|\omega_k]\omega') = 0$ for $k > 0$. Also $\widetilde{\theta}_{\varepsilon[X]}$ is a quasi-isomorphism and similar calculation described above enable us to get the equation

$$(\Phi_X \Gamma_1)(\alpha)(\nu \otimes [\omega_1]) = (-1)^{nd+d+(d+1)|\nu|}(\Psi_X J_1^* \Theta_1)(\alpha)(\nu \otimes [\omega_1]).$$

That is, the diagram (5-1) is commutative up to sign, completing the proof. □

We here recall a *minimal Sullivan model* for a simply connected space X with finite type. It is a free commutative differential graded algebra over $\mathbb{Q}$ of the form $(\Lambda V, d)$ with $V = \bigoplus_{i\geq 2} V^i$ where each V^i is of finite dimension and d is decomposable; that is, $d(V) \subset \Lambda^{\geq 2} V$. Moreover, $(\Lambda V, d)$ is equipped with a quasi-isomorphism $(\Lambda V, d) \xrightarrow{\simeq} A_{\mathrm{PL}}(X)$ to the commutative differential graded algebra $A_{\mathrm{PL}}(X)$ of differential polynomial forms on X [6, Section 12]. Observe that, as algebras, $H^*(\Lambda V, d) \cong H^*(A_{\mathrm{PL}}(X)) \cong H^*(X;\mathbb{Q})$. Let $f\colon X \to Y$ be a map between spaces of finite type. Then there exists a commutative differential graded algebra map $\tilde{f}$ from a minimal Sullivan model $(\Lambda V_Y, d)$ for Y to a minimal Sullivan model $(\Lambda V_X, d)$ for X which makes the diagram

$$\begin{array}{ccc} A_{\mathrm{PL}}(Y) & \xrightarrow{A_{\mathrm{PL}}(f)} & A_{\mathrm{PL}}(X) \\ \simeq\uparrow & & \uparrow\simeq \\ \Lambda V_Y & \xrightarrow{\tilde{f}} & \Lambda V_X \end{array}$$

commutative up to homotopy. We call $\tilde{f}$ a *Sullivan model* for f.

Proposition 5.1 *Let ΛV_X and ΛV_Y be a minimal Sullivan model for X and Y, respectively, and $\tilde{f}$ a Sullivan model for f. Then, the cochain map*

$$J_1\colon \mathrm{Der}^*(\Lambda V_Y, \Lambda V_X; \tilde{f}) \longrightarrow C^{*+1}(\Lambda V_Y; \Lambda V_X)$$

is injective in homology.

For giving a proof of Proposition 5.1, we introduce a semifree resolution of ΛV_Y as a left $\Lambda V_Y \otimes \Lambda V_Y$–module that is different from the two-sided bar resolution and give some lemmas. We consider the commutative differential graded algebra $\Lambda V_Y \otimes \Lambda V_Y \otimes \Lambda(sV_Y)$ with the differential d defined by

$$d(v \otimes 1 \otimes \bar{1}) = dv \otimes 1 \otimes \bar{1}, \quad d(1 \otimes v \otimes \bar{1}) = 1 \otimes dv \otimes \bar{1},$$

$$d(1 \otimes 1 \otimes sv) = (v \otimes 1 - 1 \otimes v) \otimes \bar{1} - \sum_{i=1}^{\infty} \frac{(sd)^i}{i!}(v \otimes 1 \otimes \bar{1}).$$

Here $\bar{1}$ is the unit of $\Lambda(sV_Y)$, and s is the unique degree -1 derivation of the algebra $\Lambda V_Y \otimes \Lambda V_Y \otimes \Lambda(sV_Y)$ defined by

$$s(v \otimes 1 \otimes \bar{1}) = 1 \otimes 1 \otimes sv = s(1 \otimes v \otimes \bar{1}), \; s(1 \otimes 1 \otimes sv) = 0.$$

