On the derivation algebra of the free Lie algebra and trace maps

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We mainly study the derivation algebra of the free Lie algebra and the Chen Lie algebra generated by the abelianization H of a free group, and trace maps. To begin with, we give the irreducible decomposition of the derivation algebra as a $\mathrm{GL}(n,\mathbf{Q})$–module via the Schur–Weyl duality and some tensor product theorems for $\mathrm{GL}(n,\mathbf{Q})$. Using them, we calculate the irreducible decomposition of the images of the Johnson homomorphisms of the automorphism group of a free group and a free metabelian group.

Next, we consider some applications of trace maps: Morita's trace map and the trace map for the exterior product of H. First, we determine the abelianization of the derivation algebra of the Chen Lie algebra as a Lie algebra, and show that the abelianization is given by the degree one part and Morita's trace maps. Second, we consider twisted cohomology groups of the automorphism group of a free nilpotent group. In particular, we show that the trace map for the exterior product of H defines a nontrivial twisted second cohomology class of it.

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1 Introduction

For a free group F_n with basis $x_1,\dots,x_n$, set $H := F_n^{\mathrm{ab}}$, the abelianization of F_n. The kernel of the natural homomorphism $\rho\colon \mathrm{Aut}\,F_n \to \mathrm{Aut}\,H$ induced from the abelianization of $F_n \to H$ is called the IA–automorphism group of F_n, and denoted by IA_n. Although IA_n plays important roles in various studies of $\mathrm{Aut}\,F_n$, the group structure of IA_n is quite complicated in general. For example, no presentation for IA_n is known. Furthermore, Krstić and McCool [24] showed that IA_3 is not finitely presentable. For $n \geq 4$, it is not known whether IA_n is finitely presentable or not.

To study the deep group structure of IA_n, it is sometimes useful to consider the Johnson filtration. For the lower central series $\Gamma_n(k)$ of F_n, the action of $\mathrm{Aut}\,F_n$ on the nilpotent quotient $F_n/\Gamma_n(k)$ induces a natural homomorphism $\mathrm{Aut}\,F_n \to \mathrm{Aut}\,(F_n/\Gamma_n(k+1))$. Its kernel $\mathcal{A}_n(k)$ then defines a descending central filtration $\mathrm{IA}_n = \mathcal{A}_n(1) \supset \mathcal{A}_n(2) \supset \cdots$.

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This filtration is called the Johnson filtration of $\mathrm{Aut}\, F_n$. Each of the graded quotients $\mathrm{gr}^k(\mathcal{A}_n) := \mathcal{A}_n(k)/\mathcal{A}_n(k+1)$ naturally has a $\mathrm{GL}(n,\mathbf{Z})$–module structure, and is considered to be a sequence of approximations of IA_n. To study $\mathrm{gr}^k(\mathcal{A}_n)$, the Johnson homomorphisms

$$\tau_k\colon \mathrm{gr}^k(\mathcal{A}_n) \hookrightarrow H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1)$$

of $\mathrm{Aut}\, F_n$ are defined where $H^* := \mathrm{Hom}_{\mathbf{Z}}(H,\mathbf{Z})$. Historically, the Johnson filtration was originally studied by Andreadakis [1] in 1960s, and the Johnson homomorphisms by D Johnson [15] in 1980s who determined the abelianization of the Torelli subgroup of the mapping class group of a surface in [16]. Now, there is a broad range of remarkable results for the Johnson homomorphisms of the mapping class group. (For example, see Hain [14] and Morita [28; 30; 31].) Since each of τ_k is $\mathrm{GL}(n,\mathbf{Z})$–equivariant injective, to clarify the structure of the image of τ_k is one of the most basic problems. By pioneering work of Andreadakis [1], it is known that τ_1 is an isomorphism. It is known that $\mathrm{Coker}(\tau_{2,\mathbf{Q}}) = S^2 H_{\mathbf{Q}}$ and $\mathrm{Coker}(\tau_{3,\mathbf{Q}}) = S^3 H_{\mathbf{Q}} \oplus \Lambda^3 H_{\mathbf{Q}}$ by Pettet [33] and Satoh [35] respectively. Here $\tau_{k,\mathbf{Q}} := \tau_k \otimes \mathrm{id}_{\mathbf{Q}}$ and $H_{\mathbf{Q}} := H \otimes_{\mathbf{Z}} \mathbf{Q}$. In general, however, it is quite a difficult problem to determine even the rank of the image of τ_k for $k \geq 4$.

Now, let $\mathcal{A}'_n(1), \mathcal{A}'_n(2), \dots$ be the lower central series of IA_n. Since the Johnson filtration is central, $\mathcal{A}'_n(k) \subset \mathcal{A}_n(k)$ for each $k \geq 1$. It is conjectured that $\mathcal{A}'_n(k) = \mathcal{A}_n(k)$ for each $k \geq 1$ by Andreadakis who showed $\mathcal{A}'_2(k) = \mathcal{A}_2(k)$ and $\mathcal{A}'_3(3) = \mathcal{A}_3(3)$. It is known that $\mathcal{A}'_n(2) = \mathcal{A}_n(2)$ due to Bachmuth [3] and that $\mathcal{A}'_n(3)$ has at most finite index in $\mathcal{A}_n(3)$ due to Pettet [33]. Set $\mathrm{gr}^k(\mathcal{A}'_n) := \mathcal{A}'_n(k)/\mathcal{A}'_n(k+1)$. Then we can also define a $\mathrm{GL}(n,\mathbf{Z})$–equivariant homomorphism

$$\tau'_k\colon \mathrm{gr}^k(\mathcal{A}'_n) \to H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1)$$

in the same way as for τ_k. We also call τ'_k the k–th Johnson homomorphism. (For the definitions of τ_k and τ'_k, see Section 2.6.) In [38], we determine the cokernel of the stable rational Johnson homomorphism $\tau'_{k,\mathbf{Q}} := \tau'_k \otimes \mathrm{id}_{\mathbf{Q}}$. More precisely, we prove that for any $k \geq 2$ and $n \geq k+2$,

$$\mathrm{Coker}(\tau'_{k,\mathbf{Q}}) \cong \mathcal{C}_n^{\mathbf{Q}}(k)$$

where $\mathcal{C}_n^{\mathbf{Q}}(k) := \mathcal{C}_n(k) \otimes_{\mathbf{Z}} \mathbf{Q}$, and $\mathcal{C}_n(k)$ is the quotient module of $H^{\otimes k}$ by the action of the cyclic group Cyc_k of order k on the components. Here the action Cyc_k on $H^{\otimes k}$ means the restriction of the ordinary action of the symmetric group of $\mathfrak{S}_k$ on $H^{\otimes k}$ on the components to the cyclic group Cyc_k. Then we have

$$\mathcal{C}_n(k) \cong H^{\otimes k} \big/ \langle a_1 \otimes a_2 \otimes \cdots \otimes a_k - a_2 \otimes a_3 \otimes \cdots \otimes a_k \otimes a_1 \mid a_i \in H \rangle.$$

In general the target of τ_k is considered to be the degree k part of the derivation algebra $\mathrm{Der}^+(\mathcal{L}_n)$ of the free Lie algebra $\mathcal{L}_n$ generated by H. The first aim of the paper is to give irreducible decompositions of $\mathrm{GL}(n,\mathbf{Q})$–modules $\mathcal{C}_n^{\mathbf{Q}}(k)$ and $\mathrm{Der}^+(\mathcal{L}_{n,\mathbf{Q}})(k) = \mathrm{Der}^+(\mathcal{L}_n)(k)\otimes_{\mathbf{Z}}\mathbf{Q}$. Our proof is based on the Schur–Weyl duality for $\mathrm{GL}(n,\mathbf{Q})$ and $\mathfrak{S}_k$.

Theorem 1 (Propositions 4.1 and 4.7 and Corollary 4.8) (1) *The multiplicities of irreducible* $\mathrm{GL}(n,\mathbf{Q})$*–modules* L^λ *with highest weight* λ *in* $\mathcal{C}_n^{\mathbf{Q}}(k)$ *are given by*

$$[L^\lambda : \mathcal{C}_n^{\mathbf{Q}}(k)] = \begin{cases} [\mathbf{triv}_k : \mathrm{Res}^{\mathfrak{S}_k}_{\mathrm{Cyc}_k} S^\lambda] & \text{if } \lambda \text{ is a partition of } k, \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathfrak{S}_k$ *is the symmetric group of degree* k, S^λ *is its irreducible module associated to a partition* λ *of* k, Cyc_k *is a cyclic subgroup of* $\mathfrak{S}_k$ *generated by a cyclic permutation of order* k *and* $\mathbf{triv}_k$ *is the trivial representation of* Cyc_k.

(2) *For any* $k \geq 1$ *and* $n \geq k+2$, *as a* $\mathrm{GL}(n,\mathbf{Q})$*–module, we have a direct decomposition*

$$\mathrm{Der}^+(\mathcal{L}_{n,\mathbf{Q}})(k) = (H_{\mathbf{Q}})^{\otimes k} \oplus \bigoplus_{\mu;\, \ell(\mu)\leq n} [L^\mu : \mathcal{L}_n(k)] L^{\{\mu;(1)\}},$$

where in the second term, the sum runs over all partitions μ *such that its length* $\ell(\mu)$ *is smaller than or equal to* n, *and* $L^{\{\mu;(1)\}}$ *is the irreducible* $\mathrm{GL}(n,\mathbf{Q})$*–module* $\det^{-1}\otimes L^\mu$.

We remark that, as a $\mathrm{GL}(n,\mathbf{Q})$–module, $\mathcal{C}_n^{\mathbf{Q}}(k)$ is isomorphic to the invariant part $a_n(k) := (H_{\mathbf{Q}}^{\otimes k})^{\mathrm{Cyc}_k}$ of $H_{\mathbf{Q}}^{\otimes k}$ by the action of Cyc_k. Namely, the cokernel $\mathrm{Coker}(\tau'_{k,\mathbf{Q}})$ is isomorphic to Kontsevich's $a_n(k)$ as a $\mathrm{GL}(n,\mathbf{Q})$–module. In our notation $a_n(k)$ is considered for any $n \geq 2$ in contrast to Kontsevich's notation for even $n = 2g$. (See Kontsevich [21; 22].) We also remark that the polynomial part $(H_{\mathbf{Q}})^{\otimes k}$ of $\mathrm{Der}^+(\mathcal{L}_{n,\mathbf{Q}})(k)$ is detected by a contraction map. (See Section 2.3 for the definition of the contraction map.) Using this theorem, for given $k \geq 1$, we can calculate the irreducible decomposition of $\mathrm{Im}(\tau'_{n,\mathbf{Q}})$ and $\mathrm{Coker}(\tau'_{n,\mathbf{Q}})$ for $n \geq k+2$.

On the other hand, we also give the irreducible decompositions of the derivation algebras of the Chen Lie algebra $\mathcal{L}^M_{n,\mathbf{Q}}$ and free abelian by polynilpotent Lie algebra $\mathcal{L}^N_{n,\mathbf{Q}}$ generated by $H_{\mathbf{Q}}$. (See Sections 2.4 and 2.5 for the precise definition.) They were studied in our previous papers [37; 40] in order to investigate the cokernel of the Johnson homomorphisms $\tau'_{k,\mathbf{Q}}$. In this paper, after tensoring with $\mathbf{Q}$, we determine the irreducible decomposition of the image of the Johnson homomorphism τ_k^M of the automorphism group of a free metabelian group and of the image of the composition

map $\tau'_{k,N}$ of τ'_k and a homomorphism $H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1) \to H^* \otimes_{\mathbf{Z}} \mathcal{L}_n^N(k+1)$ induced from the natural projection $\mathcal{L}_n(k+1) \to \mathcal{L}_n^N(k+1)$. That is:

Theorem 2 (Propositions 4.10 and 4.12) (1) *For any $k \geq 1$ and $n \geq k+2$,*

$$\mathrm{Im}(\tau^M_{k,\mathbf{Q}}) \cong L^{\{(k,1),(1)\}} \oplus L^{\{(k-1,1),0\}}.$$

(2) *For any $k \geq 1$ and $n \geq k+2$,*

$$\begin{aligned}\mathrm{Im}((\tau'_{k,N})_{\mathbf{Q}}) \cong\ & 3L^{(k-1,1)} \oplus 2L^{(k-2,2)} \oplus 2L^{(k-2,1^2)} \oplus L^{(k-3,2,1)} \oplus L^{(k-3,1^3)} \\ & \oplus L^{\{(k,1);(1)\}} \oplus L^{\{(k-1,2);(1)\}} \oplus L^{\{(k-1,1^2);(1)\}} \\ & \oplus L^{\{(k-2,2,1);(1)\}} \oplus L^{\{(k-2,1^3);(1)\}},\end{aligned}$$

where $\tau^M_{k,\mathbf{Q}} := \tau^M_k \otimes \mathrm{id}_{\mathbf{Q}}$ and $(\tau'_{k,N})_{\mathbf{Q}} := \tau'_{k,N} \otimes \mathrm{id}_{\mathbf{Q}}$.

For any irreducible $\mathrm{GL}(n,\mathbf{Q})$–module L^λ with highest weight λ, we see

$$[L^\lambda : \mathrm{Im}(\tau^M_{k,\mathbf{Q}})] \leq [L^\lambda : \mathrm{Im}((\tau'_{k,N})_{\mathbf{Q}})] \leq [L^\lambda : \mathrm{Im}(\tau'_{k,\mathbf{Q}})] \leq [L^\lambda : \mathrm{Im}(\tau_{k,\mathbf{Q}})].$$

So we can regard $\mathrm{Im}(\tau^M_{k,\mathbf{Q}})$ and $\mathrm{Im}((\tau'_{k,N})_{\mathbf{Q}})$ as lower bounds on $\mathrm{Im}(\tau'_{k,\mathbf{Q}})$ and $\mathrm{Im}(\tau_{k,\mathbf{Q}})$.

In the rest of the paper, we consider some applications of trace maps. In general, the trace maps are used to study the cokernel of the Johnson homomorphisms. One of the most important trace maps is Morita's trace map

$$\mathrm{Tr}_{[k]} := f_{[k]} \circ \Phi_1^k \colon H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1) \to S^k H.$$

Introducing this map, Morita showed that $S^k H_{\mathbf{Q}}$ appears in the irreducible decomposition of $\mathrm{Coker}(\tau_{k,\mathbf{Q}})$. Using the part (1) of Theorem 1, we see that the multiplicity of $S^k H_{\mathbf{Q}}$ in the irreducible decomposition of each of $\mathrm{Coker}(\tau'_{n,\mathbf{Q}})$ and $\mathrm{Coker}(\tau_{n,\mathbf{Q}})$ is just one. In Section 5, we determine the abelianization of the derivation algebra $\mathrm{Der}^+(\mathcal{L}_n^M)$ of the Chen Lie algebra using Morita's trace maps. That is:

Theorem 3 (Theorem 5.5) *For $n \geq 4$, we have*

$$(\mathrm{Der}^+(\mathcal{L}_n^M))^{\mathrm{ab}} \cong (H^* \otimes_{\mathbf{Z}} \Lambda^2 H) \oplus \bigoplus_{k \geq 2} S^k H.$$

More precisely, this isomorphism is given by the degree one part and Morita's trace maps $\mathrm{Tr}_{[k]}$.

We should remark that this result is the Chen Lie algebra version of Morita's conjecture:

$$(\mathrm{Der}^+(\mathcal{L}_n))^{\mathrm{ab}} \cong (H^* \otimes_{\mathbf{Z}} \Lambda^2 H) \oplus \bigoplus_{k\geq 2} S^k H$$

for the free Lie algebra $\mathcal{L}_n$ for any $n \geq 3$. (See also Morita [31].)

Next we consider another important trace map

$$\mathrm{Tr}_{[1^k]} := f_{[1^k]} \circ \Phi_1^k \colon H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1) \to \Lambda^k H,$$

called the trace map for the exterior product $\Lambda^k H$. In [35], we show that $\Lambda^k H_{\mathbf{Q}}$ appears in $\mathrm{Coker}(\tau'_{k,\mathbf{Q}})$ for odd k and $3 \leq k \leq n$, and determine $\mathrm{Coker}(\tau_{3,\mathbf{Q}})$ using $\mathrm{Tr}_{[3]}$ and $\mathrm{Tr}_{[1^3]}$. In Section 6, we prove that the trace map $\mathrm{Tr}_{[1^k]}$ defines a nontrivial twisted second cohomology class of the automorphism group $\mathrm{Aut}\, N_{n,k}$ of a free nilpotent group $N_{n,k} := F_n/\,\Gamma_n(k+1)$ with coefficients in $\Lambda^k H_{\mathbf{Q}}$ for any $k \geq 2$ and $n \geq k$. To show this, we prove that $H^1(\mathrm{Aut}\, N_{n,k}, \Lambda^k H_{\mathbf{Q}})$ is trivial. Then we consider the cohomological five term exact sequence of a group extension

$$0 \to \mathrm{Hom}_{\mathbf{Z}}(H, \mathcal{L}_n(k+1)) \to \mathrm{Aut}\, N_{n,k+1} \to \mathrm{Aut}\, N_{n,k} \to 1$$

introduced by Andreadakis [1]. (See also Morita [29, Proposition 2.3].)

