The Goodwillie tower for S^1 and Kuhn's Theorem

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We analyze the homological behavior of the attaching maps in the 2–local Goodwillie tower of the identity evaluated at S^1. We show that they exhibit the same homological behavior as the James–Hopf maps used by N Kuhn to prove the 2–primary Whitehead conjecture. We use this to prove a calculus form of the Whitehead conjecture: the Whitehead sequence is a contracting homotopy for the Goodwillie tower of S^1 at the prime 2.

1 Introduction and statement of results

The aim of this paper is to explain the relationship between the Goodwillie tower of the identity evaluated on S^1 and the Whitehead conjecture (proved by N Kuhn [12]). Such a relationship has been conjectured by Arone, Dwyer, Lesh, Kuhn and Mahowald (see [5; 4; 7]).

The author has learned that similar theorems to the main theorems of this paper (Theorem 1.8 and Corollary 1.10) were proved recently by Arone, Dwyer and Lesh [4] by very different methods. The two proofs were discovered independently and essentially at the same time.

Throughout this paper we freely use the terminology of Goodwillie's homotopy calculus of functors [11] and Weiss's orthogonal calculus [18]. We use the notation:

$$\{P_i(F)\} = \text{Goodwillie tower of } F,$$
$$D_i(F) = i\text{–th layer of the Goodwillie tower},$$
$$\mathbb{D}_i(F) = \text{infinite delooping of } F \text{ (a spectrum valued functor)},$$
$$\partial_i(F) = i\text{–th Goodwillie derivative of } F \text{ (a } \Sigma_i\text{–spectrum)},$$
$$P_i^W, \ D_i^W, \ \mathbb{D}_i^W = \text{the corresponding constructions in Weiss calculus.}$$

When $F = \text{Id}$, we omit it from the notation. We use $E^\vee$ to denote the Spanier–Whitehead dual of a spectrum E. We let $\text{conn}(X)$ denote the connectivity of a space X. As usual, we let QX denote the space $\Omega^\infty \Sigma^\infty X$. All homology and cohomology is

implicitly taken with $\mathbb{F}_2$ coefficients. *Everything in this paper is implicitly localized at the prime* 2.

Let $\mathrm{Sp}^n(S)$ denote the n–th symmetric power of the sphere spectrum. There are natural inclusions $\mathrm{Sp}^n(S) \hookrightarrow \mathrm{Sp}^{n+1}(S)$, and, by the Dold–Thom theorem, the colimit is given by

$$\mathrm{Sp}^\infty(S) = \varinjlim \mathrm{Sp}^n(S) \simeq H\mathbb{Z}.$$

Thus the symmetric products of the sphere spectrum may be regarded as giving an increasing filtration of the integral Eilenberg–Mac Lane spectrum. Nakaoka [15] showed that (2–locally) the quotients $\mathrm{Sp}^n(S)/\mathrm{Sp}^{n-1}(S)$ are nontrivial only when $n = 2^k$. The nontrivial quotients are therefore given by the spectra

$$L(k) := \Sigma^{-k} \mathrm{Sp}^{2^k}(S)/\mathrm{Sp}^{2^{k-1}}(S).$$

These spectra were studied extensively by Kuhn, Mitchell and Priddy [14], and occur in the stable splittings of classifying spaces. Applying π_* to the symmetric powers filtration gives rise to an exact couple, and hence a homological type spectral sequence

$$(\mathbf{1.1}) \qquad\qquad E^1_{k,t} = \pi_t L(k) \Rightarrow \pi_{k+t} H\mathbb{Z}.$$

Kuhn's theorem [12], known as the "Whitehead conjecture," states that this spectral sequence collapses at E_2, where it is concentrated on the $k = 0$ line.

Arone and Mahowald [6] proved that the layers of the (2–local) Goodwillie tower of the identity functor evaluated on spheres satisfy $D_i(S^n) \simeq *$ unless $i = 2^k$. Arone and Dwyer [3] proved there are equivalences

$$(\mathbf{1.2}) \qquad\qquad \Sigma^k \mathbb{D}_{2^k}(S^1) \simeq \Sigma L(k).$$

Applying π_* to the fiber sequences

$$\Omega^\infty \mathbb{D}_{2^k}(S^1) \to P_{2^k}(S^1) \to P_{2^{k-1}}(S^1)$$

results in an exact couple, giving the *Goodwillie spectral sequence* for S^1:

$$(\mathbf{1.3}) \qquad\qquad E_1^{k,t} = \pi_t D_{2^k}(S^1) \Rightarrow \pi_t(S^1).$$

Spectral sequences (1.1) and (1.3) both converge to $\mathbb{Z}$, and, by (1.2), have isomorphic E_1–terms. They differ in that one is of homological type, and one is of cohomological type, and thus in particular their d_1–differentials go in opposite directions. Kuhn's theorem leads to the following natural question: does the Goodwillie spectral sequence for S^1 also collapse at its E_2–page? More specifically, do the d_1–differentials in each of the spectral sequences serve as contracting chain homotopies for the E_1–pages of the other? The aim of this paper is to prove that indeed this is the case.

To more precisely state the main theorem of this paper, we need to recall exactly what Kuhn proved in [12]. In his proof of the 2–primary Whitehead conjecture, Kuhn formed a Kahn–Priddy sequence

$$(1.4) \qquad S^1 \underset{}{\overset{}{\Longleftarrow}} \Omega^\infty \Sigma L(0) \underset{\delta_0}{\overset{d_0}{\rightleftarrows}} \Omega^\infty \Sigma L(1) \underset{\delta_1}{\overset{d_1}{\rightleftarrows}} \Omega^\infty \Sigma L(2) \underset{\delta_2}{\overset{d_2}{\rightleftarrows}} \cdots.$$

The maps d_k are the infinite loop space maps induced by the composites

$$L(k+1) = \Sigma^{-k-1} \operatorname{Sp}^{2^{k+1}}(S)/\operatorname{Sp}^{2^k}(S) \overset{\partial}{\to} \Sigma^{-k} \operatorname{Sp}^{2^k}(S)/\operatorname{Sp}^{2^{k-1}}(S) = L(k).$$

The maps d_k may be regarded as the attaching maps between consecutive layers of the symmetric powers filtration on $H\mathbb{Z}$, and induce on π_* the d_1–differentials in the spectral sequence (1.1). To define the maps δ_k, Kuhn constructed summands X_k of the suspension spectra $\Sigma^\infty(S^1)^{\wedge 2^k}_{h\Sigma_2^{\wr k}}$, and showed that these summands are equivalent to $\Sigma L(k)$ [12, Corollary 1.7]. In particular, we have retractions

$$(1.5)$$

$$\Sigma L(k) \xrightarrow{\quad \text{id} \quad} \Sigma L(k)$$

$$\iota_k \searrow \qquad \nearrow p_k$$

$$\Sigma^\infty(S^1)^{\wedge 2^k}_{h\Sigma_2^{\wr k}}.$$

The maps δ_k in (1.4) are given by the composites

$$\Omega^\infty \Sigma L(k) \xrightarrow{\Omega^\infty \iota_k} Q(S^1)^{\wedge 2^k}_{h\Sigma_2^{\wr k}} \xrightarrow{\text{JH}} Q(S^1)^{\wedge 2^{k+1}}_{h\Sigma_2^{\wr(k+1)}} \xrightarrow{\Omega^\infty p_{k+1}} \Omega^\infty \Sigma L(k+1).$$

Here, JH is the James–Hopf map. Kuhn showed that the sum

$$d_k \delta_k + \delta_{k-1} d_{k-1}$$

is a self-equivalence of $\Omega^\infty \Sigma L(k)$. This amounts to an analysis of the diagram

$$(1.6)$$

where the infinite loop spaces E_k fit into fiber sequences

$$E_k \longrightarrow \Omega^\infty \Sigma L(k) \xrightarrow{\tilde{d}_{k-1}} E_{k-1}$$

and the maps h_k are given by the composites

$$E_k \to \Omega^\infty \Sigma L(k) \xrightarrow{\delta_k} \Omega^\infty \Sigma L(k+1).$$

To prove the Whitehead conjecture, Kuhn proved the following theorem.

