Representation stability for the cohomology of the moduli space $\mathcal{M}_g^n$

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Let $\mathcal{M}_g^n$ be the moduli space of Riemann surfaces of genus g with n labeled marked points. We prove that, for $g \geq 2$, the cohomology groups $\{H^i(\mathcal{M}_g^n;\mathbb{Q})\}_{n=1}^{\infty}$ form a sequence of S_n–representations which is representation stable in the sense of Church–Farb [7]. In particular this result applied to the trivial S_n–representation implies rational "puncture homological stability" for the mapping class group Mod_g^n. We obtain representation stability for sequences $\{H^i(\mathrm{PMod}^n(M);\mathbb{Q})\}_{n=1}^{\infty}$, where $\mathrm{PMod}^n(M)$ is the mapping class group of many connected orientable manifolds M of dimension $d \geq 3$ with centerless fundamental group; and for sequences $\{H^i\big(B\mathrm{PDiff}^n(M);\mathbb{Q}\big)\}_{n=1}^{\infty}$, where $B\mathrm{PDiff}^n(M)$ is the classifying space of the subgroup $\mathrm{PDiff}^n(M)$ of diffeomorphisms of M that fix pointwise n distinguished points in M.

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1 Introduction

Notation Let $\Sigma_{g,r}$ be a compact orientable surface of genus $g \geq 0$ with $r \geq 0$ boundary components and let $p_1, \dots, p_n$ be distinct points in the interior of $\Sigma_{g,r}$. The *mapping class group* $\mathrm{Mod}_{g,r}^n$ is the group of isotopy classes of orientation-preserving self-diffeomorphisms of $\Sigma_{g,r}^n := \Sigma_{g,r} - \{p_1, \dots, p_n\}$ that restrict to the identity on the boundary components. The *pure mapping class group* $\mathrm{PMod}_{g,r}^n$ is defined analogously by asking that the punctures remain fixed pointwise. If $r = 0$ or $n = 0$, we omit it from the notation.

The homology groups of the pure mapping class group PMod_g^n are of interest (among other reasons) due to their relation with the topology of the moduli space $\mathcal{M}_g^n$ of genus g Riemann surfaces with n labeled marked points (that is, n–pointed non-singular projective curves of genus g). The space $\mathcal{M}_g^n$ is a rational model for the classifying space $B\,\mathrm{PMod}_g^n$ for $g \geq 2$. Hence

$$H^*(\mathcal{M}_g^n;\mathbb{Q}) \approx H^*(\mathrm{PMod}_g^n;\mathbb{Q}). \tag{1}$$

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We refer the reader to Farb–Margalit [8], Hain–Looijenga [10], Kirwan [19] and Harer [15] for more about the relation between $\mathcal{M}_g^n$ and PMod_g^n.

One basic question is to understand how, for a fixed $i \geq 0$, the cohomology groups $H^i(\mathrm{PMod}_{g,r}^n; \mathbb{Q})$ change as we vary the parameters g, r and n, in particular when the parameters are very large with respect to i. It is a classical result by Harer [13] that the group $H^i(\mathrm{PMod}_{g,r}^n; \mathbb{Z})$ depends only on n provided that g is large enough. The major goal of this paper is to understand how the cohomology $H^i(\mathrm{PMod}_{g,r}^n; \mathbb{Q})$ changes as we vary the number of punctures n.

1.1 Genus and puncture homological stability

It is known that the groups $\mathrm{PMod}_{g,r}^n$ and $\mathrm{Mod}_{g,r}^n$ satisfy "genus homological stability":

> *For fixed $i, n \geq 0$ the groups $H_i(\mathrm{PMod}_{g,r}^n; \mathbb{Z})$ and $H_i(\mathrm{Mod}_{g,r}^n; \mathbb{Z})$ do not depend on the parameters g and r, for $g \gg i$.*

This was first proved in the 1980's by Harer [13] and the stable ranges have been improved since then by the work of several people (see Wahl's survey [23]).

An additional stabilization map can be defined by increasing the number of punctures. In the case of surfaces with non-empty boundary, we can consider a map $\Sigma_{g,r}^n \to \Sigma_{g,r}^{n+1}$ by gluing a punctured cylinder to one of the boundary components of $\Sigma_{g,r}^n$. This map gives a homomorphism

$$\mu_n\colon \mathrm{Mod}_{g,r}^n \to \mathrm{Mod}_{g,r}^{n+1}.$$

In [18, Proposition 1.5], Hatcher and Wahl proved that the map μ_n induces an isomorphism in $H_i(-;\mathbb{Z})$ if $n \geq 2i+1$ (for fixed $g \geq 0$ and $r > 0$). Puncture stability for closed surfaces follows, as it is known that

$$H_i(\mathrm{Mod}_{g,1}^n; \mathbb{Z}) \approx H_i(\mathrm{Mod}_g^n; \mathbb{Z}) \text{ for } g \geq \tfrac{3}{2}i$$

(see Wahl [23, Theorem 1.2]). Hanbury proved this "puncture homological stability" for non-orientable surfaces in [11] with techniques that can also be applied to the orientable case. When the surface is a punctured disk this is Arnold's classical stability theorem for the cohomology of braid groups B_n [1]. Together, puncture and genus stability imply that the homology of the mapping class group of an orientable surface stabilizes with respect to connected sum with any surface.

On the other hand, for the pure mapping class groups, attaching a punctured cylinder to $\Sigma_{g,r}^n$ also induces homomorphisms

$$\mu_n\colon \mathrm{PMod}_{g,r}^n \to \mathrm{PMod}_{g,r}^{n+1},$$

when $r > 0$. Hence we can ask whether $\mathrm{PMod}_{g,r}^n$ satisfies or not puncture homological stability.

The homology groups of $\mathrm{PMod}_{g,r}^n$ are largely unknown, apart from some low dimensional cases such as:

$$H_1(\mathrm{PMod}_{g,r}^n;\mathbb{Z}) = 0 \text{ for } g \geq 3$$

(see Farb–Margalit [8, Theorem 5.2] for a proof). Furthermore,

$$H_2(\mathrm{PMod}_{g,r}^n;\mathbb{Z}) \approx H_2(\mathrm{Mod}_{g,r+n};\mathbb{Z}) \oplus \mathbb{Z}^n \text{ for } g \geq 3$$

(this is Korkmaz–Stipsicz [20, Corollary 4.5], but the original computation for $g \geq 5$ is due to Harer [12]).

Even if the case of the first homology group is not representative, we notice that the rank of $H_2(\mathrm{PMod}_{g,r}^n;\mathbb{Z})$ blows up as $n \to +\infty$. Moreover, the pure braid groups $P_n \approx \mathrm{PMod}_{0,1}^n$ fail in each dimension $i \geq 1$ to satisfy homological stability (see Church–Farb [7, Section 4]). This suggests to us the failure of puncture homological stability in the general case.

For large g, Bödigheimer and Tillmann's results in [4], combined with Madsen–Weiss, give explicit calculations, although we do not discuss them in this paper.

1.2 Main result

We want to compare $H^i(\mathrm{PMod}_{g,r}^n;\mathbb{Q})$ as the number of punctures n varies. The natural inclusion $\Sigma_{g,r}^{n+1} \hookrightarrow \Sigma_{g,r}^n$ induces the *forgetful map*

$$f_n\colon \mathrm{PMod}_{g,r}^{n+1} \to \mathrm{PMod}_{g,r}^n .$$

Notice that f_n is a left inverse for the map μ_n above, when $r > 0$, but can be defined even for surfaces without boundary. This map allows us to relate the corresponding cohomology groups:

$$f_n^*\colon H^*(\mathrm{PMod}_{g,r}^n;\mathbb{Q}) \to H^*(\mathrm{PMod}_{g,r}^{n+1};\mathbb{Q}).$$

Observe that f_n^* is also induced by the *forgetful morphism* between moduli spaces $\mathcal{M}_g^{n+1} \to \mathcal{M}_g^n$.

The key idea is to consider the natural action of the symmetric group S_n on $\mathcal{M}_g^n$ given by permuting the n labeled marked points. Thus we can regard $H^i(\mathcal{M}_g^n;\mathbb{Q})$ as rational S_n–representations and compare them through the maps f_n^i. Moreover, we notice that the map f_n^i is equivariant with respect to the standard inclusion $S_n \hookrightarrow S_{n+1}$. In Section 3 below we explicitly compute the S_n–representation $H^2(\mathrm{PMod}_g^n;\mathbb{Q})$ and its decomposition into irreducibles.

Roughly speaking, we say that a sequence of S_n–representations $\{V_n\}$ with linear maps $\phi_n\colon V_n \to V_{n+1}$ equivariant with respect to $S_n \hookrightarrow S_{n+1}$ is *representation stable* if for sufficiently large n the following conditions hold: the maps ϕ_n are injective; the image $\phi_n(V_n)$ generates V_{n+1} as an S_{n+1}–module, and the decomposition of V_n into irreducibles can be described independently of n. This notion was introduced by Church–Farb in [7]. The precise definition of representation stability is stated in Section 2.1 below.

