Free degrees of homeomorphisms on compact surfaces

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For a compact surface M, the free degree $\mathfrak{fr}(M)$ of homeomorphisms on M is the minimum positive integer n with property that for any self homeomorphism ξ of M, at least one of the iterates $\xi, \xi^2, \dots, \xi^n$ has a fixed point. This is to say $\mathfrak{fr}(M)$ is the maximum of least periods among all periodic points of self homeomorphisms on M. We prove that $\mathfrak{fr}(F_{g,b}) \le 24g - 24$ for $g \ge 2$ and $\mathfrak{fr}(N_{g,b}) \le 12g - 24$ for $g \ge 3$.

55M20; 37E30

1 Background

Let M be a compact surface and ξ be a self homeomorphism on M. The free degree $\mathfrak{fr}(\xi)$ of ξ is the maximum positive integer n such that $\xi^1, \xi^2, \dots, \xi^{n-1}$ are all fixed point free. For a set S which consists of self homeomorphisms on M, we denote the free degree of S by $\mathfrak{fr}(S) = \max\{\mathfrak{fr}(\phi) \mid \phi \in S\}$. We denote the free degree of all homeomorphisms by $\mathfrak{fr}(M)$ and the free degree of all orientation preserving homeomorphisms by $\mathfrak{fr}^+(M)$.

We write F_g for an orientable closed surface of genus g and N_g for a nonorientable closed surface of genus g (ie a connected sum of g projective planes), write respectively $F_{g,b}$ and $N_{g,b}$ for an orientable and nonorientable surface with b boundary components.

J Nielsen [4] studied $\mathfrak{fr}^+(F_g)$ in the 1940s, showing that

$$\mathfrak{fr}^+(F_g) = \begin{cases} 2 \text{ or } 3 & \text{if } g = 2, \\ 2g-2 & \text{if } g > 2. \end{cases}$$

The exact value $\mathfrak{fr}^+(F_2) = 2$ was determined by W Dicks and J Llibre [1] in 1996.

In the 1990s, S Wang [7] obtained results on all homeomorphisms and on nonorientable closed surfaces. One of his results is

$$\mathfrak{fr}(F_g) = \begin{cases} 4 & \text{if } g = 2, \\ 2g-2 & \text{if } g > 2. \end{cases}$$

Published: 5 September 2011 DOI: 10.2140/agt.2011.11.2437

In this paper, we consider $\mathfrak{fr}(M)$ when M is a connected compact surface with boundaries. Our main result is:

Theorem 1.1 *For $F_{g,b}$ and $N_{g,b}$ orientable and nonorientable genus g surfaces with b boundary components, the free degrees satisfy:*

$$\max_b \mathfrak{fr}(F_{g,b}) \begin{cases} = \infty & \text{if } g = 0, 1, \\ \leq 24g - 24 & \text{if } g \geq 2. \end{cases}$$

$$\max_b \mathfrak{fr}(N_{g,b}) \begin{cases} = \infty & \text{if } g = 1, 2, \\ \leq 12g - 24 & \text{if } g \geq 3. \end{cases}$$

This means that for given g, the free degree $\mathfrak{fr}(F_{g,b})$ and $\mathfrak{fr}(N_{g,b})$ have an uniform upper bound which is independent of the number of boundary components.

2 Nielsen fixed point theory

In this section, we shall review some facts in Nielsen fixed point theory; see Jiang [2] for more details.

Given any self map $f\colon X \to X$, the fixed point set of f is divided into a disjoint union of some subsets, each is said to be a *fixed point class* of f. A fixed point class is an isolated fixed point set, and hence has well-defined fixed point index. The sum of all indices is the Lefschetz number $L(f)$. The number of essential (nonzero indices) fixed point class is defined to be the Nielsen number $N(f)$. One of the key result in Nielsen fixed point theory is:

Proposition 2.1 (Jiang [2, page 19, 4.7 Theorem]) *Let X be a compact polyhedron. Then, any self-map in the homotopy class of $f\colon X \to X$ has at least $N(f)$ fixed points.*

This result refines the Lefschetz fixed point theorem: $L(f) \neq 0$ implies the fixed point set of f is nonempty.

Apply these basic properties of this two invariants. We have the following.

Proposition 2.2

$$\mathfrak{fr}(f) \leq \min\{n \mid N(f^n) > 0\} \leq \min\{n \mid L(f^n) \neq 0\}.$$

This proposition is one of main tool in our present paper.

3 Standard forms of surface homeomorphisms

According to Nielsen–Thurston classification theorem of surface homeomorphisms, any surface homeomorphism is isotopic to either a periodic, pseudo-Anosov or reducible one (see Thurston [5]). In this section, we recall the "standard" homeomorphisms introduced by B Jiang and J Guo [3]. Some adjustments are made for our purpose. The local behavior at periodic points is addressed.

Let p be a positive integer, k an integer, and λ a real number with $\lambda > 1$. We define

(1) $r^+_{(p,k,\lambda)}$: a self map on $\mathbb{C}$ given by $r^+_{(p,k,\lambda)}(\rho e^{\theta i}) = \rho e^{(\theta + 2k\pi/p)i}$;

(2) $r^-_{(p,k,\lambda)}$: a self map on $\mathbb{C}$ given by $r^-_{(p,k,\lambda)}(\rho e^{\theta i}) = \rho e^{-(\theta + 2k\pi/p)i}$;

(3) $\eta_{(p,k,\lambda)}$: a self map on $\mathbb{C}$ which is the time-one map of the vector field v defined by $v(\rho e^{\theta i}) = (2\ln\lambda/p)\rho e^{(1-p)\theta i}$;

(4) $\eta'_{(p,k,\lambda)}$: a self map on $\mathbb{C} - \mathrm{int}(D)$ which is the time-one map of the vector field v' defined by $v'(\rho e^{\theta i}) = (2\ln\lambda/p)((\rho - 1)e^{(1-p)\theta i} + e^{(\theta-\pi/2)i}\sin(p\theta))$, where D is the unit disk in complex plane $\mathbb{C}$.

