Free degrees of homeomorphisms on compact surfaces

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For a compact surface M, the free degree $\mathfrak{fr}(M)$ of homeomorphisms on M is the minimum positive integer n with property that for any self homeomorphism ξ of M, at least one of the iterates $\xi, \xi^2, \dots, \xi^n$ has a fixed point. This is to say $\mathfrak{fr}(M)$ is the maximum of least periods among all periodic points of self homeomorphisms on M. We prove that $\mathfrak{fr}(F_{g,b}) \leq 24g-24$ for $g \geq 2$ and $\mathfrak{fr}(N_{g,b}) \leq 12g-24$ for $g \geq 3$.

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1 Background

Let M be a compact surface and ξ be a self homeomorphism on M. The free degree $\mathfrak{fr}(\xi)$ of ξ is the maximum positive integer n such that $\xi^1, \xi^2, \dots, \xi^{n-1}$ are all fixed point free. For a set S which consists of self homeomorphisms on M, we denote the free degree of S by $\mathfrak{fr}(S) = \max\{\mathfrak{fr}(\phi) \mid \phi \in S\}$. We denote the free degree of all homeomorphisms by $\mathfrak{fr}(M)$ and the free degree of all orientation preserving homeomorphisms by $\mathfrak{fr}^+(M)$.

We write F_g for an orientable closed surface of genus g and N_g for a nonorientable closed surface of genus g (ie a connected sum of g projective planes), write respectively $F_{g,b}$ and $N_{g,b}$ for an orientable and nonorientable surface with b boundary components.

J Nielsen [4] studied $\mathfrak{fr}^+(F_g)$ in the 1940s, showing that

$$\mathfrak{fr}^+(F_g) = \begin{cases} 2 \text{ or } 3 & \text{if } g = 2, \\ 2g-2 & \text{if } g > 2. \end{cases}$$

The exact value $\mathfrak{fr}^+(F_2) = 2$ was determined by W Dicks and J Llibre [1] in 1996.

In the 1990s, S Wang [7] obtained results on all homeomorphisms and on nonorientable closed surfaces. One of his results is

$$\mathfrak{fr}(F_g) = \begin{cases} 4 & \text{if } g = 2, \\ 2g-2 & \text{if } g > 2. \end{cases}$$

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In this paper, we consider $\mathfrak{fr}(M)$ when M is a connected compact surface with boundaries. Our main result is:

Theorem 1.1 *For $F_{g,b}$ and $N_{g,b}$ orientable and nonorientable genus g surfaces with b boundary components, the free degrees satisfy:*

$$\max_b \mathfrak{fr}(F_{g,b}) \begin{cases} = \infty & \text{if } g = 0, 1, \\ \leq 24g - 24 & \text{if } g \geq 2. \end{cases}$$

$$\max_b \mathfrak{fr}(N_{g,b}) \begin{cases} = \infty & \text{if } g = 1, 2, \\ \leq 12g - 24 & \text{if } g \geq 3. \end{cases}$$

This means that for given g, the free degree $\mathfrak{fr}(F_{g,b})$ and $\mathfrak{fr}(N_{g,b})$ have an uniform upper bound which is independent of the number of boundary components.

2 Nielsen fixed point theory

In this section, we shall review some facts in Nielsen fixed point theory; see Jiang [2] for more details.

Given any self map $f\colon X \to X$, the fixed point set of f is divided into a disjoint union of some subsets, each is said to be a *fixed point class* of f. A fixed point class is an isolated fixed point set, and hence has well-defined fixed point index. The sum of all indices is the Lefschetz number $L(f)$. The number of essential (nonzero indices) fixed point class is defined to be the Nielsen number $N(f)$. One of the key result in Nielsen fixed point theory is:

Proposition 2.1 (Jiang [2, page 19, 4.7 Theorem]) *Let X be a compact polyhedron. Then, any self-map in the homotopy class of $f\colon X \to X$ has at least $N(f)$ fixed points.*

This result refines the Lefschetz fixed point theorem: $L(f) \neq 0$ implies the fixed point set of f is nonempty.

Apply these basic properties of this two invariants. We have the following.

Proposition 2.2

$$\mathfrak{fr}(f) \leq \min\{ n \mid N(f^n) > 0\} \leq \min\{ n \mid L(f^n) \neq 0\}.$$

This proposition is one of main tool in our present paper.

3 Standard forms of surface homeomorphisms

According to Nielsen–Thurston classification theorem of surface homeomorphisms, any surface homeomorphism is isotopic to either a periodic, pseudo-Anosov or reducible one (see Thurston [5]). In this section, we recall the "standard" homeomorphisms introduced by B Jiang and J Guo [3]. Some adjustments are made for our purpose. The local behavior at periodic points is addressed.

Let p be a positive integer, k an integer, and λ a real number with $\lambda > 1$. We define

(1) $r^{+}_{(p,k,\lambda)}$: a self map on $\mathbb{C}$ given by $r^{+}_{(p,k,\lambda)}(\rho e^{\theta i}) = \rho e^{(\theta+2k\pi/p)i}$;

(2) $r^{-}_{(p,k,\lambda)}$: a self map on $\mathbb{C}$ given by $r^{-}_{(p,k,\lambda)}(\rho e^{\theta i}) = \rho e^{-(\theta+2k\pi/p)i}$;

(3) $\eta_{(p,k,\lambda)}$: a self map on $\mathbb{C}$ which is the time-one map of the vector field v defined by $v(\rho e^{\theta i}) = (2\ln\lambda/p)\rho e^{(1-p)\theta i}$;

(4) $\eta'_{(p,k,\lambda)}$: a self map on $\mathbb{C} - \mathrm{int}(D)$ which is the time-one map of the vector field v' defined by $v'(\rho e^{\theta i}) = (2\ln\lambda/p)((\rho-1)e^{(1-p)\theta i} + e^{(\theta-\pi/2)i}\sin(p\theta))$, where D is the unit disk in complex plane $\mathbb{C}$.