Hence, instead of asking if f_n^i is an isomorphism or not (puncture cohomological stability), we consider the question of whether the cohomology groups of the pure mapping class group satisfy representation stability. In [7, Theorem 4.2] Church–Farb prove that the sequence $\{H^i(P_n;\mathbb{Q}), f_n^i\}_{n=1}^{\infty}$ is representation stable. Our main result shows that this is also the case for the pure mapping class group.

Theorem 1.1 *For any $i \geq 0$ and $g \geq 2$ the sequence of cohomology groups*

$$\{H^i\left(\mathrm{PMod}_g^n;\mathbb{Q}\right)\}_{n=1}^{\infty}$$

is monotone and uniformly representation stable with stable range

$$n \geq \min\{4i + 2(4g-6)(4g-5), 2i^2 + 6i\}.$$

Our arguments work for hyperbolic non-closed surfaces (Theorem 5.9). Hence Harer's homological stability and our main theorem imply that, as an S_n–representation, $H^i(\mathrm{PMod}_{g,r}^n;\mathbb{Q})$ is independent of g, r and n, provided n and g are large enough.

By (1), Theorem 1.1 can be restated as follows.

Corollary 1.2 (Representation stability for the cohomology of the moduli space $\mathcal{M}_g^n$) *For any $i \geq 0$ and $g \geq 2$ the sequence of cohomology groups $\{H^i(\mathcal{M}_g^n;\mathbb{Q})\}_{n=1}^{\infty}$ is monotone and uniformly representation stable with stable range*

$$n \geq \min\{4i + 2(4g-6)(4g-5), 2i^2 + 6i\}.$$

Remark In [4, Theorem 1.1] Bödigheimer and Tillmann proved that

$$B(\mathrm{PMod}_{\infty,r}^n)^+ \simeq B\,\mathrm{Mod}_\infty^+ \times (\mathbb{C}P^\infty)^n.$$

Together with Harer's homological stability theorem this implies that, in dimensions $* \leq g/2$,

$$\begin{aligned} H^*(\mathrm{PMod}_{g,r}^n;\mathbb{Q}) &\approx H^*(\mathrm{PMod}_{g,r};\mathbb{Q}) \otimes \left(H^*(\mathbb{C}P^\infty;\mathbb{Q})\right)^{\otimes n} \\ &\approx H^*(\mathrm{PMod}_{g,r};\mathbb{Q}) \otimes \mathbb{Q}[x_1,\ldots,x_n], \end{aligned}$$

where each x_i has degree 2. The action of the symmetric group S_n on the left hand side corresponds to permuting the n factors $\mathbb{C}P^\infty$. In other words, it is given by the action of S_n on the polynomial ring in n variables by permutation of the variables x_i. On the other hand, Church and Farb proved in [7, Section 7] that representation stability holds for the S_n–action on the polynomial ring in n variables. Hence Bödigheimer and Tillmann result implies that for $i \le g/2$ representation stability holds for $\{H^i(\mathrm{PMod}_{g,r}^n; \mathbb{Q})\}_{n=1}^\infty$. Notice that this only holds for large g with respect to i. In contrast, our Theorem 1.1 and Theorem 5.9 give uniform representation stability and monotonicity for arbitrary $g \ge 0$ such that $2g + r + s > 2$ and large n.

1.3 Puncture (co)homological stability for Mod_g^n

Our main result, Theorem 1.1, implies cohomological stability for Mod_g^n with twisted rational coefficients (see Section 5.3). For any partition λ, we denote the corresponding irreducible S_n–representation by $V(\lambda)_n$, as we explain in Section 2.1 below. A transfer argument gives the proof of the following corollary of Theorem 1.1.

Corollary 1.3 *For any partition λ, the sequence $\{H^i(\mathrm{Mod}_g^n; V(\lambda)_n)\}_{n=1}^\infty$ of twisted cohomology groups satisfies classical cohomological stability: for fixed $i \ge 0$ and $g \ge 2$, there is an isomorphism*

$$H^i(\mathrm{Mod}_g^n; V(\lambda)_n) \approx H^i\big(\mathrm{Mod}_g^{n+1}; V(\lambda)_{n+1}\big),$$

if $n \ge \min\{4i + 2(4g-6)(4g-5), 2i^2 + 6i\}$.

In [18, Proposition 1.5], Hatcher–Wahl obtained integral puncture homological stability for the mapping class group of surfaces with non-empty boundary and established a stable range linear in i. Plugging in the trivial representation $V(0)_n$ into Corollary 1.3, we recover rational puncture homological stability for Mod_g^n. The stable range that we obtain either depends on the genus g of the surface or is quadratic in i (see Corollary 5.8). Nonetheless, our approach by representation stability is completely different from the classical techniques used in the proofs of homological stability. Furthermore, we believe that our proof gives yet another example of how the notion of representation stability can give meaningful answers where classical stability fails.

1.4 Pure mapping class groups for higher dimensional manifolds

Notation Let M be a connected, smooth manifold and let $p_1, \dots, p_n$ be distinct points in the interior of M. We define the *mapping class group* to be the group

$$\mathrm{Mod}^n(M) := \pi_0(\mathrm{Diff}^n(M))$$

where $\mathrm{Diff}^n(M)$ is the subgroup of diffeomorphisms in $\mathrm{Diff}(M \;\mathrm{rel}\; \partial M)$ that leave invariant the set of points $\{p_1, \dots, p_n\}$. Similarly, we let $\mathrm{PDiff}^n(M)$ be the subgroup of diffeomorphims in $\mathrm{Diff}(M \;\mathrm{rel}\; \partial M)$ that fix the points $p_1, \dots, p_n$ pointwise and the *pure mapping class group* is the group

$$\mathrm{PMod}^n(M) := \pi_0(\mathrm{PDiff}^n(M)).$$

In this paper, the manifolds M are always orientable.

In Section 6.2 we give a proof of representation stability for the sequence

$$\left\{H^i(G^n; \mathbb{Q})\right\}_{n=1}^{\infty}$$

for any group G. This is Proposition 6.5 below. We show how to use this result and the ideas developed in this paper to establish the analogue of Theorem 1.1 and Corollary 1.3 for the pure mapping class groups of some connected manifolds of higher dimension.

Theorem 1.4 *Let M be a smooth connected manifold of dimension $d \geq 3$ such that $\pi_1(M)$ is of type FP_∞ (for example, M compact). Suppose that $\pi_1(M)$ has trivial center or that $\mathrm{Diff}(M)$ is simply connected. If $\mathrm{Mod}(M)$ is a group of type FP_∞, then for any $i \geq 0$ the sequence of cohomology groups $\left\{H^i(\mathrm{PMod}^n(M); \mathbb{Q})\right\}_{n=1}^{\infty}$ is monotone and uniformly representation stable with stable range $n \geq 2i^2 + 4i$.*

Corollary 1.5 *Let M be as in Theorem 1.4. For any partition λ, the sequence of twisted cohomology groups $\left\{H^i(\mathrm{Mod}^n(M); V(\lambda)_n)\right\}_{n=1}^{\infty}$ satisfies classical homological stability: for fixed $i \geq 0$, there is an isomorphism*

$$H^i(\mathrm{Mod}^n(M); V(\lambda)_n) \approx H^i(\mathrm{Mod}^{n+1}(M); V(\lambda)_{n+1}) \text{ if } n \geq 2i^2 + 4i.$$

Hatcher–Wahl proved integral puncture homological stability for mapping class group of connected manifolds with boundary of dimension $d \geq 2$ in [18, Proposition 1.5]. Our Corollary 1.5, applied to the trivial representation, gives rational puncture homological stability for $\mathrm{Mod}^n(M)$ for manifolds M that satisfy the hypothesis of Theorem 1.4, even if the manifold has empty boundary.

1.5 Classifying spaces for diffeomorphism groups

Ezra Getzler and Oscar Randal-Williams pointed out to me that the same ideas also give representation stability for the rational cohomology groups of the classifying space $B\,\mathrm{PDiff}^n(M)$ of the group $\mathrm{PDiff}^n(M)$ defined above.

Theorem 1.6 *Let M be a smooth, compact and connected manifold of dimension $d \geq 3$ such that $B\,\mathrm{Diff}(M \text{ rel } \partial M)$ has the homotopy type of CW-complex with finitely many cells in each dimension. Then for any $i \geq 0$ the sequence of cohomology groups $\{H^i(B\,\mathrm{PDiff}^n(M);\mathbb{Q})\}_{n=1}^{\infty}$ is monotone and uniformly representation stable with stable range $n \geq 2i^2 + 4i$.*

The details are described at the end of the paper in Section 7.