Theorem 1.7 [12] *The composites $\widetilde{d}_k \circ h_k$ are equivalences.*

Kuhn proved Theorem 1.7 by showing that $\widetilde{d}_k \circ h_k$ is a homology equivalence.

We now turn our attention to the Goodwillie tower of the identity functor. As explained in [4], precomposing the Goodwillie tower of the identity with the functor

$$\chi : V \mapsto S^V$$

gives the Weiss tower for the functor χ: we have natural equivalences of towers of functors from vector spaces to spaces

$$\{P_i^W(\chi)(V)\} \simeq \{P_i(\mathrm{Id})(S^V)\}.$$

(See [1, Lemma 1.2] and [18, Example 5.7] for the proof of an almost identical result.) Let

$$\phi_k \colon D_{2^k}(S^V) \to BD_{2^{k+1}}(S^V)$$

be the attaching map between consecutive nontrivial layers in the Weiss tower. Arone, Dwyer and Lesh prove that there exist natural transformations

$$\psi_k \colon B^k D_{2^k}(S^V) \to B^{k+1} D_{2^{k+1}}(S^V)$$

so that $$\Omega^k \psi_k = \phi_k.$$

Under the Arone–Dwyer equivalence $\Sigma L(k) \simeq \Sigma^k \mathbb{D}_{2^k}(S^1)$, we get a delooped calculus version of the Kahn–Priddy sequence

$$S^1 \rightrightarrows D_1(S^1) \underset{\psi_0}{\overset{d_0}{\rightleftarrows}} BD_2(S^1) \underset{\psi_1}{\overset{d_1}{\rightleftarrows}} B^2 D_4(S^1) \underset{\psi_2}{\overset{d_2}{\rightleftarrows}} \cdots .$$

Our main theorem is the following [4, Conjecture 1.4]:

Theorem 1.8 *The sums*

$$d_k \psi_k + \psi_{k-1} d_{k-1}$$

are equivalences.

Remark 1.9 Our proof uses no specific properties of the natural transformations ψ_k, except for the fact that they are k–fold deloopings of the natural transformations ϕ_k. Therefore Theorem 1.8 holds independently of the choice of the deloopings.

As the maps ψ_k induced the d_1–differentials in the spectral sequence (1.3) on π_*, we get the following corollary.

Corollary 1.10 *The Goodwillie spectral sequence for* S^1 *collapses at the* E_2 *page.*

Our proof of the main theorem is similar to that of Kuhn in that we analyze the diagram of fiber sequences

(1.11)

where the maps h'_k are given by the composites

$$E_k \to B^k D_{2^k}(S^1) \xrightarrow{\psi_k} B^{k+1} D_{2^{k+1}}(S^1).$$

To prove Theorem 1.8, it suffices to show that the composites $\tilde{d}_k \circ h'_k$ are equivalences. We prove this by establishing that these composites induce isomorphisms on mod 2 homology. We will do this by endowing $H_* \Omega^\infty \Sigma L(k)$ with a weight filtration, and will prove:

Theorem 1.12 *The induced maps*

$$(\psi_k)_*, \ (\delta_k)_* \colon E_0 H_* \Sigma L(k) \to E_0 H_* \Sigma L(k+1)$$

on the associated graded homology groups with respect to the weight filtration are equal.

This theorem, together with the observation that the maps d_k behave well with respect to the weight filtration, will allow us to deduce Theorem 1.8.

The homological analysis of the maps ψ_k will be performed by observing that, at least up to the weight filtration, the homological behavior of the attaching maps between i–th and $2i$–th layers of any functor F from spaces to spaces is essentially dictated by the homological behavior of the left action of the operad $\partial_*(\mathrm{Id})$ on the derivatives $\partial_*(F)$ (such left operadic module structure exists by the work of Arone and Ching [2] and Ching [9]). The 2–primary homological behavior of this action in the case of the identity functor, when evaluated on spheres, was determined by the author [7].

This paper is organized as follows. In Section 2 we study functors from spaces to spaces concentrated in degrees $[i, 2i]$, and relate the attaching maps in their Goodwillie

tower to the left action of $\partial_*(\mathrm{Id})$. In Section 3 we recall from [7] the construction of homology operations $\bar{Q}^j$ which act on the stable homology of the derivatives of any functor from spaces to spaces, and their relationship to the Arone–Mahowald computation of the stable homology of the Goodwillie tower of the identity evaluated on spheres [6]. We also recall some homology calculations of [12]. The main theorems are proved in Section 4.

2 Generalized quadratic functors

For the purposes of this section, let F be an analytic finitary homotopy functor

$$F\colon \mathrm{Top}_* \to \mathrm{Top}_*$$

for which there exists an integer $i \geq 1$ so that the Goodwillie layers $D_k(F)$ are trivial unless $i \leq k \leq 2i$. We regard such functors as "generalized quadratic functors," as the operadic structure of their derivatives bears similarities to the quadratic case of $i = 1$. In this section we analyze the relationship between the left action of $\partial_*(\mathrm{Id})$ on $\partial_*(F)$ and the attaching maps between layers of the Goodwillie tower of F.

There is only one potentially nontrivial component to the left action of $\partial_*(\mathrm{Id})$ on $\partial_*(F)$: this is the map

$$(2.1) \qquad \mu\colon \partial_2(\mathrm{Id}) \wedge \partial_i(F) \wedge \partial_i(F) \to \partial_{2i}(F).$$

We remind the reader that $\partial_2(\mathrm{Id}) \simeq S^{-1}$ (with trivial Σ_2 action).

There is a fiber sequence of functors

(2.2) $$F(X) \to P_{2i-1}(F)(X) \xrightarrow{\phi} BD_{2i}(F)(X).$$

By [4, Theorem 4.2], the functor $P_{2i-1}(F)(X)$ admits a canonical infinite delooping

$$P_{2i-1}(F)(X) \simeq \Omega^\infty \mathbb{P}_{2i-1}(F)(X),$$

where $\mathbb{P}_{2i-1}(F)$ is a spectrum valued functor. The attaching map ϕ has an adjoint

$$\tilde{\phi} \colon \Sigma^\infty \Omega^\infty \mathbb{P}_{2i-1}(F)(X) \to \Sigma D_{2i}(F)(X).$$

Viewing $\tilde{\phi}$ as a natural transformation of functors $\mathrm{Top}_* \to \mathrm{Sp}$, there is an induced natural transformation on $2i$–th layers of the Goodwillie towers of these functors:

(2.3) $$\tilde{\phi}_2 \colon \mathbb{D}_{2i}(\Sigma^\infty \Omega^\infty \mathbb{P}_{2i-1}(F))(X) \to \Sigma \mathbb{D}_{2i}(F)(X).$$

The following lemma identifies the domain of $\tilde{\phi}_2$.

Lemma 2.4 *There is a natural equivalence*

$$\mathbb{D}_{2i}(\Sigma^\infty \Omega^\infty \mathbb{P}_{2i-1}(F))(X) \simeq \mathbb{D}_i(F)(X)^{\wedge 2}_{h\Sigma_2}.$$

Proof The derivatives of the functor $\Sigma^\infty \Omega^\infty$ are well known to be given by

$$\partial_i(\Sigma^\infty \Omega^\infty) = S$$

with trivial Σ_i–action (see, for instance, [13, Example 6.2]). We have (by the chain rule [2])

$$\partial_{2i}(\Sigma^\infty \Omega^\infty \mathbb{P}_{2i-1}(F)) \simeq (\partial_*(\Sigma^\infty \Omega^\infty) \circ \partial_*(\mathbb{P}_{2i-1}F))_{2i}$$

$$\simeq \Sigma_{2i+} \wedge_{\Sigma_2 \wr \Sigma_i} \partial_i(F)^{\wedge 2}.$$

We therefore have

$$\mathbb{D}_{2i}(\Sigma^\infty \Omega^\infty \mathbb{P}_{2i-1}(F))(X) \simeq \partial_{2i}(\Sigma^\infty \Omega^\infty \mathbb{P}_{2i-1}(F)) \wedge_{h\Sigma_{2i}} X^{\wedge 2i}$$

$$\simeq [\Sigma_{2i+} \wedge_{\Sigma_2 \wr \Sigma_i} \partial_i(F)^{\wedge 2}] \wedge_{h\Sigma_{2i}} X^{\wedge 2i}$$

$$\simeq \mathbb{D}_i(F)(X)^{\wedge 2}_{h\Sigma_2}. \qquad \square$$

Lemma 2.4 allows us to regard $\tilde{\phi}_2$ as a map

$$\tilde{\phi}_2 \colon \mathbb{D}_i(F)(X)^{\wedge 2}_{h\Sigma_2} \to \Sigma \mathbb{D}_{2i}(F)(X).$$

Our main observation is the following.