Lemma 3.1 [3, 2.1] *Let ψ be a pseudo-Anosov homeomorphism on a compact surface F, having stable foliation $\mathcal{F}^s$ and unstable foliation $\mathcal{F}^u$ with dilatation λ. Then there is a smooth atlas $\mathcal{U}$ of F, consisting of one chart for each interior singularity, one chart for each boundary component, and some other charts at regular point, such that*

(1) *if $u_x\colon (U_x, x) \to (\mathbb{C}, 0)$ is the chart for an interior singularity x, then the prongs of $\mathcal{F}^s$ are $\{u^{-1}(\rho e^{(m\pi/p)i}) \mid \rho \geq 0,\ m = 1, 3, 5, \ldots, 2p-1\}$, the prongs of $\mathcal{F}^u$ are $\{u^{-1}(\rho e^{(m\pi/p)i}) \mid \rho \geq 0,\ m = 0, 2, 4, \ldots, 2p-2\}$, and there is a commutative diagram*

$$\begin{array}{ccc} U_x, x & \xrightarrow{\psi} & U_{\psi(x)}, \psi(x) \\ {\scriptstyle u_x}\big\downarrow & & \big\downarrow{\scriptstyle u_{\psi(x)}} \\ \mathbb{C}, 0 & \xrightarrow{\xi} & \mathbb{C}, 0 \end{array}$$

where $\xi = r^+_{(p,k,\lambda)} \circ \eta_{(p,k,\lambda)}$ or $\xi = r^-_{(p,k,\lambda)} \circ \eta_{(p,k,\lambda)}$ for some nonnegative integer k (the singularity x is said to be of type $(p,k)^+$ or of type $(p,k)^-$), and $u_{\psi(x)}$ is the chart in $\mathcal{U}$ for the singularity $\psi(x)$;

(2) *if* $u_A\colon (U_A, A) \to (\mathbb{C} - \mathrm{int}(D), \partial D)$ *is the chart for a boundary component* A*, then there is a commutative diagra*

$$\begin{tikzcd} U_A, A \arrow[r, "\varphi"] \arrow[d, "u_A"'] & U_{\psi(A)}, \psi(A) \arrow[d, "u_{\psi(A)}"] \\ \mathbb{C} - \mathrm{int}(D), \partial D \arrow[r, "\xi"] & \mathbb{C} - \mathrm{int}(D), \partial D \end{tikzcd}$$

where $\xi = r^+_{(p,k,\lambda)} \circ \eta'_{(p,k,\lambda)}$ or $\xi = r^-_{(p,k,\lambda)} \circ \eta'_{(p,k,\lambda)}$ *for some nonnegative integer* k *(the boundary component* A *is said to be of type* $(p,k)^+$ *or of type* $(p,k)^-$*), and* $u_{\psi(A)}$ *is the chart in* $\mathcal{U}$ *for the boundary component* $\psi(A)$*.*

The superscript $+$ or $-$ of the type indicates orientation preserving or reversing.

The local behavior and indices of isolated fixed point sets of a pseudo-Anosov homeomorphism are given as follow.

Lemma 3.2 [3, Lemma 2.1] *Let* ψ *be an orientation-preserving pseudo-Anosov homeomorphism on a compact surface* F *with* $\chi(F) < 0$*. Then there is a smooth atlas* $\mathcal{U}$ *of* F *satisfying the conclusion of Lemma 3.1, and each fixed point of* ψ *is included in one of the following cases:*

(1) *Isolated fixed point* x*.*

- (1.1) x *is of type* $(p,0)^+$ *with* $\mathrm{ind}(\psi, x) = 1 - p$*, where* $p \geq 2$*;*
- (1.2) x *is of type* $(p,k)^+$ *with* $p \nmid k$ *with* $\mathrm{ind}(\psi, x) = 1$*.*

(2) *Boundary component* C *such that* $\psi(C) = C$*.*

- (2.1) C *is of type* $(p,0)^+$*, and* $C \subset \mathrm{Fix}(\psi)$ *with* $\mathrm{ind}(\psi, C) = -p$*;*
- (2.2) C *is of type* $(p,k)^+$ *with* $p \nmid k$*, and* $C \cap \mathrm{Fix}(\varphi) = \varnothing$*, hence* $\mathrm{ind}(\psi, C) = 0$*.*

A fixed point in the interior which is not a singularity can be regarded as a "2–prong singularity", and hence is also included in this lemma. But the chart on it is not in the chosen atlas $\mathcal{U}$.