Lemma 3.1 [3, 2.1] *Let ψ be a pseudo-Anosov homeomorphism on a compact surface F, having stable foliation $\mathcal{F}^s$ and unstable foliation $\mathcal{F}^u$ with dilatation λ. Then there is a smooth atlas $\mathcal{U}$ of F, consisting of one chart for each interior singularity, one chart for each boundary component, and some other charts at regular point, such that*

(1) *if $u_x\colon (U_x, x) \to (\mathbb{C}, 0)$ is the chart for an interior singularity x, then the prongs of $\mathcal{F}^s$ are $\{u^{-1}(\rho e^{(m\pi/p)i}) \mid \rho \geq 0,\ m = 1, 3, 5, \dots, 2p-1\}$, the prongs of $\mathcal{F}^u$ are $\{u^{-1}(\rho e^{(m\pi/p)i}) \mid \rho \geq 0,\ m = 0, 2, 4, \dots, 2p-2\}$, and there is a commutative diagram*

$$\begin{array}{ccc} U_x, x & \xrightarrow{\ \psi\ } & U_{\psi(x)}, \psi(x) \\ {\scriptstyle u_x}\big\downarrow & & \big\downarrow{\scriptstyle u_{\psi(x)}} \\ \mathbb{C}, 0 & \xrightarrow{\ \xi\ } & \mathbb{C}, 0 \end{array}$$

where $\xi = r^{+}_{(p,k,\lambda)} \circ \eta_{(p,k,\lambda)}$ or $\xi = r^{-}_{(p,k,\lambda)} \circ \eta_{(p,k,\lambda)}$ for some nonnegative integer k (the singularity x is said to be of type $(p,k)^{+}$ or of type $(p,k)^{-}$), and $u_{\psi(x)}$ is the chart in $\mathcal{U}$ for the singularity $\psi(x)$;

(2) *if* $u_A\colon (U_A, A) \to (\mathbb{C} - \mathrm{int}(D), \partial D)$ *is the chart for a boundary component* A, *then there is a commutative diagra*

$$\begin{array}{ccc} U_A, A & \xrightarrow{\varphi} & U_{\psi(A)}, \psi(A) \\ {\scriptstyle u_A}\downarrow & & \downarrow{\scriptstyle u_{\psi(A)}} \\ \mathbb{C} - \mathrm{int}(D), \partial D & \xrightarrow{\xi} & \mathbb{C} - \mathrm{int}(D), \partial D \end{array}$$

where $\xi = r^{+}_{(p,k,\lambda)} \circ \eta'_{(p,k,\lambda)}$ *or* $\xi = r^{-}_{(p,k,\lambda)} \circ \eta'_{(p,k,\lambda)}$ *for some nonnegative integer* k *(the boundary component* A *is said to be of type* $(p,k)^+$ *or of type* $(p,k)^-$*), and* $u_{\psi(A)}$ *is the chart in* $\mathcal{U}$ *for the boundary component* $\psi(A)$.

The superscript $+$ or $-$ of the type indicates orientation preserving or reversing.

The local behavior and indices of isolated fixed point sets of a pseudo-Anosov homeomorphism are given as follow.

Lemma 3.2 [3, Lemma 2.1] *Let* ψ *be an orientation-preserving pseudo-Anosov homeomorphism on a compact surface* F *with* $\chi(F) < 0$. *Then there is a smooth atlas* $\mathcal{U}$ *of* F *satisfying the conclusion of Lemma 3.1, and each fixed point of* ψ *is included in one of the following cases:*

(1) *Isolated fixed point* x.

 (1.1) x *is of type* $(p,0)^+$ *with* $\mathrm{ind}(\psi, x) = 1 - p$, *where* $p \geq 2$;
 (1.2) x *is of type* $(p,k)^+$ *with* $p \nmid k$ *with* $\mathrm{ind}(\psi, x) = 1$.

(2) *Boundary component* C *such that* $\psi(C) = C$.

 (2.1) C *is of type* $(p,0)^+$, *and* $C \subset \mathrm{Fix}(\psi)$ *with* $\mathrm{ind}(\psi, C) = -p$;
 (2.2) C *is of type* $(p,k)^+$ *with* $p \nmid k$, *and* $C \cap \mathrm{Fix}(\varphi) = \varnothing$, *hence* $\mathrm{ind}(\psi, C) = 0$.

A fixed point in the interior which is not a singularity can be regarded as a "2–prong singularity", and hence is also included in this lemma. But the chart on it is not in the chosen atlas $\mathcal{U}$.