By [6, Section 15 Example 1], the map

$$\mu \cdot \overline{\varepsilon}\colon \Lambda V_Y \otimes \Lambda V_Y \otimes \Lambda(sV_Y) \longrightarrow \Lambda V_Y$$

is a semifree resolution of ΛV_Y as a left $\Lambda V_Y \otimes \Lambda V_Y$–module, where μ is the product of ΛV_Y and $\overline{\varepsilon}$ is the canonical augmentation of $\Lambda(sV_Y)$. Since the map $\varepsilon_{\Lambda V_Y}\colon \overline{\mathbf{B}}(\Lambda V_Y, \Lambda V_Y, \Lambda V_Y) \to \Lambda V_Y$ is a surjective quasi-isomorphism, by [6, Proposition 14.6], there exists a differential graded algebra map ϕ such that the following diagram is commutative:

$$\begin{array}{ccc}
\Lambda V_Y \otimes \Lambda V_Y & \hookrightarrow & \overline{\mathbf{B}}(\Lambda V_Y, \Lambda V_Y, \Lambda V_Y) \\
\big\downarrow & \overset{\phi}{\nearrow} & \big\downarrow \varepsilon_{\Lambda V_Y} \\
\Lambda V_Y \otimes \Lambda V_Y \otimes \Lambda(sV_Y) & \xrightarrow{\mu\cdot\overline{\varepsilon}} & \Lambda V_Y .
\end{array}$$

A commutativity of the diagram shows that the map ϕ is a quasi-isomorphism. We now recall a construction of ϕ. For any basis element $v \in V_Y$, we put $\phi(v \otimes 1 \otimes \overline{1}) = v\,[\,]\,1$ and $\phi(1 \otimes v \otimes \overline{1}) = 1\,[\,]\,v$. By induction on degree of V_Y, we construct $\phi(1 \otimes 1 \otimes sv)$. For any $v' \in V$ such that $dv' = 0$, we defined $\phi(1 \otimes 1 \otimes sv') = 1\,[v']\,1$. Assume that ϕ is defined in $\Lambda V_Y \otimes \Lambda V_Y \otimes \Lambda(sV_Y^{\leq |v|})$ for some basis element $v \in V$, that is, $\phi d(1 \otimes 1 \otimes sv)$ is also defined. Since $\varepsilon_{\Lambda V_Y}$ is a quasi-isomorphism, the equation

$$\varepsilon_{\Lambda V_Y} \phi d(1 \otimes 1 \otimes sv) = (\mu \cdot \overline{\varepsilon}) d(1 \otimes 1 \otimes sv) = 0 = \varepsilon_{\Lambda V_Y} d(1\,[v]\,1)$$

shows that there is $\beta \in \overline{\mathbf{B}}(\Lambda V_Y, \Lambda V_Y, \Lambda V_Y)$ such that $\phi d(1 \otimes 1 \otimes sv) - d(1\,[v]\,1) = d\beta$. Then, we put $\phi(1 \otimes 1 \otimes sv) = 1\,[v]\,1 + \beta$. The above construction of ϕ and the differential of $\Lambda V_Y \otimes \Lambda V_Y \otimes \Lambda(sV_Y)$ establishes that $d\beta$ has no term of the form $x\,[\,]\,x'$, that is, β does not have terms of the form $x\,[\omega]\,x'$. So we have the following lemma.

Lemma 5.2 *There is a quasi-isomorphism*

$$\phi\colon \Lambda V_Y \otimes \Lambda V_Y \otimes \Lambda(sV_Y) \to \overline{\mathbf{B}}(\Lambda V_Y, \Lambda V_Y, \Lambda V_Y)$$

of $\Lambda V_Y \otimes \Lambda V_Y$–modules such that the following diagram is commutative

$$\begin{array}{ccc}
\Lambda V_Y \otimes \Lambda V_Y \otimes \Lambda(sV_Y) & \xrightarrow{\phi} & \overline{\mathbf{B}}(\Lambda V_Y, \Lambda V_Y, \Lambda V_Y) \\
{\scriptstyle \varepsilon\cdot\varepsilon\cdot\mathrm{pr}}\big\downarrow & & \big\downarrow{\scriptstyle \varepsilon\cdot\mathrm{pr}'\cdot\varepsilon} \\
sV_Y & \hookrightarrow & s\Lambda V_Y ,
\end{array}$$

where $\varepsilon\colon \Lambda V_Y \to \mathbb{Q}$ is the canonical augmentation and $\mathrm{pr}\colon \Lambda(sV_Y) \to sV_Y$ *and* $\mathrm{pr}'\colon T(s\Lambda V_Y) \to s\Lambda V$ *are the canonical projections.*