Theorem 2.5 *The map*

$$\Sigma^{-1}\widetilde{\phi}_2 \colon \Sigma^{-1}\mathbb{D}_i(F)(X)^{\wedge 2}_{h\Sigma_2} \to \mathbb{D}_{2i}(F)(X)$$

is homotopic to the composite

$$\Sigma^{-1}\mathbb{D}_i(F)(X)^{\wedge 2}_{h\Sigma_2} \simeq (\partial_2(\mathrm{Id}) \wedge \partial_i(F)^{\wedge 2} \wedge X^{\wedge 2i})_{h\Sigma_2 \wr \Sigma_i}$$

$$\xrightarrow{\mu \wedge 1} (\partial_{2i}(F) \wedge X^{\wedge 2i})_{h\Sigma_{2i}}$$

$$\simeq \mathbb{D}_{2i}(F)(X).$$

The proof of Theorem 2.5 will occupy the remainder of this section, and will require a series of supporting lemmas. At the heart of the argument is the following idea: given the attaching map ϕ, compute the induced left action of $\partial_*(\mathrm{Id})$ on $\partial_*(F)$. This will result in a formula relating $\widetilde{\phi}_2$ and μ.

To compute the left action we use the machinery of Arone and Ching. For a functor $G \colon \mathrm{Top}_* \to \mathrm{Top}_*$, Arone and Ching [2] show that $\partial_*(\Sigma^\infty G)$ is a left comodule over the commutative cooperad Comm_*, and moreover show that $\partial_*(G)$ can be recovered from the cooperadic cobar construction

$$\partial_*(G) \simeq C(1_*, \mathrm{Comm}_*, \partial_*(\Sigma^\infty G)).$$

The cobar construction is Spanier–Whitehead dual to the bar construction

(2.6) $C(1_*, \mathrm{Comm}_*, \partial_*(\Sigma^\infty G)) \simeq B(1_*, \mathrm{Comm}_*, \partial^*(\Sigma^\infty G))^\vee.$

Here, following [2],

$$\partial^*(\Sigma^\infty G) := \partial_*(\Sigma^\infty G)^\vee,$$

and must be interpreted as a symmetric sequence of pro-spectra for a general functor G. Note that in the right-hand side of (2.6), we have abusively used Comm_* to also denote the commutative *operad*, as it has the same underlying symmetric sequence as the commutative cooperad. Ching's topological model for the bar construction [9]

$$B(1_*, \mathrm{Comm}_*, \partial^*(\Sigma^\infty G))$$

carries a left coaction by the cooperad

$$B(1_*, \mathrm{Comm}_*, 1_*) \simeq \partial^*(\mathrm{Id}).$$

The action of $\partial_*(\mathrm{Id})$ induced on the dual recovers the left action of $\partial_*(\mathrm{Id})$ on $\partial_*(G)$. The proof of Theorem 2.5 will follow from an analysis of how this process plays out, when applied to our functor F.

The first step to the approach outlined above is to compute $\partial_*(\Sigma^\infty F)$. The strategy is to use the fiber sequence (2.2). Note that as the functors $P_{2i-1}(F)$ and $BD_{2i}(F)$ factor through the category of spectra, the derivatives of Σ^∞ of these functors are easily deduced from the derivatives of $\Sigma^\infty \Omega^\infty$ by applying the chain rule [2].

Since F is analytic, for X sufficiently highly connected there is a natural equivalence

(**2.7**) $$\Sigma^\infty F(X) \simeq h\mathrm{Tot}\,\Sigma^\infty(BD_{2i}(F)(X)^{\times \bullet} \times P_{2i-1}(F)(X))$$

where $$T^\bullet(X) := BD_{2i}(F)(X)^{\times \bullet} \times P_{2i-1}(F)(X)$$

is the Rector cosimplicial model [17] for the homotopy fiber (2.2)

$$P_{2i-1}(F)(X) \underset{\Rrightarrow}{\overset{\Rrightarrow}{\rightrightarrows}} BD_{2i}(F)(X) \times P_{2i-1}(F)(X) \underset{\Rrightarrow}{\overset{\Rrightarrow}{\rightrightarrows}} BD_{2i}(F)(X)^{\times 2} \times P_{2i-1}(F)(X) \cdots$$

and $h\mathrm{Tot}$ denotes homotopy totalization.

In preparation for our arguments, we briefly discuss some general properties of the homotopy Tot–tower. For a cosimplicial spectrum $Z^\bullet$, this tower takes the form

$$h\,\mathrm{Tot}^0 Z^\bullet \leftarrow h\,\mathrm{Tot}^1 Z^\bullet \leftarrow h\,\mathrm{Tot}^2 Z^\bullet \leftarrow \cdots .$$

Let $\mathrm{fib}^n Z^\bullet$ denote the homotopy fiber

$$\mathrm{fib}^n Z^\bullet \to h\,\mathrm{Tot}^n Z^\bullet \to h\,\mathrm{Tot}^{n-1} Z^\bullet.$$

There are homotopy fiber sequences

$$\mathrm{fib}^n Z^\bullet \to Z^n \to \underset{\substack{[n] \twoheadrightarrow [k] \\ k < n}}{\mathrm{holim}}\, Z^k.$$

Since a surjection $[n] \twoheadrightarrow [k]$ is uniquely determined by specifying the subset of arrows of

$$[n] = (0 \to 1 \to 2 \to \cdots \to n)$$

which go to identity arrows in $[k]$, the fiber $\mathrm{fib}^n Z^\bullet$ is computed as the total homotopy fiber of an n–cubical diagram

(**2.8**) $$\mathrm{fib}^n Z^\bullet \simeq h\,\mathrm{tfiber}\{Z^{n-|S|}\}_{S \subseteq \underline{n}},$$

where the maps in the n–cubical diagram are given by codegeneracy maps of $Z^\bullet$.

Since F was assumed to be analytic, there exists ρ, q such that on sufficiently highly connected spaces X the natural transformations

$$F(X) \to P_k(F)(X)$$

are $(q - k(\rho - 1) + (k+1)\,\mathrm{conn}(X))$–connected. We will need the following connectivity estimate.

Lemma 2.9 *On sufficiently highly connected spaces X, the map*

$$\Sigma^\infty F(X) \to h\operatorname{Tot}^n \Sigma^\infty T^\bullet(X)$$

is $((n+1)(q-(2i-1)(\rho-1))+1+2i\,(n+1)\operatorname{conn}(X))$–connected.

Proof Using the splitting

$$\Sigma^\infty(Y \times Y') \simeq \Sigma^\infty Y \vee \Sigma^\infty Y' \vee \Sigma^\infty Y \wedge Y'$$

for $Y, Y' \in \operatorname{Top}_*$, one inductively computes from (2.8) that

$$\operatorname{fib}^n \Sigma^\infty T^\bullet(X) \simeq \Sigma^\infty BD_{2i}(F)(X)^{\wedge n} \wedge P_{2i-1}(F)(X)_+ \,.$$

For $\operatorname{conn}(X) \geq \rho$ the map

$$P_{2i}(F)(X) \to P_{2i-1}(F)(X)$$

is $(q-(2i-1)(\rho-1)+2i\cdot\operatorname{conn}(X))$–connected. This means that the fiber $D_{2i}(F)(X)$ is $(q-(2i-1)(\rho-1)+2i\cdot\operatorname{conn}(X)-1)$–connected, and the space $BD_{2i}(F)(X)$ is $(q-(2i-1)(\rho-1)+2i\cdot\operatorname{conn}(X))$–connected. Let X be highly enough connected to make this number positive. Then $\operatorname{fib}^n \Sigma^\infty T^\bullet$ is $(n(q-(2i-1)(\rho-1))+2i\,n\cdot\operatorname{conn}(X))$–connected. We deduce that the map

$$\Sigma^\infty F(X) \simeq h\operatorname{Tot}\Sigma^\infty T^\bullet(X) \to h\operatorname{Tot}^n \Sigma^\infty T^\bullet(X)$$

is $((n+1)(q-(2i-1)(\rho-1))+1+2i\,(n+1)\operatorname{conn}(X))$–connected. $\square$

We are now able to identify $\partial_*(\Sigma^\infty F)$ for $* \leq 2i$.