Lemma 3.3 [3, Lemma 3.1] *Any orientation-preserving homeomorphism on an annulus* $F_{0,2} \cong S^1 \times I$ *is isotopic to one of the following:*

(1) *an annular twist* $\psi(z,t) = (ze^{2(a+bt)\pi i}, t)$*, where* a *and* b *are rational numbers;*

(2) *a flip-twist* $\psi(z,t) = (\bar{z}e^{a(1-2t)\pi i}, 1-t)$*, where* a *is a rational number.*

Lemma 3.4 [3, Lemma 3.6] *Let M be a connected compact oriented surface M with $\chi(M) < 0$. Then any orientation-preserving homeomorphism on M is isotopic to a homeomorphism ψ (*in standard form*), having following properties. There is a set (*the cutting system*) $\Gamma = \{\gamma_1, \gamma_2, \dots, \gamma_k\}$ of simple closed curves on M. Each γ_j has a neighborhood $N(\gamma_j)$ homeomorphic to $S^1 \times I$ such that*

(i) *the restriction $\psi|_{N(\gamma_j)}$ of ψ on each $N(\gamma_j)$ is an annular twist or a flip-twist;*

(ii) *the restriction of ψ on each component of $M - \bigcup_{j=1}^{k} \operatorname{int} N(\gamma_j)$ is either periodic or pseudo-Anosov. (These components are said to be* pieces*, or ψ–pieces if we need to emphasis the related homeomorphism ψ.)*

Moreover, each nonempty fixed point class of ψ is a connected subset of M, being included in one of the following cases:

(1) *Isolated fixed point x.*

(1.1) *ψ is conjugate to a rotation in a neighborhood of x in periodic piece,* $\operatorname{ind}(\psi, x) = 1$;

(1.1) *a fixed point of an annular flip-twist,* $\operatorname{ind}(\psi, x) = 1$;

(1.1) *a fixed point of type $(p,0)^+$ in a pseudo-Anosov piece,* $\operatorname{ind}(\psi, x) = 1 - p$;

(1.1) *a fixed point of $(p,k)^+$ with $p \nmid k$ in a pseudo-Anosov piece,* $\operatorname{ind}(\psi, x) = 1$.

(2) *Fixed point circle C.*

(2.2) *an isolated fixed point set of an annular twist;* $\operatorname{ind}(\psi, C) = 0$;

(2.2) *a boundary component with type $(p,0)^+$ of some pseudo-Anosov piece, on the other side C is a boundary component of an annular twist;* $\operatorname{ind}(\psi, C) = -p$;

(2.2) *a boundary component of M, and also a boundary component with type $(p,0)^+$ of some pseudo-Anosov piece,* $\operatorname{ind}(\psi, C) = -p$;

(3) *Fixed point subsurface.*

Corollary 3.5 *If ψ is in standard form, then ψ^n is in standard form for any positive n. Moreover, the cutting system of ψ^n can be chosen as a subset of a cutting system of ψ.*

Proposition 3.6 *Let $\psi\colon M \to M$ be an orientation-preserving homeomorphism on a connect compact oriented surface M with $\chi(M) < 0$, and be in standard form. Let V be an invariant set of ψ^n which consists of some ψ–pieces and some neighborhoods of cutting curves. If $N((\psi|_V)^n) > 0$, then $N(\psi^n) > 0$.*

4 Periodic homeomorphisms

In this section, we shall discuss the upper bound for the free degree of periodic homeomorphisms on a compact surface.

Given a compact surface F, we write $o(F)$ for the maximal order of periodic homeomorphisms.

Lemma 4.1 (1) *$o(F_{g,b}) \leq o(F_g)$ for all b.*

(2) *$o(N_{g,b}) \leq o(N_g)$ for all b.*

Proof Regard the closed surface F_g as the quotient space of $F_{g,b}$ by collapsing each boundary component to one point. We write $q\colon F_{g,b} \to F_g$ for this natural quotient map. Let ψ be a periodic homeomorphism on $F_{g,b}$. Then it induces a periodic homeomorphism $\overline{\psi}$ on F_g, ie there is a commutative diagram

$$\begin{tikzcd} F_{g,b} \arrow[r, "\psi"] \arrow[d, "q"'] & F_{g,b} \arrow[d, "q"] \\ F_g \arrow[r, "\overline{\psi}"] & F_g . \end{tikzcd} \tag{4-1}$$

By definition, $o(F_g)$ is the maximum of the orders of the periodic map on F_g. The order of $\overline{\psi}$ is not greater than $o(F_g)$, ie there is a positive integer n with $n \leq o(F_g)$ such that $\overline{\psi}^n$ is the identity on F_g. It follows that ψ^n is the identity on $F_{g,b}$. This proves the conclusion (1). The proof of (2) is the same. □

Lemma 4.2 *Let ξ be a self homeomorphism on a connected compact surface M which is homotopic to a periodic map. If $\chi(M) \neq 0$, then there is a positive integer $n \leq o(M)$ such that $N(\xi^n) = 1$, and hence the free degree $\mathfrak{fr}(\xi)$ of ξ is no more than $o(M)$.*

Proof This lemma is trivial if $o(M)$ is infinite.

Assume that $o(M)$ is finite. Let ϕ be a periodic map homotopic to the given map ξ. By definition of maximal order, there is natural number n with $n \leq o(M)$ such that $\phi^n = \mathrm{id}$. From the homotopy invariance of Nielsen number, we have that $N(\xi^n) = N(\phi^n) = N(\mathrm{id})$. Since $\chi(M) \neq 0$, we have that $N(\xi^n) = N(\phi^n) = N(\mathrm{id}) = 1$. □

Combining our two lemmas with Wang [6, Theorem 1], we have:

Theorem 4.3 (1) *Let $\xi\colon F_{g,b} \to F_{g,b}$ be a self-map homotopic to a periodic one. Then $\mathfrak{fr}(\xi) \leq 4g + 3 + (-1)^g$ for all $g \geq 2$.*

(2) *Let $\xi\colon N_{g,b} \to N_{g,b}$ be a self-map homotopic to a periodic one. Then $\mathfrak{fr}(\xi) \leq 2g - 1 + (-1)^{g+1}$ for all $g \geq 3$.*

Let us consider the free degree of compact surface with genus 0 or 1. It is known that $\mathfrak{fr}(F_{0,0}) = 2$, and $\mathfrak{fr}(N_{1,0}) = 1$.