On the other hand, let $T_{n,k}$ be the image of the natural homomorphism $\mathrm{Aut}\, F_n \to \mathrm{Aut}\, N_{n,k}$ induced from the projection $F_n \to N_{n,k}$. The group $T_{n,k}$ is called the tame automorphism group of $N_{n,k}$. Similarly to $\mathrm{Aut}\, N_{n,k}$, observing the cohomological five term exact sequence of a group extension

$$0 \to \mathrm{gr}^k(\mathcal{A}_n) \to T_{n,k+1} \to T_{n,k} \to 1,$$

we show that the $\mathrm{GL}(n,\mathbf{Z})$–equivariant homomorphism $\mathrm{Tr}_{[1^k]} \circ \tau_k$ defines a nontrivial twisted second cohomology class of $T_{n,k}$. Namely, the main purpose in Section 6 is to show:

Theorem 4 (Propositions 6.6 and 6.8) (1) $0 \neq \mathrm{tg}(\mathrm{Tr}_{[1^k]} \circ \tau_k) \in H^2(T_{n,k}, \Lambda^k H_{\mathbf{Q}})$ *for even* k *and* $2 \leq k \leq n$,

(2) $0 \neq \mathrm{tg}(\mathrm{Tr}_{[1^k]}) \in H^2(\mathrm{Aut}\, N_{n,k}, \Lambda^k H_{\mathbf{Q}})$ *for* $k \geq 3$ *and* $n \geq k$,

where tg *means the transgression map.*

In Section 6, We also show that $H^1(T_{n,k}, H) = \mathbf{Z}$ and it is generated by Morita's crossed homomorphism. (See Proposition 6.4.)

2 Preliminaries

In this section, after fixing notation and conventions, we briefly recall some facts of the automorphism group of a free group, the free Lie algebra, the Chen Lie algebra and the automorphism group of a free nilpotent group.

2.1 Notation and conventions

Throughout the paper, we use the following notation and conventions. Let G be a group and N a normal subgroup of G.

- The abelianization of G is denoted by G^{ab}. Namely, $G^{\mathrm{ab}} = H_1(G, \mathbf{Z})$. Similarly, for any Lie algebra $\mathcal{G}$, we denote by $\mathcal{G}^{\mathrm{ab}}$ the abelianization of $\mathcal{G}$ as a Lie algebra.
- The automorphism group $\mathrm{Aut}\, G$ of G acts on G from the right unless otherwise noted. For any $\sigma \in \mathrm{Aut}\, G$ and $x \in G$, the action of σ on x is denoted by x^{σ}.
- For an element $g \in G$, we also denote the coset class of g by $g \in G/N$ if there is no confusion.
- For elements x and y of G, the commutator bracket $[x, y]$ of x and y is defined to be $[x, y] := xyx^{-1}y^{-1}$.
- For elements $g_1, \dots, g_k \in G$, a commutator of weight k of the type
$$[[\cdots[[g_1, g_2], g_3], \dots], g_k]$$
with all of its brackets to the left of all the elements occurring is called a simple k–fold commutator, and is denoted by $[g_{i_1}, g_{i_2}, \dots, g_{i_k}]$.
- For any $\mathbf{Z}$–module M and a commutative ring R, we denote $M \otimes_{\mathbf{Z}} R$ by the symbol obtained by attaching subscript or superscript R to M, like M_R or M^R. Similarly, for any $\mathbf{Z}$–linear map $f\colon A \to B$, the induced R–linear map $f \otimes_{\mathbf{Z}} \mathrm{id}_R\colon A_R \to B_R$ is denoted by f_R or f^R.

2.2 Automorphism group of a free group and its subgroups

Here we review some properties of the automorphism group of a free group. To begin with, we recall the Nielsen's finite presentation for $\mathrm{Aut}\, F_n$. In this paper, we fix a basis $x_1, \dots, x_n$ of a free group F_n of rank n. Let P, Q, S and U be automorphisms of F_n defined as follows:

	x_1	x_2	x_3	$\cdots$	x_{n-1}	x_n
P	x_2	x_1	x_3	$\cdots$	x_{n-1}	x_n
Q	x_2	x_3	x_4	$\cdots$	x_n	x_1
S	x_1^{-1}	x_2	x_3	$\cdots$	x_{n-1}	x_n
U	x_1x_2	x_2	x_3	$\cdots$	x_{n-1}	x_n

Namely, P is an automorphism induced from the permutation of x_1 and x_2, Q is induced from the cyclic permutation of the basis, and so on. Nielsen [32] obtained the first finite presentation for $\mathrm{Aut}\, F_n$ in 1924.

Theorem 2.1 (Nielsen [32]) *For $n \geq 3$, $\mathrm{Aut}\, F_n$ has a finite presentation with generators P, Q, S and U subject to relations*

(R1) $P^2 = 1$,

(R2) $(QP)^{n-1} = Q^n = 1$,

(R3) $[P, Q^{-i} P Q^i] = 1$, $(2 \leq i \leq [n/2])$,

(R4) $S^2 = 1$,

(R5) $[S, Q^{-1} P Q] = [S, QP] = 1$,

(R6) $(PS)^4 = (PSPU)^2 = 1$,

(R7) $[U, Q^{-2} P Q^2] = [U, Q^{-2} U Q^2] = 1$, $(n \geq 4)$,

(R8) $[U, Q^{-2} S Q^2] = [U, SUS] = 1$,

(R9) $[U, QPQ^{-1} PQ] = [U, PQ^{-1} SUSQP] = 1$,

(R10) $[U, PQ^{-1} PQPUPQ^{-1} PQP] = 1$,

(R11) $PUPSU = USPS$,

(R12) $(PQ^{-1}UQ)^2 UQ^{-1}U^{-1}QU^{-1} = 1$.

In Section 6, we use Nielsen's presentation to compute twisted first cohomology groups of $\mathrm{Aut}\, F_n$. Let $X^{\pm 1} := \{x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}\} \subset F_n$ be the subset of all letters of F_n. We denote by Ω_n the subgroup of $\mathrm{Aut}\, F_n$ consisting of all $\sigma \in \mathrm{Aut}\, F_n$ that effect a permutation on $X^{\pm 1}$. Then it is known that Ω_n is a finite group of order $2^n n!$ and is generated by P, Q and S [32]. The subgroup Ω_n is called the extended symmetric group of degree n.

Next we consider the natural projection induced from the abelianization of F_n. Let $H := F_n^{\mathrm{ab}}$ be the abelianization of F_n and $\rho : \mathrm{Aut}\, F_n \to \mathrm{Aut}\, H$ the natural homomorphism induced from the abelianization of $F_n \to H$. Throughout the paper, we identify $\mathrm{Aut}\, H$ with the general linear group $\mathrm{GL}(n, \mathbf{Z})$ by fixing a basis of H induced from the basis $x_1, \ldots, x_n$ of F_n. Using Nielsen's presentation, we easily see that ρ is surjective. For any $\sigma \in \mathrm{Aut}\, F_n$, we also denote $\rho(\sigma) \in \mathrm{GL}(n, \mathbf{Z})$ by σ if there is no confusion. With this notation, P, Q, S and U generate $\mathrm{GL}(n, \mathbf{Z})$. In particular, it is known that

Theorem 2.2 (Magnus [26]; see also Coxeter–Moser [10, Section 7.3]) *For $n \geq 3$, the group $\mathrm{GL}(n, \mathbf{Z})$ has a finite presentation with generators P, Q, S and U subject to relations (R1), …, (R12) and*

(R13) $(SU)^2 = 1$.

Now the kernel IA_n of ρ is called the IA–automorphism group of F_n. Magnus [26] showed that for any $n \geq 3$, IA_n is finitely generated by automorphisms

$$K_{ij}\colon x_t \mapsto \begin{cases} {x_j}^{-1} x_i x_j & t = i, \\ x_t & t \neq i, \end{cases}$$

for distinct $1 \leq i,\ j \leq n$, and

$$K_{ijl}\colon x_t \mapsto \begin{cases} x_i[x_j, x_l] & t = i, \\ x_t & t \neq i, \end{cases}$$

for distinct $1 \leq i,\ j,\ l \leq n$ and $j < l$. Recently, Cohen and Pakianathan [8; 9], Farb [11] and Kawazumi [18] independently showed

$$\mathrm{IA}_n^{\mathrm{ab}} \cong H^* \otimes_{\mathbf{Z}} \Lambda^2 H \tag{1}$$

as a $\mathrm{GL}(n, \mathbf{Z})$–module where $H^* := \mathrm{Hom}_{\mathbf{Z}}(H, \mathbf{Z})$ is the $\mathbf{Z}$–linear dual group of H. In particular, from their result, we see that $\mathrm{IA}_n^{\mathrm{ab}}$ is a free abelian group of rank $2n^2(n-1)$ with basis the coset classes of the Magnus generators K_{ij} and K_{ijl}.

We denote by $\overline{\Omega}_n$ the image of Ω_n by the natural projection $\rho\colon \mathrm{Aut}\, F_n \to \mathrm{Aut}\, H$. Since IA_n is torsion free, $\overline{\Omega}_n$ is isomorphic to Ω_n. Namely, $\overline{\Omega}_n$ is a finite group of order $2^n n!$ generated by P, Q and S.

2.3 The free Lie algebra $\mathcal{L}_n$ and its derivations

In this subsection, we recall the free Lie algebra generated by H, and its derivation algebra. Let $\Gamma_n(1) \supset \Gamma_n(2) \supset \cdots$ be the lower central series of a free group F_n defined by the rule

$$\Gamma_n(1) := F_n, \quad \Gamma_n(k) := [\Gamma_n(k-1), F_n],\ k \geq 2.$$

We denote by $\mathcal{L}_n(k) := \Gamma_n(k)/\Gamma_n(k+1)$ the k–th graded quotient of the lower central series of F_n, and by $\mathcal{L}_n := \bigoplus_{k \geq 1} \mathcal{L}_n(k)$ the associated graded sum. It is classically well known due to Witt [41] that each $\mathcal{L}_n(k)$ is a free abelian group of rank

$$\mathrm{rank}_{\mathbf{Z}}(\mathcal{L}_n(k)) = \frac{1}{k} \sum_{d|k} \mathrm{Möb}(d) n^{k/d}, \tag{2}$$

where Möb is the Möbius function. The graded sum $\mathcal{L}_n$ naturally has a graded Lie algebra structure induced from the commutator bracket on F_n, and called the free Lie algebra generated by H. (See Reutenauer [34] for basic material concerning the free Lie algebra.) For each $k \geq 1$, $\mathrm{Aut}\, F_n$ naturally acts on $\mathcal{L}_n(k)$. Since the action of IA_n on $\mathcal{L}_n(k)$ is trivial, that of $\mathrm{GL}(n, \mathbf{Z}) = \mathrm{Aut}\, F_n/\mathrm{IA}_n$ on $\mathcal{L}_n(k)$ is well-defined.

Next, we consider an embedding of the free Lie algebra into the tensor algebra. Let

$$T(H) := \mathbf{Z} \oplus H \oplus H^{\otimes 2} \oplus \cdots$$

be the tensor algebra of H over $\mathbf{Z}$. Then $T(H)$ is the universal enveloping algebra of the free Lie algebra $\mathcal{L}_n$, and the natural map $\iota\colon \mathcal{L}_n \to T(H)$ defined by

$$[X, Y] \mapsto X \otimes Y - Y \otimes X$$

for X, $Y \in \mathcal{L}_n$ is an injective graded Lie algebra homomorphism. We denote by $\iota_k\colon \mathcal{L}_n(k) \to H^{\otimes k}$ the homomorphism of degree k part of ι, and consider $\mathcal{L}_n(k)$ as a submodule $H^{\otimes k}$ through ι_k.

Here, we recall the derivation algebra of the free Lie algebra. Let $\mathrm{Der}(\mathcal{L}_n)$ be the graded Lie algebra of derivations of $\mathcal{L}_n$. Namely,

$$\mathrm{Der}(\mathcal{L}_n) := \{f\colon \mathcal{L}_n \xrightarrow{\mathbf{Z}-\text{linear}} \mathcal{L}_n \mid f([a,b]) = [f(a), b] + [a, f(b)],\ a, b \in \mathcal{L}_n\}.$$

For $k \geq 0$, the degree k part of $\mathrm{Der}(\mathcal{L}_n)$ is defined to be

$$\mathrm{Der}(\mathcal{L}_n)(k) := \{f \in \mathrm{Der}(\mathcal{L}_n) \mid f(a) \in \mathcal{L}_n(k+1),\ a \in H\}.$$

Then, we have

$$\mathrm{Der}(\mathcal{L}_n) = \bigoplus_{k \geq 0} \mathrm{Der}(\mathcal{L}_n)(k),$$

and can consider $\mathrm{Der}(\mathcal{L}_n)(k)$ as

$$\mathrm{Hom}_{\mathbf{Z}}(H, \mathcal{L}_n(k+1)) = H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1)$$

for each $k \geq 1$ by the universality of the free Lie algebra. Let $\mathrm{Der}^+(\mathcal{L}_n)$ be a graded Lie subalgebra of $\mathrm{Der}(\mathcal{L}_n)(k)$ with positive degree. (See Bourbaki [5, Section 8 of Chapter II].) Similarly, we define a graded Lie algebra $\mathrm{Der}^+(\mathcal{L}_{n,\mathbf{Q}})$ over $\mathbf{Q}$. Then, we have $\mathrm{Der}^+(\mathcal{L}_{n,\mathbf{Q}}) = \mathrm{Der}^+(\mathcal{L}_n) \otimes_{\mathbf{Z}} \mathbf{Q}$.

For $k \geq 1$, let $\varphi^k\colon H^* \otimes_{\mathbf{Z}} H^{\otimes(k+1)} \to H^{\otimes k}$ be the contraction map defined by

$$x_i^* \otimes x_{j_1} \otimes \cdots \otimes x_{j_{k+1}} \mapsto x_i^*(x_{j_1}) \cdot x_{j_2} \otimes \cdots \otimes \cdots \otimes x_{j_{k+1}}.$$

For the natural embedding $\iota_{k+1}\colon \mathcal{L}_n(k+1) \to H^{\otimes(k+1)}$, we obtain a $\mathrm{GL}(n, \mathbf{Z})$–equivariant homomorphism

$$\Phi^k = \varphi^k \circ (\mathrm{id}_{H^*} \otimes \iota_{k+1})\colon H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1) \to H^{\otimes k}.$$

We also call Φ^k a contraction map. In Proposition 4.7, we study the irreducible decomposition of $\mathrm{Der}^+(\mathcal{L}_{n,\mathbf{Q}})(k)$ as a $\mathrm{GL}(n, \mathbf{Z})$–module using the contraction map.

Finally, we review the trace maps. For any $k \geq 2$, let $f_{[k]}\colon H^{\otimes k} \to S^k H$ be the natural projection defined by

$$x_{i_1} \otimes \cdots \otimes x_{i_k} \mapsto x_{i_1} \cdots x_{i_k}.$$

Then the composition map

$$\mathrm{Tr}_{[k]} := f_{[k]} \circ \Phi^k\colon H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1) \to S^k H$$

is a $\mathrm{GL}(n,\mathbf{Z})$–equivariant surjective homomorphism, and is called Morita's trace map. Morita studied the cokernel of the Johnson homomorphism of $\mathrm{Aut}\, F_n$ using Morita's trace map, and showed that $S^k H_{\mathbf{Q}}$ appears in the irreducible decomposition of it for any $k \geq 2$.

Let $f_{[1^k]}\colon H^{\otimes k} \to \Lambda^k H$ be the natural projection defined by

$$x_{i_1} \otimes \cdots \otimes x_{i_k} \mapsto x_{i_1} \wedge \cdots \wedge x_{i_k}.$$

Then the composition map

$$\mathrm{Tr}_{[1^k]} := f_{[1^k]} \circ \Phi^k\colon H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1) \to \Lambda^k H$$

is called the trace map for $\Lambda^k H$. Using the trace map $\mathrm{Tr}_{[1^k]}$, we showed that $\Lambda^k H_{\mathbf{Q}}$ appears in the cokernel of the Johnson homomorphism restricted to the lower central series of IA_n if $3 \leq k \leq n$ and k is odd. (See the second author's paper [35] for details.) In Section 6, we show that $\mathrm{Tr}_{[1^k]}$ defines a nontrivial twisted second cohomology class of the automorphism group of a free nilpotent group.

2.4 Chen Lie algebra $\mathcal{L}_n^M$ and its derivations

Here we recall the Chen Lie algebra generated by H, and its derivation algebra. Let $F_n^M := F_n/[[F_n,F_n],[F_n,F_n]]$ be a free metabelian group of rank n. Let $\Gamma_n^M(1) \supset \Gamma_n^M(2) \supset \cdots$ be the lower central series of a free group F_n^M defined by the rule

$$\Gamma_n^M(1) := F_n^M, \quad \Gamma_n^M(k) := [\Gamma_n^M(k-1), F_n^M],\ k \geq 2.$$

We denote by $\mathcal{L}_n^M(k) := \Gamma_n^M(k)/\Gamma_n^M(k+1)$ the k–th graded quotient of the lower central series of F_n^M, and by $\mathcal{L}_n^M := \bigoplus_{k\geq 1} \mathcal{L}_n^M(k)$ the associated graded sum. The graded sum $\mathcal{L}_n^M$ naturally has a graded Lie algebra structure induced from the commutator bracket on F_n^M by the same argument as the free Lie algebra $\mathcal{L}_n$. The Lie algebra $\mathcal{L}_n^M$ is called the free metabelian Lie algebra or the Chen Lie algebra, generated by H.