Lemma 3.3 [3, Lemma 3.1] *Any orientation-preserving homeomorphism on an annulus* $F_{0,2} \cong S^1 \times I$ *is isotopic to one of the following:*

(1) *an annular twist* $\psi(z,t) = (ze^{2(a+bt)\pi i}, t)$, *where* a *and* b *are rational numbers;*

(2) *a flip-twist* $\psi(z,t) = (\bar{z}e^{a(1-2t)\pi i}, 1-t)$, *where* a *is a rational number.*

Lemma 3.4 [3, Lemma 3.6] *Let M be a connected compact oriented surface M with $\chi(M) < 0$. Then any orientation-preserving homeomorphism on M is isotopic to a homeomorphism ψ (*in standard form*), having following properties. There is a set (the* cutting system*) $\Gamma = \{\gamma_1, \gamma_2, \dots, \gamma_k\}$ of simple closed curves on M. Each γ_j has a neighborhood $N(\gamma_j)$ homeomorphic to $S^1 \times I$ such that*

(i) *the restriction $\psi|_{N(\gamma_j)}$ of ψ on each $N(\gamma_j)$ is an annular twist or a flip-twist;*

(ii) *the restriction of ψ on each component of $M - \bigcup_{j=1}^{k} \operatorname{int} N(\gamma_j)$ is either periodic or pseudo-Anosov. (These components are said to be* pieces*, or ψ–pieces if we need to emphasis the related homeomorphism ψ.)*

Moreover, each nonempty fixed point class of ψ is a connected subset of M, being included in one of the following cases:

(1) *Isolated fixed point x.*

(1.1) *ψ is conjugate to a rotation in a neighborhood of x in periodic piece,* $\operatorname{ind}(\psi, x) = 1$;

(1.1) *a fixed point of an annular flip-twist,* $\operatorname{ind}(\psi, x) = 1$;

(1.1) *a fixed point of type $(p,0)^+$ in a pseudo-Anosov piece,* $\operatorname{ind}(\psi, x) = 1 - p$;

(1.1) *a fixed point of $(p,k)^+$ with $p \nmid k$ in a pseudo-Anosov piece,* $\operatorname{ind}(\psi, x) = 1$.

(2) *Fixed point circle C.*

(2.2) *an isolated fixed point set of an annular twist;* $\operatorname{ind}(\psi, C) = 0$;

(2.2) *a boundary component with type $(p,0)^+$ of some pseudo-Anosov piece, on the other side C is a boundary component of an annular twist;* $\operatorname{ind}(\psi, C) = -p$;

(2.2) *a boundary component of M, and also a boundary component with type $(p,0)^+$ of some pseudo-Anosov piece,* $\operatorname{ind}(\psi, C) = -p$;

(3) *Fixed point subsurface.*

Corollary 3.5 *If ψ is in standard form, then ψ^n is in standard form for any positive n. Moreover, the cutting system of ψ^n can be chosen as a subset of a cutting system of ψ.*

Proposition 3.6 *Let $\psi\colon M \to M$ be an orientation-preserving homeomorphism on a connect compact oriented surface M with $\chi(M) < 0$, and be in standard form. Let V be an invariant set of ψ^n which consists of some ψ–pieces and some neighborhoods of cutting curves. If $N((\psi|_V)^n) > 0$, then $N(\psi^n) > 0$.*

4 Periodic homeomorphisms

In this section, we shall discuss the upper bound for the free degree of periodic homeomorphisms on a compact surface.

Given a compact surface F, we write $o(F)$ for the maximal order of periodic homeomorphisms.

Lemma 4.1 (1) *$o(F_{g,b}) \le o(F_g)$ for all b.*

(2) *$o(N_{g,b}) \le o(N_g)$ for all b.*

Proof Regard the closed surface F_g as the quotient space of $F_{g,b}$ by collapsing each boundary component to one point. We write $q\colon F_{g,b} \to F_g$ for this natural quotient map. Let ψ be a periodic homeomorphism on $F_{g,b}$. Then it induces a periodic homeomorphism $\bar{\psi}$ on F_g, ie there is a commutative diagram

$$\begin{array}{ccc} F_{g,b} & \xrightarrow{\psi} & F_{g,b} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q} \\ F_g & \xrightarrow{\bar{\psi}} & F_g . \end{array} \tag{4-1}$$

By definition, $o(F_g)$ is the maximum of the orders of the periodic map on F_g. The order of $\bar{\psi}$ is not greater than $o(F_g)$, ie there is a positive integer n with $n \le o(F_g)$ such that $\bar{\psi}^n$ is the identity on F_g. It follows that ψ^n is the identity on $F_{g,b}$. This proves the conclusion (1). The proof of (2) is the same. □

Lemma 4.2 *Let ξ be a self homeomorphism on a connected compact surface M which is homotopic to a periodic map. If $\chi(M) \ne 0$, then there is a positive integer $n \le o(M)$ such that $N(\xi^n) = 1$, and hence the free degree $\mathfrak{fr}(\xi)$ of ξ is no more than $o(M)$.*

Proof This lemma is trivial if $o(M)$ is infinite.

Assume that $o(M)$ is finite. Let ϕ be a periodic map homotopic to the given map ξ. By definition of maximal order, there is natural number n with $n \le o(M)$ such that $\phi^n = \mathrm{id}$. From the homotopy invariance of Nielsen number, we have that $N(\xi^n) = N(\phi^n) = N(\mathrm{id})$. Since $\chi(M) \ne 0$, we have that $N(\xi^n) = N(\phi^n) = N(\mathrm{id}) = 1$. □

Combining our two lemmas with Wang [6, Theorem 1], we have:

Theorem 4.3 (1) *Let $\xi\colon F_{g,b} \to F_{g,b}$ be a self-map homotopic to a periodic one. Then $\mathfrak{fr}(\xi) \le 4g + 3 + (-1)^g$ for all $g \ge 2$.*

(2) *Let $\xi\colon N_{g,b} \to N_{g,b}$ be a self-map homotopic to a periodic one. Then $\mathfrak{fr}(\xi) \le 2g - 1 + (-1)^{g+1}$ for all $g \ge 3$.*

Let us consider the free degree of compact surface with genus 0 or 1. It is known that $\mathfrak{fr}(F_{0,0}) = 2$, and $\mathfrak{fr}(N_{1,0}) = 1$.