As for the cases $F_{0,b}$, $F_{1,b}$, $N_{1,b}$ and $N_{2,b}$, we have the following examples.

Example 4.4 Regard $F_{0,2}$ as

$$S^1 \times I = \{(z,t) \mid |z| = 1,\ 0 \le t \le 1\}.$$

Given any positive integer k, a periodic homeomorphism ξ_k is defined by $\xi_k(z,t) = (ze^{2\pi i/k}, t)$.

Pick a small open disk W in $F_{0,2}$. Note that $G = F_{0,2} - \bigsqcup_{j=1}^{\infty} \xi_k^j(W)$ is homeomorphic to $F_{0,k+2}$. The restriction $\xi_k|_G$ of ξ_k on G is also a periodic homeomorphism. Each point is a periodic point of period k. Hence, $\mathfrak{fr}(\xi_k|_G) = k$. This implies that $\mathfrak{fr}(F_{0,k+2}) \ge k$.

Let $\tau\colon F_{0,2} \to F_{0,2}$ be an involution given by $\tau(z,t) = (-z, 1-t)$. It gives a $\mathbb{Z}_2$ action on $F_{0,2}$, and the orbit space $F_{0,2}/\tau$ is homeomorphic to the Möbius band $N_{1,1}$. If k is even, ξ_k induces a periodic homeomorphism $\overline{\xi}_k\colon F_{0,2}/\tau \to F_{0,2}/\tau$ with period $k/2$. Note that each point on $\overline{\xi}_k$ has period $k/2$. We have that $\mathfrak{fr}(\overline{\xi}_k|_{G/\tau}) = k/2$. Since $G/\tau \cong N_{1,k/2+1}$, we also have that $\mathfrak{fr}(N_{1,k/2+1}) \ge k/2$.

Example 4.5 Regard $F_{1,0}$ as

$$S^1 \times S^1 = \{(z,w) \mid |z| = |w| = 1\} \subset \mathbb{C}^2.$$

Given any positive integer k, a periodic homeomorphism ξ_k is defined by $\xi_k(z,w) = (ze^{2\pi i/k}, w)$.

Pick a small open disk W in $F_{1,0}$. Note that

$$G = F_{1,0} - \bigsqcup_{j=1}^{\infty} \xi_k^j(W) = F_{1,0} - \bigsqcup_{j=1}^{k} \xi_k^j(W)$$

is homeomorphic to $F_{1,k}$. The restriction $\xi_k|_G$ of ξ_k on G is also a periodic homeomorphism. Each point is a periodic point of period k. Hence, $\mathfrak{fr}(\xi_k|_G) = k$. This implies that $\mathfrak{fr}(F_{1,k}) \ge k$.

Let $\tau\colon F_{1,0} \to F_{1,0}$ be an involution given by $\tau(z,w) = (-z, \overline{w})$. It gives a $\mathbb{Z}_2$ action on $F_{1,0}$, and the orbit space $F_{1,0}/\tau$ is homeomorphic to the Klein bottle $N_{2,0}$. If k is even, ξ_k induces a periodic homeomorphism $\overline{\xi}_k\colon F_{1,0}/\tau \to F_{1,0}/\tau$ with period $k/2$. Note that each point on $\overline{\xi}_k$ has period $k/2$. We have that $\mathfrak{fr}(\overline{\xi}_k|_{G/\tau}) = k/2$. Since $G/\tau \cong N_{2,k/2}$, we also have that $\mathfrak{fr}(N_{2,k/2}) \ge k/2$.

Moreover, by using irrational angle rotation, we can show that there is a homeomorphism on $F_{0,2}$ (the annulus) without any periodic point, and a homeomorphism on $N_{1,1}$ (the Möbius band) without any periodic point. This implies that both $\mathfrak{fr}(F_{0,2})$ and $\mathfrak{fr}(N_{1,1})$ are infinite. But, by classical fixed point theorem, any map on $F_{0,1}$ (the disk) must have a fixed point. Hence, $\mathfrak{fr}(F_{0,1}) = 1$.

5 Special homeomorphisms on surfaces with small genus

In this section, we consider the free degree of some homeomorphisms on the surfaces of genus 0 or 1.

Lemma 5.1 *Let $\xi\colon F_{0,b} \to F_{0,b}$ be an orientation preserving homeomorphism on a sphere with b boundary components. If there are at least three boundary components on $F_{0,b}$ which are invariant under ξ, then $L(\xi) \neq 0$, hence $\mathfrak{fr}(\xi) = 1$.*

Proof See Figure 1.

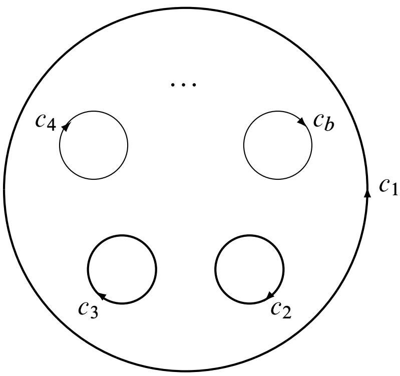

Figure 1: Bases of $H_1(F_{0,b}, \mathbb{Q})$

The boundary components c_1, c_2, c_3 are invariant under ξ. Choose the classes $[c_2], [c_3], [c_4], \dots, [c_b]$ as a basis of $H_1(F_{0,b}, \mathbb{Q})$. The induced isomorphism ξ_{*1} on $H_1(F_{0,b}, \mathbb{Q})$ by ξ has the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & B_1 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & B_l \end{pmatrix},$$

where $B_1, \dots, B_l$ are permutation matrices.