1.6 Outline of the proof of Theorem 1.1

The proof of Theorem 1.1 is presented in Section 5 and relies on the existence of the Birman exact sequence which realizes $\pi_1(C_n(\Sigma_g))$ as a subgroup of PMod_g^n. Here $C_n(\Sigma_{g,r})$ denotes the configuration space of n distinct ordered points in the interior of $\Sigma_{g,r}$. Then for each n we can consider the associated Hochschild–Serre spectral sequence $E_*(n)$, which allows us to relate $H^*(\mathrm{PMod}_g^n;\mathbb{Q})$ with $H^*(\pi_1(C_n(\Sigma_g));\mathbb{Q})$. Following ideas of Church in [6], we use an inductive argument to show that the terms in each page of the spectral sequence are uniformly representation stable and thus we conclude the result in Theorem 1.1 from the E_∞–page.

The notion of *monotonicity* for a sequence of S_n–representations introduced in [6] is key in our inductive argument on the pages of the spectral sequence. The base of the induction is monotonicity and representation stability for the terms in the E_2–page of the Hochschild–Serre spectral sequence. In order to prove this, we introduce, in Section 4 below, the notion of a consistent sequence of rational S_n–representations *compatible with G–actions* and prove the following general result which we hope will be useful in future computations.

Theorem 1.7 (Representation stability with changing coefficients) *Let G be a group of type FP_∞. Consider a consistent sequence $\{V_n,\phi_n\}_{n=1}^{\infty}$ of finite dimensional rational representations of S_n compatible with G–actions. If the sequence $\{V_n,\phi_n\}_{n=1}^{\infty}$ is monotone and uniformly representation stable with stable range $n \geq N$, then for any integer $p \geq 0$, the sequence $\{H^p(G;V_n),\phi_n^*\}_{n=1}^{\infty}$ is monotone and uniformly representation stable with the same stable range.*

Monotonicity and uniform representation stability for the E_2–page both follow from Theorem 1.7, as a consequence of the following result by Church [6, Theorem 1].

Theorem 1.8 (Church) *For any connected orientable manifold M of finite type and any $q \geq 0$, the cohomology groups $\{H^q(C_n(M);\mathbb{Q})\}$ of the ordered configuration space $C_n(M)$ are monotone and uniformly representation stable, with stable range $n \geq 2q$ if $\dim M \geq 3$ and stable range $n \geq 4q$ if $\dim M = 2$.*

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2 Preliminaries

The precise definition of representation stability and monotonicity are stated below. We also recall some useful facts about group extensions and cohomology of groups.

2.1 Representation stability and monotonicity

Recall that a *rational S_n–representation* is a $\mathbb{Q}$–vector space equipped with a linear S_n–action. The irreducible representations of S_n are classified by partitions $\lambda = (\lambda_1 \geq \cdots \geq \lambda_l)$ of n (with $\lambda_1 + \cdots + \lambda_l = n$). We denote the corresponding irreducible S_n–representation by V_λ. Every V_λ is defined over $\mathbb{Q}$ and any S_n–representation decomposes over $\mathbb{Q}$ into a direct sum of irreducibles (Fulton–Harris [9] is a standard reference).

If λ is any partition of k, then for any $n \geq k + \lambda_1$ the *padded partition* $\lambda[n]$ of n is given by $\lambda[n] = (n-k, \lambda_1, \cdots, \lambda_l)$. Keeping the notation from Church–Farb [7] we set $V(\lambda)_n = V_{\lambda[n]}$ for any $n \geq k + \lambda_1$. Every irreducible S_n–representation is of the form $V(\lambda)_n$ for a unique partition λ.

The notion of representation stability for different families of groups was first defined in Church–Farb [7]. We recall this notion for the case of S_n–representations.

Definition 2.1 A sequence $\{V_n\}_{n=1}^{\infty}$ of finite dimensional rational S_n–representations with linear maps $\phi_n \colon V_n \to V_{n+1}$ is said to be *uniformly representation stable with stable range $n \geq N$* if the following conditions are satisfied for all $n \geq N$:

0 **Consistent sequence** The maps $\phi_n \colon V_n \to V_{n+1}$ are equivariant with respect to the natural inclusion $S_n \hookrightarrow S_{n+1}$.

I **Injectivity** The maps $\phi_n \colon V_n \to V_{n+1}$ are injective.

II **Surjectivity** The S_{n+1}–span of $\phi_n(V_n)$ equals V_{n+1}.

III **Uniformly multiplicity stable with range** $n \geq N$ For each partition λ, the multiplicities $c_\lambda(V_n)$ of $V(\lambda)_n$ in V_n are constant for all $n \geq N$.

The notion of monotonicity introduced by Church [6] will be key in our argument.

Definition 2.2 A consistent sequence $\{V_n\}_{n=1}^{\infty}$ of S_n–representations with injective maps $\phi_n\colon V_n \hookrightarrow V_{n+1}$ is *monotone* for $n \geq N$ if for each subspace $W < V_n$ isomorphic to $V(\lambda)_n^{\oplus k}$, the S_{n+1}–span of $\phi_n(W)$ contains $V(\lambda)_{n+1}^{\oplus k}$ as a subrepresentation for $n \geq N$.

Now we point out the properties of monotone sequences that are useful for our purpose. These results are proven in [6, Sections 2.1 and 2.2].

Proposition 2.3 *Given $\{W_n\} < \{V_n\}$, if the sequence $\{V_n\}$ is monotone then so is $\{W_n\}$. If $\{V_n\}$ and $\{W_n\}$ are monotone and uniformly representation stable with stable range $n \geq N$, then $\{V_n/W_n\}$ is monotone and representation stable for $n \geq N$. Conversely, if $\{W_n\}$ and $\{V_n/W_n\}$ are monotone and uniformly representation stable with stable range $n \geq N$, then $\{V_n\}$ is monotone and uniformly representation stable for $n \geq N$.*

Proposition 2.4 *Let $\{V_n\}$ and $\{W_n\}$ be monotone sequences for $n \geq N$, and assume that $\{V_n\}$ is uniformly representation stable for $n \geq N$. Then for any consistent sequence of maps $f_n\colon V_n \to W_n$ that makes the following diagram commutative*

$$\begin{array}{ccc} V_n & \xrightarrow{f_n} & W_n \\ {\scriptstyle\phi_n}\downarrow & & \downarrow{\scriptstyle\psi_n} \\ V_{n+1} & \xrightarrow{f_{n+1}} & W_{n+1}, \end{array}$$

the sequences $\{\ker f_n\}$ and $\{\operatorname{im} f_n\}$ are monotone and uniformly representation stable for $n \geq N$.

The previous propositions apply also to $V(\lambda)_n$ for a single partition λ. In particular to the case of the trivial representation $V(0)_n$.

Proposition 2.5 *For a fixed partition λ, assuming monotonicity just for $V(\lambda)_n^{\otimes k}$, Propositions 2.3 and 2.4 hold if we replace "uniform representation stability" by "the multiplicity of $V(\lambda)_n$ is stable".*

2.2 On the cohomology of group extensions

A *group extension* of a group Q by a group H is a short exact sequence of groups

$$1 \to H \to G \to Q \to 1. \tag{2}$$

Given a G–module M, the conjugation action $(h,m) \mapsto (ghg^{-1}, g\cdot m)$ of G on (H,M) induces an action of $G/H \cong Q$ on $H^*(H;M)$ as follows. Let $F \to \mathbb{Z}$ be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ and consider the diagonal action of G in the cochain complex $\mathcal{H}om(F,M)$ given by $f \mapsto [x \mapsto g\cdot f(g^{-1}\cdot x)]$, for $f \in \mathcal{H}om(F,M)$ and $g \in G$. This action restricts to the subcomplex $\mathcal{H}om_H(F,M)$ where H acts trivially by definition, hence we get an induced action of $Q \cong G/H$ on $\mathcal{H}om_H(F,M)$. But the cohomology of this complex is $H^*(H;M)$, giving the desired action of Q on $H^*(H;M)$.

The *cohomology Hochschild–Serre spectral sequence* for the group extension (2) is a first quadrant spectral sequence converging to $H^*(G;M)$ whose E_2 page is of the form

$$E_2^{p,q} = H^p(Q; H^q(H;M)).$$

Furthermore, from the construction of the Hochschild–Serre spectral sequence it can be shown that this spectral sequence is natural in the following sense. Assume we have group extensions (I) and (II) and group homomorphisms f_H and f_G making the following diagram commute

$$\begin{array}{ccccccccc} 1 & \longrightarrow & H_1 & \longrightarrow & G_1 & \longrightarrow & Q & \longrightarrow & 1 & \quad \text{(I)} \\ & & \downarrow{\scriptstyle f_H} & & \downarrow{\scriptstyle f_G} & & \Vert{\scriptstyle \mathrm{id}} & & & \\ 1 & \longrightarrow & H_2 & \longrightarrow & G_2 & \longrightarrow & Q & \longrightarrow & 1 & \quad \text{(II)} \end{array}$$

Then the induced map

$$f_H^*\colon H^*(H_2;\mathbb{Q}) \to H^*(H_1;\mathbb{Q})$$

is Q–equivariant. Moreover, if ${}'E_*$ and ${}''E_*$ denote the Hochschild–Serre spectral sequences corresponding to the extensions (I) and (II), we have

(1) Induced maps $(f_H)_r^*\colon {}''E_r^{p,q} \to {}'E_r^{p,q}$ that commute with the differentials.