Lemma 2.10 *There are equivalences*

$$\partial_k(\Sigma^\infty F) \simeq \partial_k(F), \quad \text{for } i \leq k < 2i,$$

$$\partial_{2i}(\Sigma^\infty F) \simeq \operatorname{fiber}\Big(\Sigma 2i + \underset{\Sigma_2 \wr \Sigma_i}{\wedge} \partial_i(F)^{\wedge 2} \xrightarrow{\partial_{2i}(\tilde\phi_2)} \Sigma\partial_{2i}(F)\Big).$$

Proof Recall from the proof of Lemma 2.4 that we used the chain rule to deduce

$$\partial_{2i}(\Sigma^\infty P_{2i-1}(F)) \simeq \Sigma 2i + \underset{\Sigma_2 \wr \Sigma_i}{\wedge} \partial_i(F)^{\wedge 2}.$$

The same argument shows that

$$\partial_k(\Sigma^\infty P_{2i-1}(F)) \simeq \partial_k(F) \quad \text{for } i \leq k < 2i,$$

$$\partial_{2i}(\Sigma^\infty BD_{2i}(F)) \simeq \Sigma\partial_{2i}(F).$$

By Lemma 2.9, the functors $\Sigma^\infty F$ and $h\operatorname{Tot}^0 \Sigma^\infty T^\bullet$ agree to order $2i-1$ and the functors $\Sigma^\infty F$ and $h\operatorname{Tot}^1 \Sigma^\infty T^\bullet$ agree to order $2i$. It follows [11] that

$$\partial_k(\Sigma^\infty F) \simeq h\operatorname{Tot}^0 \partial_k(\Sigma^\infty T^\bullet), \quad k < 2i,$$

$$\partial_{2i}(\Sigma^\infty F) \simeq h\operatorname{Tot}^1 \partial_{2i}(\Sigma^\infty T^\bullet).$$

This immediately implies the first equivalence of the lemma.

To prove the second equivalence, we must compute $h\operatorname{Tot}^1 \partial_{2i}(\Sigma^\infty T^\bullet)$. The ∂_{2i} computations above imply that

(2.11)
$$\partial_{2i}(\Sigma^\infty T^s) \simeq \underbrace{\Sigma\partial_{2i}(F) \vee \cdots \vee \Sigma\partial_{2i}(F)}_{s} \vee \Sigma^{2i} + {}_{\Sigma_2 \wr \Sigma_i} \wedge \partial_i(F)^{\wedge 2}.$$

We claim that under the equivalences (2.11), the last coface map in the cosimplicial Σ_{2i}–spectrum $\partial_{2i}(\Sigma^\infty T^\bullet)$ from level 0 to level 1 is given by

$$d^1 = \partial_{2i}(\widetilde{\phi}_2) \times 1,$$

and the codegeneracy map from level 1 to level 0 is the map which collapses out the wedge summand $\Sigma\partial_{2i}(F)$. The second equivalence of the lemma follows immediately from this claim.

To establish the claim concerning the cosimplicial structure maps above, observe that the d^1 map from level 0 to level 1 in the cosimplicial functor $\Sigma^\infty T^\bullet(X)$ is the composite

$$\delta\colon \Sigma^\infty P_{2i-1}(F)(X) \xrightarrow{\Sigma^\infty \Delta} \Sigma^\infty (P_{2i-1}(F)(X) \times P_{2i-1}(F)(X))$$

$$\xrightarrow{\Sigma^\infty \phi \times 1} \Sigma^\infty (BD_{2i}(F)(X) \times P_{2i-1}(F)(X)).$$

The induced map $\partial_{2i}(\delta)$ is determined by the composites with the projections onto the wedge summands of

$$\partial_{2i}(\Sigma^\infty(BD_{2i}(F) \times P_{2i-1}(F))) \simeq \Sigma\partial_{2i}(F) \vee \Sigma^{2i} + {}_{\Sigma_2 \wr \Sigma_i} \wedge \partial_i(F)^{\wedge 2}_{h\Sigma_2}.$$

Composing δ with the projection onto the second factor gives the identity, and this implies that the second component of $\partial_{2i}(\delta)$ is the identity. Composing δ with the projection onto the first factor is the natural transformation

$$\Sigma^\infty \phi\colon \Sigma^\infty P_{2i-1}(F)(X) \to \Sigma^\infty BD_{2i}(F)(X).$$

Using the fact that the adjoint $\widetilde{\phi}$ is the composite

$$\Sigma^\infty \Omega^\infty \mathbb{P}_{2i-1}(F)(X) \xrightarrow{\Sigma^\infty \phi} \Sigma^\infty \Omega^\infty \Sigma \mathbb{D}_{2i}(F)(X) \xrightarrow{\epsilon} \Sigma \mathbb{D}_{2i}(F)(X),$$

together with the fact that ϵ is a ∂_{2i}–equivalence, we deduce that the first component of $\partial_{2i}(\delta)$ is $\partial_{2i}(\widetilde{\phi}_2)$, as desired. The claim concerning the codegeneracy of $\partial_{2i}(\Sigma^\infty T^\bullet)$ follows immediately from the fact that the codegeneracy from level 1 to level 0 of the cosimplicial functor $T^\bullet(X)$ projects away the first component. $\qquad\square$

The last equivalence of Lemma 2.10 gives a fiber sequence of Σ_{2i}–spectra:

$$(2.12) \qquad \partial_{2i}(F) \xrightarrow{\eta} \partial_{2i}(\Sigma^\infty F) \xrightarrow{\xi} \Sigma_{2i+} \underset{\Sigma_2 \wr \Sigma_i}{\wedge} \partial_i(F)^{\wedge 2} \xrightarrow{\partial_{2i}(\widetilde{\phi}_2)} \Sigma \partial_{2i}(F).$$

Our next task is to understand the left coaction of Comm_* on $\partial_*(\Sigma^\infty F)$ in low degrees in terms of the attaching map ϕ.

Lemma 2.13 *Under the equivalence $\partial_i(F) \simeq \partial_i(\Sigma^\infty F)$, the map ξ of (2.12) agrees with the left Comm_*–comodule structure map*

$$\partial_{2i}(\Sigma^\infty F) \to \Sigma_{2i+} \underset{\Sigma_2 \wr \Sigma_i}{\wedge} \mathrm{Comm}_2 \wedge \partial_i(\Sigma^\infty F)^{\wedge 2}.$$

Proof The left coaction of Comm_* on

$$\partial_*(\Sigma^\infty P_{2i-1}(F)) = \partial_*(\Sigma^\infty \Omega^\infty \mathbb{P}_{2i-1}(F))$$

is easily deduced from the chain rule [2], together with the fact that under the equivalence

$$\partial_*(\Sigma^\infty \Omega^\infty) \simeq \mathrm{Comm}_*,$$

the left coaction of Comm_* on $\partial_*(\Sigma^\infty \Omega^\infty)$ is given by the left coaction of Comm_* on itself. In particular, the coaction map corresponding to the partition $2i = i + i$ is given by the composite (of equivalences)

$$\partial_{2i}(\Sigma^\infty P_{2i-1}(F)) \xrightarrow{\simeq} \Sigma_{2i+} \underset{\Sigma_2 \wr \Sigma_i}{\wedge} \partial_i(F)^{\wedge 2} \xrightarrow{\simeq} \Sigma_{2i+} \underset{\Sigma_2 \wr \Sigma_i}{\wedge} \partial_i(\Sigma^\infty P_{2i-1}(F))^{\wedge 2}.$$