So $\mathrm{tr}(\xi_{*1}) \geq 2$ and $L(\xi) = 1 - \mathrm{tr}(\xi_{*1}) \neq 0$. $\square$

Lemma 5.2 *Let $\xi\colon F_{1,b} \to F_{1,b}$ be an orientation preserving homeomorphism on a torus with $b \geq 1$ boundary components. If there is at least one boundary component of $F_{1,b}$ which is invariant under ξ, then $\mathfrak{fr}(\xi) \leq 6$.*

Proof See Figure 2. The boundary component c_1 is invariant under ξ. Choose $[x],[y],[c_2],\dots,[c_b]$ as bases of $H_1(F_{1,b},\mathbb{Q})$. The induced isomorphism ξ_{*1} on $H_1(F_{1,b},\mathbb{Q})$ by ξ has the form

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & \cdots & \cdots \\ a_{21} & a_{22} & \cdots & \cdots & \cdots \\ 0 & 0 & B_1 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & B_l \end{pmatrix},$$

where $B_1,\dots,B_l$ are permutation matrices. This matrix is similar to

$$\begin{pmatrix} \lambda & * & \cdots & \cdots & \cdots \\ 0 & \lambda^{-1} & \cdots & \cdots & \cdots \\ 0 & 0 & C_1 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & C_l \end{pmatrix}.$$

Here if C_j is of rank n, then

$$C_j = \begin{pmatrix} e^{2\pi/ni} & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & e^{2(n-1)\pi/ni} & \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Suppose the number of rank 1,2,3,4,6 C_j's is m,t,s,q,r. If $L(\xi^1)=L(\xi^2)=L(\xi^3)=L(\xi^4)=L(\xi^6)=0$, then we have the following identities.

$$\lambda+\lambda^{-1}+m=1 \tag{5-1}$$

$$\lambda^2+\lambda^{-2}+m+2t=1 \tag{5-2}$$

$$\lambda^3+\lambda^{-3}+m+3s=1 \tag{5-3}$$

$$\lambda^4+\lambda^{-4}+m+2t+4q=1 \tag{5-4}$$

$$\lambda^6+\lambda^{-6}+m+2t+3s+6r=1 \tag{5-5}$$

So $(1-m)^2=(\lambda+\lambda^{-1})^2=\lambda^2+\lambda^{-2}+2=3-m-2t$. We have $m=2,\ t=0$ or $m=1,\ t=1$ or $m=0,\ t=1$.

If $m = 2, t = 0$, then $\lambda + \lambda^{-1} + 1 = 0$ induces $\lambda = e^{\frac{2\pi}{3}i}$ or $e^{-\frac{2\pi}{3}i}$. We have $\lambda^3 + \lambda^{-3} = 2$ which contradicts (5-3).

If $m = 1, t = 1$, then $\lambda + \lambda^{-1} = 0$ induces $\lambda = i$ or $-i$. We have $\lambda^4 + \lambda^{-4} = 2$ which contradicts (5-4).

If $m = 0, t = 1$, then $\lambda + \lambda^{-1} - 1 = 0$ induces $\lambda = e^{\pi/3i}$ or $e^{-\pi/3i}$. We have $\lambda^6 + \lambda^{-6} = 2$ which contradicts (5-5).

The argument above shows that at least one of the $L(\xi^1)$, $L(\xi^2)$, $L(\xi^3)$, $L(\xi^4)$, $L(\xi^6)$ is not equal to 0. Thus $\mathfrak{fr}(\xi) \leq 6$. □

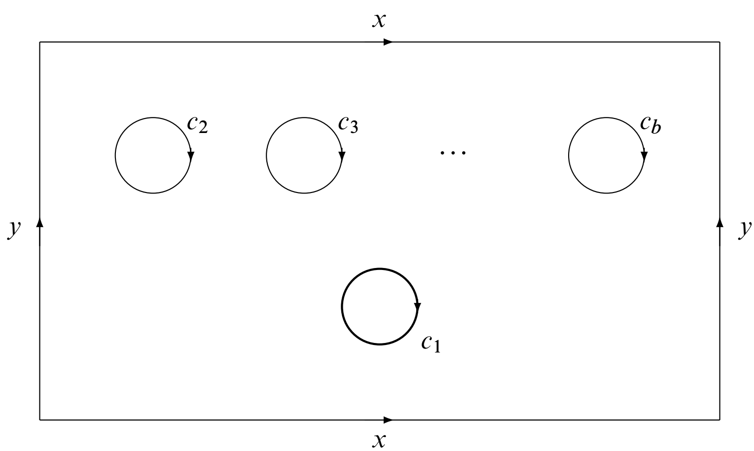

Figure 2: Bases of $H_1(F_{1,b}, \mathbb{Q})$

6 Pseudo-Anosov homeomorphisms

We consider the free degrees of pseudo-Anosov homeomorphisms in this section.

Lemma 6.1 *Let $\mathcal{F}$ be a singular foliation on a compact surface $F_{g,b}$. Then*

$$\sum_{m=1}^{\infty}\left(\left(1-\frac{m}{2}\right)\mathrm{Pr}_m^{\mathrm{int}}(\mathcal{F})-\frac{m}{2}\,\mathrm{Pr}_m^{\mathrm{bd}}(\mathcal{F})\right)=\chi(F_{g,b})=2-2g-b,$$

where $\mathrm{Pr}_m^{\mathrm{int}}(\mathcal{F})$ is the number of m–prong singularities in the interior of $F_{g,b}$ and $\mathrm{Pr}_m^{\mathrm{bd}}(\mathcal{F})$ is the number of boundary components of $F_{g,b}$ with m–prong singularities.