Since $(F_n^M)^{\mathrm{ab}} = H$, $\mathrm{Aut}\,(F_n^M)^{\mathrm{ab}} = \mathrm{Aut}(H) = \mathrm{GL}(n,\mathbf{Z})$. By an argument similar to the free Lie algebra, it turns out that each of the graded quotients $\mathcal{L}_n^M(k)$ is a

$\mathrm{GL}(n,\mathbf{Z})$–module. For $1 \le k \le 3$, we have $\mathcal{L}_n(k) = \mathcal{L}_n^M(k)$. It is also classically known due to Chen [7] that each $\mathcal{L}_n^M(k)$ is a free abelian group of rank

$$\operatorname{rank}_{\mathbf{Z}}(\mathcal{L}_n^M(k)) = (k-1)\binom{n+k-2}{k} \tag{3}$$

with basis

$$\{[x_{i_1}, x_{i_2}, \dots, x_{i_k}] \mid i_1 > i_2 \le i_3 \le \cdots \le i_k\}. \tag{4}$$

Let $\mathrm{Der}^+(\mathcal{L}_n^M)$ be the graded Lie algebra of derivations of $\mathcal{L}_n^M$ with positive degree. The degree k part of $\mathrm{Der}^+(\mathcal{L}_n^M)$ is considered as

$$\mathrm{Der}^+(\mathcal{L}_n^M)(k) = H^* \otimes_{\mathbf{Z}} \mathcal{L}_n^M(k+1).$$

Similarly, we define a graded Lie algebra $\mathrm{Der}^+(\mathcal{L}_{n,\mathbf{Q}}^M)$ over $\mathbf{Q}$. In Section 4.3, we give the irreducible decomposition of $\mathrm{Der}^+(\mathcal{L}_{n,\mathbf{Q}}^M)$.

Now, Morita's trace map $\mathrm{Tr}_{[k]}$ naturally factors through a surjective homomorphism $H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k) \to H^* \otimes_{\mathbf{Z}} \mathcal{L}_n^M(k)$. Namely, $\mathrm{Tr}_{[k]}$ induces a $\mathrm{GL}(n,\mathbf{Z})$–equivariant homomorphism

$$\mathrm{Tr}_{[k]}^M \colon H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1) \to S^k H.$$

(See [37, Section 3.2].) We also call it Morita's trace map. In Section 5, we determine the abelianization of $\mathrm{Der}^+(\mathcal{L}_n^M)$ as a Lie algebra using Morita's trace map $\mathrm{Tr}_{[k]}^M$.

2.5 Lie algebra $\mathcal{L}_n^N$ and its derivations

Let F_n^N be the quotient group of F_n by $[\Gamma_n(3), \Gamma_n(3)][[\Gamma_n(2), \Gamma_n(2)], \Gamma_n(2)]$. Let $\Gamma_n^N(1) \supset \Gamma_n^N(2) \supset \cdots$ be the lower central series of a free group F_n^N and $\mathcal{L}_n^N(k) := \Gamma_n^N(k)/\Gamma_n^N(k+1)$ its graded quotients. Similarly, the graded sum $\mathcal{L}_n^N := \bigoplus_{k\ge 1} \mathcal{L}_n^N(k)$ has a graded Lie algebra structure induced from the commutator bracket of F_n^N.

In our paper [40], we have determined the $\mathbf{Z}$–module structure of $\mathcal{L}_n^N(k)$, and showed that its rank is given by

$$\operatorname{rank}_{\mathbf{Z}}(\mathcal{L}_n^N(k)) = (k-1)\binom{k+n-2}{k} + \frac{1}{2}n(n-1)(k-3)\binom{n+k-4}{k-2}, \tag{5}$$

and

$$\begin{aligned}&\{[x_{i_1}, x_{i_2}, \dots, x_{i_k}] \mid i_1 > i_2 \le i_3 \le \cdots \le i_k\}\\ &\quad \cup \{[x_{i_1}, \dots, x_{i_{k-2}}, [x_{i_{k-1}}, x_{i_k}]] \mid i_1 > i_2 \le i_3 \le \cdots \le i_{k-2},\ i_{k-1} > i_k\}.\end{aligned} \tag{6}$$

is a basis of it.

We [40] used these facts to investigate the cokernel of the Johnson homomorphism. Let $\mathrm{Der}^+(\mathcal{L}_n^N)$ be the graded Lie algebra of derivations of $\mathcal{L}_n^N$ with positive degree. The degree k part of $\mathrm{Der}^+(\mathcal{L}_n^N)$ is considered as

$$\mathrm{Der}^+(\mathcal{L}_n^N)(k) = H^* \otimes_{\mathbf{Z}} \mathcal{L}_n^N(k+1).$$

Similarly, we define a graded Lie algebra $\mathrm{Der}^+(\mathcal{L}_{n,\mathbf{Q}}^N)$ over $\mathbf{Q}$. In Section 4.4, we give the irreducible decomposition of $\mathrm{Der}^+(\mathcal{L}_{n,\mathbf{Q}}^N)$.

2.6 Johnson homomorphisms

For each $k \geq 1$, let $N_{n,k} := F_n/\,\Gamma_n(k+1)$ of F_n be the free nilpotent group of class k and rank n, and $\mathrm{Aut}\, N_{n,k}$ its automorphism group. Since the subgroup $\Gamma_n(k+1)$ is characteristic in F_n, the group $\mathrm{Aut}\, F_n$ naturally acts on $N_{n,k}$. This action induces a homomorphism

$$\rho_k \colon \mathrm{Aut}\, F_n \to \mathrm{Aut}\, N_{n,k}.$$

Let $\mathcal{A}_n(k)$ be the kernel of ρ_k. Then the groups $\mathcal{A}_n(k)$ define a descending central filtration

$$\mathrm{IA}_n = \mathcal{A}_n(1) \supset \mathcal{A}_n(2) \supset \cdots,$$

called the Johnson filtration of $\mathrm{Aut}\, F_n$. Set $\mathrm{gr}^k(\mathcal{A}_n) := \mathcal{A}_n(k)/\mathcal{A}_n(k+1)$. For each $k \geq 1$, the group $\mathrm{Aut}\, F_n$ acts on $\mathrm{gr}^k(\mathcal{A}_n)$ by conjugation. This action induces that of $\mathrm{GL}(n,\mathbf{Z}) = \mathrm{Aut}\, F_n/\mathrm{IA}_n$ on it.

In order to study the $\mathrm{GL}(n,\mathbf{Z})$–module structure of $\mathrm{gr}^k(\mathcal{A}_n)$, the Johnson homomorphisms of $\mathrm{Aut}\, F_n$ are defined as follows. For each $k \geq 1$, define a homomorphism $\widetilde{\tau}_k \colon \mathcal{A}_n(k) \to \mathrm{Hom}_{\mathbf{Z}}(H, \mathcal{L}_n(k+1))$ by

$$\sigma \mapsto (x \mapsto x^{-1}x^{\sigma}), \quad x \in H.$$

Then the kernel of $\widetilde{\tau}_k$ is just $\mathcal{A}_n(k+1)$. Hence it induces an injective homomorphism

$$\tau_k \colon \mathrm{gr}^k(\mathcal{A}_n) \hookrightarrow \mathrm{Hom}_{\mathbf{Z}}(H, \mathcal{L}_n(k+1)) = H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1).$$

The homomorphism τ_k is $\mathrm{GL}(n,\mathbf{Z})$–equivariant, and is called the k–th Johnson homomorphism of $\mathrm{Aut}\, F_n$. Furthermore, we remark that the sum of the Johnson homomorphisms forms a Lie algebra homomorphism as follows. Let $\mathrm{gr}(\mathcal{A}_n) := \bigoplus_{k\geq 1} \mathrm{gr}^k(\mathcal{A}_n)$ be the graded sum of $\mathrm{gr}^k(\mathcal{A}_n)$. The graded sum $\mathrm{gr}(\mathcal{A}_n)$ has a graded Lie algebra structure induced from the commutator bracket on IA_n. Then the sum of the Johnson homomorphisms

$$\tau := \bigoplus_{k\geq 1} \tau_k \colon \mathrm{gr}(\mathcal{A}_n) \to \mathrm{Der}^+(\mathcal{L}_n)$$

is a graded Lie algebra homomorphism.

It is known that τ_1 gives the abelianization of IA_n by an independent work of Cohen and Pakianathan [8; 9], Farb [11] and Kawazumi [18] as mentioned above. Namely, $\mathrm{gr}^1(\mathcal{A}_n) \cong \mathrm{IA}_n^{\mathrm{ab}}$. Furthermore, we have exact sequences

$$0 \to \mathrm{gr}^2(\mathcal{A}_n) \xrightarrow{\tau_2} \mathrm{Hom}_{\mathbf{Z}}(H, \mathcal{L}_n(3)) \xrightarrow{\mathrm{Tr}_{[2]}} S^2 H \to 0,$$

$$0 \to \mathrm{gr}^3_{\mathbf{Q}}(\mathcal{A}_n) \xrightarrow{\tau_{3,\mathbf{Q}}} \mathrm{Hom}_{\mathbf{Q}}(H_{\mathbf{Q}}, \mathcal{L}_{n,\mathbf{Q}}(3)) \xrightarrow{\mathrm{Tr}^{\mathbf{Q}}_{[3]} \oplus \mathrm{Tr}^{\mathbf{Q}}_{[1^3]}} S^3 H_{\mathbf{Q}} \oplus \Lambda^3 H_{\mathbf{Q}} \to 0.$$

as $\mathrm{GL}(n,\mathbf{Z})$–modules. (See [35] for details.) In general, however, the $\mathrm{GL}(n,\mathbf{Z})$–module structure of $\mathrm{gr}^k_{\mathbf{Q}}(\mathcal{A}_n)$ is not determined for $k \geq 4$.

To give a lower bound on the image of the Johnson homomorphisms τ_k, or equivalently an upper bound on the cokernel of τ_k, it is sometimes useful to consider the restriction of $\widetilde{\tau}_k$ to the lower central series of IA_n. Let $\mathcal{A}'_n(k)$ be the lower central series of IA_n with $\mathcal{A}'_n(1) = \mathrm{IA}_n$. Since the Johnson filtration is central, $\mathcal{A}'_n(k) \subset \mathcal{A}_n(k)$ for each $k \geq 1$. Set $\mathrm{gr}^k(\mathcal{A}'_n) := \mathcal{A}'_n(k)/\mathcal{A}'_n(k+1)$. Then $\mathrm{GL}(n,\mathbf{Z})$ naturally acts on each of $\mathrm{gr}^k(\mathcal{A}'_n)$, and the restriction of $\widetilde{\tau}_k$ to $\mathcal{A}'_n(k)$ induces a $\mathrm{GL}(n,\mathbf{Z})$–equivariant homomorphism

$$\tau'_k \colon \mathrm{gr}^k(\mathcal{A}'_n) \to H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1).$$

We also call τ'_k the Johnson homomorphism of $\mathrm{Aut}\, F_n$. We remark that if we denote by $i_k \colon \mathrm{gr}^k(\mathcal{A}'_n) \to \mathrm{gr}^k(\mathcal{A}_n)$ the homomorphism induced from the inclusion $\mathcal{A}'_n(k) \hookrightarrow \mathcal{A}_n(k)$, then $\tau'_k = \tau_k \circ i_k$ for each $k \geq 1$.

Let $\mathcal{C}_n(k)$ be a quotient module of $H^{\otimes k}$ by the action of cyclic group Cyc_k of order k on the components. Then we have

$$\mathcal{C}_n(k) \cong H^{\otimes k} \big/ \langle a_1 \otimes a_2 \otimes \cdots \otimes a_k - a_2 \otimes a_3 \otimes \cdots \otimes a_k \otimes a_1 \mid a_i \in H \rangle.$$

In our paper [38], we showed that for any $k \geq 2$ and $n \geq k+2$,

$$\mathrm{Coker}(\tau'_{k,\mathbf{Q}}) \cong \mathcal{C}^{\mathbf{Q}}_n(k). \tag{7}$$

Now, by the same argument as τ_k, we can define the Johnson homomorphisms τ^M_k of $\mathrm{Aut}\, F^M_n$ as follows. Let $\mathcal{A}^M_n(k)$ be the kernel of a natural homomorphism $\mathrm{Aut}\, F^M_n \to \mathrm{Aut}\,(F_n/\Gamma^M_n(k+1))$, and $\mathrm{gr}^k(\mathcal{A}^M) := \mathcal{A}^M_n(k)/\mathcal{A}^M_n(k+1)$ its graded quotient. Then a $\mathrm{GL}(n,\mathbf{Z})$–equivariant injective homomorphism

$$\tau^M_k \colon \mathrm{gr}^k(\mathcal{A}^M_n) \hookrightarrow H^* \otimes_{\mathbf{Z}} \mathcal{L}^M_n(k+1)$$

is defined by $\sigma \mapsto (x \mapsto x^{-1}x^{\sigma})$. In our paper [37], we showed that

$$0 \to \mathrm{gr}^k(\mathcal{A}^M_n) \xrightarrow{\tau^M_k} \mathrm{Hom}_{\mathbf{Z}}(H, \mathcal{L}^M_n(k+1)) \xrightarrow{\mathrm{Tr}^M_{[k]}} S^k H \to 0$$

is an exact sequence of $\mathrm{GL}(n,\mathbf{Z})$–modules for each $k \geq 2$ and $n \geq 4$.

2.7 Automorphism group of a free nilpotent group

In this section we recall some properties of the automorphism group $\mathrm{Aut}\, N_{n,k}$ of a free nilpotent group $N_{n,k}$. First, we consider generators of $\mathrm{Aut}\, N_{n,k}$. For any $\sigma \in \mathrm{Aut}\, F_n$, we also denote $\rho_k(\sigma) \in \mathrm{Aut}\, N_{n,k}$ by σ if there is no confusion. Andreadakis [1] showed that ρ_2 is surjective, and that ρ_k is not surjective for $k \geq 3$. Hence $\mathrm{Aut}\, N_{n,2}$ is generated by the Nielsen's generators P, Q, S and U.

For $k \geq 3$, Goryaga [13] showed that $\mathrm{Aut}\, N_{n,k}$ is finitely generated for $n \geq 3 \cdot 2^{k-2} + k$ and $k \geq 2$. In 1984, Andreadakis [2] showed that $\mathrm{Aut}\, N_{n,k}$ is generated by P, Q, S, U and $k-2$ other elements for $n \geq k \geq 2$. In this paper, we use the following result of Bryant and Gupta. Let θ be an automorphism of $N_{n,k}$, defined by

$$\theta\colon x_t \mapsto \begin{cases} [x_1,[x_2,x_1]]x_1 & t = 1, \\ x_t & t \neq 1. \end{cases}$$

Then Bryant and Gupta [6] showed that for $k \geq 3$ and $n \geq k-1$, the group $\mathrm{Aut}\, N_{n,k}$ is generated by P, Q, S, U and θ. In Section 6, we use these generators to compute the first cohomology group of $\mathrm{Aut}\, N_{n,k}$. We remark that no presentation for $\mathrm{Aut}\, N_{n,k}$ is known except for $\mathrm{Aut}\, N_{2,k}$ for $k = 1, 2$ and 3 due to Lin [25].

Next, we consider a relation between $\mathrm{Aut}\, N_{n,k}$ and $\mathrm{Aut}\, N_{n,k+1}$. For $k \geq 1$, we have a natural central group extension

$$0 \to \mathcal{L}_n(k+1) \to N_{n,k+1} \to N_{n,k} \to 1.$$

In this paper, we identify $\mathcal{L}_n(k+1)$ with its image in $N_{n,k+1}$. Namely, $\mathcal{L}_n(k+1)$ is equal to the $(k+1)$–st term of the lower central series of $N_{n,k+1}$. In particular, $\mathcal{L}_n(k+1)$ is a characteristic subgroup of $N_{n,k+1}$. Hence the projection $N_{n,k+1} \to N_{n,k}$ induces a homomorphism $\psi_k\colon \mathrm{Aut}\, N_{n,k+1} \to \mathrm{Aut}\, N_{n,k}$. In order to investigate the kernel of ψ_k, we consider the degree k part of the derivation algebra of the free Lie algebra. For any $f \in \mathrm{Hom}_{\mathbf{Z}}(H, \mathcal{L}_n(k+1))$, define a map $\widetilde{f}\colon N_{n,k+1} \to N_{n,k+1}$ by

$$x^{\widetilde{f}} = ([x])^f x$$

where $[x] \in H$ is the image of x in H under the natural projection $N_{n,k+1} \to H$. Then $\widetilde{f}$ is an automorphism of $N_{n,k+1}$, and a map $\mathrm{Hom}_{\mathbf{Z}}(H, \mathcal{L}_n(k+1)) \to \mathrm{Aut}\, N_{n,k+1}$ defined by $f \mapsto \widetilde{f}$ is an injective homomorphism whose image coincides with the kernel of ψ_k. Namely, for $k \geq 1$, we have a group extension

$$0 \to \mathrm{Hom}_{\mathbf{Z}}(H, \mathcal{L}_n(k+1)) \to \mathrm{Aut}\, N_{n,k+1} \to \mathrm{Aut}\, N_{n,k} \to 1 \tag{8}$$

introduced by Andreadakis [1]. (For details, see also Morita [29, Proposition 2.3].)

Let $T_{n,k}$ be the image of the homomorphism $\rho_k\colon \operatorname{Aut} F_n \to \operatorname{Aut} N_{n,k}$ for each $k \geq 1$. Clearly, the group $T_{n,k}$ is generated by P, Q, S and U, and is called the tame automorphism group of $N_{n,k}$. The exact sequence (8) induces

$$0 \to \mathrm{gr}^k(\mathcal{A}_n) \to T_{n,k+1} \to T_{n,k} \to 1. \tag{9}$$

In Section 6, we use these two group extensions to study twisted cohomology groups of $T_{n,k}$ and $\operatorname{Aut} N_{n,k}$.

3 Representation theory of $\mathrm{GL}(n,\mathbf{Q})$ and the symmetric group $\mathfrak{S}_k$

In this section, we prepare some results in representation theory for $\mathrm{GL}(n,\mathbf{Q})$, namely Cartan and Weyl's highest weight theory, several tensor product theorems and the Schur–Weyl duality for $\mathrm{GL}(n,\mathbf{Q})$ and the symmetric group $\mathfrak{S}_k$. At the end of this section, we briefly recall Kraśkiewicz and Weyman's combinatorial description for the branching rules of irreducible $\mathfrak{S}_k$–modules to a cyclic subgroup Cyc_k of order k.