The natural transformation of functors

$$F \to P_{2i-1}(F)$$

induces a map of left Comm_*–comodules

$$\partial_*(\Sigma^\infty F) \to \partial_*(\Sigma^\infty P_{2i-1}(F)).$$

In particular, there is a commutative diagram

$$
\begin{array}{ccc}
\partial_{2i}(\Sigma^\infty F) & \xrightarrow{\ \xi\ } & \Sigma_{2i+} \underset{\Sigma_2 \wr \Sigma_i}{\wedge} \partial_i(F)^{\wedge 2} \\
\downarrow & & \downarrow = \\
\Sigma_{2i+} \underset{\Sigma_2 \wr \Sigma_i}{\wedge} \partial_i(\Sigma^\infty F)^{\wedge 2} & \xrightarrow{\ \simeq\ } & \Sigma_{2i+} \underset{\Sigma_2 \wr \Sigma_i}{\wedge} \partial_i(F)^{\wedge 2}
\end{array}
$$

where the vertical arrows are Comm_*–comodule structure maps. We conclude that the map ξ in (2.12) encodes the primary Comm_*–comodule structure map, as desired. $\square$

Proof of Theorem 2.5   By [2], we have

$$(2.14) \qquad\qquad \partial_*(F) \simeq C(1_*, \mathrm{Comm}_*, \partial_*(\Sigma^\infty F)).$$

In particular, we have

$$
\partial_{2i}(F) \simeq C(1_*, \mathrm{Comm}_*, \partial_*(\Sigma^\infty F))_{2i}
$$
$$
\simeq \mathrm{fiber}\Big(\partial_{2i}(\Sigma^\infty F) \xrightarrow{\xi} \Sigma_{2i+} \underset{\Sigma_2 \wr \Sigma_i}{\wedge} \partial_i(F)^{\wedge 2}\Big).
$$

This equivalence was already recorded in the fiber sequence (2.12), but now it implicitly records more structure, as (2.14) is an equivalence of left $\partial_*(\mathrm{Id})$–modules. Indeed, we now compute from (2.14) the $\partial_*(\mathrm{Id})$–module structure of $\partial_*(F)$ in terms of the attaching map ϕ.
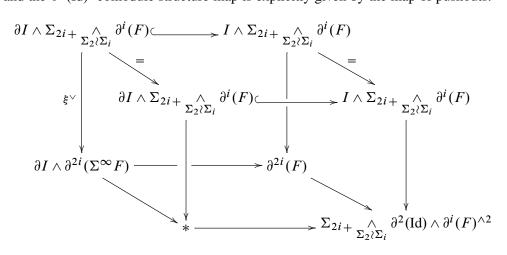
To accomplish this, we work with dual derivatives, and then dualize. We have

$$
\partial^* F = B(1_*, \mathrm{Comm}_*, \partial^*(\Sigma^\infty F)).
$$

Using the Ching model for the bar construction [9], we have a pushout

$$
\begin{array}{ccc}
\partial I \wedge \Sigma_{2i+} \underset{\Sigma_2 \wr \Sigma_i}{\wedge} \partial^i(F) & \hookrightarrow & I \wedge \Sigma_{2i+} \underset{\Sigma_2 \wr \Sigma_i}{\wedge} \partial^i(F) \\
{\scriptstyle \xi^\vee}\downarrow & & \downarrow \\
\partial I \wedge \partial^{2i}(\Sigma^\infty F) & \longrightarrow & B(1_*, \mathrm{Comm}_*, \partial^*(\Sigma^\infty F))_{2i}
\end{array}
$$

and the $\partial^*(\mathrm{Id})$–comodule structure map is explicitly given by the map of pushouts:

$$\partial I \wedge \Sigma_{2i+} \underset{\Sigma_2 \wr \Sigma_i}{\wedge} \partial^i(F) \hookrightarrow I \wedge \Sigma_{2i+} \underset{\Sigma_2 \wr \Sigma_i}{\wedge} \partial^i(F)$$

In particular, we deduce that the coaction map

$$\mu^{\vee} \colon \partial^{2i}(F) \to \Sigma_{2i+} \underset{\Sigma_2 \wr \Sigma_i}{\wedge} \partial^2(\mathrm{Id}) \wedge \partial^i(F)^{\wedge 2}$$

is precisely the connecting morphism $(\Sigma^{-1}\partial_{2i}(\widetilde{\phi}_2))^{\vee}$ in the cofiber sequence dual to the fiber sequence (2.12):

$$\Sigma_{2i+} \underset{\Sigma_2 \wr \Sigma_i}{\wedge} \partial^i(F)^{\wedge 2} \xrightarrow{\xi^{\vee}} \partial^{2i}(\Sigma^{\infty}F) \xrightarrow{\eta^{\vee}} \partial^{2i}(F) \xrightarrow{(\Sigma^{-1}\partial_{2i}(\widetilde{\phi}_2))^{\vee}} \Sigma\Big(\Sigma_{2i+} \underset{\Sigma_2 \wr \Sigma_i}{\wedge} \partial^i(F)^{\wedge 2}\Big).$$

Dualizing, we deduce that

$$\mu = \Sigma^{-1}\partial_{2i}(\widetilde{\phi}_2)$$

and the theorem follows. $\qquad\qquad\square$

3 Homology of the layers

In this section we briefly recall some facts about the homology of the layers of the Goodwillie tower of the identity evaluated on spheres. This computation is due to Arone and Mahowald [6], but we will need to take advantage of the interpretation presented in [7].

Let $F\colon \mathrm{Top}_* \to \mathrm{Top}_*$ be a reduced finitary homotopy functor. In [7], the author introduced operations

$$\bar{Q}^j \colon H_*(\mathbb{D}_i(F)(X)) \to H_{*+j-1}(\mathbb{D}_{2i}(F)(X)).$$

These operations were defined as follows: the left action of $\partial_*(\mathrm{Id})$ on $\partial_*(F)$ yields a map

$$\mu\colon \Sigma^{-1}\partial_i(F)^{\wedge 2} \simeq \partial_2(\mathrm{Id}) \wedge \partial_i(F)^{\wedge 2} \to \partial_{2i}(F).$$

This map induces a map

$$\mu'\colon \Sigma^{-1}\mathbb{D}_i(F)(X)^{\wedge 2}_{h\Sigma_2} \to \mathbb{D}_{2i}(F)(X).$$

The operations are given by

(3.1) $$\bar{Q}^j(x) := \mu'_* \sigma^{-1} Q^j x$$

for $x \in H_*(\mathbb{D}_i(F)(X))$.

In [7], the Arone–Mahowald computation is interpreted in terms of these operations, and it is shown that

$$H_*(\mathbb{D}_{2^k}(S^n)) = \mathbb{F}_2\{\bar{Q}^{i_1} \cdots \bar{Q}^{i_k}\iota_n : i_s \geq 2i_{s+1} + 1, \ i_k \geq n\}.$$

Recall that $H_*(S^1)^{\wedge 2^k}_{h\Sigma_2^{\wr k}}$ contains a direct summand

$$\tilde{\mathcal{R}}_1(k) = \mathbb{F}_2\{Q^{i_1} \wr \cdots \wr Q^{i_k}\iota_1 : i_s \geq i_{s+1} + \cdots + i_k + 1\}.$$

In [12], certain idempotents e_k are constructed to act on $\tilde{\mathcal{R}}_1(k)$ (in [12], these idempotents are denoted D_{k-1}, but we use the notation e_k in this paper so as to not create confusion with the notation used for the layers of the Goodwillie tower). These idempotents split off the summand $H_*(\Sigma L(k))$. Kuhn shows that

$$H_*(\Sigma L(k)) = \mathbb{F}_2\{e_k(Q^{i_1} \wr \cdots \wr Q^{i_k}\iota_1) : i_s \geq 2i_{s+1} + 1, \ i_k \geq 1\}.$$

Lemma 3.2 *Under the equivalence* $\Sigma L(k) \simeq \Sigma^k \mathbb{D}_{2^k}(S^1)$ *of* (1.2), *we have a bijection between the two bases*

$$e_k(Q^{i_1} \wr \cdots \wr Q^{i_k}\iota_1) \leftrightarrow \sigma^k \bar{Q}^{i_1} \cdots \bar{Q}^{i_k}\iota_1.$$

Proof In Section 1.4 of [7] an algebra $\bar{\mathcal{R}}_n$ of operations $\bar{Q}^j$ is defined, with relations

(1) $\quad \bar{Q}^r\bar{Q}^s = \displaystyle\sum_t \left[\binom{s-r+t}{s-t} + \binom{s-r+t}{2t-r}\right]\bar{Q}^{r+s-t}\bar{Q}^t,$

(2) $\quad \bar{Q}^{j_1} \cdots \bar{Q}^{j_k} = 0$ if $j_1 < j_2 + \cdots + j_k + n$.