Proof Consider the closed surface F_g, a singular foliation can be regarded as a path field on F_g. Both have the same singularity set. An m–prong singularity has index $1-m/2$. $\square$

Lemma 6.2 *Let $\xi\colon F_{g,b}\to F_{g,b}$ be a self homeomorphism on $F_{g,b}(g\geq 2)$ which is homotopic to an orientation preserving pseudo-Anosov homeomorphism ψ. If the stable (and hence unstable) singular foliation of ψ has no 1–prong singularity, then $\mathfrak{fr}(\xi)\leq 8g-8$.*

Proof Since Nielsen number is a homotopy invariant, we only need to prove that there exists a positive integer $n\leq 8g-8$ such that $N(\psi^n)>0$.

Let $\mathcal{F}^s$ and $\mathcal{F}^u$ be respectively stable and unstable singular foliations of the pseudo-Anosov homeomorphism ψ. We denote the number of m–prong singularities of $\mathcal{F}^s$ by $\mathrm{Pr}_m(\mathcal{F}^s)$. Regard the closed surface F_g as the quotient space of $F_{g,b}$ by collapsing each boundary component to one point. Then ψ induces an orientation preserving homeomorphism $\overline{\psi}$ on F_g, satisfying the commutative diagram

$$\begin{array}{ccc} F_{g,b} & \xrightarrow{\psi} & F_{g,b} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q} \\ F_g & \xrightarrow{\overline{\psi}} & F_g \end{array}.$$

Since $\mathcal{F}^s$ has no 1–prong singularity, the quotient map q gives a singular foliation $q(\mathcal{F}^s)$ on F_g. Thus, $q(\mathcal{F}^s)$ and $q(\mathcal{F}^u)$ are respectively stable and unstable singular foliations of the pseudo-Anosov homeomorphism $\overline{\psi}$.

By the results of Nielsen [4] with Dicks and Llibre [1] (see Wang [7, Remark (1)]), we know that $\mathfrak{fr}(\overline{\psi})=2g-2$. This implies that there must be an integer n_0 with $n_0\leq 2g-2$ such that $\overline{\psi}^{n_0}$ has a fixed point $\overline{x}_0$. Since $\overline{\psi}^{n_0}$ is a pseudo-Anosov homeomorphism on the closed surface F_g, the point $\overline{x}_0$ consists of a fixed point class of $\overline{\psi}^{n_0}$ with $\mathrm{ind}(\overline{\psi}^{n_0},\overline{x}_0)\neq 0$. If $q^{-1}(\overline{x}_0)$ is a singleton, the point $q^{-1}(\overline{x}_0)$ is an isolated fixed point of ψ^{n_0} with nonzero index. Since ψ is in standard form, this isolated fixed point is an essential fixed point class of ψ^{n_0}. Thus, we have that $N(\psi^{n_0})>0$.

If it is not a singleton, $q^{-1}(\overline{x}_0)$ is a boundary component of $F_{g,b}$. Thus, $q^{-1}(\overline{x}_0)$ is an invariant circle of type (p_0,k_0) for ψ^{n_0}. Note that $q^{-1}(\overline{x}_0)$ is a fixed point circle for $\psi^{p_0n_0}$ of type (p_0,p_0k_0), ie of type $(p_0,0)$. By Lemma 3.2, we have that $\mathrm{ind}(\psi^{p_0n_0},q^{-1}(\overline{x}_0))=-p_0\neq 0$. It follows that $N(\psi^{p_0n_0})>0$. It is sufficient to show that $p_0n_0\leq 8g-8$. There are two cases:

Case 1 $p_0 = 2$ or 3. We have that $p_0 n_0 \leq 3(2g-2) = 6g-6$.

Case 2 $p_0 \geq 4$. Applying Lemma 6.1 to the foliation $q(\mathcal{F}^s)$ on F_g, we have

$$2-2g = \chi(F_g) = \sum_{p=1}^{\infty}\left(1-\frac{p}{2}\right)\mathrm{Pr}_p^{\mathrm{int}}(q(\mathcal{F}^s)) \leq \left(1-\frac{p_0}{2}\right)\mathrm{Pr}_{p_0}^{\mathrm{int}}(q(\mathcal{F}^s)),$$

because $q(\mathcal{F}^s)$ has no 1–prong singularity. It follows that

$$p_0\,\mathrm{Pr}_{p_0}^{\mathrm{int}}(q(\mathcal{F}^s)) \leq (4g-4)\left(1+\frac{2}{p_0-2}\right) \leq 8g-8.$$

Since ψ permutes the boundaries of p_0–prong, we have that

$$n_0 \leq \mathrm{Pr}_{p_0}^{\mathrm{bd}}(\mathcal{F}^s) \leq \mathrm{Pr}_{p_0}^{\mathrm{bd}}(\mathcal{F}^s) + \mathrm{Pr}_{p_0}^{\mathrm{int}}(\mathcal{F}^s) = \mathrm{Pr}_{p_0}^{\mathrm{int}}(q(\mathcal{F}^s)),$$

and hence $p_0 n_0 \leq 8g-8$. □

7 Main results

Lemma 7.1 *Let $F_{g,b}$ be a connected compact surface of genus g with b boundary components, where $g \geq 2$. Then for any orientation-preserving homeomorphism $\psi\colon F_{g,b} \to F_{g,b}$, there is a positive integer n with $n \leq 12g-12$ such that $N(\psi^n) > 0$.*

Proof The procedure of our proof will be fulfilled by using a reduction on the pairs (g,b) according to the lexicographic order. That is, we say $(g',b') < (g'',b'')$ if either $g' < g''$ or $g' = g''$ and $b' < b''$.