3.1 Partitions and symmetric functions

A partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ is a sequence of decreasing nonnegative integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$. We denote the set of partitions by $\mathcal{P}$. Set $|\lambda| := \lambda_1 + \lambda_2 + \cdots$. If a partition satisfies $|\lambda| = m$, then we call λ a partition of m and write $\lambda \vdash m$. The conjugate partition of λ is the partition $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$ defined by $\lambda'_i := \sharp\{j \mid \lambda_j \geq i\}$. Put $\ell(\lambda) = \sharp\{i \mid \lambda_i \neq 0\}$, we call it the length of λ.

We can see a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ as the set of points $(i,j) \in \mathbf{Z}^2$ such that $1 \leq j \leq \lambda_i$. Usually it is convenient to replace nodes by boxes. We call such a diagram the Young diagram of shape λ. We also denote the diagram of a partition λ by the same symbol λ. For two partitions λ and μ satisfying $\lambda \supset \mu$, we say that λ is obtained by adding a vertical strip to μ if the difference $\lambda \backslash \mu$ does not contain more than one box in each row. An vertical n–strip means a vertical strip consisting n boxes. If we can obtain a partition λ by adding a box x to a partition μ, we call the box x an addable box to μ. We also call the box x a removable box from λ.

For a partition λ of m, a semistandard (resp. standard) tableaux of shape λ is an array $T = (T_{ij})$ of positive integers $1, 2, \ldots, m$ of shape λ that is weakly (resp. strictly) increasing in every row and strictly increasing in every column.

For a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$, we define a polynomial in n–variables by

$$s_\lambda(x_1, \dots, x_n) := \frac{\det(x_i^{\lambda_j + n - j})_{1 \le i,j \le n}}{\det(x_i^{n-j})_{1 \le i,j \le n}}.$$

This is a homogeneous symmetric polynomial of degree $|\lambda|$. We call it the Schur polynomial associated to λ. For two partitions $\mu = (\mu_1, \dots, \mu_n)$ and $\nu = (\nu_1, \dots, \nu_n)$, we define the Littlewood–Richardson coefficients $\mathrm{LR}^\lambda_{\mu\nu}$ by

$$s_\mu(x_1, \dots, x_n) \cdot s_\nu(x_1, \dots, x_n) = \sum_\lambda \mathrm{LR}^\lambda_{\mu\nu}\, s_\lambda(x_1, \dots, x_n).$$

Then $\mathrm{LR}^\lambda_{\mu\nu}$ becomes a nonnegative integer. The following property is well known.

Proposition 3.1 *Let λ, μ, ν be partitions.*

(1) $\mathrm{LR}^\lambda_{\mu,\nu} = \mathrm{LR}^\lambda_{\nu,\mu}$.

(2) $\mathrm{LR}^\lambda_{(0),\nu} = \mathrm{LR}^\lambda_{\nu,(0)} = \delta_{\lambda,\nu}$.

(3) $\mathrm{LR}^\lambda_{\mu,\nu} = 0$ *unless* $|\lambda| = |\mu| + |\nu|$.

(4) $\mathrm{LR}^\lambda_{\lambda,\nu} = \mathrm{LR}^\lambda_{\nu,\lambda} = \delta_{\nu,(0)}$.

3.2 Highest weight theory for GL(n, Q)

Let $T_n := \{\mathrm{diag}(t_1, \dots, t_n) \mid t_j \neq 0,\ 1 \le j \le n\}$ be the maximal torus of $\mathrm{GL}(n, \mathbf{Q})$. We define one-dimensional representations ε_i of T_n by $\varepsilon_i(\mathrm{diag}(t_1, \dots, t_n)) = t_i$. Then

$$P_{\mathrm{GL}(n,\mathbf{Q})} := \{\lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n \mid \lambda_i \in \mathbf{Z},\ 1 \le i \le n\} \cong \mathbf{Z}^n,$$
$$P^+_{\mathrm{GL}(n,\mathbf{Q})} := \{\lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n \in P_{\mathrm{GL}(n,\mathbf{Q})} \mid \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n\}$$

gives the weight lattice and the set of dominant integral weights of $\mathrm{GL}(n, \mathbf{Q})$ respectively. In the following, for simplicity, we often write $G = \mathrm{GL}(n, \mathbf{Q})$, $T = T_n$, $P = P_{\mathrm{GL}(n,\mathbf{Q})}$ and $P^+ = P^+_{\mathrm{GL}(n,\mathbf{Q})}$. Furthermore, we write $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbf{Z}^n$ for $\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n \in P$ or P^+ if there is no confusion.

For a rational representation V of G, there exists an irreducible decomposition $V = \bigoplus_{\lambda \in P} V_\lambda$ as a T–module where $V_\lambda := \{v \in V \mid tv = t_1^{\lambda_1} t_2^{\lambda_2} \cdots t_n^{\lambda_n} v \text{ for any } t \in T\}$. We call this decomposition a weight decomposition of V with respect to T. If $V_\lambda \neq \{0\}$, then we call λ a weight of V. For a weight λ, a nonzero vector $v \in V_\lambda$ is called a weight vector of weight λ.

Let U be the subgroup of G consisting of all upper unitriangular matrices in G. For a rational representation V of G, we set $V^U := \{v \in V \mid uv = v \text{ for all } u \in U\}$. We

call a nonzero vector $v \in V^U$ a maximal vector of V. This subspace V^U is T–stable. Thus, as a T–module, V^U has an irreducible decomposition $V^U = \bigoplus_{\lambda \in P} V^U_\lambda$ where $V^U_\lambda := V^U \cap V_\lambda$.

Theorem 3.2 (Cartan and Weyl's highest weight theory) (1) *Any rational representation of V is completely reducible.*

(2) *Suppose V is an irreducible rational representation of G. Then V^U is one-dimensional, and the weight λ of $V^U = V^U_\lambda$ belongs to P^+. We call this λ the highest weight of V, and any nonzero vector $v \in V^U_\lambda$ is called a highest weight vector of V.*

(3) *For any $\lambda \in P^+$, there exists a unique (up to isomorphism) irreducible rational representation L^λ of G with highest weight λ. Moreover, for two $\lambda, \mu \in P^+$, $L^\lambda \cong L^\mu$ if and only if $\lambda = \mu$.*

(4) *The set of isomorphism classes of irreducible rational representations of G is parameterized by the set P^+ of dominant integral weights.*

(5) *Let V be a rational representation of G and χ_V a character of V as a T–module. Then for two rational representation V and W, they are isomorphic as G–modules if and only if $\chi_V = \chi_W$.*

Remark 3.3 We can parameterize the set of isomorphism classes of irreducible rational representations of G by P^+. On the other hand, we define the determinant representation by $\det\colon \mathrm{GL}(n,\mathbf{Q}) \ni X \to \det X \in \mathbf{Q}^\times$. Its highest weight is $(1,1,\dots,1) \in P^+$. For an integer e, we denote by $\det^e$ the irreducible $\mathrm{GL}(n,\mathbf{Q})$–representation of highest weight $(e,e,\dots,e) \in P^+$. This is the one dimensional representation defined by $\mathrm{GL}(n,\mathbf{Q}) \ni X \to (\det X)^e \in \mathbf{Q}^\times$. If $\lambda \in P^+$ satisfies $\lambda_n < 0$, then $L^\lambda \cong \det^{-\lambda_n} \otimes L^{(\lambda_1-\lambda_n,\lambda_2-\lambda_n,\dots,0)}$. Therefore we can parameterize the set of isomorphism classes of irreducible rational representations of G by the set $\{(\lambda, e)\}$ where λ is a partition such that $\ell(\lambda) \le n$ and $e \in \mathbf{Z}_{<0}$. Moreover the set of isomorphism classes of polynomial irreducible representations is parameterized by the set of partitions λ such that $\ell(\lambda) \le n$.

Note that the dual representation of $L^{(\lambda_1,\lambda_2,\dots,\lambda_n)}$ is isomorphic to $L^{(-\lambda_n,\dots,-\lambda_2,-\lambda_1)}$. Especially, the natural representation $H_\mathbf{Q} = \mathbf{Q}^n$ of G and its dual representation $H^*_\mathbf{Q}$ are irreducible with highest weight $(1,0,\dots,0)$ and $(0,\dots,0,-1)$ respectively. We also have $H^*_\mathbf{Q} \cong \det^{-1} \otimes L^{(1,\dots,1,0)}$.

There is another parameterization of irreducible rational representations of G. For any $\lambda \in P^+$, we define two partitions λ_+ and λ_- by

$$\lambda_+ := (\max(\lambda_1,0),\dots,\max(\lambda_n,0)),$$
$$\lambda_- := (-\min(\lambda_n,0),\dots,-\min(\lambda_1,0)).$$

Then there is a bijection

$$P^+ \to \{(\mu;\nu) \in \mathcal{P}\times\mathcal{P} \mid \ell(\mu)+\ell(\nu) \le n\}.$$

defined by $\lambda \mapsto (\lambda_+;\lambda_-)$. Using this bijection, we can parameterize the isomorphism classes of irreducible rational representations of G by the set $\{L^{\{\lambda_+;\lambda_-\}}\}$. Note that under this notation, we have $(L^{\{\lambda;\mu\}})^* \cong L^{\{\mu;\lambda\}}$.

Theorem 3.4 (Weyl's character and dimension formula for $\mathrm{GL}(n,\mathbf{Q})$)

(1) *For a partition λ, let χ_λ be a character of an irreducible polynomial representation L^λ. Then we have $\chi_\lambda(t) = s_\lambda(t_1,\dots,t_n)$ for a diagonal matrix $\mathrm{diag}(t_1,\dots,t_n) \in T$.*

(2) *For a partition λ, the dimension of the irreducible polynomial representation L^λ coincides with the number of semistandard tableaux on λ.*

3.3 Decompositions of tensor products

In this subsection, we recall some decomposition formulae of tensor products.

Theorem 3.5 (Pieri's formula) *Let μ be a partition such that $\ell(\mu) \le n$. Then*

$$L^{(1^k)} \otimes L^\mu \cong \bigoplus_\lambda L^\lambda,$$

where λ runs over the set of partitions obtained by adding a vertical k–strip to μ such that $\ell(\lambda) \le n$.

Theorem 3.6 (Koike [20, Theorem 2.4]) *For four partitions ξ,η,σ and τ such that $\ell(\xi)+\ell(\eta)+\ell(\sigma)+\ell(\tau) \le n$, we have*

$$L^{\{\xi;\eta\}} \otimes L^{\{\sigma;\tau\}} \cong \bigoplus_{\substack{\lambda,\mu\in\mathcal{P},\\ \ell(\lambda)+\ell(\mu)\le n}} \mathrm{LR}^{\{\lambda;\mu\}}_{\{\xi;\eta\},\{\sigma;\tau\}}\, L^{\{\lambda;\mu\}}.$$

Here the coefficient $\mathrm{LR}^{\{\lambda;\mu\}}_{\{\xi;\eta\},\{\sigma;\tau\}}$ is defined to be

$$\sum_{\alpha,\beta,\gamma,\delta\in\mathcal{P}} \Bigg(\sum_{\kappa\in\mathcal{P}} \mathrm{LR}^\xi_{\kappa\alpha}\,\mathrm{LR}^\tau_{\kappa\beta}\Bigg)\Bigg(\sum_{\pi\in\mathcal{P}} \mathrm{LR}^\eta_{\pi\gamma}\,\mathrm{LR}^\sigma_{\pi\delta}\Bigg)\,\mathrm{LR}^\lambda_{\alpha\delta}\,\mathrm{LR}^\mu_{\beta\gamma}\,.$$

Corollary 3.7 *Assume that $1+\ell(\sigma) \le n$. Then we have*

$$L^{\{0;(1)\}} \otimes L^{\{\sigma;0\}} \cong L^{\{\sigma;(1)\}} \oplus \bigoplus_{\lambda,\ell(\lambda)\le n} \mathrm{LR}^\sigma_{(1),\lambda}\, L^{\{\lambda;0\}}.$$

Proof Under the notation of Theorem 3.5, we suppose $\xi=\tau=(0)$. By Proposition 3.1, we have $\mathrm{LR}^{\zeta}_{\kappa\alpha}\,\mathrm{LR}^{\tau}_{\kappa\beta}=0$ unless $\kappa=\alpha=\beta=(0)$. Hence we obtain

$$\mathrm{LR}^{\{\lambda;\mu\}}_{\{(0);(1)\},\{\sigma;(0)\}}=\sum_{\gamma,\delta}\Bigg(\sum_{\pi}\mathrm{LR}^{(1)}_{\pi\gamma}\,\mathrm{LR}^{\sigma}_{\pi,\delta}\Bigg)\mathrm{LR}^{\lambda}_{(0)\delta}\,\mathrm{LR}^{\mu}_{(0)\gamma}\,.$$

Since $\mathrm{LR}^{\lambda}_{(0)\delta}\,\mathrm{LR}^{\mu}_{(0)\gamma}=0$ unless $\delta=\lambda$ and $\gamma=\mu$ by Proposition 3.1, we have

$$\mathrm{LR}^{\{\lambda;\mu\}}_{\{(0);(1)\},\{\sigma;(0)\}}=\sum_{\pi}\mathrm{LR}^{(1)}_{\pi,\mu}\,\mathrm{LR}^{\sigma}_{\pi\lambda}\,.$$

Since $\mathrm{LR}^{(1)}_{\pi,\mu}=0$ unless $\pi=(1)$ or (0), we have

$$\mathrm{LR}^{\{\lambda;\mu\}}_{\{(0);(1)\},\{\sigma;(0)\}}=\mathrm{LR}^{(1)}_{(1),\mu}\,\mathrm{LR}^{\sigma}_{(1)\lambda}+\mathrm{LR}^{(1)}_{(0),\mu}\,\mathrm{LR}^{\sigma}_{(0)\lambda}\,.$$

By Proposition 3.1, we have $\mathrm{LR}^{(1)}_{(1),\mu}=\delta_{\mu,(0)}$, $\mathrm{LR}^{(1)}_{(0),\mu}=\delta_{\mu,(1)}$ and $\mathrm{LR}^{\sigma}_{(0)\lambda}=\delta_{\lambda,\sigma}$. Therefore we obtain

$$\mathrm{LR}^{\{\lambda;\mu\}}_{\{(0);(1)\},\{\sigma;(0)\}}=\delta_{\mu,(1)}\delta_{\sigma,\lambda}+\mathrm{LR}^{\sigma}_{(1)\lambda}\,\delta_{\mu,(0)}$$

and our claim. □

Corollary 3.8 (Multiplicities of trivial representation) *If $\ell(\xi)+\ell(\eta)+\ell(\sigma)+\ell(\tau)\le n$, then $[L^{(0)}:L^{\{\xi;\eta\}}\otimes L^{\{\sigma;\tau\}}]=\delta_{\xi,\tau}\delta_{\eta,\sigma}$ where $\delta_{a,b}$ is Kronecker's delta.*

Proof Under the notation of Theorem 3.5, we suppose $\lambda=\mu=(0)$. By Proposition 3.1, we have

$$\mathrm{LR}^{\{(0);(0)\}}_{\{\xi;\eta\},\{\sigma;\tau\}}=0$$

unless $\alpha=\beta=\gamma=\delta=(0)$. Thus we have

$$\mathrm{LR}^{\{(0);(0)\}}_{\{\xi;\eta\},\{\sigma;\tau\}}=\Bigg(\sum_{\kappa}\mathrm{LR}^{\xi}_{\kappa(0)}\,\mathrm{LR}^{\tau}_{\kappa(0)}\Bigg)\Bigg(\sum_{\pi}\mathrm{LR}^{\xi}_{\pi(0)}\,\mathrm{LR}^{\tau}_{\pi(0)}\Bigg).$$

By Proposition 3.1, the righthand side is 0 unless $\kappa=\xi=\tau$ and $\pi=\eta=\sigma$. Thus we obtain

$$\mathrm{LR}^{\{(0);(0)\}}_{\{\xi;\eta\},\{\sigma;\tau\}}=\delta_{\xi,\tau}\delta_{\eta,\sigma}$$

and our claim. □

3.4 Schur–Weyl duality

For the natural representation $H_{\mathbf{Q}} \cong L^{(1,0,\dots,0)}$ of $\mathrm{GL}(n,\mathbf{Q})$, we consider the m–th tensor product representation $\mathrm{GL}(n,\mathbf{Q}) \to \mathrm{GL}(H^{\otimes m})$. The symmetric group $\mathfrak{S}_m$ acts on $(H_{\mathbf{Q}})^{\otimes m}$ by $\sigma\cdot(v_1\otimes\cdots\otimes v_m) = v_{\sigma(1)}\otimes\cdots v_{\sigma(m)}$ for $\sigma\in\mathfrak{S}_m$. Since these two actions are commutative, we can decompose $H^{\otimes m}$ as a $\mathrm{GL}(n,\mathbf{Q})\times\mathfrak{S}_m$–module. Let us recall this irreducible decomposition, called the Schur–Weyl duality for $\mathrm{GL}(n,\mathbf{Q})$ and $\mathfrak{S}_m$.