(2) The map $(f_G)^*\colon H^*(G_2;\mathbb{Q}) \to H^*(G_1;\mathbb{Q})$ preserves the natural filtrations of $H^*(G_1;\mathbb{Q})$ and $H^*(G_2;\mathbb{Q})$ inducing a map on the succesive quotients of the filtrations which is the map

$$(f_H)_\infty^*\colon {}''E_\infty^{p,q} \to {}'E_\infty^{p,q}.$$

(3) The map $(f_H)_2^*\colon {}''E_2^{p,q} \to {}'E_2^{p,q}$ is the one induced by the group homomorphisms id: $Q \to Q$ and $f_H\colon H_1 \to H_2$.

For an explicit description of the Hochschild–Serre spectral sequence we refer the reader to Brown [5] and Mac Lane [21] (where it is called the Lyndon spectral sequence).

3 The second cohomology $H^2(\mathcal{M}_g^n;\mathbb{Q})$

In this section we understand the consistent sequence of S_n–representations

$$\{H^2\big(\mathrm{PMod}_g^n;\mathbb{Q}\big), f_n^2\}$$

to give an explicit discussion of the phenomenon of representation stability proved on Theorem 1.1.

The second cohomology group is given by:

$$H^2(\mathcal{M}_{g,n};\mathbb{Q}) \approx H^2\big(\mathrm{PMod}_g^n;\mathbb{Q}\big) \approx H^2(\mathrm{Mod}_{g,n};\mathbb{Q}) \oplus \mathbb{Q}^n, \text{ for } g \geq 3. \tag{3}$$

We want to compare $H^2(\mathrm{PMod}_g^n;\mathbb{Q})$ through the forgetful maps

$$f_n^2\colon H^2\big(\mathrm{PMod}_g^n;\mathbb{Q}\big) \to H^2\big(\mathrm{PMod}_g^{n+1};\mathbb{Q}\big).$$

We already know that f_n^2 is never an isomorphism (failure of homological stability). Instead, we consider $H^2\big(\mathrm{PMod}_g^n;\mathbb{Q}\big)$ as an S_n–representation and we investigate how those representations depend on the parameter n. When $g \geq 4$, $H^2(\mathrm{Mod}_{g,n};\mathbb{Q}) \approx \mathbb{Q}$ (see Harer [12]) and the S_n–action on this summand is trivial. On the other hand, the summand $\mathbb{Q}^n$ is generated by classes $\tau_i \in H^2(\mathrm{PMod}_g^n;\mathbb{Q})$ $(i = 1,\dots,n)$ corresponding to the central extensions $\mathrm{PMod}(X_i)$:

$$1 \to \mathbb{Z} \to \mathrm{PMod}(X_i) \to \mathrm{PMod}_g^n \to 1.$$

The right map above is induced from the inclusion $X_i := \Sigma_g - N_\epsilon(p_i) \hookrightarrow \Sigma_g^n$, where $N_\epsilon(p_i) = \big\{x \in \Sigma_g^n : d(x,p_i) < \epsilon\big\}$ for a small $\epsilon > 0$. Notice that $X_i \simeq \Sigma_{g,1}^{n-1}$. The kernel is generated by a Dehn twist around the boundary component, which is the simple loop $\partial N_\epsilon(p_i)$ around the puncture p_i in Σ_g^n. Observe that a permutation of the punctures induces a corresponding permutation of the surfaces $\{X_1,\dots,X_n\}$, hence of the classes τ_i in $H^2\big(\mathrm{PMod}_g^n;\mathbb{Q}\big)$.

We can also think of τ_i as the first Chern class of the line bundle $\mathbb{L}_i$ over $\mathcal{M}_g^n$ defined as follows: at a point in $\mathcal{M}_g^n$, that is, a Riemann surface X with marked points $p_1,\dots,p_n$, the fiber of $\mathbb{L}_i$ is the cotangent space to X at p_i. In fact, the τ–classes are the image of the ψ–classes under the surjective homomorphism $H^2(\overline{\mathcal{M}}_g^n;\mathbb{Q}) \to$

$H^2(\mathcal{M}_g^n;\mathbb{Q})$, where $\overline{\mathcal{M}}_g^n$ is the Deligne–Mumford compactification of $\mathcal{M}_g^n$ (see Hain–Looijenga [10]). A permutation of the marked points induces the same permutation of the classes τ_i in $H^2(\mathcal{M}_g^n;\mathbb{Q})$. Therefore, S_n acts on the summand $\mathbb{Q}^n$ in (3) by permuting the generators.

Thus, for $g \geq 4$ and $n \geq 3$, the decomposition of (3) into irreducibles is given by

$$H^2\big(\mathrm{PMod}_g^n;\mathbb{Q}\big) \approx V(0)_n \oplus V(0)_n \oplus V(1)_n,$$

where, following our notation from Section 2.1, $V(0)_n$ is the trivial S_n–representation and $V(1)_n$ is the standard S_n–representation. Notice that, even though the dimension of $H^2\big(\mathrm{PMod}_g^n;\mathbb{Q}\big)$ blows up as n increases, the decomposition into irreducibles stabilizes. In terms of definition of representation stability stated in Section 2.1, we have shown that the sequence of S_n–representations $\big\{H^2\big(\mathrm{PMod}_g^n;\mathbb{Q}\big)\big\}$ is *uniformly multiplicity stable* with stable range $n \geq 3$. This indicates to us that representation stability of the cohomology groups of PMod_g^n may be the phenomena to expect.

4 Representation stability for $H^*(G;V_n)$

We discuss here when representation stability for a sequence $\{V_n\}$ of G–modules will imply representation stability for the cohomology of a group G with coefficients V_n. This is Theorem 1.7 below and it is a key ingredient for the base of the induction in the proof of Theorem 1.1.

Definition 4.1 Let G be a group. We will say that a sequence of rational vector spaces V_n with given maps $\phi_n\colon V_n \to V_{n+1}$ is *consistent* and *compatible with G–actions* if it satisfies the following:

> **Consistent sequence** Each V_n is a rational S_n–representation and the map $\phi_n\colon V_n \to V_{n+1}$ is equivariant with respect to the inclusion $S_n \hookrightarrow S_{n+1}$.
>
> **Compatible with G–actions** Each V_n is a G–module and the maps $\phi_n\colon V_n \to V_{n+1}$ are G–maps. The G–action commutes with the S_n–action.

Notice that for a sequence as in the previous definition and $p \geq 0$, we have that $\{H^p(G;V_n);\phi_n^*\}$ is a consistent sequence of rational S_n–representations. Here

$$\phi_n^*\colon H^p(G;V_n) \to H^p(G;V_{n+1})$$

denotes the map induced by $\phi_n\colon V_n \to V_{n+1}$.

Theorem 1.7 (Representation stability with changing coefficients) *Let G be a group of type FP_∞. Consider a consistent sequence $\{V_n, \phi_n\}_{n=1}^\infty$ of finite dimensional rational representations of S_n compatible with G–actions. If the sequence $\{V_n, \phi_n\}_{n=1}^\infty$ is monotone and uniformly representation stable with stable range $n \geq N$, then for any non-negative integer p, the sequence $\{H^p(G; V_n), \phi_n^*\}_{n=1}^\infty$ is monotone and uniformly representation stable with the same stable range.*

Proof Take $E \to \mathbb{Z}$ a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ of finite type. This means that each E_p is a free G–module of finite rank, say $E_p \approx (\mathbb{Z}G)^{d_p}$ generated by $x_1, \dots, x_{d_p}$.

There is an S_n–action on the chain complex $\mathcal{H}om(E, V_n)$ given by $\sigma \cdot h \colon x \mapsto \sigma \cdot h(x)$ for any $h \in \mathcal{H}om(E, V_n)$ and $\sigma \in S_n$. Since the S_n–action and the G–action on V_n commute, this action restricts to a well-defined S_n–action on $\mathcal{H}om_G(E, V_n)$ which makes each $\mathcal{H}om_G(E, V_n)^p := \mathrm{Hom}_G(E_p, V_n)$ into a rational S_n–representation.

Observe that any G–homomorphism $h \colon E_p \to V_n$ is completely determined by the d_p–tuple $(h(x_1), \dots, h(x_{d_p}))$. Then the assignment $h \mapsto (h(x_1), \dots, h(x_{d_p}))$ gives us an isomorphism

$$\mathcal{H}om_G(E, V_n)^p \approx V_n^{\oplus d_p}$$

not just of rational vector spaces, but of S_n–representations. Notice that since V_n is finite dimensional, $\mathcal{H}om_G(E, V_n)^p$ also has finite dimension. Moreover, under this isomorphism the map

$$\phi_n^p := \mathcal{H}om_G(E, \phi_n)^p \colon \mathcal{H}om_G(E, V_n)^p \to \mathcal{H}om_G(E, V_{n+1})^p$$

is just $(\phi_n)^{\oplus d_p} \colon V_n^{\oplus d_p} \to V_{n+1}^{\oplus d_p}$. From Proposition 2.3, it follows that the sequence $\{\mathcal{H}om_G(E, V_n)^p; \phi_n^p\}$ is monotone and uniformly representation stable for $n \geq N$.