As for the cases $F_{0,b}$, $F_{1,b}$, $N_{1,b}$ and $N_{2,b}$, we have the following examples.

Example 4.4 Regard $F_{0,2}$ as

$$S^1 \times I = \{(z,t) \mid |z| = 1,\ 0 \le t \le 1\}.$$

Given any positive integer k, a periodic homeomorphism ξ_k is defined by $\xi_k(z,t) = (ze^{2\pi i/k}, t)$.

Pick a small open disk W in $F_{0,2}$. Note that $G = F_{0,2} - \bigsqcup_{j=1}^{\infty} \xi_k^j(W)$ is homeomorphic to $F_{0,k+2}$. The restriction $\xi_k|_G$ of ξ_k on G is also a periodic homeomorphism. Each point is a periodic point of period k. Hence, $\mathfrak{fr}(\xi_k|_G) = k$. This implies that $\mathfrak{fr}(F_{0,k+2}) \ge k$.

Let $\tau\colon F_{0,2} \to F_{0,2}$ be an involution given by $\tau(z,t) = (-z, 1-t)$. It gives a $\mathbb{Z}_2$ action on $F_{0,2}$, and the orbit space $F_{0,2}/\tau$ is homeomorphic to the Möbius band $N_{1,1}$. If k is even, ξ_k induces a periodic homeomorphism $\overline{\xi}_k\colon F_{0,2}/\tau \to F_{0,2}/\tau$ with period $k/2$. Note that each point on $\overline{\xi}_k$ has period $k/2$. We have that $\mathfrak{fr}(\overline{\xi}_k|_{G/\tau}) = k/2$. Since $G/\tau \cong N_{1,k/2+1}$, we also have that $\mathfrak{fr}(N_{1,k/2+1}) \ge k/2$.

Example 4.5 Regard $F_{1,0}$ as

$$S^1 \times S^1 = \{(z,w) \mid |z| = |w| = 1\} \subset \mathbb{C}^2.$$

Given any positive integer k, a periodic homeomorphism ξ_k is defined by $\xi_k(z,w) = (ze^{2\pi i/k}, w)$.

Pick a small open disk W in $F_{1,0}$. Note that

$$G = F_{1,0} - \bigsqcup_{j=1}^{\infty} \xi_k^j(W) = F_{1,0} - \bigsqcup_{j=1}^{k} \xi_k^j(W)$$

is homeomorphic to $F_{1,k}$. The restriction $\xi_k|_G$ of ξ_k on G is also a periodic homeomorphism. Each point is a periodic point of period k. Hence, $\mathfrak{fr}(\xi_k|_G) = k$. This implies that $\mathfrak{fr}(F_{1,k}) \ge k$.

Let $\tau\colon F_{1,0} \to F_{1,0}$ be an involution given by $\tau(z,w) = (-z, \overline{w})$. It gives a $\mathbb{Z}_2$ action on $F_{1,0}$, and the orbit space $F_{1,0}/\tau$ is homeomorphic to the Klein bottle $N_{2,0}$. If k is even, ξ_k induces a periodic homeomorphism $\overline{\xi}_k\colon F_{1,0}/\tau \to F_{1,0}/\tau$ with period $k/2$. Note that each point on $\overline{\xi}_k$ has period $k/2$. We have that $\mathfrak{fr}(\overline{\xi}_k|_{G/\tau}) = k/2$. Since $G/\tau \cong N_{2,k/2}$, we also have that $\mathfrak{fr}(N_{2,k/2}) \ge k/2$.

Moreover, by using irrational angle rotation, we can show that there is a homeomorphism on $F_{0,2}$ (the annulus) without any periodic point, and a homeomorphism on $N_{1,1}$ (the Möbius band) without any periodic point. This implies that both $\mathfrak{fr}(F_{0,2})$ and $\mathfrak{fr}(N_{1,1})$ are infinite. But, by classical fixed point theorem, any map on $F_{0,1}$ (the disk) must have a fixed point. Hence, $\mathfrak{fr}(F_{0,1}) = 1$.

5 Special homeomorphisms on surfaces with small genus

In this section, we consider the free degree of some homeomorphisms on the surfaces of genus 0 or 1.

Lemma 5.1 *Let $\xi\colon F_{0,b} \to F_{0,b}$ be an orientation preserving homeomorphism on a sphere with b boundary components. If there are at least three boundary components on $F_{0,b}$ which are invariant under ξ, then $L(\xi) \neq 0$, hence $\mathfrak{fr}(\xi) = 1$.*

Proof See Figure 1.

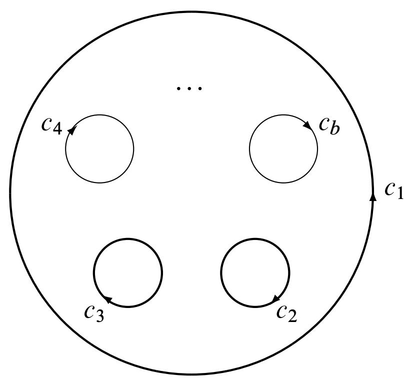

Figure 1: Bases of $H_1(F_{0,b}, \mathbb{Q})$

The boundary components c_1, c_2, c_3 are invariant under ξ. Choose the classes $[c_2], [c_3], [c_4], \dots, [c_b]$ as a basis of $H_1(F_{0,b}, \mathbb{Q})$. The induced isomorphism ξ_{*1} on $H_1(F_{0,b}, \mathbb{Q})$ by ξ has the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & B_1 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & B_l \end{pmatrix},$$

where $B_1, \dots, B_l$ are permutation matrices.