Theorem 3.9 (Schur–Weyl duality for $\mathrm{GL}(n,\mathbf{Q})$ and $\mathfrak{S}_m$)

(1) *Let λ be a partition of m such that $\ell(\lambda)\le n$. There exists a nonzero maximal vector v_λ with weight λ satisfying the following three conditions:*
   (i) *The $\mathfrak{S}_m$–invariant subspace $S^\lambda := \sum_{\sigma\in\mathfrak{S}_m}\mathbf{Q}\sigma v_\lambda$ gives an irreducible representation of $\mathfrak{S}_m$.*
   (ii) *The subspace $(H_{\mathbf{Q}}^{\otimes m})^U_\lambda$ of weight λ coincides with the subspace S^λ.*
   (iii) *The $\mathrm{GL}(n,\mathbf{Q})$–module generated by v_λ is isomorphic to the irreducible representation L^λ of $\mathrm{GL}(n,\mathbf{Q})$ associated to λ.*

(2) *We have the irreducible decomposition*

$$H_{\mathbf{Q}}^{\otimes m} \cong \bigoplus_{\lambda=(\lambda_1\ge\cdots\ge\lambda_n\ge 0)\vdash m} L^\lambda\boxtimes S^\lambda$$

as $\mathrm{GL}(n,\mathbf{Q})\times\mathfrak{S}_m$–modules. Here $L^\lambda\boxtimes S^\lambda$ is the external tensor product representation of the $\mathrm{GL}(n,\mathbf{Q})$–module L^λ and the $\mathfrak{S}_m$–module S^λ.

(3) *Suppose $n\ge m$. Then $\{S^\lambda \mid \lambda\vdash m\}$ gives a set of complete representatives of irreducible representations of $\mathfrak{S}_m$.*

3.5 Combinatorial description of branching laws from $\mathfrak{S}_m$ to Cyc_m

Let Cyc_m be a cyclic group of order m. Take a generator σ_m of Cyc_m and a primitive m–th root of unity $\zeta_m\in\mathbf{C}$. In this section, we consider representations of the cyclic group Cyc_m over an intermediate field $\mathbf{Q}(\zeta_m)\subset\mathbf{K}\subset\mathbf{C}$.

Define one-dimensional representations (or characters) $\chi_m^j\colon \mathrm{Cyc}_m\to\mathbf{K}^\times$ by $\chi_m^j(\sigma_m) := \zeta_m^j$ for $0\le j\le m-1$. In particular, we denote the trivial representation χ_m^0 by $\mathbf{triv}_m$. The set of isomorphism classes of irreducible representations of Cyc_m is given by $\{\chi_m^j,\ 0\le j\le m-1\}$.

Consider Cyc_m as a subgroup of $\mathfrak{S}_m$ by the embedding $\sigma_m^i\mapsto(1\,2\,\cdots\,m)^i$ for $0\le i\le m-1$. Let us recall Kraśkiewicz and Weyman's combinatorial description for

the branching rules of irreducible $\mathfrak{S}_m$–modules S^λ to the cyclic subgroup Cyc_m. To do this, first we define a major index of a standard tableau. For a standard tableau T, we define the descent set of T to be the set of entries i in T such that $i+1$ is located in a lower row than that which i is located. We denote by $D(T)$ the descent set of T. The major index of T is defined by

$$\mathrm{maj}(T) := \sum_{i \in D(T)} i.$$

If $D(T) = \varnothing$, we set $\mathrm{maj}(T) = 0$.

Theorem 3.10 (Kraśkiewicz–Weyman [23]; Reutenauer [34, Theorem 8.8, 8.9]; Garsia [12, Theorem 8.4]) *The multiplicity of χ_m^j in $\mathrm{Res}^{\mathfrak{S}_m}_{\mathrm{Cyc}_m} S^\lambda$ is equal to the number of standard tableau with shape λ satisfying $\mathrm{maj}(T) \equiv j$ modulo m.*

Example 3.11 For $m \geq 2$, Table 1 shows on the multiplicities of $\mathbf{triv}_m = \chi_j^0$ and χ_j^1.

4 Irreducible decomposition of the derivation algebras over Q

In this section, we give the irreducible decompositions of each of the degree k parts of the derivation algebras $\mathcal{C}_n^{\mathbf{Q}}$, $\mathrm{Der}^+(\mathcal{L}_{n,\mathbf{Q}})$, $\mathrm{Der}^+(\mathcal{L}^M_{n,\mathbf{Q}})$ and $\mathrm{Der}^+(\mathcal{L}^N_{n,\mathbf{Q}})$. For describing the multiplicity of L^λ, we will use a representation of Cyc_m over an intermediate field $\mathbf{Q}(\zeta_m) \subset \mathbf{K} \subset \mathbf{C}$. But we should note that our irreducible decompositions of $\mathcal{C}_n^{\mathbf{Q}}(k)$, $\mathcal{L}_{n,\mathbf{Q}}(k)$, $\mathrm{Der}^+(\mathcal{L}_{n,\mathbf{Q}})(k)$, $\mathrm{Der}^+(\mathcal{L}^M_{n,\mathbf{Q}})(k)$ and $\mathrm{Der}^+(\mathcal{L}^N_{n,\mathbf{Q}})(k)$ hold over $\mathbf{Q}$.

4.1 Decomposition of $\mathcal{C}_n^{\mathbf{Q}}(k)$

For the natural representation $H_{\mathbf{Q}} \cong L^{(1,0,\dots,0)}$ of $\mathrm{GL}(n,\mathbf{Q})$, the module $\mathcal{C}_n^{\mathbf{Q}}(k)$ can be considered as a quotient module of $H_{\mathbf{Q}}^{\otimes k}$ by the action of the cyclic subgroup Cyc_k of $\mathfrak{S}_k$ on the components:

$$\mathcal{C}_n^{\mathbf{Q}}(k) \cong H_{\mathbf{Q}}^{\otimes k} \big/ \mathbf{Q}\text{-span}\langle v - \sigma_k^i v \mid 1 \leq i \leq k-1,\ \forall v \in H_{\mathbf{Q}}^{\otimes k}\rangle,$$

where σ_k is a generator of Cyc_k. In this subsection, we give the irreducible decomposition of $\mathcal{C}_n^{\mathbf{Q}}(k)$.

Since the actions of $\mathrm{GL}(n,\mathbf{Q})$ and $\mathfrak{S}_k$ commute, the space $\mathcal{C}_n^{\mathbf{Q}}(k)$ is a $\mathrm{GL}(n,\mathbf{Q})$–module. Let $\mathrm{pr}\colon H_{\mathbf{Q}}^{\otimes k} \twoheadrightarrow \mathcal{C}_n^{\mathbf{Q}}(k)$ be the natural projection. The map pr is a $\mathrm{GL}(n,\mathbf{Q})$–equivariant homomorphism.

λ	T	major index	mult. of $\mathbf{triv}_m$	mult. of χ^1_m
(m)	$\begin{array}{\|c\|c\|c\|c\|}\hline 1 & 2 & \cdots & m \\ \hline \end{array}$	0	1	0
$(m-1,1)$	$\begin{array}{\|c\|c\|c\|c\|}\hline 1 & 2 & \cdots & m \\ \hline p \\ \cline{1-1} \end{array}$ $(2 \le p \le m)$	$p-1$	0	1
(1^m)	$\begin{array}{\|c\|}\hline 1 \\ \hline 2 \\ \hline \vdots \\ \hline m \\ \hline \end{array}$	$\dfrac{m(m-1)}{2}$ $\equiv \begin{cases} 0 & \text{if } m\text{:odd} \\ -m/2 & \text{if } m\text{:even} \end{cases}$	$\begin{cases} 1 & m:\text{odd} \\ 0 & m:\text{even} \end{cases}$	$\begin{cases} 1 & m=2 \\ 0 & m\neq 2 \end{cases}$
$(2,1^{m-2})$	$\begin{array}{\|c\|c\|}\hline 1 & p \\ \hline 2 \\ \cline{1-1} \vdots \\ \cline{1-1} m \\ \cline{1-1} \end{array}$ $(2 \le p \le m)$	$\dfrac{m(m-1)}{2} - (p-1)$ $\equiv \begin{cases} 1-p & \text{if } m\text{:odd} \\ 1-p-m/2 & \text{if } m\text{:even} \end{cases}$	$\begin{cases} 1 & m:\text{even} \\ 0 & m:\text{odd} \end{cases}$	$\begin{cases} 1 & m\neq 2 \\ 0 & m=2 \end{cases}$

Table 1: The multiplicities of $\mathbf{triv}_m = \chi^0_j$ and χ^1_j for $m \ge 2$

Proposition 4.1 (Irreducible decomposition of $\mathcal{C}^{\mathbf{Q}}_n(k)$) *For $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in P^+$, the multiplicity of L^λ in $\mathcal{C}^{\mathbf{Q}}_n(k)$ as a $\mathrm{GL}(n,\mathbf{Q})$–module is given by*

$$[L^\lambda : \mathcal{C}^{\mathbf{Q}}_n(k)] = \begin{cases} [\mathbf{triv}_k : \mathrm{Res}^{\mathfrak{S}_k}_{\mathrm{Cyc}_k} S^\lambda] = \dfrac{1}{k} \displaystyle\sum_{\sigma \in \mathrm{Cyc}_k} \chi^\lambda(\sigma) & \text{if } \lambda \vdash k, \\ 0 & \text{otherwise.} \end{cases}$$

Here, $[\mathbf{triv}_k : \mathrm{Res}^{\mathfrak{S}_k}_{\mathrm{Cyc}_k} S^\lambda]$ means the multiplicity of the trivial representation $\mathbf{triv}_k$ of Cyc_k in the restriction of the irreducible $\mathfrak{S}_k$–module S^λ.

Proof By the Schur–Weyl duality, complete reducibility and Schur's lemma, the subspace $\mathcal{C}^{\mathbf{Q}}_n(k)^U_\lambda$ coincides with $\mathrm{pr}(S^\lambda)$. Therefore $[L^\lambda : \mathcal{C}^{\mathbf{Q}}_n(k)] = \dim_{\mathbf{Q}} \mathrm{pr}(S^\lambda)$. On the other hand, the restriction $\mathrm{Res}^{\mathfrak{S}_k}_{\mathrm{Cyc}_k} S^\lambda$ has a direct isotypic decomposition

$S^\lambda \otimes_{\mathbf{Q}} \mathbf{K} = \bigoplus_{j=0}^{k-1} V_j$ over $\mathbf{K}$ where V_j is a certain direct sum of the irreducible representation χ_j of Cyc_k, namely $V_j = \{v \in S^\lambda \mid \sigma_k v = \zeta_k^j v\}$ for $0 \le j \le k-1$.

Then, $\mathrm{Ker}(\mathrm{pr}) \otimes_{\mathbf{Q}} \mathbf{K} = \bigoplus_{j\neq 0} V_j$. In fact, if $0 \neq v \in V_j$ for $j \neq 0$, we have $v - \sigma_k v = (1-\zeta_k)v \in \mathrm{Ker}(\mathrm{pr})$. Since $1-\zeta_k^j \neq 0$ for $1 \le j \le k-1$, $v \in \mathrm{Ker}(\mathrm{pr}) \otimes_{\mathbf{Q}} \mathbf{K}$. Thus we obtain $V_j \subset \mathrm{Ker}(\mathrm{pr}) \otimes_{\mathbf{Q}} \mathbf{K}$ for $j \neq 0$. Conversely, for any $v \in H_{\mathbf{K}}^{\otimes k}$, there exists $v_0 \in V_0$ and $v' \in \bigoplus_{j\neq 0} V_j$ such that $v = v_0 + v'$. Since $\bigoplus_{j\neq 0} V_j$ is Cyc_k–stable, $v - \sigma v = (v_0 + v') - (v_0 + \sigma v') = v' - \sigma v' \in \bigoplus_{j\neq 0} V_j$ for any $\sigma \in \mathrm{Cyc}_k$. Hence $\mathrm{Ker}(\mathrm{pr}) \otimes_{\mathbf{Q}} \mathbf{K} \subset \bigoplus_{j\neq 0} V_j$.

Therefore we obtain $\mathrm{pr}(S^\lambda) \cong V_0$ as a vector space, and $[L^\lambda : \mathcal{C}_n^{\mathbf{Q}}(k)] = \dim_{\mathbf{Q}} \mathrm{pr}(S^\lambda) = \dim_{\mathbf{K}} V_0 = [\mathbf{triv}_k : \mathrm{Res}^{\mathfrak{S}_k}_{\mathrm{Cyc}_k} S^\lambda]$. The second equality of the claim follows from the ordinary character theory of finite groups. □

For the symmetric product $V = S^k H_{\mathbf{Q}} = L^{(k)}$ or the exterior product $V = \Lambda^k H_{\mathbf{Q}} = L^{(1^k)}$, we calculate the multiplicity $[V : \mathcal{C}_n^{\mathbf{Q}}(k)]$ as follows:

Corollary 4.2 (Explicit results as a $\mathrm{GL}(n,\mathbf{Q})$–module)

(1) $[L^{(k)} : \mathcal{C}_n^{\mathbf{Q}}(k)] = 1$.

(2) $[L^{(1^k)} : \mathcal{C}_n^{\mathbf{Q}}(k)] = \begin{cases} 1 & k \text{ is odd and } k \le n, \\ 0 & \text{otherwise.} \end{cases}$

(3) $[L^{(2,1^{k-2})} : \mathcal{C}_n^{\mathbf{Q}}(k)] = \begin{cases} 1 & k \text{ is even and } k \le n, \\ 0 & \text{otherwise.} \end{cases}$

Proof The claims follows from direct computation for irreducible decompositions of $\mathrm{Res}^{\mathfrak{S}_k}_{\mathrm{Cyc}_k} S^{(k)}$, $\mathrm{Res}^{\mathfrak{S}_k}_{\mathrm{Cyc}_k} S^{(1^k)}$ and $\mathrm{Res}^{\mathfrak{S}_k}_{\mathrm{Cyc}_k} S^{(2,1^{k-2})}$ respectively. But the claim also follows from Kraśkiewicz and Weyman's combinatorial description. (See Theorem 3.10 and Example 3.11.) □

4.2 Decomposition of $\mathrm{Der}^+(\mathcal{L}_{n,\mathbf{Q}})$

In this subsection, we consider the irreducible decomposition of the derivation algebra of the free Lie algebra $\mathcal{L}_{n,\mathbf{Q}}$. The degree m part $\mathcal{L}_{n,\mathbf{Q}}(m)$ of $\mathcal{L}_{n,\mathbf{Q}}$ is a submodule of $H_{\mathbf{Q}}^{\otimes m}$ as a representation of $\mathrm{GL}(n,\mathbf{Q})$. The irreducible decomposition of $\mathcal{L}_{n,\mathbf{Q}}(m)$ is obtained by the following theorem.

Theorem 4.3 (Kljačko [19]) (1) *Let us consider a subspace L_m of $H_{\mathbf{K}}^{\otimes m}$ generated by all elements $v \in H_{\mathbf{K}}^{\otimes m}$ such that $\sigma_m v = \zeta_m v$. Then L_m becomes a $\mathrm{GL}(n,\mathbf{K})$–submodule of $H_{\mathbf{K}}^{\otimes m}$. Moreover as $\mathrm{GL}(n,\mathbf{K})$–module, we have $L_m \cong \mathcal{L}_{n,\mathbf{Q}}(m) \otimes_{\mathbf{Q}} \mathbf{K}$.*

(2) *For $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_n)\in P^+$, the multiplicity of L^λ in $\mathcal{L}_{n,\mathbf{Q}}(m)$ as a $\mathrm{GL}(n,\mathbf{Q})$–module is given by*

$$[L^\lambda:\mathcal{L}_{n,\mathbf{Q}}(m)]=\begin{cases}[\chi_1:\mathrm{Res}^{\mathfrak{S}_m}_{\mathrm{Cyc}_m}S^\lambda] & \text{if } \lambda_n\geq 0 \text{ (λ is a partition)},\\ 0 & \text{otherwise.}\end{cases}$$

Remark 4.4 We can obtain a more explicit description of the right hand side by using the Möbius function as follows:

$$[\chi_1:\mathrm{Res}^{\mathfrak{S}_m}_{\mathrm{Cyc}_m}S^\lambda]=\frac{1}{m}\sum_{g\in\mathrm{Cyc}_m}\overline{\chi_1(g)}\chi^\lambda(g)=\frac{1}{m}\sum_{d|n}\mathrm{Möb}(d)\chi^\lambda(\sigma_m^{n/d}).$$

Next, we consider the irreducible decomposition of $H^*_{\mathbf{Q}}\otimes\mathcal{L}_{n,\mathbf{Q}}(m)$ as a $\mathrm{GL}(n,\mathbf{Q})$–module.

Proposition 4.5 (Irreducible decomposition of $H^*_{\mathbf{Q}}\otimes\mathcal{L}_{n,\mathbf{Q}}(m)$) *For any partition $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_n)$, the multiplicity of the irreducible polynomial representation L^λ in $H^*_{\mathbf{Q}}\otimes\mathcal{L}_{n,\mathbf{Q}}(m)$ is given by*

$$[L^\lambda:H^*_{\mathbf{Q}}\otimes\mathcal{L}_{n,\mathbf{Q}}(m)]=\sum_\mu[L^\mu:\mathcal{L}_{n,\mathbf{Q}}(m)],$$

where μ runs over all partitions obtained by removing a vertical $(n-1)$–strip from $(\lambda_1+1,\dots,\lambda_n+1)$.

Proof Recall that $H^*_{\mathbf{Q}}\cong\det^{-1}\otimes L^{(1^{n-1},0)}$. Since the highest weight of the irreducible rational representation $\det^{-1}$ is $(-1,\dots,-1)$, we have $\det^{-1}\otimes L^\mu\cong L^{\mu-(1^n)}$. Thus we obtain

$$[L^\lambda:H^*_{\mathbf{Q}}\otimes\mathcal{L}_{n,\mathbf{Q}}(m)]=[L^{(\lambda_1+1,\dots,\lambda_n+1)}:L^{(1,1,\dots,1,0)}\otimes\mathcal{L}_{n,\mathbf{Q}}(m)].$$

By the Pieri's formula given by Theorem 3.5, we have $L^{(1^{n-1},0)}\otimes L^\mu=\bigoplus_\nu L^\nu$, where ν runs over all partitions obtained by adding a vertical $(n-1)$–strip to μ such that $\ell(\nu)\leq n$. Thus we conclude the claim. □

Here we consider the multiplicities of the symmetric product and the exterior product of $H_{\mathbf{Q}}$ in $H^*_{\mathbf{Q}}\otimes\mathcal{L}_{n,\mathbf{Q}}(m)$.