The differentials δ_p^n of the cochain complex $\mathcal{H}om_G(E, V_n)$ are a consistent sequence of maps, meaning that the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{H}om_G(E, V_n)^p & \xrightarrow{\phi_n^p} & \mathcal{H}om_G(E, V_{n+1})^p \\
\Big\downarrow{\scriptstyle \delta_p^n} & & \Big\downarrow{\scriptstyle \delta_p^{n+1}} \\
\mathcal{H}om_G(E, V_n)^{p+1} & \xrightarrow{\phi_n^{p+1}} & \mathcal{H}om_G(E, V_{n+1})^{p+1}
\end{array}$$

From Proposition 2.4 the subsequences $\{\ker \delta_p^n\}$ and $\{\mathrm{im}\, \delta_p^n\}$ are monotone and uniformly representation stable for $n \geq N$. Finally Proposition 2.3 gives the desired result for $H^p(G; V_n) := \ker \delta_p^n / \mathrm{im}\, \delta_{p+1}^n$. □

Since $H^0(G;V_n)$ is equal to the G–invariants V_n^G, as a particular case of Theorem 1.7, we get the following.

Corollary 4.2 *The sequence of G–invariants $\{V_n^G,\phi_n\}$ is monotone and uniformly representation stable with the same stable range as $\{V_n,\phi_n\}$.*

5 Representation stability for $H^*(\mathrm{PMod}_g^n;\mathbb{Q})$

In this section we prove our main result Theorem 1.1 and some consequences of it. We will focus on the sequence of pure mapping class groups PMod_g^n and its cohomology with rational coefficients. We consider the case $g\geq 2$.

5.1 The ingredients for the proof of the main theorem

Here we describe three of the four main ingredients needed in our proof of Theorem 1.1 in Section 5.2. The ingredient (iv) is Theorem 1.8 (see Church [6, Theorem 1]).

5.1.1 The Birman exact sequence Our approach relies on the existence of a nice short exact sequence, introduced by Birman in 1969, that relates the pure mapping class group with the pure braid group of the surface: the *Birman exact sequence* $(\mathrm{Bir1}_n)$.

Let $C_n(\Sigma_g)$ be the configuration space of Σ_g and $\mathfrak{p}=(p_1,\cdots,p_n)\in C_n(\Sigma_g)$ the punctures or marked points in Σ_g^n. The map in $(\mathrm{Bir1}_n)$ that realizes $\pi_1(C_n(\Sigma_g),\mathfrak{p})$ as a subgroup of PMod_g^n is the *point-pushing map* Push. For an element $\gamma\in\pi_1(C_n(\Sigma_g),\mathfrak{p})$, consider the isotopy defined by "pushing" the n–tuple $(p_1,\cdots,p_n)$ along γ. Then $\mathrm{Push}(\gamma)$ is represented by the diffeomorphism at the end of the isotopy. The map f in $(\mathrm{Bir1}_n)$ is a *forgetful morphism* induced by the inclusion $\Sigma_g^n\hookrightarrow\Sigma_g$.

Taking the quotient $(\mathrm{Bir1}_n)$ by the S_n–action there, we obtain the Birman exact sequence $(\mathrm{Bir2}_n)$. The relation between these two sequences is illustrated in the following diagram.

(4)
$$
\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \pi_1(C_n(\Sigma_g)) & \xrightarrow{\text{Push}} & \mathrm{PMod}_g^n & \xrightarrow{f} & \mathrm{Mod}_g & \longrightarrow & 1 \qquad (\mathrm{Bir1}_n) \\
 & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q} & & \| {\scriptstyle \mathrm{id}} & & \\
1 & \longrightarrow & \pi_1(B_n(\Sigma_g)) & \xrightarrow{\text{Push}} & \mathrm{Mod}_g^n & \xrightarrow{f} & \mathrm{Mod}_g & \longrightarrow & 1 \qquad (\mathrm{Bir2}_n) \\
 & & \downarrow & & \downarrow & & & & \\
 & & S_n & \overset{\mathrm{id}}{=\!=\!=} & S_n & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}
$$

The columns in this diagram relate the groups $\pi_1(C_n(\Sigma_g))$ and PMod_g^n with the groups $\pi_1(B_n(\Sigma_g))$ and Mod_g^n, respectively, in the same way as the pure braid group P_n is related to the braid group B_n by the short exact sequence

$$1 \to P_n \to B_n \to S_n \to 1.$$

Proofs of the exactness of the sequences in diagram (4) can be found in Birman [3] and Farb–Margalit [8]. The exactness of $(\mathrm{Bir1}_1)$ and $(\mathrm{Bir2}_n)$ requires $g \geq 2$.

Observe that from the short exact sequence $(\mathrm{Bir1}_n)$ we get an action of Mod_g on $H^*(\pi_1(C_n(\Sigma_g));\mathbb{Q})$. The second column in diagram (4) defines an S_n–action on $H^*(\mathrm{PMod}_g^n;\mathbb{Q})$ which restricts to the S_n–action on $H^*(\pi_1(C_n(\Sigma_g));\mathbb{Q})$ defined by the short exact sequence in the first column. The induced map

$$\mathrm{Push}^*\colon H^*\big(\mathrm{PMod}_g^n;\mathbb{Q}\big) \to H^*(\pi_1(C_n(\Sigma_g));\mathbb{Q})$$

is a S_n–map between rational S_n–representations. Moreover, from the commutativity of diagram (4) we have the following.

Proposition 5.1 *The actions of S_n and Mod_g on $H^*(\pi_1(C_n(\Sigma_g));\mathbb{Q})$ commute.*

5.1.2 The Hochschild–Serre spectral sequence We denote the Hochschild–Serre spectral sequence associated to the short exact sequence $(\mathrm{Bir1}_n)$ by $E_*(n)$, where the E_2–page is given by:

$$E_2^{p,q}(n) = H^p(\mathrm{Mod}_g; H^q(\pi_1(C_n(\Sigma_g));\mathbb{Q})),$$

and the spectral sequence converges to $H^{p+q}\big(\mathrm{PMod}_g^n;\mathbb{Q}\big)$. This spectral sequence gives a natural filtration of $H^i\big(\mathrm{PMod}_g^n;\mathbb{Q}\big)$:

$$0 \le F_i^i(n) \le F_{i-1}^i(n) \le \cdots \le F_1^i(n) \le F_0^i(n) = H^i\big(\mathrm{PMod}_g^n;\mathbb{Q}\big), \tag{5}$$

where the successive quotients are $F_p^i(n)/F_{p+1}^i(n) \cong E_\infty^{p,i-p}(n)$.

The following lemma is due to Harer [14, Theorem 4.1] and establishes that Mod_g satisfies the finiteness conditions that our argument requires.

Lemma 5.2 *For $2g+s+r>2$, the mapping class group $\mathrm{Mod}_{g,r}^s$ is a virtual duality group with virtual cohomological dimension $d(g,r,s)$, where $d(g,0,0)=4g-5$, $d(g,r,s)=4g+2r+s-4$, $g>0$ and $r+s>0$, and $d(0,r,s)=2r+s-3$. In particular, $\mathrm{Mod}_{g,r}^s$ is a group of type FP_∞, and for any rational $\mathrm{Mod}_{g,r}^s$–module M, we have $H^p\big(\mathrm{Mod}_{g,r}^s;M\big)=0$ for $p>d(g,r,s)$.*

We now see that the terms of the spectral sequence $E_*(n)$ are finite dimensional S_n–representations.

Proposition 5.3 *For $2\le r\le\infty$, each $E_r^{p,q}(n)$ is a finite dimensional rational S_n–representation and the differentials*

$$d_r^{p,q}(n)\colon E_r^{p,q}(n) \to E_r^{p+r,q-r+1}(n)$$

are S_n–maps.

Proof Let $\sigma\in S_n$ and take $\widetilde{\sigma}\in \mathrm{Push}(\pi_1(B_n(\Sigma_g))<\mathrm{Mod}_g^n$ (see $(\mathrm{Bir2}_n)$). Denote by $c(\widetilde{\sigma})$ the conjugation by $\widetilde{\sigma}$. Diagram (4) then gives

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(C_n(\Sigma_g)) & \longrightarrow & \mathrm{PMod}_g^n & \longrightarrow & \mathrm{Mod}_g & \longrightarrow & 1\\
 & & \downarrow{\scriptstyle c(\widetilde{\sigma})} & & \downarrow{\scriptstyle c(\widetilde{\sigma})} & & \|{\scriptstyle \mathrm{id}} & & \\
1 & \longrightarrow & \pi_1(C_n(\Sigma_g)) & \longrightarrow & \mathrm{PMod}_g^n & \longrightarrow & \mathrm{Mod}_g & \longrightarrow & 1
\end{array}$$

The induced maps $c(\widetilde{\sigma})_r^*\colon E_r^{p,q}(n)\to E_r^{p,q}(n)$ do not depend on the lift of $\sigma\in S_n$ and, by naturality of the Hochschild–Serre spectral sequence, they commute with the differentials. Hence we get an S_n–action on each $E_r^{p,q}(n)$ for $2\le r\le\infty$ that commutes with the differentials. Moreover, naturality also implies that the S_n–action on $H^*(\mathrm{PMod}_g^n;\mathbb{Q})$ induces the corresponding S_n–action on $E_\infty^{p,q}(n)$.