So $\operatorname{tr}(\xi_{*1}) \geq 2$ and $L(\xi) = 1 - \operatorname{tr}(\xi_{*1}) \neq 0$. □

Lemma 5.2 *Let $\xi\colon F_{1,b} \to F_{1,b}$ be an orientation preserving homeomorphism on a torus with $b \geq 1$ boundary components. If there is at least one boundary component of $F_{1,b}$ which is invariant under ξ, then $\mathfrak{fr}(\xi) \leq 6$.*

Proof See Figure 2. The boundary component c_1 is invariant under ξ. Choose $[x], [y], [c_2], \dots, [c_b]$ as bases of $H_1(F_{1,b}, \mathbb{Q})$. The induced isomorphism ξ_{*1} on $H_1(F_{1,b}, \mathbb{Q})$ by ξ has the form

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & \cdots & \cdots \\ a_{21} & a_{22} & \cdots & \cdots & \cdots \\ 0 & 0 & B_1 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & B_l \end{pmatrix},$$

where $B_1, \dots, B_l$ are permutation matrices. This matrix is similar to

$$\begin{pmatrix} \lambda & * & \cdots & \cdots & \cdots \\ 0 & \lambda^{-1} & \cdots & \cdots & \cdots \\ 0 & 0 & C_1 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & C_l \end{pmatrix}.$$

Here if C_j is of rank n, then

$$C_j = \begin{pmatrix} e^{2\pi/ni} & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & e^{2(n-1)\pi/ni} & \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Suppose the number of rank 1,2,3,4,6 C_j's is m, t, s, q, r. If $L(\xi^1) = L(\xi^2) = L(\xi^3) = L(\xi^4) = L(\xi^6) = 0$, then we have the following identities.

$$\lambda + \lambda^{-1} + m = 1 \tag{5-1}$$

$$\lambda^2 + \lambda^{-2} + m + 2t = 1 \tag{5-2}$$

$$\lambda^3 + \lambda^{-3} + m + 3s = 1 \tag{5-3}$$

$$\lambda^4 + \lambda^{-4} + m + 2t + 4q = 1 \tag{5-4}$$

$$\lambda^6 + \lambda^{-6} + m + 2t + 3s + 6r = 1 \tag{5-5}$$

So $(1-m)^2 = (\lambda + \lambda^{-1})^2 = \lambda^2 + \lambda^{-2} + 2 = 3 - m - 2t$. We have $m = 2,\ t = 0$ or $m = 1,\ t = 1$ or $m = 0,\ t = 1$.

If $m=2, t=0$, then $\lambda+\lambda^{-1}+1=0$ induces $\lambda=e^{\frac{2\pi}{3}i}$ or $e^{-\frac{2\pi}{3}i}$. We have $\lambda^3+\lambda^{-3}=2$ which contradicts (5-3).

If $m=1, t=1$, then $\lambda+\lambda^{-1}=0$ induces $\lambda=i$ or $-i$. We have $\lambda^4+\lambda^{-4}=2$ which contradicts (5-4).

If $m=0, t=1$, then $\lambda+\lambda^{-1}-1=0$ induces $\lambda=e^{\pi/3i}$ or $e^{-\pi/3i}$. We have $\lambda^6+\lambda^{-6}=2$ which contradicts (5-5).

The argument above shows that at least one of the $L(\xi^1)$, $L(\xi^2)$, $L(\xi^3)$, $L(\xi^4)$, $L(\xi^6)$ is not equal to 0. Thus $\mathfrak{fr}(\xi)\le 6$. □

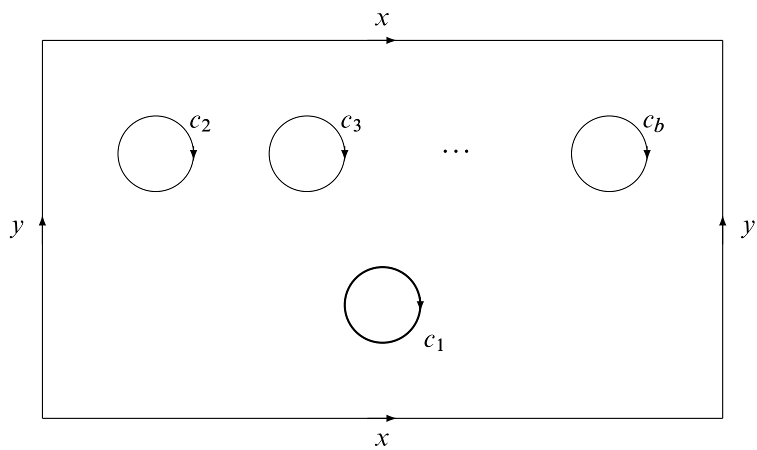

Figure 2: Bases of $H_1(F_{1,b},\mathbb{Q})$

6 Pseudo-Anosov homeomorphisms

We consider the free degrees of pseudo-Anosov homeomorphisms in this section.

Lemma 6.1 *Let $\mathcal{F}$ be a singular foliation on a compact surface $F_{g,b}$. Then*

$$\sum_{m=1}^{\infty}\left(\left(1-\frac{m}{2}\right)\mathrm{Pr}_m^{\mathrm{int}}(\mathcal{F})-\frac{m}{2}\,\mathrm{Pr}_m^{\mathrm{bd}}(\mathcal{F})\right)=\chi(F_{g,b})=2-2g-b,$$

where $\mathrm{Pr}_m^{\mathrm{int}}(\mathcal{F})$ is the number of m–prong singularities in the interior of $F_{g,b}$ and $\mathrm{Pr}_m^{\mathrm{bd}}(\mathcal{F})$ is the number of boundary components of $F_{g,b}$ with m–prong singularities.