Here, and throughout this section, mod 2 binomial coefficients $\binom{a}{b} \in \mathbb{F}_2$ are defined for all $a, b \in \mathbb{Z}$ by

$$\binom{a}{b} = \text{coefficient of } t^b \text{ in } (1+t)^a.$$

Let $\bar{\mathcal{R}}_n(k)$ be the summand additively generated by length k sequences of operations. It is shown in [7] that $H_* \Sigma^k \mathbb{D}_{2^k}(S^1)$ is precisely the quotient of $\widetilde{\mathcal{R}}_1(k)$ by relation (1) above, and therefore

$$H_* \mathbb{D}_{2^k}(S^1) = \bar{\mathcal{R}}_1(k)\{\iota_1\}.$$

Kuhn's idempotents e_k are defined in [12] as certain iterates of idempotents

$$T_s \colon \widetilde{\mathcal{R}}_1 \to \widetilde{\mathcal{R}}_1, \quad 1 \le s \le k - 1,$$

where

$$T_s(Q^{i_1} \wr \cdots \wr Q^{i_k}) =$$
$$\sum_t \left[\binom{i_{s+1} - i_s + t}{i_{s+1} - t} + \binom{i_{s+1} - i_s + t}{2t - i_s} \right] Q^{i_1} \wr \cdots \wr \bar{Q}^{i_s + i_{s+1} - t} \wr Q^t \wr \cdots \wr Q^{i_k}.$$

Let

(3.3) $$\nu_k \colon \widetilde{\mathcal{R}}_1(k) \to \bar{\mathcal{R}}_1(k)$$

be the canonical surjection. Clearly $\nu_k T_s = \nu_k$, and therefore $\nu_k e_k = \nu_k$. In particular

$$\nu_k e_k Q^{i_1} \wr \cdots \wr Q^{i_k} = \bar{Q}^{i_1} \cdots \bar{Q}^{i_k}. \qquad \square$$

We remark that the above lemma, and the homomorphisms used in its proof, allow us to easily describe the effect on homology

$$(\iota_k)_* \colon H_* \Sigma^k \mathbb{D}_{2^k}(S^1) \to H_*(S^1)^{\wedge 2^k}_{h\Sigma^{\wr k}_2},$$
$$(p_k)_* \colon H_*(S^1)^{\wedge 2^k}_{h\Sigma^{\wr k}_2} \to H_* \Sigma^k \mathbb{D}_{2^k}(S^1)$$

of the splitting maps (1.5) in terms of the $\bar{Q}$–basis. Namely, we have

(3.4) $$(\iota_k)_* \sigma^k \bar{Q}^{i_1} \cdots \bar{Q}^{i_k} \iota_1 = e_k(Q^{i_1} \wr \cdots \wr Q^{i_k} \iota_1),$$

(3.5) $$(p_k)_* Q^{i_1} \wr \cdots \wr Q^{i_k} \iota_1 = \sigma^k \bar{Q}^{i_1} \cdots \bar{Q}^{i_k} \iota_1.$$

Since (1.2) and (1.5) give the spectrum $\Sigma^k \mathbb{D}_{2^k}(S^1)$ as a summand of the suspension spectrum $\Sigma^\infty(S^1)^{\wedge 2^k}_{h\Sigma^{\wr k}_2}$, we can explicitly describe the homology of its zeroth space [10]:

$$H_*(B^k D_{2^k}(S^1)) = \mathcal{F}\big(\mathbb{F}_2\{\sigma^k \bar{Q}^{i_1} \cdots \bar{Q}^{i_k} \iota_1 : i_s \ge 2i_{s+1} + 1, \ i_k \ge 1\}\big).$$

Here, $\mathcal{F}$ is the functor

$$\mathcal{F} \colon \text{graded } \mathbb{F}_2\text{–vector spaces} \to \text{allowable } \mathcal{R}\text{–algebras}$$

which associates to a graded $\mathbb{F}_2$–vector space V the free allowable algebra over the Dyer–Lashof algebra.

We endow $H_*(B^k D_{2^k}(S^1))$ with a (decreasing) weight filtration by declaring that

$$w(x) = 2^k \quad \text{for } x \in H_* \Sigma^k \mathbb{D}_{2^k}(S^1),$$
$$w(Q^i x) = 2 \cdot w(x),$$
$$w(x * y) = w(x) + w(y).$$

This weight filtration is related to Goodwillie calculus in the following manner. As indicated in the proof of Lemma 2.4, the functor $\Sigma^\infty \Omega^\infty$ has derivatives

$$\partial_i(\Sigma^\infty \Omega^\infty) = S$$

with trivial Σ_i–action. For connected spectra E, the Goodwillie tower $P_i(\Sigma^\infty \Omega^\infty)(E)$ converges, giving a spectral sequence

(3.6) $$E_1^{i,*} = H_*(E_{h\Sigma_i}^{\wedge i}) \Rightarrow H_*(\Omega^\infty E).$$

In [13, Example 6.1], it is explained that the Goodwillie tower for $\Sigma^\infty \Omega^\infty$ splits when evaluated on connected suspension spectra. In these cases the spectral sequence (3.6) degenerates. By naturality, this also holds for summands of connected suspension spectra. The weight filtration is simply an appropriate scaling of the filtration in this spectral sequence.

The induced morphisms

$$H_* B^k D_{2^k}(S^1) \underset{(\delta_k)_*}{\overset{(d_k)_*}{\rightleftarrows}} H_* B^{k+1} D_{2^{k+1}}(S^1)$$

were computed in [12]: we end this section be recalling these explicit descriptions.

Suppose that $$Q^{j_1} \cdots Q^{j_\ell} \sigma^{k+1} \bar{Q}^{i_1} \cdots \bar{Q}^{i_{k+1}} \iota_1$$

is an algebra generator of $H_* B^{k+1} D_{2^{k+1}}(S^1)$. Writing

$$e_{k+1} Q^{i_1} \wr \cdots \wr Q^{i_{k+1}} = \sum Q^{i'_1} \wr \cdots \wr Q^{i'_{k+1}},$$

we have

$$(d_k)_* Q^{j_1} \cdots Q^{j_\ell} \sigma^{k+1} \bar{Q}^{i_1} \cdots \bar{Q}^{i_{k+1}} \iota_1 = \sum Q^{j_1} \cdots Q^{j_\ell} Q^{i'_1} \sigma^k \bar{Q}^{i'_2} \cdots \bar{Q}^{i'_{k+1}}.$$

Furthermore, as d_k is an infinite loop map, $(d_k)_*$ is a map of algebras. We see that $(d_k)_*$ preserves the weight filtration. In fact, $(d_k)_*$ is isomorphic to its own associated graded (with respect to the monomial basis).