By Lemma 5.2, the group Mod_g is of type FP_∞. Totaro showed in [22, Theorem 4] that the cohomology ring $H^*(\pi_1(C_n(\Sigma_g));\mathbb{Q})$ is generated by cohomology classes from the rings $H^*(\Sigma_g;\mathbb{Q})$ and $H^*(P_n;\mathbb{Q})$. In particular, his result implies that $H^q(\pi_1(C_n(\Sigma_g));\mathbb{Q})$ is a finite dimensional $\mathbb{Q}$–vector space for $q \geq 0$. It follows that

$$E_2^{p,q}(n) = H^p(\mathrm{Mod}_g; H^q(\pi_1(C_n(\Sigma_g));\mathbb{Q}))$$

is a finite dimensional $\mathbb{Q}$–vector space, and likewise for the subquotients $E_r^{p,q}(n)$. □

5.1.3 The forgetful map For the pure braid group, there is a natural map $f_n\colon P_{n+1} \to P_n$ given by "forgetting" the last strand. Similarly, the inclusion $\Sigma_g^{n+1} \hookrightarrow \Sigma_g^n$ induces a homomorphism

$$f_n\colon \mathrm{PMod}_g^{n+1} \to \mathrm{PMod}_g^n$$

that we call the *forgetful map*. We can also think of this map as the one induced by "forgetting a marked point" in Σ_g^n. When restricted to the subgroup $\mathrm{Push}(\pi_1(C_{n+1}(\Sigma_g)))$ it corresponds to the homomorphism in fundamental groups induced by the map $C_{n+1}(\Sigma_g) \to C_n(\Sigma_g)$ given by "forgetting the last coordinate". This gives rise to the commutative diagram (3) that relates the exact sequences $(\mathrm{Bir1}_{n+1})$ and $(\mathrm{Bir1}_n)$.
(6)

$$\begin{array}{ccccccccccc}
 & & 1 & & 1 & & & & & & \\
 & & \downarrow & & \downarrow & & & & & & \\
 & & \pi_1(\Sigma_g^n) & \overset{\mathrm{id}}{=\!=\!=} & \pi_1(\Sigma_g^n) & & & & & & \\
 & & \downarrow & & \downarrow & & & & & & \\
1 & \longrightarrow & \pi_1(C_{n+1}(\Sigma_g)) & \longrightarrow & \mathrm{PMod}_g^{n+1} & \longrightarrow & \mathrm{Mod}_g & \longrightarrow & 1 & & (\mathrm{Bir}_{n+1}) \\
 & & {\scriptstyle f_n}\downarrow & & {\scriptstyle f_n}\downarrow & & {\scriptstyle \mathrm{id}}\Vert & & & & \\
1 & \longrightarrow & \pi_1(C_n(\Sigma_g)) & \longrightarrow & \mathrm{PMod}_g^n & \longrightarrow & \mathrm{Mod}_g & \longrightarrow & 1 & & (\mathrm{Bir1}_n) \\
 & & \downarrow & & \downarrow & & & & & & \\
 & & 1 & & 1 & & & & & &
\end{array}$$

Diagram (6) and our remarks in Section 2.2 imply the following.

Proposition 5.4 *The induced maps*

$$f_n^*\colon H^*(\pi_1(C_n(\Sigma_g));\mathbb{Q}) \to H^*(\pi_1(C_{n+1}(\Sigma_g);\mathbb{Q})$$

are Mod_g*–maps.*

Moreover, diagram (6) and naturality of the Hochschild–Serre spectral sequence give us:

(1) Induced maps $(f_n)_r^*\colon E_r^{p,q}(n) \to E_r^{p,q}(n+1)$ that commute with the differentials. This means that the differentials $d_r^{p,q}(n)$ are consistent maps in the sense of Proposition 2.4.

(2) The map $(f_n)^*\colon H^*(\mathrm{PMod}_g^n;\mathbb{Q}) \to H^*(\mathrm{PMod}_g^{n+1};\mathbb{Q})$ preserves the filtrations (5) inducing a map on the succesive quotients $E_\infty^{p,q}(n)$ which is the map $(f_n)_\infty^*\colon E_\infty^{p,q}(n) \to E_\infty^{p,q}(n+1)$.

(3) The map $(f_n)_2^*\colon E_2^{p,q}(n) \to E_2^{p,q}(n+1)$ is the one induced by the group homomorphisms $\mathrm{id}\colon \mathrm{Mod}_g \to \mathrm{Mod}_g$ and $f_n\colon \pi_1(C_{n+1}(\Sigma_g)) \to \pi_1(C_n(\Sigma_g))$.

5.2 The proof of the main theorem (Theorem 1.1)

In order to prove Theorem 1.1 we use an inductive argument on the pages of the spectral sequence described in Section 5.1 (ii). The following lemma gives us the base of the induction.

Lemma 5.5 *For each $p \geq 0$ and $q \geq 0$, the consistent sequence of rational S_n–representations*

$$\left\{E_2^{p,q}(n) = H^p(\mathrm{Mod}_g; H^q(\pi_1(C_n(\Sigma_g));\mathbb{Q}))\right\}$$

is monotone and uniformly representation stable with stable range $n \geq 4q$.

Proof Let $q \geq 0$. Since $C_n(\Sigma_g)$ is aspherical, by Theorem 1.8 of Church we have that the consistent sequence of rational S_n–representations $\{H^q(\pi_1(C_n(\Sigma_g));\mathbb{Q})\}$ with the forgetful maps

$$f_n\colon H^q(\pi_1(C_n(\Sigma_g));\mathbb{Q}) \to H^q(\pi_1(C_{n+1}(\Sigma_g));\mathbb{Q})$$

is monotone and uniformly representation stable with stable range $n \geq 4q$. Moreover, Propositions 5.1 and 5.4 imply that the sequence is compatible with the Mod_g–action. The group Mod_g is FP_∞ (Lemma 5.2). Hence we can apply Theorem 1.7. □

From Lemma 5.5, we follow the same type of inductive argument from [6, Section 3] that Church uses in order to prove his main result [6, Theorem 1]. Here we get monotonicity and uniform representation stability for all the pages of the spectral sequence $E_*(n)$. We include the proofs here for completeness.

Lemma 5.6 *The sequence $\{E_r^{p,q}(n)\}$ is monotone and uniformly representation stable with stable range $n \geq 4q + 2(r-1)(r-2)$.*

Proof The proof is done by induction on r where the base case $r=2$ is given by Lemma 5.5. Assume that $\{E_r^{p,q}(n)\}$ is monotone and uniformly representation stable for $n \geq 4\big(q+\sum_{k=1}^{r-2} k\big)$.

As noted before, the differentials

$$d_r^{p,q}(n)\colon E_r^{p,q}(n) \to E_r^{p+r,q-r+1}(n)$$

are a consistent sequence of maps in the sense of Proposition 2.4. Then $\{\ker d_r^{p,q}(n)\}$ is monotone and uniformly representation stable for $n \geq 4\big(q+\sum_{k=1}^{r-2} k\big)$. Moreover $\{\operatorname{im} d_r^{p-r,q+r-1}(n)\}$ is monotone and uniformly representation stable for $n \geq 4\big(q+(r-1)+\sum_{k=1}^{r-2} k\big)$. Therefore by Proposition 2.3 the next page in the spectral sequence

$$E_r^{p,q}(n) \cong \ker d_r^{p,q}(n)/\operatorname{im} d_r^{p-r,q+r-1}$$

is monotone and uniformly representation stable for $n \geq 4\big(q+\sum_{k=1}^{r-1} k\big)$. □

Lemma 5.7 *For every $p,q \geq 0$ and every $n \geq 2$, we have $E_\infty^{p,q}(n) = E_R^{p,q}(n)$, where*

$$R = 4g-4 = \operatorname{vcd}(\mathrm{Mod}_g)+1.$$

Proof The Hochschild–Serre spectral sequence $E_*(n)$ is a first-quadrant spectral sequence. Moreover, from Lemma 5.2 it follows that for every $p > 4g-5$

$$0 = H^p(\mathrm{Mod}_g; H^q(\pi_1(C_n(\Sigma_g)) = E_2^{p,q}(n) = E_r^{p,q}(n).$$

Therefore for $R = 4g-4$, $q \geq 0$ and $0 \leq p \leq 4g-5$, we have that

$$E_R^{p-R,q+R-1}(n) = 0$$

since $p-R<0$, and

$$E_R^{p+R,q-R+1}(n) = 0$$

since $p+R > 4g-5$. Then the differentials $d_R^{p,q}$ and $d_R^{p-R,q+R-1}$ are zero and hence

$$E_{R+1}^{p,q}(n) = \ker d_R^{p,q}/\operatorname{im} d_R^{p-R,q+R-1} = E_R^{p,q}(n).$$ □

Having built up, we are now able to prove our main result: uniform representation stability of $\big\{H^i\big(\mathrm{PMod}_{g,r}^n;\mathbb{Q}\big)\big\}_{n=1}^\infty$.