Proof Consider the closed surface F_g, a singular foliation can be regarded as a path field on F_g. Both have the same singularity set. An m–prong singularity has index $1-m/2$. □

Lemma 6.2 *Let $\xi\colon F_{g,b} \to F_{g,b}$ be a self homeomorphism on $F_{g,b} (g \geq 2)$ which is homotopic to an orientation preserving pseudo-Anosov homeomorphism ψ. If the stable (and hence unstable) singular foliation of ψ has no 1–prong singularity, then $\mathfrak{fr}(\xi) \leq 8g-8$.*

Proof Since Nielsen number is a homotopy invariant, we only need to prove that there exists a positive integer $n \leq 8g-8$ such that $N(\psi^n) > 0$.

Let $\mathcal{F}^s$ and $\mathcal{F}^u$ be respectively stable and unstable singular foliations of the pseudo-Anosov homeomorphism ψ. We denote the number of m–prong singularities of $\mathcal{F}^s$ by $\mathrm{Pr}_m(\mathcal{F}^s)$. Regard the closed surface F_g as the quotient space of $F_{g,b}$ by collapsing each boundary component to one point. Then ψ induces an orientation preserving homeomorphism $\bar{\psi}$ on F_g, satisfying the commutative diagram

$$\begin{tikzcd} F_{g,b} \arrow[r, "\psi"] \arrow[d, "q"'] & F_{g,b} \arrow[d, "q"] \\ F_g \arrow[r, "\bar{\psi}"] & F_g . \end{tikzcd}$$

Since $\mathcal{F}^s$ has no 1–prong singularity, the quotient map q gives a singular foliation $q(\mathcal{F}^s)$ on F_g. Thus, $q(\mathcal{F}^s)$ and $q(\mathcal{F}^u)$ are respectively stable and unstable singular foliations of the pseudo-Anosov homeomorphism $\bar{\psi}$.

By the results of Nielsen [4] with Dicks and Llibre [1] (see Wang [7, Remark (1)]), we know that $\mathfrak{fr}(\bar{\psi}) = 2g-2$. This implies that there must be an integer n_0 with $n_0 \leq 2g-2$ such that $\bar{\psi}^{n_0}$ has a fixed point $\bar{x}_0$. Since $\bar{\psi}^{n_0}$ is a pseudo-Anosov homeomorphism on the closed surface F_g, the point $\bar{x}_0$ consists of a fixed point class of $\bar{\psi}^{n_0}$ with $\mathrm{ind}(\bar{\psi}^{n_0}, \bar{x}_0) \neq 0$. If $q^{-1}(\bar{x}_0)$ is a singleton, the point $q^{-1}(\bar{x}_0)$ is an isolated fixed point of ψ^{n_0} with nonzero index. Since ψ is in standard form, this isolated fixed point is an essential fixed point class of ψ^{n_0}. Thus, we have that $N(\psi^{n_0}) > 0$.

If it is not a singleton, $q^{-1}(\bar{x}_0)$ is a boundary component of $F_{g,b}$. Thus, $q^{-1}(\bar{x}_0)$ is an invariant circle of type (p_0, k_0) for ψ^{n_0}. Note that $q^{-1}(\bar{x}_0)$ is a fixed point circle for $\psi^{p_0 n_0}$ of type $(p_0, p_0 k_0)$, ie of type $(p_0, 0)$. By Lemma 3.2, we have that $\mathrm{ind}(\psi^{p_0 n_0}, q^{-1}(\bar{x}_0)) = -p_0 \neq 0$. It follows that $N(\psi^{p_0 n_0}) > 0$. It is sufficient to show that $p_0 n_0 \leq 8g-8$. There are two cases:

Case 1 $p_0 = 2$ or 3. We have that $p_0 n_0 \leq 3(2g-2) = 6g-6$.

Case 2 $p_0 \geq 4$. Applying Lemma 6.1 to the foliation $q(\mathcal{F}^s)$ on F_g, we have

$$2-2g = \chi(F_g) = \sum_{p=1}^{\infty} \left(1-\frac{p}{2}\right) \mathrm{Pr}_p^{\mathrm{int}}(q(\mathcal{F}^s)) \leq \left(1-\frac{p_0}{2}\right) \mathrm{Pr}_{p_0}^{\mathrm{int}}(q(\mathcal{F}^s)),$$

because $q(\mathcal{F}^s)$ has no 1–prong singularity. It follows that

$$p_0 \, \mathrm{Pr}_{p_0}^{\mathrm{int}}(q(\mathcal{F}^s)) \leq (4g-4)\left(1+\frac{2}{p_0-2}\right) \leq 8g-8.$$

Since ψ permutes the boundaries of p_0–prong, we have that

$$n_0 \leq \mathrm{Pr}_{p_0}^{\mathrm{bd}}(\mathcal{F}^s) \leq \mathrm{Pr}_{p_0}^{\mathrm{bd}}(\mathcal{F}^s) + \mathrm{Pr}_{p_0}^{\mathrm{int}}(\mathcal{F}^s) = \mathrm{Pr}_{p_0}^{\mathrm{int}}(q(\mathcal{F}^s)),$$

and hence $p_0 n_0 \leq 8g-8$. □

7 Main results

Lemma 7.1 *Let $F_{g,b}$ be a connected compact surface of genus g with b boundary components, where $g \geq 2$. Then for any orientation-preserving homeomorphism $\psi \colon F_{g,b} \to F_{g,b}$, there is a positive integer n with $n \leq 12g-12$ such that $N(\psi^n) > 0$.*

Proof The procedure of our proof will be fulfilled by using a reduction on the pairs (g,b) according to the lexicographic order. That is, we say $(g',b') < (g'',b'')$ if either $g' < g''$ or $g' = g''$ and $b' < b''$.