Corollary 4.6 (Explicit results as $\mathrm{GL}(n,\mathbf{Q})$–modules)

(1) $[L^{(m-1,0,\dots,0)}:H^*_{\mathbf{Q}}\otimes\mathcal{L}_{n,\mathbf{Q}}(m)]=1$.

(2) $[L^{(1^{m-1},0)}:H^*_{\mathbf{Q}}\otimes\mathcal{L}_{n,\mathbf{Q}}(m)]=1$ *for* $1\leq 1+m\leq n$.

Proof (1) By Proposition 4.5 and Theorem 4.3, we have

$$\begin{aligned}[L^{(m-1,0,\dots,0)} : H_{\mathbf{Q}}^* \otimes \mathcal{L}_{n,\mathbf{Q}}(m)] &= [L^{(m,1,\dots,1)} : L^{(1^{n-1},0)} \otimes \mathcal{L}_{n,\mathbf{Q}}(m)] \\ &= [L^{(m,0,\dots,0)} : \mathcal{L}_{n,\mathbf{Q}}(m)] + [L^{(m-1,1,0,\dots,0)} : \mathcal{L}_{n,\mathbf{Q}}(m)] \\ &= [\chi_m^1 : \mathrm{Res}^{\mathfrak{S}_m}_{\mathrm{Cyc}_m} S^{(m)}] + [\chi_m^1 : \mathrm{Res}^{\mathfrak{S}_m}_{\mathrm{Cyc}_m} S^{(m-1,1)}].\end{aligned}$$

The claim follows from Example 3.11.

(2) By Proposition 4.5 and Theorem 4.3, we have

$$\begin{aligned}[L^{(1^{m-1},0)} : H_{\mathbf{Q}}^* \otimes \mathcal{L}_{n,\mathbf{Q}}(m)] &= [L^{(2^{m-1},1^{n-m+1})} : L^{(1^{n-1},0)} \otimes \mathcal{L}_{n,\mathbf{Q}}(m)] \\ &= [L^{(1^m,0)} : \mathcal{L}_{n,\mathbf{Q}}(m)] + [L^{(2,1^{m-2},0)} : \mathcal{L}_{n,\mathbf{Q}}(m)] \\ &= [\chi_m^1 : \mathrm{Res}^{\mathfrak{S}_m}_{\mathrm{Cyc}_m} S^{(1^m)}] + [\chi_m^1 : \mathrm{Res}^{\mathfrak{S}_m}_{\mathrm{Cyc}_m} S^{(2,1^{m-2})}].\end{aligned}$$

The claim also follows from Example 3.11. □

From the corollary above, we verify that the multiplicities of $S^k H$ and $\Lambda^k H$ in $\mathrm{Der}^+(\mathcal{L}_{n,\mathbf{Q}})(k) = H_{\mathbf{Q}}^* \otimes \mathcal{L}_{n,\mathbf{Q}}(k+1)$ are both one. We can write down each of a maximal vector of $S^k H_{\mathbf{Q}}$ and $\Lambda^k H_{\mathbf{Q}}$ in $\mathrm{Der}^+(\mathcal{L}_{n,\mathbf{Q}})(k)$. A maximal vector of $S^k H_{\mathbf{Q}}$ is given by

$$v_{(k)} := \sum_{i=2}^{n} x_i^* \otimes [x_i, x_1, \dots, x_1] \in \mathrm{Der}^+(\mathcal{L}_{n,\mathbf{Q}})(k)$$

and that of $\Lambda^k H_{\mathbf{Q}}$ is given by

$$v_{(1^k)} := \sum_{\sigma \in \mathfrak{S}} \sum_{l \neq \sigma(1)} \mathrm{sgn}(\sigma) x_l^* \otimes [x_l, x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(k)}] \in \mathrm{Der}^+(\mathcal{L}_{n,\mathbf{Q}})(k).$$

We leave the proof to the reader as exercises.

More generally, we show that the multiplicity of L^λ coincides with $[L^\lambda : H_{\mathbf{Q}}^{\otimes k}] = \dim S^\lambda$. In other words, the following theorem and corollary describes the kernel of the contraction map $\Phi_{\mathbf{Q}}^k$.

Proposition 4.7 *As a* $\mathrm{GL}(n,\mathbf{Q})$*–module, we have a direct decomposition*

$$H_{\mathbf{Q}}^* \otimes \mathcal{L}_{n,\mathbf{Q}}(k+1) \cong H_{\mathbf{Q}}^{\otimes k} \oplus W$$

for some subrepresentation W *such that all irreducible components of* W *are non-polynomial representations. In particular, the kernel of the contraction map* $\Phi_{\mathbf{Q}}^k$ *is isomorphic to* W*.*

Proof We shall prove that

$$[L^\lambda : H_{\mathbf{Q}}^* \otimes \mathcal{L}_{n,\mathbf{Q}}(k+1)] = \dim S^\lambda \ (= [L^\lambda : H_{\mathbf{Q}}^{\otimes k}])$$

for any partition λ of k. By Proposition 4.5, we have

$$[L^\lambda : H_{\mathbf{Q}}^* \otimes \mathcal{L}_{n,\mathbf{Q}}(k+1)] = \sum_\mu [L^\mu : \mathcal{L}_{n,\mathbf{Q}}(k+1)],$$

where μ runs over all elements in the set of partitions obtained by removing a vertical $(n-1)$–strip from $(\lambda_1+1,\dots,\lambda_n+1)$. But for a partition λ, this set coincides with the set of partitions obtained by adding one box to λ. Thus, by Theorem 4.3, we have

$$\begin{aligned}[L^\lambda : H_{\mathbf{Q}}^* \otimes \mathcal{L}_{n,\mathbf{Q}}(k+1)] &= \sum_x [L^{\lambda\sqcup\{x\}} : \mathcal{L}_{n,\mathbf{Q}}(k+1)] \\ &= \sum_x [\chi_1 : \operatorname{Res}^{\mathfrak{S}_{k+1}}_{\mathrm{Cyc}_{k+1}} S^{\lambda\sqcup\{x\}}] \\ &= \Big[\chi_1 : \bigoplus_x \operatorname{Res}^{\mathfrak{S}_{k+1}}_{\mathrm{Cyc}_{k+1}} S^{\lambda\sqcup\{x\}}\Big],\end{aligned}$$

where x runs over all addable boxes to λ.

Recall that $\operatorname{Ind}^{\mathfrak{S}_{k+1}}_{\mathfrak{S}_k} S^\lambda \cong \bigoplus_x S^{\lambda\sqcup\{x\}}$ where x runs over the set of addable boxes to λ. Therefore we obtain

$$\begin{aligned}[L^\lambda : H_{\mathbf{Q}}^* \otimes \mathcal{L}_{n,\mathbf{Q}}(k+1)] &= \big[\chi_1 : \operatorname{Res}^{\mathfrak{S}_{k+1}}_{\mathrm{Cyc}_{k+1}} \operatorname{Ind}^{\mathfrak{S}_{k+1}}_{\mathfrak{S}_k} S^\lambda\big] \\ &= \big[S^\lambda : \operatorname{Res}^{\mathfrak{S}_{k+1}}_{\mathfrak{S}_k} \operatorname{Ind}^{\mathfrak{S}_{k+1}}_{\mathrm{Cyc}_{k+1}} \chi_1\big].\end{aligned}$$

by the Frobenius reciprocity. We shall use Mackey's decomposition theorem for $\mathfrak{S}_{k+1}$, $\mathfrak{S}_k$ and Cyc_{k+1}. But since $\mathfrak{S}_k \cup \mathrm{Cyc}_k$ generates $\mathfrak{S}_{k+1}$, in this case the $(\mathfrak{S}_k, \mathrm{Cyc}_{k+1})$–coset is trivial. Moreover $\mathfrak{S}_k \cap \mathrm{Cyc}_{k+1} = \{1\}$. Thus by Mackey's decomposition theorem, we have

$$\big[S^\lambda : \operatorname{Res}^{\mathfrak{S}_{k+1}}_{\mathfrak{S}_k} \operatorname{Ind}^{\mathfrak{S}_{k+1}}_{\mathrm{Cyc}_{k+1}} \chi_{\xi_{k+1}}\big] = \big[S^\lambda : \operatorname{Ind}^{\mathfrak{S}_k}_{\{1\}} \operatorname{Res}^{\mathrm{Cyc}_{k+1}}_{\{1\}} \chi_{\xi_{k+1}}\big].$$

Here $\operatorname{Ind}^{\mathfrak{S}_k}_{\{1\}} \operatorname{Res}^{\mathrm{Cyc}_{k+1}}_{\{1\}} \chi_{\xi_{k+1}} = \operatorname{Ind}^{\mathfrak{S}_k}_{\{1\}}(\mathbf{triv})$ is isomorphic to the regular representation of $\mathfrak{S}_k$. Since the multiplicity of S^λ in the regular representation of $\mathfrak{S}_k$ is equal to the dimension of S^λ, we conclude $[L^\lambda : H_{\mathbf{Q}}^* \otimes \mathcal{L}_{n,\mathbf{Q}}(k+1)] = \dim S^\lambda$. □

Corollary 4.8 *Under the notation above, we have*

$$W \cong \bigoplus_{\mu;\ell(\mu)\le n} [L^\mu : \mathcal{L}_{n,\mathbf{Q}}(k+1)] L^{\{\mu;(1)\}}.$$

Proof This follows from Corollary 3.7. □

In [38], we have obtained a $\mathrm{GL}(n,\mathbf{Q})$–equivariant exact sequence

$$0 \to \mathrm{Im}(\tau'_{k,\mathbf{Q}}) \to \mathrm{Der}^+(\mathcal{L}_{n,\mathbf{Q}})(k) \to \mathcal{C}^{\mathbf{Q}}_n(k) \to 0$$

for any $n \geq k+2$. Hence if we fix an integer $k \geq 2$, for any $n \geq k+2$ we can calculate the irreducible decompositions of $\mathcal{C}^{\mathbf{Q}}_n(k)$ and $\mathrm{Im}(\tau'_{k,\mathbf{Q}})$ using Proposition 4.1, Theorem 4.3, Proposition 4.5, Proposition 4.7 and Corollary 4.8. We give tables of the irreducible decompositions of $\mathcal{C}^{\mathbf{Q}}_n(k)$ and $\mathrm{Im}(\tau'_{k,\mathbf{Q}})$ in Tables 2 and 3.

k	$\mathcal{C}^{\mathbf{Q}}_n(k) = \mathrm{Coker}(\tau'_{k,\mathbf{Q}}),\ n \geq k+2$	
1	0	Andreadakis [1]
2	(2)	Pettet [33]
3	$(3) \oplus (1^3)$	Satoh [35]
4	$(4) \oplus (2,2) \oplus (2,1^2)$	Satoh [39]
5	$(5) \oplus (3,2) \oplus 2(3,1^2) \oplus (2^2,1) \oplus (1^5)$	
6	$(6) \oplus 2(4,2) \oplus 2(4,1^2) \oplus (3^2) \oplus 2(3,2,1)$ $\oplus (3,1^3) \oplus 2(2^3) \oplus (2^2,1^2) \oplus (2,1^4)$	
7	$(7) \oplus 2(5,2) \oplus 3(5,1^2) \oplus 2(4,3) \oplus 5(4,2,1)$ $\oplus 2(4,1^3) \oplus 3(3^2,1) \oplus 3(3,2^2) \oplus 5(3,2,1^2)$ $\oplus 3(3,1^4) \oplus 2(2^3,1) \oplus 2(2^2,1^3) \oplus (1^7)$	

Table 2: Decompositions of $\mathcal{C}^{\mathbf{Q}}_n(k)$. For simplicity, we write (λ) for an irreducible polynomial representation $L^{(\lambda)}$.

4.3 Decomposition of $\mathrm{Der}^+(\mathcal{L}^M_{n,\mathbf{Q}})$

We have a basis (4) of $\mathcal{L}^M_n(k)$ for each $k \geq 1$. Note that an element $[x_2, x_1, \ldots, x_1]$ is a maximal vector with weight $(k-1,1)$ in $\mathcal{L}^M_{n,\mathbf{Q}}(k)$. The number of elements satisfying $i_1 > i_2 \leq i_3 \leq \cdots \leq i_k$ coincides with the number of semistandard tableau of shape $(k-1,1)$. Thus we see $\mathcal{L}^M_{n,\mathbf{Q}}(k)$ is isomorphic to the irreducible representation $L^{(k-1,1)}$ as a $\mathrm{GL}(n,\mathbf{Q})$–module by Theorem 3.4. Using Corollary 3.7, we have:

Proposition 4.9 *For any $k \geq 1$ and $n \geq k+2$,*

$$\mathrm{Der}^+(\mathcal{L}^M_{n,\mathbf{Q}})(k) = H^*_{\mathbf{Q}} \otimes \mathcal{L}^M_{n,\mathbf{Q}}(k+1) \cong L^{\{(k,1);(1)\}} \oplus L^{\{(k);0\}} \oplus L^{\{(k-1,1);0\}}.$$

k	polynomial part of $\mathrm{Im}(\tau'_{k,\mathbf{Q}})$	nonpolynomial part of $\mathrm{Im}(\tau'_{k,\mathbf{Q}})$
1	(1)	$(1,1)$
2	(1^2)	$(2,1)$
3	$2(2,1)$	$(3,1)\oplus(2,1^2)$
4	$3(3,1)\oplus(2^2)\oplus 2(2,1^2)\oplus(1^4)$	$(4,1)\oplus(3,2)\oplus(3,1^2)$ $\oplus(2^2,1)\oplus(2,1^3)$
5	$4(4,1)\oplus 4(3,2)\oplus 4(3,1^2)$ $\oplus 4(2^2,1)\oplus 4(2,1^3)$	$(5,1)\oplus(4,2)\oplus 2(4,1^2)\oplus(3^2)$ $\oplus 3(3,2,1)\oplus(3,1^3)$ $\oplus 2(2^2,1^2)\oplus(2,1^4)$
6	$5(5,1)\oplus 7(4,2)\oplus 8(4,1^2)\oplus 4(3^2)$ $\oplus 14(3,2,1)\oplus 9(3,1^3)\oplus 3(2^3)$ $\oplus 8(2^2,1^2)\oplus 4(2,1^4)\oplus(1^6)$	$(6,1)\oplus 2(5,2)\oplus 2(5,1^2)\oplus 2(4,3)$ $\oplus 5(4,2,1)\oplus 3(4,1^3)\oplus 3(3^2,1)$ $\oplus 3(3,2^2)\oplus 5(3,2,1^2)\oplus 2(3,1^4)$ $\oplus 2(2^3,1)\oplus 2(2^2,1^3)\oplus(2,1^5)$
7	$6(6,1)\oplus 12(5,2)\oplus 12(5,1^2)\oplus 12(4,3)$ $\oplus 30(4,2,1)\oplus 18(4,1^3)\oplus 18(3^2,1)$ $\oplus 18(3,2^3)\oplus 30(3,2,1^2)\oplus 12(3,1^4)$ $\oplus 12(2^3,1)\oplus 12(2^2,1^3)\oplus 6(2,1^5)$	$(7,1)\oplus 2(6,2)\oplus 3(6,1^2)\oplus 4(5,3)$ $\oplus 8(5,2,1)\oplus 4(5,1^3)\oplus(4^2)$ $\oplus 9(4,3,1)\oplus 6(4,2^2)\oplus 12(4,2,1^2)$ $\oplus 4(4,1^4)\oplus 6(3^2,2)\oplus 9(3,2^2,1)$ $\oplus 8(3,2,1^3)\oplus 3(3,1^5)\oplus(2^4)$ $\oplus 4(2^3,1^2)\oplus 2(2^2,1^4)\oplus(2,1^6)$

Table 3: Decompositions of $\mathrm{Im}(\tau'_{k,\mathbf{Q}})$. (λ) means an irreducible polynomial representation $L^{(\lambda)}$ in the polynomial part, and (μ) means an irreducible nonpolynomial representation $L^{\{\mu;(1)\}}$ in the nonpolynomial part.

Note $L^{\{(k);0\}}$ is nothing but $S^k H_{\mathbf{Q}}$. In [37], we showed that the cokernel of τ_k^M is isomorphic to $S^k H$ for any $n \geq 4$ and $k \geq 2$. So by Proposition 4.9, we obtain:

Proposition 4.10 *For any $k \geq 1$ and $n \geq k+2$,*

$$\mathrm{Im}(\tau^M_{k,\mathbf{Q}}) \cong L^{\{(k,1);(1)\}} \oplus L^{\{(k-1,1);0\}}.$$

4.4 Decomposition of $\mathrm{Der}^+(\mathcal{L}^N_{n,\mathbf{Q}})$

We have a basis (6) in $\mathcal{L}^N_n(k)$ consisting of weight vectors. Using this basis, we see the character of $\mathcal{L}^N_{n,\mathbf{Q}}(k)$ coincides with that of $L^{(k-1,1)} \oplus (L^{(k-3,1)} \otimes L^{(1^2)})$. Thus,

$$\begin{aligned}\mathcal{L}^N_{n,\mathbf{Q}}(k) &\cong L^{(k-1,1)} \oplus (L^{(k-3,1)} \otimes L^{(1^2)}) \\ &\cong L^{(k-1,1)} \oplus L^{(k-2,2)} \oplus L^{(k-2,1^2)} \oplus L^{(k-3,2,1)} \oplus L^{(k-3,1^3)}\end{aligned}$$

by Theorem 3.4 and Pieri's formula (see Theorem 3.5).