Consider the canonical isomorphism

$$\zeta\colon \mathrm{Hom}_{\Lambda V_Y\otimes\Lambda V_Y}(\Lambda V_Y\otimes\Lambda V_Y\otimes\Lambda(sV_Y),\Lambda V_X)\longrightarrow \mathrm{Hom}_{\mathbb{Q}}(\Lambda(sV_Y),\Lambda V_X).$$

and define $\bar{D}=\zeta D\zeta^{-1}$, where D is the differential of

$$\mathrm{Hom}_{\Lambda V_Y\otimes\Lambda V_Y}(\Lambda V_Y\otimes\Lambda V_Y\otimes\Lambda(sV_Y),\Lambda V_X).$$

Then, for $\psi\in\mathrm{Hom}_{\mathbb{Q}}(\Lambda(sV_Y),\Lambda V_X)$ and $sv_1sv_2\cdots sv_p\in\Lambda(sV_Y)$,

$$\bar{D}(\psi)(sv_1sv_2\cdots sv_p)=d\psi(sv_1sv_2\cdots sv_p)$$
$$+(-1)^{|\psi|}\sum_{i=1}^{p}\sum_{v_i}\sum_{k=1}^{p}\pm\omega_{i_1}\cdots\omega_{i_{k-1}}\omega_{i_{k+1}}\cdots\omega_{i_p}\psi(sv_1\cdots sv_{i-1}s\omega_{i_k}sv_{i+1}\cdots sv_p),$$

where $dv_i=\sum_{v_i}\omega_{i_1}\omega_{i_2}\cdots\omega_{i_p}$ and the sign $\pm$ is the Koszul sign convention. In fact, for example $p=1$ and $v=v_1\in V$ with $dv=\sum_v\omega_1\cdots\omega_p$,

$$\begin{aligned}\bar{D}(\psi)(sv)&=d\zeta^{-1}(\psi)(1\otimes 1\otimes sv)-(-1)^{|\psi|}\zeta^{-1}(\psi)d(1\otimes 1\otimes sv)\\&=d\psi(sv)+(-1)^{|\psi|}\zeta^{-1}(\psi)\Bigl(\sum_{i=1}^{\infty}\frac{(sd)^i}{i!}(v\otimes 1\otimes\bar{1})\Bigr)\\&=d\psi(sv)+(-1)^{|\psi|}\sum_{v}\sum_{j=1}^{p}\pm\,\omega_1\cdots\omega_{j-1}\omega_{j+1}\cdots\omega_p\psi(s\omega_j)\\&\qquad+\zeta^{-1}(\psi)\Bigl(\sum_{i=2}^{\infty}\frac{(sd)^i}{i!}(v\otimes 1\otimes\bar{1})\Bigr).\end{aligned}$$

An induction on the degree of v gives that $\zeta^{-1}(\psi)((sd)^2(v\otimes 1\otimes\bar{1}))=0$. Therefore, we see that $\mathrm{Hom}_{\mathbb{Q}}(\Lambda(sV_Y),\Lambda V_X)$ decomposes into a direct sum of complexes

$$(\mathrm{Hom}_{\mathbb{Q}}(\Lambda(sV_Y),\Lambda V_X),\bar{D})=\bigoplus_{p\geq 0}(\mathrm{Hom}_{\mathbb{Q}}(\Lambda^p(sV_Y),\Lambda V_X),\bar{D}). \tag{5-2}$$

Note that the decomposition is a Hochschild cohomology version of Vigué's work [20].