Suppose that $$Q^{j_1} \cdots Q^{j_\ell} \sigma^k \bar{Q}^{i_1} \cdots \bar{Q}^{i_k} \iota_1$$

is an algebra generator of $H_* B^k D_{2^k}(S^1)$. Then we have

$$(\delta_k)_* Q^{j_1} \cdots Q^{j_\ell} \sigma^k \bar{Q}^{i_1} \cdots \bar{Q}^{i_k} \iota_1 = \sum_s Q^{j_1} \cdots \bar{Q}^{j_s} \cdots Q^{j_\ell} \sigma^k \bar{Q}^{i_1} \cdots \bar{Q}^{i_k} \iota_1.$$

Here, we move the $\bar{Q}^i$ past the Q^j's using the mixed Adem relation

$$\bar{Q}^r Q^s = \sum_t \left[\binom{s-r+t}{s-t} + \binom{s-r+t}{2t-r} \right] Q^{r+s-t} \bar{Q}^t.$$

In particular, $(\delta_k)_*$ preserves the weight filtration on algebra generators. While the map $(\delta_k)_*$ is *not* a map of algebras, Kuhn shows that its associated graded with respect to the weight filtration *is* a map of algebras [12, Proposition 2.7].

4 Homological behavior of ψ_k

In this section we will prove Theorem 1.12, and then explain how it implies Theorem 1.8.

The map δ_k is given by the composite

$$(4.1) \qquad \Omega^\infty \Sigma^k \mathbb{D}_{2^k}(S^1) \xrightarrow{\text{JH}} \Omega^\infty (\Sigma^k \mathbb{D}_{2^k}(S^1))_{h\Sigma_2}^{\wedge 2} \xrightarrow{\Omega^\infty \alpha_k} \Omega^\infty \Sigma^{k+1} \mathbb{D}_{2^{k+1}}(S^1)$$

where α_k is the composite (see (1.2) and (1.5))

$$(\Sigma^k \mathbb{D}_{2^k}(S^1))_{h\Sigma_2}^{\wedge 2} \xrightarrow{(\iota_k)_{h\Sigma_2}^{\wedge 2}} \Sigma^\infty (S^1)_{h\Sigma_2^{\wr k+1}}^{\wedge 2^{k+1}} \xrightarrow{p_{k+1}} \Sigma^{k+1} \mathbb{D}_{2^{k+1}}(S^1)$$

and the James–Hopf map JH is defined by the splitting of $\Sigma^\infty \Omega^\infty \Sigma^k \mathbb{D}_{2^k}(S^1)$ induced by the retract of the Goodwillie towers:

$$P_i(\Sigma^\infty \Omega^\infty)(\Sigma^k \mathbb{D}_{2^k}(S^1)) \xrightarrow{\quad \text{id} \quad} P_i(\Sigma^\infty \Omega^\infty)(\Sigma^k \mathbb{D}_{2^k}(S^1))$$

$$\searrow_{(\iota_k)_*} \qquad \nearrow_{(p_k)_*}$$

$$P_i(\Sigma^\infty \Omega^\infty)(\Sigma^\infty (S^1)_{h\Sigma_2^{\wr k}}^{\wedge 2^k})$$

Consider the natural transformation of functors from vector spaces to spectra given by the adjoint of ψ_k:

$$\widetilde{\psi}_k \colon \Sigma^\infty \Omega^\infty \Sigma^k \mathbb{D}_{2^k}(S^V) \to \Sigma^{k+1} \mathbb{D}_{2^{k+1}}(S^V).$$

On the level of the 2^{k+1}–st layers of the corresponding Weiss towers, $\widetilde{\psi}_k$ induces a map

$$[\widetilde{\psi}_k]_2 \colon (\Sigma^k \mathbb{D}_{2^k}(S^V))_{h\Sigma_2}^{\wedge 2} \simeq \mathbb{D}_{2^{k+1}}^W (\Sigma^\infty \Omega^\infty \Sigma^k \mathbb{D}_{2^k} \circ \chi)(V) \to \Sigma^{k+1} \mathbb{D}_{2^{k+1}}(S^V).$$

The proof of Theorem 1.12 will rest on the following two lemmas.

Lemma 4.2 *The natural transformation ψ_k, when evaluated on S^1, admits a factorization*

$$\Omega^\infty \Sigma^k \mathbb{D}_{2^k}(S^1) \xrightarrow{\text{JH}} \Omega^\infty (\Sigma^k \mathbb{D}_{2^k}(S^1))_{h\Sigma_2}^{\wedge 2} \xrightarrow{\Omega^\infty [\widetilde{\psi}_k]_2} \Omega^\infty \Sigma^{k+1} \mathbb{D}_{2^{k+1}}(S^1).$$

Proof of Lemma 4.2 Since the functor $\Sigma^{k+1} \mathbb{D}_{2^{k+1}}(S^V)$ is of degree 2^{k+1} in V, the adjoint $\widetilde{\psi}_k$ factors as

$$\Sigma^\infty \Omega^\infty \Sigma^k \mathbb{D}_{2^k}(S^V) \to P_{2^{k+1}}^W (\Sigma^\infty \Omega^\infty \Sigma^k \mathbb{D}_{2^k} \circ \chi)(V) \xrightarrow{\tau_k} \Sigma^{k+1} \mathbb{D}_{2^{k+1}}(S^V).$$

Specializing to the case of $V = \mathbb{R}$, and using the splitting

$$P_{2^{k+1}}^W (\Sigma^\infty \Omega^\infty \Sigma^k \mathbb{D}_{2^k} \circ \chi)(\mathbb{R}) \simeq P_2(\Sigma^\infty \Omega^\infty)(\Sigma^k \mathbb{D}_{2^k}(S^1))$$
$$\simeq \Sigma^k \mathbb{D}_{2^k}(S^1) \vee (\Sigma^k \mathbb{D}_{2^k}(S^1))_{h\Sigma_2}^{\wedge 2},$$

we see that in this case τ_k may be decomposed as

$$\tau_k = [\widetilde{\psi}_k]_1 \vee [\widetilde{\psi}_k]_2.$$

Using [16, Corollary 5.4], we see that

$$[\Sigma^k \mathbb{D}_{2^k}(S^1), \Sigma^{k+1} \mathbb{D}_{2^{k+1}}(S^1)] \cong [L(k), L(k+1)] = 0.$$

Therefore $[\widetilde{\psi}_k]_1 \simeq *$, and the lemma follows. $\square$

Lemma 4.3 *The induced maps*

$$(\alpha_k)_*, \ ([\widetilde{\psi}_k]_2)_* \colon H_*(\Sigma^k \mathbb{D}_{2^k}(S^1))_{h\Sigma_2}^{\wedge 2} \to H_* \Sigma^{k+1} \mathbb{D}_{2^{k+1}}(S^1)$$

are equal.

Proof of Lemma 4.3 The map

$$(\alpha_k)_* \colon H_*(\Sigma^k \mathbb{D}_{2^k}(S^1))_{h\Sigma_2}^{\wedge 2} \to H_* \Sigma^{k+1} \mathbb{D}_{2^{k+1}}(S^1)$$

can be computed using (3.4), (3.5), and the relation $\nu_k T_s = \nu_k$ established in the proof of Lemma 3.2. One finds that $(\alpha_k)_*$ is given by

$$(\alpha_k)_* Q^j \sigma^k \bar{Q}^{i_1} \cdots \bar{Q}^{i_k} \iota_1 = \sigma^{k+1} \bar{Q}^j \bar{Q}^{i_1} \cdots \bar{Q}^{i_k} \iota_1.$$

We just need to show that the same holds for $[\widetilde{\psi}_k]_2$.