Using Corollary 3.7, we have:

Proposition 4.11 *Let W_1 and W_2 be the polynomial part and the nonpolynomial part of the irreducible decomposition of* $\mathrm{Der}^+(\mathcal{L}^N_{n,\mathbf{Q}})(k) = H^*_{\mathbf{Q}} \otimes \mathcal{L}^N_{n,\mathbf{Q}}(k+1)$ *respectively. Then we have*

$$W_1 \cong L^{(k)} \oplus 3L^{(k-1,1)} \oplus 2L^{(k-2,2)} \oplus 3L^{(k-2,1^2)} \oplus L^{(k-3,2,1)} \oplus L^{(k-3,1^3)}$$
$$W_2 \cong L^{\{(k,1);(1)\}} \oplus L^{\{(k-1,2);(1)\}} \oplus L^{\{(k-1,1^2);(1)\}} \oplus L^{\{(k-2,2,1);(1)\}} \oplus L^{\{(k-2,1^3);(1)\}}.$$

In our paper [40], we investigate the cokernel of the composition map

$$\tau'_{k,N}\colon \mathrm{gr}^k(\mathcal{A}'_n) \xrightarrow{\tau'_k} H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1) \to H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1),$$

where the second map is induced from the natural projection $\mathcal{L}_n(k+1) \to \mathcal{L}^N_n(k+1)$. In particular, we showed that

$$\mathrm{Coker}((\tau'_{k,N})_{\mathbf{Q}}) \cong L^{(k)} \oplus L^{(k-2,1^2)}$$

as a $\mathrm{GL}(n,\mathbf{Q})$–modules. Hence we see that:

Proposition 4.12 *For any $k \geq 1$ and $n \geq k+2$,*

$$\mathrm{Im}((\tau'_{k,N})_{\mathbf{Q}}) \cong 3L^{(k-1,1)} \oplus 2L^{(k-2,2)} \oplus 2L^{(k-2,1^2)} \oplus L^{(k-3,2,1)} \oplus L^{(k-3,1^3)} \oplus W_2.$$

Here we mention a relation between $\tau'_{k,N}$ and the k–th Johnson homomorphism of $\mathrm{Aut}\, F^N_n$. Let IA^N_n be the IA–automorphism group of F^N_n. Then we can define the Johnson homomorphisms

$$\eta_{k,N}\colon \mathrm{gr}^k(\mathcal{A}^N_n) \to H^* \otimes_{\mathbf{Z}} \mathcal{L}^N_n(k+1) \quad \text{and} \quad \eta'_{k,N}\colon \mathrm{gr}^k(\mathcal{A}'^N_n) \to H^* \otimes_{\mathbf{Z}} \mathcal{L}^N_n(k+1)$$

by an argument similar to τ_k and τ'_k respectively. (See [37, Section 2.1.2] for details.) Then we have

$$\mathrm{Im}(\tau'_{k,N}) \subset \mathrm{Im}(\eta'_{k,N}) \subset \mathrm{Im}(\eta_{k,N}) \subset H^* \otimes_{\mathbf{Z}} \mathcal{L}^N_n(k+1).$$

5 Abelianization of $\mathrm{Der}^+(\mathcal{L}^M_n)$

In this section, we determine the abelianization of the derivation algebra of the Chen Lie algebra. To begin with, in order to give an lower bound on it, we consider Morita's trace map

$$\mathrm{Tr}_{[k]}\colon H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1) \to S^k H$$

for $k \geq 2$ as mentioned in Section 2.3. Recently, using these trace maps Morita constructed a surjective graded Lie algebra homomorphism

$$\Theta := \mathrm{id}_1 \oplus \bigoplus_{k\geq 2} \mathrm{Tr}_{[k]} : \mathrm{Der}^+(\mathcal{L}_n) \to (H^* \otimes_{\mathbf{Z}} \Lambda^2 H) \oplus \bigoplus_{k\geq 2} S^k H,$$

where id_1 is the identity map on the degree one part $H^* \otimes_{\mathbf{Z}} \Lambda^2 H$ of $\mathrm{Der}^+(\mathcal{L}_n)$, and the target is understood to be an abelian Lie algebra. In particular, Morita showed that Θ gives the abelianization of $\mathrm{Der}^+(\mathcal{L}_n)$ up to degree $n(n-1)$, based on a theorem of Kassabov in [17]. (See [31, Theorem 25] for details.)

On the other hand, Θ naturally induces a surjective graded Lie algebra homomorphism

$$\Theta^M := \mathrm{id}_1 \oplus \bigoplus_{k\geq 2} \mathrm{Tr}^M_{[k]} : \mathrm{Der}^+(\mathcal{L}^M_n) \to (H^* \otimes_{\mathbf{Z}} \Lambda^2 H) \oplus \bigoplus_{k\geq 2} S^k H,$$

and hence

$$\mathrm{Der}^+(\mathcal{L}^M_n)^{\mathrm{ab}} \to (H^* \otimes_{\mathbf{Z}} \Lambda^2 H) \oplus \bigoplus_{k\geq 2} S^k H.$$

In order to prove this is an isomorphism, it suffices to show that for any $k \geq 2$, the degree k part of $\mathrm{Der}^+(\mathcal{L}^M_n)^{\mathrm{ab}}$ is generated by

$$\binom{n+k-1}{k} = \mathrm{rank}_{\mathbf{Z}} S^k H$$

elements as an abelian group.

Let $(\mathrm{Der}^+(\mathcal{L}^M_n))^{\mathrm{ab}}(k)$ be the degree k part of $\mathrm{Der}^+(\mathcal{L}^M_n)^{\mathrm{ab}}$. Then as a $\mathbf{Z}$–module, $(\mathrm{Der}^+(\mathcal{L}^M_n))^{\mathrm{ab}}(k)$ is generated by

$$\mathfrak{E} := \{x_i^* \otimes [x_{i_1}, x_{i_2}, \ldots, x_{i_{k+1}}] \mid 1 \leq i, i_j \leq n\},$$

where $x_1^*, \ldots, x_n^*$ is the dual basis of H^* with respect to $x_1, \ldots, x_n \in H$. To reduce the generators in $\mathfrak{E}$, we prepare some lemmas. The lemmas below essentially follow from the facts obtained in our previous paper [37] (refer there for the proofs).

Lemma 5.1 *Let $l \geq 2$ and $n \geq 2$. For any element $[x_{i_1}, x_{i_2}, x_{j_1}, \ldots, x_{j_l}] \in \mathcal{L}^M_n(l+2)$ and any $\lambda \in \mathfrak{S}_l$,*

$$[x_{i_1}, x_{i_2}, x_{j_1}, \ldots, x_{j_l}] = [x_{i_1}, x_{i_2}, x_{j_{\lambda(1)}} \ldots, x_{j_{\lambda(l)}}].$$

Lemma 5.2 *Let $k \geq 2$ and $n \geq 4$. For any i and $i_1, i_2, \ldots, i_{k+1} \in \{1, 2 \ldots, n\}$, if $i_1, i_2 \neq i$,*

$$x_i^* \otimes [x_{i_1}, x_{i_2}, \ldots, x_{i_{k+1}}] = 0 \in (\mathrm{Der}^+(\mathcal{L}^M_n))^{\mathrm{ab}}(k).$$

Lemma 5.3 *Let $k \geq 2$ and $n \geq 4$. For any i and $i_1, i_2, \dots, i_k \in \{1, 2 \dots, n\}$ such that $i_1, i_2 \neq i$, and any transposition $\lambda = (m\, m+1) \in \mathfrak{S}_k$,*

$$x_i^* \otimes [x_i, x_{i_1}, \dots, x_{i_k}] = x_i^* \otimes [x_i, x_{i_{\lambda(1)}}, \dots, x_{i_{\lambda(k)}}] \in (\mathrm{Der}^+(\mathcal{L}_n^M))^{\mathrm{ab}}(k).$$

Lemma 5.4 *Let $k \geq 2$ and $n \geq 4$. For any $i_2, \dots, i_{k+1} \in \{1, 2, \dots, n\}$, we have*

$$x_i^* \otimes [x_i, x_{i_2}, \dots, x_{i_{k+1}}] = x_j^* \otimes [x_j, x_{i_2}, \dots, x_{i_{k+1}}] \in (\mathrm{Der}^+(\mathcal{L}_n^M))^{\mathrm{ab}}(k)$$

for any $i \neq i_2$ and $j \neq i_2, i_{k+1}$.

Using Lemmas 5.2 and 5.4, we see that $(\mathrm{Der}^+(\mathcal{L}_n^M))^{\mathrm{ab}}(k)$ is generated by

$$\{x_i^* \otimes [x_{i_1}, x_{i_2}, \dots, x_{i_{k+1}}] \mid 1 \leq i, i_j \leq n,\ i \neq i_2, i_{k+1}\}.$$

Furthermore, by Lemma 5.4 again,

$$x_i^* \otimes [x_{i_1}, x_{i_2}, \dots, x_{i_{k+1}}], \quad i \neq i_2, i_{k+1}$$

does not depend on the choice of i such that $i \neq i_2, i_{k+1}$. Hence we can set

$$s(i_1, \dots, i_k) := x_i^* \otimes [x_{i_1}, x_{i_2}, \dots, x_{i_{k+1}}] \in (\mathrm{Der}^+(\mathcal{L}_n^M))^{\mathrm{ab}}(k)$$

for $i \neq i_1,\ i_k$. On the other hand, take any transposition $\sigma = (m\, m+1) \in \mathfrak{S}_k$. If $2 \leq m \leq k-2$, we see $s(i_1, \dots, i_k) = s(i_{\sigma(1)}, \dots, i_{\sigma(k)})$ by Lemma 5.1. If $m = k-1$, there exists some $1 \leq j \leq n$ such that $j \neq i_1, i_{k-1}, i_k$ since $n \geq 4$. Then we have

$$\begin{aligned} s(i_1, \dots, i_k) &= x_j^* \otimes [x_j, x_{i_1}, \dots, x_{i_k}] = x_j^* \otimes [x_j, x_{i_1}, \dots, x_{i_{k-1}}, x_{i_k}] \\ &= s(i_1, \dots, i_k, i_{k-1}) = s(i_{\sigma(1)}, \dots, i_{\sigma(k)}) \end{aligned}$$

by Lemma 5.3. Similarly, we verify that $s(i_1, \dots, i_k) = s(i_{\sigma(1)}, \dots, i_{\sigma(k)})$ if $m = 1$. Therefore we conclude that for $n \geq 4$ and $k \geq 2$, $(\mathrm{Der}^+(\mathcal{L}_n^M))^{\mathrm{ab}}(k)$ is generated by

$$\{s(i_1, \dots, i_k) \mid 1 \leq i_1 \leq \cdots \leq i_k \leq n\}.$$

Namely, we obtain:

Theorem 5.5 *For $n \geq 4$, we have*

$$(\mathrm{Der}^+(\mathcal{L}_n^M))^{\mathrm{ab}} \cong (H^* \otimes_{\mathbf{Z}} \Lambda^2 H) \oplus \bigoplus_{k \geq 2} S^k H.$$

More precisely, this isomorphism is given by the degree one part and Morita's trace maps $\mathrm{Tr}_{[k]}$.

This theorem induces a Lie algebra exact sequence

$$0 \to \bigoplus_{k\geq 2} \mathrm{gr}^k(\mathcal{A}_n^M) \xrightarrow{\oplus_{k\geq 2}\tau_k^M} \mathrm{Der}^+(\mathcal{L}_n^M) \xrightarrow{\Theta^M} (H^* \otimes_{\mathbf{Z}} \Lambda^2 H) \oplus \bigoplus_{k\geq 2} S^k H \to 0.$$

6 Twisted cohomology groups with coefficients in $\Lambda^l H_{\mathbf{Q}}$

In general, for any $\mathrm{GL}(n,\mathbf{Z})$–module M, we can naturally regard M as an $\mathrm{Aut}\, F_n$–module and an $\mathrm{Aut}\, N_{n,k}$–module through the surjective homomorphisms $\mathrm{Aut}\, F_n \to \mathrm{GL}(n,\mathbf{Z})$ and $\mathrm{Aut}\, N_{n,k} \to \mathrm{GL}(n,\mathbf{Z})$ respectively. Here we consider the case where $M = \Lambda^l H_{\mathbf{Q}}$ for $l \geq 1$. In this section, we study the twisted first and second cohomology groups of $T_{n,k}$ and $\mathrm{Aut}\, N_{n,k}$ with coefficients in $\Lambda^l H_{\mathbf{Q}}$. In particular, we show that the trace map $\mathrm{Tr}^{\mathbf{Q}}_{[1^k]}$ for $\Lambda^k H_{\mathbf{Q}}$ defines a nontrivial cohomology class in $H^2(T_{n,k}, \Lambda^k H_{\mathbf{Q}})$ for even k and $2 \leq k \leq n$, and $H^2(\mathrm{Aut}\, N_{n,k}, \Lambda^k H_{\mathbf{Q}})$ for any $3 \leq k \leq n$.

In the following, for a group G and a G–module M, we write $Z^1(G,M)$ for the abelian group of crossed homomorphisms from G to M. We denote by δ the coboundary operator in group cohomology theory. We also remark that for any finite group G and G–module M,

$$H^p(G, M \otimes_{\mathbf{Z}} \mathbf{Q}) = 0, \quad p \geq 1.$$

6.1 Twisted first cohomologies of $\mathrm{GL}(n,\mathbf{Z})$ and $\mathrm{Aut}\, F_n$

Here we show that for $n \geq 3$ the first cohomologies of $\mathrm{GL}(n,\mathbf{Z})$ and $\mathrm{Aut}\, F_n$ with coefficients in $\Lambda^l H_{\mathbf{Q}}$ are trivial except for $H^1(\mathrm{Aut}\, F_n, H_{\mathbf{Q}}) = \mathbf{Q}$.

Proposition 6.1 *For $n \geq 3$ and $l \geq 1$, $H^1(\mathrm{GL}(n,\mathbf{Z}), \Lambda^l H_{\mathbf{Q}}) = 0$.*

Proof Take any crossed homomorphism $f\colon \mathrm{GL}(n,\mathbf{Z}) \to \Lambda^l H_{\mathbf{Q}}$ and consider the restriction of $f_{\overline{\Omega}_n}$ to the subgroup $\overline{\Omega}_n$ of $\mathrm{GL}(n,\mathbf{Z})$. Since $\overline{\Omega}_n$ is a finite group, $H^1(\overline{\Omega}_n, \Lambda^l H_{\mathbf{Q}}) = 0$. Hence there exists some $x \in \Lambda^l H_{\mathbf{Q}}$ such that $f_{\overline{\Omega}_n} = \delta x$. Consider a crossed homomorphism

$$f' := f - \delta x\colon \mathrm{GL}(n,\mathbf{Z}) \to \Lambda^l H_{\mathbf{Q}}.$$

Then we see that $f'(\sigma) = 0$ for $\sigma = P$, Q and S. Hence it suffices to show $f'(U) = 0$.

Set $$f'(U) := \sum_{1\leq i_1<\cdots<i_l\leq n} a_{i_1,\ldots,i_l} e_{i_1} \wedge \cdots \wedge e_{i_l} \in \Lambda^l H_{\mathbf{Q}}$$

for $a_{i_1,\dots,i_l} \in \mathbf{Q}$. Consider a relation $U\,Q^{-(j-1)}SQ^{j-1} = Q^{-(j-1)}SQ^{j-1}\,U$ in $\mathrm{GL}(n,\mathbf{Z})$. Since f' is a crossed homomorphism, f' satisfies

$$(Q^{-(j-1)}SQ^{j-1}-1)\,f'(U) = (U-1)\,f'(Q^{-(j-1)}SQ^{j-1}) = 0 \tag{10}$$

since $f'(Q) = f'(S) = 0$.

Case 1 If $l \geq 3$, for any $a_{i_1,\dots,i_l}$, we see $3 \leq i_l \leq n$. By observing the coefficients of $e_{i_1} \wedge \cdots \wedge e_{i_l}$ in (10) for $j = i_l$, we obtain $2a_{i_1,\dots,i_l} = 0$, and hence $a_{i_1,\dots,i_l} = 0$.

Case 2 Assume $l = 2$. We can see $a_{i_1,i_2} = 0$ for any $1 \leq i_1 < i_2 \leq n$, except for $(i_1,i_2) = (1,2)$, by the same argument as above. To show $a_{1,2} = 0$, consider the relation (R11): $PUPSU = USPS$. Since f' is a crossed homomorphism, f' satisfies

$$\begin{aligned} f'(P) + Pf'(U) + PUf'(P) + PUPf'(S) + PUPSf'(U) \\ = f'(U) + Uf'(S) + USf'(P) + USPf'(S), \end{aligned}$$

and hence

$$Pf'(U) + PUPSf'(U) = f'(U) \tag{11}$$

by $f'(P) = f'(S) = 0$. Observing the coefficients of $e_1 \wedge e_2$, we have $3a_{1,2} = 0$, and hence $a_{1,2} = 0$.

Case 3 Finally, assume $l = 1$. Since $a_i = 0$ for $3 \leq i \leq n$ by the same argument as above, it suffices to show $a_1 = a_2 = 0$. By observing the coefficients of e_2 in (11), we see $a_1 = 0$. Similarly, from the relation (R13): $(SU)^2 = 1$, we have

$$f'(S) + Sf'(U) + SUf'(S) + SUSf'(U) = 0$$

and

$$f'(U) + USf'(U) = 0$$

by $f'(S) = 0$. Observing the coefficients of e_2 in this equation, we obtain $a_2 = 0$.

This shows that $f' \equiv 0$ as a crossed homomorphism for any case. Namely, $f = \delta x$. Hence we obtain the required results. This completes the proof of Proposition 6.1. □

By the same argument as Proposition 6.1, we have:

Proposition 6.2 *For $n \geq 3$ and $l \geq 2$, $H^1(\mathrm{Aut}\,F_n, \Lambda^l H_{\mathbf{Q}}) = 0$.*

Here we should remark that $H^1(\mathrm{Aut}\,F_n, H) = \mathbf{Z}$ for any $n \geq 2$. (See our paper [36].)