By the homotopy invariance of Nielsen number, we may assume that ψ is in standard form.

Case 1 $b = 0$. From the proof of Wang [7, Theorem 1], we know that the Lefschetz number $L(\psi^n)$ is nonzero for some n satisfying $n \leq 4$ if $g = 2$; $n \leq 2g-2$ if $g \geq 3$. This implies that $N(\psi^n) > 0$ for such an n.

Case 2 ψ is periodic. We have $N(\psi^n) = 1$ for some $n \leq 4g+3+(-1)^g$.

Case 3 ψ is a pseudo-Anosov map. Note that the homeomorphism ψ permutes the boundary components of type $(1,0)^+$. Let l_0 be the minimal length of orbits of ψ–action on the set of all boundary components of type $(1,0)^+$. We have three subcases according to the value of l_0.

Subcase 3.1 $l_0 = 0$. This means logically that the number of boundary components of $F_{g,b}$ with type $(1,0)$ is zero. This is done in Lemma 6.2.

Subcase 3.2 $0 < l_0 \le 12g - 12$. Let C_0 be a boundary component with type $(1,0)^+$ such that $\psi^{l_0}(C_0) = C_0$. By Lemma 3.4, C_0 is a fixed point class of ψ^{l_0}, having fixed point index -1. This implies that $N(\psi^{l_0}) > 0$.

Subcase 3.3 $l_0 > 12g - 12$. We collapse each boundary component of type $(1,0)^+$ to one point. The homeomorphism ψ induces a homeomorphism $\bar{\psi}$ on the resulting surface $F_{g,b'}$. We write $q\colon F_{g,b} \to F_{g,b'}$ for this natural quotient map. Then we have a commutative diagram

$$\begin{array}{ccc} F_{g,b} & \xrightarrow{\psi} & F_{g,b} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q} \\ F_{g,b'} & \xrightarrow{\bar{\psi}} & F_{g,b'} \end{array}.$$

Of course, $\bar{\psi}$ is not in standard form. By definition of l_0, we have that $\mathrm{Fix}(\psi^m) = \mathrm{Fix}(\bar{\psi}^m)$ for any m with $0 < m \le 12g - 12$. Since q is a homeomorphism near this fixed point set, any isolated fixed point set have the same fixed point indices. In other word, $\mathrm{ind}(\psi^m, F) = \mathrm{ind}(\bar{\psi}^m, F)$ for any fixed point class F of ψ^m with $0 < m \le 12g - 12$. Since q is a surjective map and since q only collapses $(1,0)^+$ boundary components, any fixed point class of $\bar{\psi}^m$ will be an union of some fixed point classes of ψ^m. Any essential fixed point class of $\bar{\psi}^m$ contains at least one essential fixed point class of ψ^m if $m \le 12g-12$. Thus, it is sufficient to prove that $N(\bar{\psi}^m) > 0$ for some m with $m \le 12g - 12$. This is just the inductive assumption.

Case 4 ψ is reducible. Let P_0 be a reduced piece with the biggest genus among all pieces. Assume that $P_0 \cong F_{g_0,b_0}$.

Thus, either $g_0 < g$ or $g_0 = g$ and $b_0 < b$. Note that ψ permutes all pieces.

We consider three subcases according to the value of g_0.

Case 4.1 $g_0 \ge 2$. We write l_0 for the orbit length of P_0 under the action of ψ. That is $\psi^{l_0}(P_0) = P_0$, and $\psi^j(P_0) \ne P_0$ for $j = 1, 2, \dots, l_0 - 1$. Clearly, $(\psi|_{P_0})^{l_0}$ is a homeomorphism on $P_0 \cong F_{g_0,b_0}$. By assumption of reduction, there is a positive number n_0 with $n_0 \le 12g_0 - 12$ such that $N((\psi|_{P_0})^{l_0})^{n_0})) > 0$, ie $N(\psi^{l_0 n_0}|_{P_0}) > 0$. By Proposition 3.6, we have that $N(\psi^{l_0 n_0}) > 0$. Clearly,

$$l_0 n_0 \le l_0(12g_0 - 12) \le 12g - 12l_0 \le 12g - 12.$$

Case 4.2 $g_0 = 0$ or 1. Consider the quotient map $q\colon F_{g,b} \to F_g$ and the induced homeomorphism satisfying the commutative diagram (4-1). Let $\Gamma = \{\gamma_1, \gamma_2, \dots, \gamma_k\}$ be the cutting system for ψ. We assume that $q(\gamma_j)$ is essential in F_g for $j = 1, 2, \dots k'$, and inessential for $j = k' + 1, \dots, k$. We write $\Gamma' = \{\gamma_1, \gamma_2, \dots, \gamma_{k'}\}$. Then each component of $F_g - q(\Gamma')$ is an union of one component of $F_g - q(\Gamma)$ and other components of $F_g - q(\Gamma)$ are disks. This implies that $k' > 0$ and the maximal genus of the components of $F_g - q(\Gamma')$ is still g_0. Since each curve $q(\gamma_j)$ in $q(\Gamma')$ is essential in F_g, the Euler characteristic number of each component of $F_g - q(\Gamma')$ is negative. Consider two sub-subcases.