By the homotopy invariance of Nielsen number, we may assume that ψ is in standard form.

Case 1 $b = 0$. From the proof of Wang [7, Theorem 1], we know that the Lefschetz number $L(\psi^n)$ is nonzero for some n satisfying $n \leq 4$ if $g = 2$; $n \leq 2g-2$ if $g \geq 3$. This implies that $N(\psi^n) > 0$ for such an n.

Case 2 ψ is periodic. We have $N(\psi^n) = 1$ for some $n \leq 4g+3+(-1)^g$.

Case 3 ψ is a pseudo-Anosov map. Note that the homeomorphism ψ permutes the boundary components of type $(1,0)^+$. Let l_0 be the minimal length of orbits of ψ–action on the set of all boundary components of type $(1,0)^+$. We have three subcases according to the value of l_0.

Subcase 3.1 $l_0 = 0$. This means logically that the number of boundary components of $F_{g,b}$ with type $(1,0)$ is zero. This is done in Lemma 6.2.

Subcase 3.2 $0 < l_0 \le 12g-12$. Let C_0 be a boundary component with type $(1,0)^+$ such that $\psi^{l_0}(C_0) = C_0$. By Lemma 3.4, C_0 is a fixed point class of ψ^{l_0}, having fixed point index -1. This implies that $N(\psi^{l_0}) > 0$.

Subcase 3.3 $l_0 > 12g-12$. We collapse each boundary component of type $(1,0)^+$ to one point. The homeomorphism ψ induces a homeomorphism $\bar{\psi}$ on the resulting surface $F_{g,b'}$. We write $q\colon F_{g,b} \to F_{g,b'}$ for this natural quotient map. Then we have a commutative diagram

$$\begin{array}{ccc} F_{g,b} & \xrightarrow{\psi} & F_{g,b} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q} \\ F_{g,b'} & \xrightarrow{\bar{\psi}} & F_{g,b'} . \end{array}$$

Of course, $\bar{\psi}$ is not in standard form. By definition of l_0, we have that $\mathrm{Fix}(\psi^m) = \mathrm{Fix}(\bar{\psi}^m)$ for any m with $0 < m \le 12g-12$. Since q is a homeomorphism near this fixed point set, any isolated fixed point set have the same fixed point indices. In other word, $\mathrm{ind}(\psi^m, F) = \mathrm{ind}(\bar{\psi}^m, F)$ for any fixed point class F of ψ^m with $0 < m \le 12g-12$. Since q is a surjective map and since q only collapses $(1,0)^+$ boundary components, any fixed point class of $\bar{\psi}^m$ will be an union of some fixed point classes of ψ^m. Any essential fixed point class of $\bar{\psi}^m$ contains at least one essential fixed point class of ψ^m if $m \le 12g-12$. Thus, it is sufficient to prove that $N(\bar{\psi}^m) > 0$ for some m with $m \le 12g-12$. This is just the inductive assumption.

Case 4 ψ is reducible. Let P_0 be a reduced piece with the biggest genus among all pieces. Assume that $P_0 \cong F_{g_0,b_0}$.

Thus, either $g_0 < g$ or $g_0 = g$ and $b_0 < b$. Note that ψ permutes all pieces.

We consider three subcases according to the value of g_0.

Case 4.1 $g_0 \ge 2$. We write l_0 for the orbit length of P_0 under the action of ψ. That is $\psi^{l_0}(P_0) = P_0$, and $\psi^j(P_0) \ne P_0$ for $j = 1, 2, \dots, l_0 - 1$. Clearly, $(\psi|_{P_0})^{l_0}$ is a homeomorphism on $P_0 \cong F_{g_0,b_0}$. By assumption of reduction, there is a positive number n_0 with $n_0 \le 12g_0 - 12$ such that $N(((\psi|_{P_0})^{l_0})^{n_0})) > 0$, ie $N(\psi^{l_0 n_0}|_{P_0}) > 0$. By Proposition 3.6, we have that $N(\psi^{l_0 n_0}) > 0$. Clearly,

$$l_0 n_0 \le l_0(12g_0 - 12) \le 12g - 12l_0 \le 12g - 12.$$

Case 4.2 $g_0 = 0$ or 1. Consider the quotient map $q\colon F_{g,b} \to F_g$ and the induced homeomorphism satisfying the commutative diagram (4-1). Let $\Gamma = \{\gamma_1, \gamma_2, \dots, \gamma_k\}$ be the cutting system for ψ. We assume that $q(\gamma_j)$ is essential in F_g for $j = 1, 2, \dots k'$, and inessential for $j = k'+1, \dots, k$. We write $\Gamma' = \{\gamma_1, \gamma_2, \dots, \gamma_{k'}\}$. Then each component of $F_g - q(\Gamma')$ is an union of one component of $F_g - q(\Gamma)$ and other components of $F_g - q(\Gamma)$ are disks. This implies that $k' > 0$ and the maximal genus of the components of $F_g - q(\Gamma')$ is still g_0. Since each curve $q(\gamma_j)$ in $q(\Gamma')$ is essential in F_g, the Euler characteristic number of each component of $F_g - q(\Gamma')$ is negative. Consider two sub-subcases.