Theorem 1.1 *For any $i \geq 0$ and $g \geq 2$ the sequence of cohomology groups*

$$\big\{H^i\big(\mathrm{PMod}_g^n;\mathbb{Q}\big)\big\}_{n=1}^\infty$$

is monotone and uniformly representation stable with stable range

$$n \geq \min\{4i+2(4g-6)(4g-5), 2i^2+6i\}.$$

Proof Each of the successive quotients of the natural filtration (5) of $H^i\big(\mathrm{PMod}_g^n;\mathbb{Q}\big)$ give us a sequence

$$\big\{F_p^i(n)/F_{p+1}^i(n) \approx E_\infty^{p,i-p}(n)\big\}$$

which, by Lemmas 5.6 and 5.7, is monotone and uniformly representation stable with stable range $n \geq 4(i-p)+2(4g-6)(4g-5)$. This is the case, in particular, for $F_{i-1}^i(n)/F_i^i(n)$ and $F_i^i(n) \approx E_\infty^{i,0}(n)$. Then by Proposition 2.3 we have that $F_{i-1}^i(n)$ is monotone and uniformly representation stable. Reverse induction and Proposition 2.3 imply that the sequences $\big\{F_p^i(n)\big\}$ $(0 \leq p \leq i)$ are monotone and uniformly representation stable with the same stable range. In particular this is true for $F_0^i(n) = H^i\big(\mathrm{PMod}_g^n;\mathbb{Q}\big)$.

Observe that

$$4(i-p)+2(4g-6)(4g-5)+4p \geq 4(i-p)+2(4g-6)(4g-5)$$

for all $0 \leq p \leq i$, which give us the desired stable range.

Finally, we notice that for a fixed $i \geq 0$, the group $H^i\big(\mathrm{PMod}_g^n;\mathbb{Q}\big)$ only depends on the terms $E_\infty^{p,i-p}(n) = E_{i+2}^{p,i-p}(n)$, $i \geq p \geq 0$. Hence from Lemma 5.6 we get a stable range that does not depend on the genus g. However, this stable range is quadratic on i: the sequence $\big\{H^i\big(\mathrm{PMod}_g^n;\mathbb{Q}\big)\big\}$ is monotone and uniformly representation stable for $n \geq 4i+2(i+1)(i) = (2i)(i+3)$. □

5.3 Rational homological stability for Mod_g^n

From the short exact sequence in the second column of diagram (1), we have that any rational S_n–representation can be regarded as a representation of Mod_g^n by composing with the projection $\mathrm{Mod}_g^n \to S_n$. As a consequence of Theorem 1.1 we get cohomological stability for Mod_g^n with twisted coefficients.

Corollary 1.3 *For any partition λ, the sequence $\big\{H^i\big(\mathrm{Mod}_g^n;V(\lambda)_n\big)\big\}_{n=1}^\infty$ of twisted cohomology groups satisfies classical cohomological stability: for fixed $i \geq 0$ and $g \geq 2$, there is an isomorphism*

$$H^i\big(\mathrm{Mod}_g^n;V(\lambda)_n\big) \approx H^i\big(\mathrm{Mod}_g^{n+1};V(\lambda)_{n+1}\big),$$

if $n \geq \min\{4i+2(4g-6)(4g-5), n \geq 2i^2+6i\}$.

Proof This is just the argument by Church–Farb in [7, Corollary 4.4]. The group PMod_g^n is a finite index subgroup of Mod_g^n and the coefficients $V(\lambda)_n$ are rational vector spaces, therefore the transfer map (see Brown [5]) give us an isomorphism

$$H^i\big(\mathrm{Mod}_g^n;V(\lambda)_n\big) \approx H^i\big(\mathrm{PMod}_g^n;V(\lambda)_n\big)^{S_n}.$$

Moreover, $V(\lambda)_n$ is a trivial PMod_g^n–representation, since the action of Mod_g^n on $V(\lambda)_n$ factors through S_n. Hence, from the universal coefficient theorem, we have

$$H^i\big(\mathrm{PMod}_g^n; V(\lambda)_n)\big)^{S_n} \approx \big(H^i\big(\mathrm{PMod}_g^n;\mathbb{Q}\big)\otimes V(\lambda)_n\big)^{S_n}. \tag{7}$$

For two partitions λ and μ of n the representation $V(\lambda)\otimes V(\mu)$ contains the trivial representation if and only if $\lambda=\mu$, in which case it has multiplicity 1 (see Fulton–Harris [9]). Therefore the dimension of (7) is the multiplicity of $V(\lambda)_n$ in $H^i\big(\mathrm{PMod}_g^n;\mathbb{Q}\big)$ which is constant for $n\geq 4i+2(4g-6)(4g-5)$ by Theorem 1.1. This completes the proof. □

In particular, the multiplicity of the trivial representation in $H^i\big(\mathrm{PMod}_g^n;\mathbb{Q}\big)$, which equals $H^i\big(\mathrm{Mod}_g^n;\mathbb{Q}\big)$, is constant for $n\geq 4i+2(4g-6)(4g-5)$. In fact, the stable range in this case can be slightly improved.

Corollary 5.8 *For any $i\geq 0$ and a fixed $g\geq 2$, the sequence of mapping class groups $\big\{\mathrm{Mod}_g^n\big\}_{n=1}^{\infty}$ satisfies rational cohomological stability:*

$$H^i\big(\mathrm{Mod}_g^n;\mathbb{Q}\big)\approx H^i\big(\mathrm{Mod}_g^{n+1};\mathbb{Q}\big),$$

if $n\geq \max\{i+(2g-3)(4g-5), 2i^2+4i\}$.

Proof For any n the S_n–invariants of the spectral sequence $\big(E_2^{p,q}\big)^{S_n}$ form a spectral sequence that converges to $H^{p+q}\big(\mathrm{PMod}_g^n;\mathbb{Q}\big)^{S_n}$. In fact, $\big(E_2^{p,q}\big)^{S_n}$ is just the (p,q)–term of the E_2–page of the Hochschild–Serre spectral sequence of the group extension (Bir2$_n$) converging to $H^{p+q}\big(\mathrm{Mod}_g^n;\mathbb{Q}\big)$. In Church [6, Corollary 3] a better stable range than the one in Theorem 1.8 is obtained when restricted to the S_n–invariants: the dimension of $H_q(C_n(\Sigma_g);\mathbb{Q})^{S_n}$ is constant for $n>q$. As a consequence the dimension of $(E_2^{p,q})^{S_n}$ is constant for $n\geq q$. Proposition 2.5 allows us to repeat the general argument for this spectral sequence of S_n–invariants in order to get the desired stable range. □

5.4 Non-closed surfaces

Our main result is also true if we consider a non-closed surface $\Sigma_{g,r}^s$ of genus g, with r boundary components and s punctures with $2g+r+s>2$.

Let $p_1,\dots,p_n$ be distinct points in the interior of $\Sigma_{g,r}^s$. We define the *mapping class group* $\mathrm{Mod}^n\big(\Sigma_{g,r}^s\big)$ as the group of isotopy classes of orientation-preserving self-diffeomorphisms of $\Sigma_{g,r}^s$ that permute the distinguished points $p_1,\dots,p_n$ and that restrict to the identity on the boundary components. The *pure mapping class group*

$\mathrm{PMod}^n_{g,r}$ is defined analogously by asking that the distinguished points $p_1, \dots, p_n$ remain fixed pointwise.

When $2g + r + s > 2$ we have again a Birman exact sequence (see Farb–Margalit [8]):

$$1 \to \pi_1\big(C_n\big(\Sigma_g^{r+s}\big)\big) \to \mathrm{PMod}^n\left(\Sigma^s_{g,r}\right) \to \mathrm{Mod}^s_{g,r} \to 1.$$

In particular, this includes the three punctured sphere Σ^3_0 and the punctured torus Σ^1_1.

Using this short exact sequence and Theorem 1.8 we can use the previous arguments to get representation stability for the cohomology of $\mathrm{PMod}^n\left(\Sigma^s_{g,r}\right)$, when $2g+s+r>2$.