Since $\phi_k = \Omega^k \psi_k$, the evaluation maps $\Sigma^k \Omega^k \to \mathrm{Id}$ allow one to fit the adjoints $\tilde{\phi}_k$, $\tilde{\psi}_k$ of these natural transformations into the commutative diagram

$$\begin{array}{ccc}
\Sigma^k \Sigma^\infty \Omega^\infty \mathbb{D}_{2^k}(S^V) & \xrightarrow{\Sigma^k \tilde{\phi}_k} & \Sigma^{k+1} \mathbb{D}_{2^{k+1}}(S^V) \\
{\scriptstyle E^k} \downarrow & & \downarrow {\scriptstyle =} \\
\Sigma^\infty \Omega^\infty \Sigma^k \mathbb{D}_{2^k}(S^V) & \xrightarrow[\tilde{\psi}_k]{} & \Sigma^{k+1} \mathbb{D}_{2^{k+1}}(S^V) \, .
\end{array}$$

On the level of 2^{k+1}–st Weiss layers, evaluated on $V = \mathbb{R}$, we get a diagram

$$\begin{array}{ccc}
\Sigma^k (\mathbb{D}_{2^k}(S^1))^{\wedge 2}_{h\Sigma_2} & \xrightarrow{\Sigma^k [\tilde{\phi}_k]_2} & \Sigma^{k+1} \mathbb{D}_{2^{k+1}}(S^1) \\
{\scriptstyle E^k} \downarrow & & \downarrow {\scriptstyle =} \\
(\Sigma^k \mathbb{D}_{2^k}(S^1))^{\wedge 2}_{h\Sigma_2} & \xrightarrow[{[\tilde{\psi}]_2}]{} & \Sigma^{k+1} \mathbb{D}_{2^{k+1}}(S^1)
\end{array}$$

The map

$$(E^k)_* \colon H_* \Sigma^k (\mathbb{D}_{2^k}(S^1))^{\wedge 2}_{h\Sigma_2} \to H_* (\Sigma^k \mathbb{D}_{2^k}(S^1))^{\wedge 2}_{h\Sigma_2}$$

is surjective, since Q^i–operations commute with $(E^k)_*$ (see [8, Lemma II.5.6], for example). Therefore, it suffices to compute

$$([\tilde{\phi}_k]_2)_* \colon H_* (\mathbb{D}_{2^k}(S^1))^{\wedge 2}_{h\Sigma_2} \to H_* \Sigma \mathbb{D}_{2^{k+1}}(S^1).$$

We compute this map using the technology of Section 2.

Let $P_{2^k, 2^{k+1}}(X)$ be the generalized quadratic functor defined by the fiber sequence

$$P_{2^k, 2^{k+1}}(X) \to P_{2^{k+1}}(X) \to P_{2^k-1}(X).$$

Then, as explained in Section 2, there is a fiber sequence

$$P_{2^k, 2^{k+1}}(X) \to \Omega^\infty \mathbb{P}_{2^k, 2^{k+1}-1}(X) \xrightarrow{\phi_k} \Omega^\infty \Sigma \mathbb{D}_{2^{k+1}}(X).$$

Here we have purposefully abused notation, as this new attaching map ϕ_k agrees with the old ϕ_k when X is a sphere. Associated to the adjoint of ϕ_k is a transformation

$$[\tilde{\phi}_k]_2 \colon \mathbb{D}_{2^k}(X)^{\wedge 2}_{h\Sigma_2} \to \Sigma \mathbb{D}_{2^{k+1}}(X)$$

which reduces to the previously defined $[\tilde{\phi}_k]_2$ when X is a sphere. Theorem 2.5 implies that $\Sigma^{-1}[\tilde{\phi}_k]_2$ is given by the map

$$\Sigma^{-1} \mathbb{D}_{2^k}(X)^{\wedge 2}_{h\Sigma_2} \to \mathbb{D}_{2^{k+1}}(X)$$

induced by the left action of $\partial_*(\mathrm{Id})$. Letting $X = S^1$, and using (3.1), we deduce that

$$([\widetilde{\phi}_k]_2)_* Q^j \, \overline{Q}^{i_1} \cdots \overline{Q}^{i_k} \iota_1 = \sigma \, \overline{Q}^j \, \overline{Q}^{i_1} \cdots \overline{Q}^{i_k}.$$

We therefore deduce that

$$([\widetilde{\psi}_k]_2)_* Q^j \sigma^k \, \overline{Q}^{i_1} \cdots \overline{Q}^{i_k} \iota_1 = \sigma^{k+1} \, \overline{Q}^j \, \overline{Q}^{i_1} \cdots \overline{Q}^{i_k} \iota_1,$$

and the lemma follows. $\qquad\qquad\square$

Using the above two lemmas, we may now prove Theorem 1.12 and deduce Theorem 1.8.

Proof of Theorem 1.12 Endow

$$H_* \Omega^\infty (\Sigma^k \mathbb{D}_{2^k}(S^1))^{\wedge 2}_{h\Sigma_2} = \mathcal{F} H_*(\Sigma^k \mathbb{D}_{2^k}(S^1))^{\wedge 2}_{h\Sigma_2}$$

with a weight filtration by defining

$$w(x) = 2^{k+1} \quad \text{for } x \in H_*(\Sigma^k \mathbb{D}_{2^k}(S^1))^{\wedge 2}_{h\Sigma_2},$$
$$w(Q^i x) = 2 \cdot w(x),$$
$$w(x * y) = w(x) + w(y).$$

Then, by Propositions 2.5 and 2.7 of [12], the map

$$\mathrm{JH}_* \colon H_* \Omega^\infty \Sigma^k \mathbb{D}_{2^k}(S^1) \to H_* \Omega^\infty (\Sigma^k \mathbb{D}_{2^k}(S^1))^{\wedge 2}_{h\Sigma_2}$$

preserves the weight filtration. The maps of the (collapsing) spectral sequences (3.6) induced by α_k and $[\widetilde{\psi}_k]_2$ imply that the maps

$$(\Omega^\infty \alpha_k)_*, \ (\Omega^\infty [\widetilde{\psi}_k]_2)_* \colon H_* \Omega^\infty (\Sigma^k \mathbb{D}_{2^k}(S^1))^{\wedge 2}_{h\Sigma_2} \to H_* \Omega^\infty \Sigma^{k+1} \mathbb{D}_{2^{k+1}}(S^1)$$

both preserve the weight filtration. Lemma 4.3 implies that on the level of associated graded groups, the maps $E_0(\Omega^\infty \alpha_k)_*$ and $E_0(\Omega^\infty [\widetilde{\psi}_k]_2)_*$ are equal. It follows from (4.1) and Lemma 4.3 that

$$E_0(\delta_k)_* = E_0(\psi_k)_* \colon E_0 H_* B^k D_{2^k}(S^1) \to E_0 H_* B^{k+1} D_{2^{k+1}}(S^1)$$

as desired. $\qquad\qquad\square$

Remark 4.4 The referee points out that the fact that the map JH_* preserves the weight filtration can also be easily deduced from calculus: take the induced map of Goodwillie towers on the natural transformation of functors

$$\Sigma^\infty \mathrm{JH} \colon \Sigma^\infty Q(-) \to \Sigma^\infty Q(-)^{\wedge 2}_{h\Sigma_2}$$

from spaces to spectra, and apply homology.

Proof of Theorem 1.8 Referring to Diagram (1.11), it is shown in [12] that

$$H_*(E_k) = \text{Im}(d_k)_* \subseteq H_* B^k D_{2^k}(S^1).$$

The weight filtration on $H_* B^k D_{2^k}(S^1)$ therefore induces a weight filtration on $H_* E_k$. It follows from Theorem 1.12 that

$$E_0(h'_k)_* = E_0(h_k)_* \colon E_0 H_* E_k \to E_0 H_* B^{k+1} D_{2^{k+1}}(S^1).$$

Kuhn proved Theorem 1.7 by showing that

$$E_0(\widetilde{d}_k)_* \circ E_0(h_k) = \text{Id} \colon E_0 H_* E_k \to E_0 H_* E_k.$$

We deduce that

$$E_0(\widetilde{d}_k)_* \circ E_0(h'_k) = \text{Id} \colon E_0 H_* E_k \to E_0 H_* E_k$$

and thus $\widetilde{d}_k \circ h'_k$ is a self-equivalence of E_k. Consider the induced splittings

$$B^k D_{2^k}(S^1) \simeq E_{k-1} \times E_k.$$

With respect to these splittings, d_k takes "matrix form"

$$d_k = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

and there exist self-equivalences $f_k \colon E_k \to E_k$ so that

$$\psi_k = \begin{bmatrix} * & f_k \\ * & 0 \end{bmatrix}.$$

We deduce that

$$d_k \psi_k + \psi_{k-1} d_{k-1} = \begin{bmatrix} f_{k-1} & 0 \\ * & f_k \end{bmatrix}$$

and in particular, $d_k \psi_k + \psi_{k-1} d_{k-1}$ is an equivalence. □

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