Case 4.2.1 $g_0 = 1$. Let Q_0 be a component of $F_g - q(\Gamma')$ with genus 1. Let l_0 be the orbit length of Q_0 under the action of $\overline{\psi}$, and b_0 be number of boundary components of Q_0. Note that each component of $F_g - q(\Gamma')$ has nonpositive Euler characteristic number. We obtain that $l_0(2-2-b_0) \geq 2-2g$. Since $\overline{\psi}^{l_0}(Q_0) = Q_0$, there must be a positive integer n_0 with $n_0 \leq b_0$ such that $\overline{\psi}^{l_0 n_0}$ has at least one invariant boundary component C_0 of Q_0. By commutative diagram (4-1), we have $\psi^{l_0 n_0}(q^{-1}(Q_0)) = q^{-1}(Q_0)$. By definition of q, the closure $P = \overline{q^{-1}(Q_0)}$ of $q^{-1}(Q_0)$ is homeomorphic to $F_{1,b}$ for some $b \geq b_0$. Apply Lemma 5.2 to the homeomorphism $\psi^{l_0 n_0}|_P$, there is a positive integer n with $n \leq 6$ such that $L((\psi^{l_0 n_0}|_P)^n) \neq 0$. Hence, $N((\psi^{l_0 n_0}|_P)^n) > 0$. It follows from Proposition 3.6 that $N(\psi^{l_0 n_0 n}) > 0$.

Case 4.2.2 $g_0 = 0$. In this situation, each component of $F_g - q(\Gamma')$ has genus zero, ie a disk with holes. Note that each component of $F_g - q(\Gamma')$ has nonpositive Euler characteristic number. From the additivity of Euler characteristic numbers, there must be a component Q_0 with $\chi(Q_0) < 0$, ie Q_0 has at least three boundary components. Let l_0 be the orbit length of Q_0 under the action of $\overline{\psi}$, and b_0 be number of boundary components of Q_0. Then we have $l_0(2-b_0) \geq 2-2g$. This implies that

$$l_0 b_0 \leq \frac{b_0}{b_0 - 2}(2g-2) \leq 6g-6,$$

because $b_0 \geq 3$. Since $\overline{\psi}^{l_0}$ permutes the boundary components of Q_0, there must be a positive integer n_0 with $n_0 \leq b_0$ such that $\overline{\psi}^{l_0 n_0}$ fixes set-wisely at least three boundary components. Notice that the closure $P = \overline{q^{-1}(Q_0)}$ of $q^{-1}(Q_0)$ is also a disk with holes. The homeomorphism $\psi^{l_0 n_0}|_P$ also fixes set-wisely at least three boundary components. Apply Lemma 5.1 to the homeomorphism $\psi^{l_0 n_0}|_P$, we have that $L(\psi^{l_0 n_0}|_P) \neq 0$. Hence, $N(\psi^{l_0 n_0}|_P) > 0$. It follows from Proposition 3.6 that $N(\psi^{l_0 n_0}) > 0$. □

Theorem 7.2 *For $F_{g,b}$ and $N_{g,b}$ orientable and nonorientable genus g surfaces with b boundary components, the free degrees satisfy:*

$$\max_b \mathfrak{fr}(F_{g,b}) \begin{cases} = \infty & \text{if } g = 0, 1, \\ \leq 24g - 24 & \text{if } g \geq 2. \end{cases}$$

$$\max_b \mathfrak{fr}(N_{g,b}) \begin{cases} = \infty & \text{if } g = 1, 2, \\ \leq 12g - 24 & \text{if } g \geq 3. \end{cases}$$

Proof The infiniteness has been shown in Example 4.4 and Example 4.5.

Consider a homeomorphism $\psi\colon F_{g,b} \to F_{g,b}$, where $g \geq 2$. Then ψ^2 must be orientation preserving. By Lemma 7.1, there is a positive integer n with $n \leq 12g-12$ such that $N(\psi^{2n}) = N((\psi^2)^n) > 0$. It follows that ψ^{2n} has a fixed point. Hence, $\mathfrak{fr}(\psi) \leq 24g-24$. Since ψ is an arbitrary homeomorphism on $F_{g,b}$. We obtain that $\mathfrak{fr}(F_{g,b}) \leq 24g-24$.

Let $\eta\colon F_{g-1,2b} \to N_{g,b}$ be the classical orientation covering. Write τ for the unique nontrivial covering transformation. Any homeomorphism $\psi\colon N_{g,b} \to N_{g,b}$ has two liftings ϕ and $\tau\phi$. Without loss of the generality, we may assume that ϕ is orientation preserving. By Lemma 7.1, there is a positive integer n with $n \leq 12(g-1)-12$ such that ϕ^n has a fixed point x_0. Clearly, $\psi^n(\eta(x_0)) = \eta(\phi^n(x_0)) = \eta(x_0)$, ie $\eta(x_0)$ is a fixed point of ψ^n. This implies that $\mathfrak{fr}(N_{g,b}) \leq 12(g-1)-12 = 12g-24$. □

From the proof of this theorem, we obtain:

Corollary 7.3 *For $F_{g,b}$ an orientable genus g surface with b boundary components, the orientation preserving free degree satisfies:*

$$\max_b \mathfrak{fr}^+(F_{g,b}) \begin{cases} = \infty & \text{if } g = 0, 1, \\ \leq 12g-12 & \text{if } g \geq 2. \end{cases}$$

Acknowledgments The paper was enhanced by the referee's comments. The authors would like to thank professor Shicheng Wang for drawing them to this topic. The first author is supported by NSFC grant number 11001190. The second author is supported by NSFC grant number 10931005.

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Received: 30 March 2011 Revised: 6 August 2011