6.2 Twisted cohomologies of $T_{n,k}$

Here we consider twisted cohomology groups of the tame automorphism group $T_{n,k}$ of $N_{n,k}$ for $k \geq 2$. To begin with, from Proposition 6.2, we have:

Lemma 6.3 *For any $n \geq 3$, $k \geq 2$ and $l \geq 2$, $H^1(T_{n,k}, \Lambda^l H_{\mathbf{Q}}) = 0$.*

Proof It is clear from the fact that the induced homomorphism

$$H^1(\mathrm{Aut}\, N_{n,k}, \Lambda^l H_{\mathbf{Q}}) \to H^1(\mathrm{Aut}\, F_n, \Lambda^l H_{\mathbf{Q}}) = 0$$

from the natural projection $\mathrm{Aut}\, F_n \to T_{n,k}$ is injective. □

Similarly, for $l = 1$, we have an injective homomorphism

$$H^1(T_{n,k}, H) \to H^1(\mathrm{Aut}\, F_n, H) = \mathbf{Z} \tag{12}$$

for $n \geq 2$. Hence, $H^1(T_{n,k}, H) = 0$ or $\mathbf{Z}$. In order to show $H^1(T_{n,k}, H) = \mathbf{Z}$, we consider Morita's crossed homomorphism. Let

$$\frac{\partial}{\partial x_j} \colon \mathbf{Z}[F_n] \longrightarrow \mathbf{Z}[F_n]$$

be Fox's free derivations for $1 \leq j \leq n$. (For a basic material concerning the Fox derivative, see Birman [4] for example.) Let $\mathfrak{a} \colon \mathbf{Z}[F_n] \to \mathbf{Z}[H]$ be the ring homomorphism induced from the abelianization $F_n \to H$. For any matrix $A = (a_{ij}) \in \mathrm{GL}(n, \mathbf{Z}[F_n])$, set $A^{\mathfrak{a}} = (a_{ij}^{\mathfrak{a}}) \in \mathrm{GL}(n, \mathbf{Z}[H])$. Then a map

$$r_M \colon \mathrm{Aut}\, F_n \longrightarrow \mathrm{GL}(n, \mathbf{Z}[H])$$

defined by

$$\sigma \mapsto \left(\frac{\partial x_i^{\sigma}}{\partial x_j} \right)^{\mathfrak{a}}$$

is called the Magnus representation of $\mathrm{Aut}\, F_n$. We remark that r_M is not a homomorphism but a crossed homomorphism. Namely, r_M satisfies

$$r_M(\sigma\tau) = r_M(\sigma)^{\tau_*} \cdot r_M(\tau)$$

for any σ, $\tau \in \mathrm{Aut}\, F_n$ where $r_M(\sigma)^{\tau_*}$ denotes the matrix obtained from $r_M(\sigma)$ by applying a ring homomorphism $\tau_* \colon \mathbf{Z}[H] \to \mathbf{Z}[H]$ induced from τ on each entry. (For detail for the Magnus representation, see Morita [28].)

Observing the images of Nielsen's generators by $\det \circ r_M$, we verify that $\mathrm{Im}(\det \circ r_M)$ is contained in a multiplicative abelian subgroup $\pm H$ of $\mathbf{Z}[H]$. In order to modify

the image of $\det \circ r_M$, we consider the signature of $\mathrm{Aut}\, F_n$. For any $\sigma \in \mathrm{Aut}\, F_n$, set $\mathrm{sgn}(\sigma) := \det(\rho(\sigma)) \in \{\pm 1\}$, and define a map $f_M\colon \mathrm{Aut}\, F_n \longrightarrow \mathbf{Z}[H]$ by

$$\sigma \mapsto \mathrm{sgn}(\sigma)\det(r_M(\sigma)).$$

Then the map f_M is also a crossed homomorphism whose image is contained in a multiplicative abelian subgroup H in $\mathbf{Z}[H]$. In the following, we identify the multiplicative abelian group structure of H with the additive one. Morita [27] showed that the twisted first cohomology group of a mapping class group of a surface with coefficients in H is the infinite cyclic group generated by f_M restricted to the mapping class group. In our previous paper [36], we showed that f_M is a generator of $H^1(\mathrm{Aut}\, F_n, H) = \mathbf{Z}$ for any $n \geq 2$. We call f_M Morita's crossed homomorphism.

Now, we consider the restriction of f_M to the IA–automorphism group IA_n. It is a group homomorphism whose target is an abelian group H. On the other hand, $\mathcal{A}_n(2)$ coincides with the commutator subgroup of IA_n since $\mathrm{gr}^1(\mathcal{A}_n)$ is the abelianization of IA_n as mentioned above. Hence we see that $f_M(\mathcal{A}_n(2)) = 0$. This shows that the crossed homomorphism $f_M\colon \mathrm{Aut}\, F_n \to H$ extends to the quotient group $\mathrm{Aut}\, F_n/\mathcal{A}_n(k) \cong T_{n,k}$ for any $k \geq 2$. We also call this extended crossed homomorphism Morita's crossed homomorphism. Hence the homomorphism (12) is surjective, and in fact is an isomorphism. Therefore we have:

Proposition 6.4 *For any $n \geq 2$ and $k \geq 2$, $H^1(T_{n,k}, H) = \mathbf{Z}$ which is generated by Morita's crossed homomorphism.*

Furthermore, we see:

Lemma 6.5 *For any $n \geq 2$, and $k \geq 2$, the natural projection $T_{n,k+1} \to T_{n,k}$ induces an isomorphism*

$$H^1(T_{n,k}, H) \cong H^1(T_{n,k+1}, H).$$

Proof Clearly, the induced homomorphism $H^1(T_{n,k}, H) \to H^1(T_{n,k+1}, H)$ from the natural projection $T_{n,k+1} \to T_{n,k}$ maps the cohomology class of Morita's crossed homomorphism in $H^1(T_{n,k}, H)$ to that in $H^1(T_{n,k+1}, H)$. Hence we see the required results. □

Now, from the cohomological five term exact sequence of the group extension (9), for $k \geq 2$ and $l \geq 1$, we have

$$\begin{aligned} 0 \to H^1(T_{n,k}, \Lambda^l H_{\mathbf{Q}}) &\to H^1(T_{n,k+1}, \Lambda^l H_{\mathbf{Q}}) \\ &\to H^1(\mathrm{gr}^k(\mathcal{A}_n), \Lambda^l H_{\mathbf{Q}})^{\mathrm{GL}(n,\mathbf{Z})} \to H^2(T_{n,k}, \Lambda^l H_{\mathbf{Q}}) \to H^2(T_{n,k+1}, \Lambda^l H_{\mathbf{Q}}). \end{aligned}$$

From Lemmas 6.3 and 6.5, we see that

$$0 \to H^1(\mathrm{gr}^k(\mathcal{A}_n), \Lambda^l H_{\mathbf{Q}})^{\mathrm{GL}(n,\mathbf{Z})} \to H^2(T_{n,k}, \Lambda^l H_{\mathbf{Q}}) \to H^2(T_{n,k+1}, \Lambda^l H_{\mathbf{Q}})$$

is exact.

On the other hand, we have a $\mathrm{GL}(n,\mathbf{Z})$–equivariant homomorphism $\mathrm{Tr}_{[1^k]} \circ \tau_k$, namely $\mathrm{Tr}_{[1^k]} \circ \tau_k \in H^1(\mathrm{gr}^k(\mathcal{A}_n), \Lambda^l H_{\mathbf{Q}})^{\mathrm{GL}(n,\mathbf{Z})}$. In [35], we showed that $\mathrm{Tr}_{[1^k]} \circ \tau_k$ is surjective for even k and $2 \le k \le n$. In particular, we have:

Proposition 6.6 *For even k and $2 \le k \le n$, if* tg *is the transgression map, then* $0 \neq \mathrm{tg}(\mathrm{Tr}_{[1^k]} \circ \tau_k) \in H^2(T_{n,k}, \Lambda^k H_{\mathbf{Q}})$.

6.3 Twisted cohomologies of Aut $N_{n,k}$

In this subsection, for $k \ge 3$, we consider twisted cohomology groups of $\mathrm{Aut}\, N_{n,k}$ with coefficients in $\Lambda^l H_{\mathbf{Q}}$.

Proposition 6.7 *For $k \ge 3$, $n \ge k-1$ and $l \ge 3$, $H^1(\mathrm{Aut}\, N_{n,k}, \Lambda^l H_{\mathbf{Q}}) = 0$.*

Proof Take any crossed homomorphism $f\colon \mathrm{Aut}\, N_{n,k} \to \Lambda^l H_{\mathbf{Q}}$. Let g be the image of f under the homomorphism

$$Z^1(\mathrm{Aut}\, N_{n,k}, \Lambda^l H_{\mathbf{Q}}) \to Z^1(\mathrm{Aut}\, F_n, \Lambda^l H_{\mathbf{Q}})$$

induced from the natural homomorphism $\mathrm{Aut}\, F_n \to \mathrm{Aut}\, N_{n,k}$. By Proposition 6.2, $H^1(\mathrm{Aut}\, F_n, \Lambda^l H_{\mathbf{Q}}) = 0$, so there exists some $x \in \Lambda^l H_{\mathbf{Q}}$ such that $g = \delta x \in Z^1(\mathrm{Aut}\, F_n, \Lambda^l H_{\mathbf{Q}})$.

Set $f' := f - \delta x \in Z^1(\mathrm{Aut}\, N_{n,k}, \Lambda^l H_{\mathbf{Q}})$. Then we have

$$f'(\sigma) = f(\sigma) - \delta x = g(\sigma) - \delta x = 0$$

for $\sigma = P$, Q, S and U. Hence it suffices to show $f'(\theta) = 0$. Set

$$f'(\theta) := \sum_{1 \le i_1 < \cdots < i_l \le n} b_{i_1,\ldots,i_l} e_{i_1} \wedge \cdots \wedge e_{i_l} \in \Lambda^l H_{\mathbf{Q}}$$

for $b_{i_1,\ldots,i_l} \in \mathbf{Q}$. Consider the relation $\theta\, Q^{-(j-1)} S Q^{j-1} = Q^{-(j-1)} S Q^{j-1}\, \theta$ in $\mathrm{Aut}\, N_{n,k}$. Since f' is a crossed homomorphism, f' satisfies

$$(Q^{-(j-1)} S Q^{j-1} - 1)\, f'(\theta) = (\theta - 1)\, f'(Q^{-(j-1)} S Q^{j-1}) = 0 \tag{13}$$

since $f'(Q) = f'(S) = 0$.

Since $l \geq 3$, for any $b_{i_1,\dots,i_l}$, we see $3 \leq i_l \leq n$. By observing the coefficients of $e_{i_1} \wedge \cdots \wedge e_{i_l}$ in (13) for $j = i_l$, we obtain $2b_{i_1,\dots,i_l} = 0$, and hence $b_{i_1,\dots,i_l} = 0$.

Thus $f' \equiv 0$, and hence $f = \delta x$. This completes the proof of Proposition 6.7. □

By considering the cohomological five term exact sequence of the group extension (8), we obtain

$$\begin{aligned} 0 \to H^1(\mathrm{Aut}\, N_{n,k}, \Lambda^l H_{\mathbf{Q}}) &\to H^1(\mathrm{Aut}\, N_{n,k+1}, \Lambda^l H_{\mathbf{Q}}) \\ &\to H^1(H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1), \Lambda^l H_{\mathbf{Q}})^{\mathrm{GL}(n,\mathbf{Z})} \\ &\to H^2(\mathrm{Aut}\, N_{n,k}, \Lambda^l H_{\mathbf{Q}}) \to H^2(\mathrm{Aut}\, N_{n,k+1}, \Lambda^l H_{\mathbf{Q}}) \end{aligned}$$

for $k \geq 3$, $n \geq k-1$ and $l \geq 3$. From Proposition 6.7, we have an exact sequence

$$\begin{aligned} 0 \to H^1(H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1), \Lambda^k H_{\mathbf{Q}})^{\mathrm{GL}(n,\mathbf{Z})} \\ \to H^2(\mathrm{Aut}\, N_{n,k}, \Lambda^k H_{\mathbf{Q}}) \to H^2(\mathrm{Aut}\, N_{n,k+1}, \Lambda^k H_{\mathbf{Q}}) \end{aligned}$$

for $k \geq 3$ and $n \geq k-1$. As the trace map $\mathrm{Tr}_{[1^k]} \in H^1(H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1), \Lambda^k H_{\mathbf{Q}})^{\mathrm{GL}(n,\mathbf{Z})}$ is surjective for any $3 \leq k \leq n$, we have:

Proposition 6.8 *For $k \geq 3$ and $n \geq k$, we see $0 \neq \mathrm{tg}(\mathrm{Tr}_{[1^k]}) \in H^2(\mathrm{Aut}\, N_{n,k}, \Lambda^k H_{\mathbf{Q}})$ where* tg *is the transgression map.*

Remark 6.9 Finally, we remark on the multiplicity of the $\mathrm{GL}(n,\mathbf{Q})$–trivial part in $\mathrm{Hom}_{\mathbf{Z}}(H^*_{\mathbf{Q}} \otimes_{\mathbf{Z}} \mathcal{L}_{n,\mathbf{Q}}(k+1), \Lambda^k H_{\mathbf{Q}})$ and $\mathrm{Hom}_{\mathbf{Z}}(\mathrm{gr}^k_{\mathbf{Q}}(\mathcal{A}_n), \Lambda^k H_{\mathbf{Q}})$.

First of all, if $n \geq k+2$ and an irreducible representation $L^{\{\lambda;\mu\}}$ appears in the decomposition of $(H^*_{\mathbf{Q}} \otimes_{\mathbf{Z}} \mathcal{L}_{n,\mathbf{Q}}(k+1))^*$, then $\ell(\lambda) + \ell(\mu) \leq k+2$ by Corollary 3.7. Note that $\Lambda^k H_{\mathbf{Q}} = L^{(1^k)}$ and the multiplicity of $(\Lambda^k H_{\mathbf{Q}})^* = L^{\{0;(1^k)\}}$ is exactly one in $(H^*_{\mathbf{Q}} \otimes_{\mathbf{Z}} \mathcal{L}_{n,\mathbf{Q}}(k+1))^*$ by the part (2) of Corollary 4.6.

The multiplicity of the $\mathrm{GL}(n,\mathbf{Q})$–trivial part $L^{(0)}$ in $(H^*_{\mathbf{Q}} \otimes_{\mathbf{Z}} \mathcal{L}_{n,\mathbf{Q}}(k+1))^* \otimes \Lambda^k H_{\mathbf{Q}}$ is exactly one under the condition $n \geq 2k+2$ by Corollary 3.8. Thus we obtain

$$\mathrm{Hom}_{\mathbf{Z}}(H^*_{\mathbf{Q}} \otimes_{\mathbf{Z}} \mathcal{L}_{n,\mathbf{Q}}(k+1), \Lambda^k H_{\mathbf{Q}})^{\mathrm{GL}(n,\mathbf{Q})} \cong \mathbf{Q}$$

for $n \geq 2k+2$. Hence, $\mathrm{Hom}_{\mathbf{Z}}(H^*_{\mathbf{Q}} \otimes_{\mathbf{Z}} \mathcal{L}_{n,\mathbf{Q}}(k+1), \Lambda^k H_{\mathbf{Q}})^{\mathrm{GL}(n,\mathbf{Q})}$ is generated by $\mathrm{Tr}^{\mathbf{Q}}_{[1^k]}$ for $n \geq 2k+2$.

Recall that

$$\mathrm{Im}(\tau'_{k,\mathbf{Q}}) \subset \mathrm{Im}(\tau_{k,\mathbf{Q}}) \subset H^*_{\mathbf{Q}} \otimes \mathcal{L}_{n,\mathbf{Q}}(k+1) \quad \text{and} \quad \mathrm{Im}(\tau_{k,\mathbf{Q}}) \cong \mathrm{gr}^k_{\mathbf{Q}}(\mathcal{A}_n)$$

as $\mathrm{GL}(n,\mathbf{Q})$–modules. Since we have a nonzero element

$$\mathrm{Tr}^{\mathbf{Q}}_{[1^k]} \circ \tau_{k,\mathbf{Q}} \in \mathrm{Hom}_{\mathbf{Z}}(\mathrm{gr}^k_{\mathbf{Q}}(\mathcal{A}_n), \Lambda^l H_{\mathbf{Q}})^{\mathrm{GL}(n,\mathbf{Q})},$$

we obtain

$$\mathrm{Hom}_{\mathbf{Z}}(\mathrm{gr}^k_{\mathbf{Q}}(\mathcal{A}_n), \Lambda^k H_{\mathbf{Q}})^{\mathrm{GL}(n,\mathbf{Q})} = \mathbf{Q}$$

for even k and $2k+2 \leq n$. Hence $\mathrm{Hom}_{\mathbf{Z}}(\mathrm{gr}^k(\mathcal{A}_n), \Lambda^k H_{\mathbf{Q}})^{\mathrm{GL}(n,\mathbf{Q})}$ is generated by the element $\mathrm{Tr}^{\mathbf{Q}}_{[1^k]} \circ \tau_{k,\mathbf{Q}}$.

At the present stage, however, it seems to difficult to determine the precise structures of $H^1(H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1), \Lambda^k H_{\mathbf{Q}})^{\mathrm{GL}(n,\mathbf{Z})}$ and $H^1(\mathrm{gr}^k(\mathcal{A}_n), \Lambda^k H_{\mathbf{Q}})^{\mathrm{GL}(n,\mathbf{Z})}$ in general.

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