Proof of Proposition 5.1 By Lemma 5.2, the following diagram of complexes is commutative:

$$\begin{array}{ccc}C^*(\Lambda V_Y,\Lambda V_X) & \xrightarrow{\ \zeta\phi^*\ } & \mathrm{Hom}^*_{\mathbb{Q}}(\Lambda(sV_Y),\Lambda V_X)\\ {\scriptstyle J_1}\uparrow & & \uparrow\\ \mathrm{Der}^{*-1}(\Lambda V_Y,\Lambda V_X;\tilde{f}) & \xrightarrow{\ \zeta_1\ } & \mathrm{Hom}^*_{\mathbb{Q}}(sV_Y,\Lambda V_X),\end{array}$$

where ζ_1 is the canonical degree 1 isomorphism of complexes defined by $\zeta_1(\theta)(sv)=(-1)^{|\theta|}\theta(v)$ for $\theta\in\mathrm{Der}^{*-1}(\Lambda V_Y,\Lambda V_X;\tilde{f})$ and $v\in V_Y$. Therefore, the decomposition (5-2) shows that J_1 is injective in homology. □

Before proving Corollary 1.3, we recall the definition of the isomorphism

$$\Phi\colon \pi_n(\mathrm{map}(X,Y;f))\otimes\mathbb{Q}\to H^{-n}(\mathrm{Der}^*(\Lambda V_Y,\Lambda V_X;\widetilde{f}))$$

defined by [1; 14]. Let $\alpha\in\pi_n(\mathrm{map}(X,Y;f))\otimes\mathbb{Q}$ and $g\colon S^n\times X\to Y$ be the adjoint of α. Denote by $\widetilde{g}\colon \Lambda V_X\to\Lambda V_{S^n}\otimes\Lambda V_Y$ a Sullivan model for g. Since S^n is formal, there is a quasi-isomorphism $\phi\colon \Lambda V_{S^n}\to(H^*(S^n;\mathbb{Q}),0)$ and, for any $v\in\Lambda V$, we may write

$$(\phi\otimes 1)\widetilde{g}(v)=1\otimes\widetilde{f}(v)+e_n\otimes v'.$$

Then we put $\Phi(\alpha)(v)=v'$.

Proof of Corollary 1.3 By the definition of Θ_1 and Φ, we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{HH}^{-n+1}(TV;C^*(X)) & \xrightarrow{\;\cong\;} & \mathrm{HH}^{-n+1}(\Lambda V_Y;\Lambda V_X) \\ \uparrow{\scriptstyle J_1^*} & & \uparrow{\scriptstyle J_1^*} \\ H^{-n}\,\mathrm{Der}^*(TV,C^*(X);f^*\circ\rho) & & H^{-n}(\mathrm{Der}^*(\Lambda V_Y,\Lambda V_X;\widetilde{f})) \\ & {\nwarrow}^{\Theta_1} \qquad {\nearrow}^{\Phi} & \\ & \pi_n(\mathrm{map}(X,Y;f))\otimes\mathbb{Q}, & \end{array}$$

where the isomorphism at the top of the above diagram is the map induced by chains of natural quasi-isomorphisms [6, Corollary 10.10]

$$TV\xrightarrow{\simeq}C^*(Y)\xrightarrow{\simeq}\cdots\xleftarrow{\simeq}A_{\mathrm{PL}}(Y)\xleftarrow{\simeq}\Lambda V_Y,$$
$$C^*(X)\xrightarrow{\simeq}\cdots\xleftarrow{\simeq}A_{\mathrm{PL}}(X)\xleftarrow{\simeq}\Lambda V_X.$$

Since Φ is an isomorphism, the commutativity of (5-1) and Proposition 5.1 show the assertion. □

6 Noncommutativity for $\mathbf{H}_*(L_fY;\mathbb{Q})$

We retain the notation described in the section above. Let X be a simply connected d–dimensional closed oriented manifold, Y a simply connected space with finite type and $f\colon X\to Y$ a based space. We see that the shifted homology $\mathbf{H}_*(L_fY)$ has a graded algebra structure by Gruher and Salvatore [10]. As an application for the main result, we have the following proposition.

Proposition 6.1 *If the rational homotopy group $\pi_{\geq 2}(\mathrm{map}(X,Y;f))\otimes\mathbb{Q}$ has a nontrivial Whitehead product, then $\mathbf{H}_*(L_fY;\mathbb{Q})$ is a noncommutative graded algebra.*