Case 4.2.1 $g_0 = 1$. Let Q_0 be a component of $F_g - q(\Gamma')$ with genus 1. Let l_0 be the orbit length of Q_0 under the action of $\overline{\psi}$, and b_0 be number of boundary components of Q_0. Note that each component of $F_g - q(\Gamma')$ has nonpositive Euler characteristic number. We obtain that $l_0(2-2-b_0) \geq 2-2g$. Since $\overline{\psi}^{l_0}(Q_0) = Q_0$, there must be a positive integer n_0 with $n_0 \leq b_0$ such that $\overline{\psi}^{l_0 n_0}$ has at least one invariant boundary component C_0 of Q_0. By commutative diagram (4-1), we have $\psi^{l_0 n_0}(q^{-1}(Q_0)) = q^{-1}(Q_0)$. By definition of q, the closure $P = \overline{q^{-1}(Q_0)}$ of $q^{-1}(Q_0)$ is homeomorphic to $F_{1,b}$ for some $b \geq b_0$. Apply Lemma 5.2 to the homeomorphism $\psi^{l_0 n_0}|_P$, there is a positive integer n with $n \leq 6$ such that $L((\psi^{l_0 n_0}|_P)^n) \neq 0$. Hence, $N((\psi^{l_0 n_0}|_P)^n) > 0$. It follows from Proposition 3.6 that $N(\psi^{l_0 n_0 n}) > 0$.

Case 4.2.2 $g_0 = 0$. In this situation, each component of $F_g - q(\Gamma')$ has genus zero, ie a disk with holes. Note that each component of $F_g - q(\Gamma')$ has nonpositive Euler characteristic number. From the additivity of Euler characteristic numbers, there must be a component Q_0 with $\chi(Q_0) < 0$, ie Q_0 has at least three boundary components. Let l_0 be the orbit length of Q_0 under the action of $\overline{\psi}$, and b_0 be number of boundary components of Q_0. Then we have $l_0(2-b_0) \geq 2-2g$. This implies that

$$l_0 b_0 \leq \frac{b_0}{b_0 - 2}(2g-2) \leq 6g-6,$$

because $b_0 \geq 3$. Since $\overline{\psi}^{l_0}$ permutes the boundary components of Q_0, there must be a positive integer n_0 with $n_0 \leq b_0$ such that $\overline{\psi}^{l_0 n_0}$ fixes set-wisely at least three boundary components. Notice that the closure $P = \overline{q^{-1}(Q_0)}$ of $q^{-1}(Q_0)$ is also a disk with holes. The homeomorphism $\psi^{l_0 n_0}|_P$ also fixes set-wisely at least three boundary components. Apply Lemma 5.1 to the homeomorphism $\psi^{l_0 n_0}|_P$, we have that $L(\psi^{l_0 n_0}|_P) \neq 0$. Hence, $N(\psi^{l_0 n_0}|_P) > 0$. It follows from Proposition 3.6 that $N(\psi^{l_0 n_0}) > 0$. □

Theorem 7.2 *For $F_{g,b}$ and $N_{g,b}$ orientable and nonorientable genus g surfaces with b boundary components, the free degrees satisfy:*

$$\max_b \mathfrak{fr}(F_{g,b}) \begin{cases} = \infty & \text{if } g = 0, 1, \\ \leq 24g - 24 & \text{if } g \geq 2. \end{cases}$$

$$\max_b \mathfrak{fr}(N_{g,b}) \begin{cases} = \infty & \text{if } g = 1, 2, \\ \leq 12g - 24 & \text{if } g \geq 3. \end{cases}$$

Proof The infiniteness has been shown in Example 4.4 and Example 4.5.

Consider a homeomorphism $\psi\colon F_{g,b} \to F_{g,b}$, where $g \geq 2$. Then ψ^2 must be orientation preserving. By Lemma 7.1, there is a positive integer n with $n \leq 12g-12$ such that $N(\psi^{2n}) = N((\psi^2)^n) > 0$. It follows that ψ^{2n} has a fixed point. Hence, $\mathfrak{fr}(\psi) \leq 24g-24$. Since ψ is an arbitrary homeomorphism on $F_{g,b}$. We obtain that $\mathfrak{fr}(F_{g,b}) \leq 24g-24$.

Let $\eta\colon F_{g-1,2b} \to N_{g,b}$ be the classical orientation covering. Write τ for the unique nontrivial covering transformation. Any homeomorphism $\psi\colon N_{g,b} \to N_{g,b}$ has two liftings ϕ and $\tau\phi$. Without loss of the generality, we may assume that ϕ is orientation preserving. By Lemma 7.1, there is a positive integer n with $n \leq 12(g-1)-12$ such that ϕ^n has a fixed point x_0. Clearly, $\psi^n(\eta(x_0)) = \eta(\phi^n(x_0)) = \eta(x_0)$, ie $\eta(x_0)$ is a fixed point of ψ^n. This implies that $\mathfrak{fr}(N_{g,b}) \leq 12(g-1)-12 = 12g-24$. □

From the proof of this theorem, we obtain:

Corollary 7.3 *For $F_{g,b}$ an orientable genus g surface with b boundary components, the orientation preserving free degree satisfies:*

$$\max_b \mathfrak{fr}^+(F_{g,b}) \begin{cases} = \infty & \text{if } g = 0, 1, \\ \leq 12g-12 & \text{if } g \geq 2. \end{cases}$$

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