Proof By [21, Chapter X, Theorem (7.10)], $\pi_{\geq 2}(\mathrm{map}(X,Y;f))\otimes\mathbb{Q}$ has a nontrivial Whitehead product if and only if there is a nontrivial Samelson product on $\pi_{\geq 1}(\Omega\,\mathrm{map}(X,Y;f))\otimes\mathbb{Q}$. We denote $\langle\beta_1,\beta_2\rangle$ by the nontrivial Samelson product for some β_1 and β_2. Then, by [21, Chapter X, Theorem (6.3)], we have the equality $h(\langle\beta_1,\beta_2\rangle)=h(\beta_1)h(\beta_2)-(-1)^{|\beta_1||\beta_2|}h(\beta_2)h(\beta_1)$, where h is the Hurewicz map. We note that a graded algebra structure on $H_*(\Omega\,\mathrm{map}(X,Y;f);\mathbb{Q})$ is determined by the H-space structure on $\Omega\,\mathrm{map}(X,Y;f)$. Since the map $g\colon \Omega\,\mathrm{map}(X,Y;f)\times X\to L_fY$ is a morphism of fiberwise monoids from the projection $\Omega\,\mathrm{map}(X,Y;f)\times X\to X$ to the map $\chi\colon L_fY\to X$, by [10, Theorem 4.1 (ii)], the map $\Gamma\colon H_*(\Omega\,\mathrm{map}(X,Y;f);\mathbb{Q})\to \mathbf{H}_*(L_fY;\mathbb{Q})$ stated in Section 1 is an algebra map. Therefore, we see that

$$\Gamma_1(\langle\beta_1,\beta_2\rangle)=\Gamma_1(\beta_1)\Gamma_1(\beta_2)-(-1)^{|\beta_1||\beta_2|}\Gamma_1(\beta_2)\Gamma_1(\beta_1)$$

and Corollary 1.3 shows that $\Gamma_1(\beta_1)\Gamma_1(\beta_2)\neq(-1)^{|\beta_1||\beta_2|}\Gamma_1(\beta_2)\Gamma_1(\beta_1)$. □

In the rest of this section, we give a example of $\mathbf{H}_*(L_fY;\mathbb{Q})$ which is noncommutative.

Example 6.2 Let $\mathbb{C}P^n$ be the complex projective space and $i\colon \mathbb{C}P^{n-1}\hookrightarrow\mathbb{C}P^n$ the inclusion for $n\geq 2$. Recall that the commutative differential graded algebra $M(\mathbb{C}P^n):=(\Lambda(x_2,x_{2n+1}),dx_{2n+1}=x_2^{n+1})$ is a minimal Sullivan model for $\mathbb{C}P^n$ and a map

$$\tilde{\imath}\colon M(\mathbb{C}P^n)\longrightarrow M(\mathbb{C}P^{n-1}),\quad \tilde{\imath}(x_2)=x_2,\ \tilde{\imath}(x_{2n+1})=x_2x_{2n-1}$$

is a Sullivan model for i, where the degree of x_j is j. By [2, Theorem 2], a $\tilde{\imath}$–derivation of degree -3, $[\theta,\theta]$, defined by

$$[\theta,\theta]\colon M(\mathbb{C}P^n)\longrightarrow M(\mathbb{C}P^{n-1}),\quad [\theta,\theta](x_2)=0,\ [\theta,\theta](x_{2n+1})=x_2^{n-1}$$

is a nontrivial Whitehead product of

$$H^{-3}(\mathrm{Der}^*(M(\mathbb{C}P^n),M(\mathbb{C}P^{n-1});\tilde{\imath}))\cong\pi_3(\mathrm{map}(\mathbb{C}P^{n-1},\mathbb{C}P^n;i))\otimes\mathbb{Q},$$

where θ is a $\tilde{\imath}$–derivation of degree -2 defined by $\theta(x_2)=1$ and $\theta(x_{2n+1})=0$. The existence of a nonzero Whitehead product in $\pi_*(\mathrm{map}(\mathbb{C}P^{n-1},\mathbb{C}P^n;i))\otimes\mathbb{Q}$ is also showed by the results of Møller and Raussen [17, Example 3.4]. They proved that $\mathrm{map}(\mathbb{C}P^{n-1},\mathbb{C}P^n;i)$ is of the rational homotopy type of $S^2\times S^5\times S^7\times\cdots\times S^{2n+1}$ and the nonzero Whitehead product comes from the S^2 factor. Therefore, by Proposition 6.1, $\mathbf{H}_*(L_i\mathbb{C}P^n;\mathbb{Q})$ is a noncommutative algebra.

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