Theorem 5.9 *For any $i \geq 0$ and $2g+s+r>2$ the sequence*

$$\left\{H^i\left(\mathrm{PMod}^n(\Sigma^s_{g,r});\mathbb{Q}\right)\right\}_{n=1}^{\infty}$$

is monotone and uniformly representation stable with stable range

$$n \geq \min\{4i + 2\big(d(g,r,s)\big)\big(d(g,r,s)-1\big), 2i^2+6i\}.$$

Furthermore for any partition λ and any fixed $i \geq 0$ and $2g+s+r>2$, there is an isomorphism

$$H^i\left(\mathrm{Mod}^n(\Sigma^s_{g,r}); V(\lambda)_n\right) \approx H^i\left(\mathrm{Mod}^{n+1}\left(\Sigma^s_{g,r}\right); V(\lambda)_{n+1}\right),$$

if $n \geq \min\{4i + 2(d(g,r,s))(d(g,r,s)-1), 2i^2+6i\}$.

Here $d(g,r,s)$ denotes the virtual cohomological dimension of $\mathrm{Mod}^s_{g,r}$ as in Lemma 5.2.

In the case of trivial coefficients $V(0)_n = \mathbb{Q}$ we recover puncture stability for the rational cohomology groups of $\mathrm{Mod}^n(\Sigma^s_{g,r})$ for $2g+s+r>2$.

6 Pure mapping class groups of higher dimensional manifolds

We now explain how the key ideas from before can be applied to obtain representation stability for the cohomology of pure mapping class groups of higher dimensional manifolds.

6.1 Representation stability for $H^*(\mathrm{PMod}^n(M);\mathbb{Q})$

Let M be a connected, smooth manifold and consider the mapping class group $\mathrm{Mod}^n(M)$ and the pure mapping class group $\mathrm{PMod}^n(M)$ as defined in the introduction. We now show how, in some cases, the previous techniques and Proposition 6.5 from Section 6.2 can be used to prove representation stability for $\left\{H^i(\mathrm{PMod}^n(M);\mathbb{Q}), f^i_n\right\}$.

Notation We denote by $C_n(M)$ (resp. $B_n(M)$) the configuration space of n distinct ordered (resp. unordered) points in the interior of any manifold M. We refer to $p_1,\dots,p_n$ as the "punctures" or the "marked points". We will usually take the n–tuple $\mathfrak{p}=(p_1,\dots,p_n)\in C_n(M)$ as the base point of $\pi_1(C_n(M))$ (resp. $\pi_1(B_n(M))$). The group $P_n:=\pi_1(C_n(\mathbb{R}^2),\mathfrak{p})\approx \mathrm{PMod}_{0,1}^n$ is the *pure braid group* and the *braid group* is $B_n:=\pi_1(B_n(\mathbb{R}^2),\mathfrak{p})\approx \mathrm{Mod}_{0,1}^n$.

The inclusion

$$\big(M-\{p_1,\dots,p_n,p_{n+1}\}\big)\hookrightarrow\big(M-\{p_1,\dots,p_n\}\big)$$

induces the *forgetful homomorphism*

$$f_n\colon \mathrm{PMod}^{n+1}(M)\to \mathrm{PMod}^n(M).$$

Recall that one of the main ingredients needed in our proof of Theorem 1.1 is the existence of a Birman exact sequence that allows us to relate $\pi_1(C_n(M),\mathfrak{p})$ with $\mathrm{PMod}^n(M)$. First we notice that, when the dimension of M is $d\geq 3$, the group $\pi_1(C_n(M))$ can be completely understood in terms of $\pi_1(M)$.

Lemma 6.1 *Let M be a smooth connected manifold of dimension $d\geq 3$. Then for any $n\geq 1$ the inclusion map $C_n(M)\hookrightarrow M^n$ induces an isomorphism $\pi_1(C_n(M),\mathfrak{p})\approx \pi_1(M^n,\mathfrak{p})\approx\prod_{i=1}^n\pi_1(M,p_i)$.*

The case for closed manifolds is due to Birman [2, Theorem 1]. As Allen Hatcher explained to me, there are many manifolds for which there is a Birman exact sequence.

Lemma 6.2 (Existence of a Birman Exact Sequence) *Let M be a smooth connected manifold of dimension $d\geq 3$. If the fundamental group $\pi_1(M)$ has trivial center or* $\mathrm{Diff}(M)$ *is simply connected, then there exists a Birman exact sequence*

$$1\longrightarrow\pi_1(C_n(M))\longrightarrow \mathrm{PMod}^n(M)\longrightarrow \mathrm{Mod}(M)\longrightarrow 1. \tag{8}$$

Proof The evaluation map

$$\mathrm{ev}\colon \mathrm{Diff}(M)\to C_n(M),$$

given by $f\mapsto (f(p_1),\dots,f(p_n))$ is a fibration with fiber $\mathrm{PDiff}^n(M)$. Consider the associated long exact sequence in homotopy groups

$$\cdots\longrightarrow\pi_1(\mathrm{Diff}(M))\longrightarrow\pi_1(C_n(M))\overset{\delta}{\longrightarrow}\pi_0(\mathrm{PDiff}^n(M))\longrightarrow\pi_0(\mathrm{Diff}(M))\longrightarrow 1.$$

If $\mathrm{Diff}(M)$ is simply connected, then the existence of the short exact sequence (8) follows. On the other hand, we may consider the map

$$\psi\colon \pi_0(\mathrm{PDiff}^n(M)) \to \mathrm{Aut}[\pi_1(C_n(M))]$$

given by $[f] \mapsto [\gamma \mapsto f \circ \gamma]$.

The composition

$$\pi_1(C_n(M)) \xrightarrow{\delta} \pi_0(\mathrm{PDiff}^n(M)) \xrightarrow{\psi} \mathrm{Aut}[\pi_1(C_n(M))]$$

sends $\sigma \in \pi_1(C_n(M))$ to the inner automorphism $c(\sigma)$ given by conjugation by σ. If the dimension $d \geq 3$ and $\pi_1(M)$ has trivial center, then so does $\pi_1(C_n(M))$ by Lemma 6.1. In this case, the boundary map δ is injective and we get the desired Birman exact sequence (8). □

The E_2–page of the Hochschild–Serre spectral sequence associated to (8) is then

$$E_2^{p,q}(n) = H^p(\mathrm{Mod}(M); H^q(\pi_1(C_n(M)); \mathbb{Q})).$$

By Lemma 6.1

$$H^q(\pi_1(C_n(M)); \mathbb{Q})) = H^q(\pi_1(M)^n; \mathbb{Q}).$$

Moreover, by Proposition 6.5 below, if the group $\pi_1(M)$ is of type FP_∞, the consistent sequence $\{H^q(\pi_1(M)^n; \mathbb{Q})\}_{n=1}^{\infty}$ is monotone and uniformly representation stable, with stable range $n \geq 2q$. Hence when $\mathrm{Mod}(M)$ is also of type FP_∞ (for example, M is compact), Theorem 1.7 and the same inductive argument on the successive pages of spectral sequence yield the following:

Lemma 6.3 *For every $i \geq 0$ and every $n \geq 2$, the consistent sequence of rational S_n–representations*

$$\{E_2^{i-q,q}(n) = H^{i-q}(\mathrm{Mod}(M); H^q(\pi_1(C_n(M)); \mathbb{Q}))\}_{n=1}^{\infty}$$

is monotone and uniformly representation stable with stable range $n \geq 2q$. Furthermore $E_\infty^{i-q,q}(n) = E_{i+2}^{i-q,q}(n)$, which is monotone and uniformly representation stable with stable range

$$n \geq 2q + 2(i+1)(i).$$

Observe that now we have all the ingredients needed in order to reproduce our arguments from Section 5.2 and prove Theorem 1.4 and Corollary 1.5.

6.2 Representation stability of $H^*(G^n;\mathbb{Q})$

Given a group G, we may consider the sequence of groups $\{G^n = \prod_{i=1}^n G\}$ with the corresponding S_n–action given by permuting the factors. The natural homomorphism $G^{n+1} \to G^n$ by forgetting the last coordinate is equivariant with respect to the inclusion $S_n \hookrightarrow S_{n+1}$. For a fixed $q \geq 0$ the induced maps

$$\phi_n\colon H^q(G^n;\mathbb{Q}) \to H^q(G^{n+1};\mathbb{Q})$$

give us a consistent sequence of S_n–representations. If G is of type FP_∞, we have finite dimensional representations. Monotonicity and uniform representation stability of this sequence are a particular case of Church [6, Proposition 3.1] (corresponding to the first row in the spectral sequence). Since this result gives us the inductive hypothesis for the proof of Theorem 1.4, we present here a complete proof for the reader's convenience.

For a fixed S_l–representation V and each $n \geq l$, we denote by $V_\alpha \boxtimes \mathbb{Q}$ the corresponding $(S_l \times S_{n-l})$–representation, where the factor S_{n-l} acts trivially. We can then consider the sequence of S_n–representation $\{\operatorname{Ind}_{S_l\times S_{n-l}}^{S_n} V_\alpha \boxtimes \mathbb{Q}\}$ with the natural inclusions

$$\iota_n\colon \operatorname{Ind}_{S_l\times S_{n-l}}^{S_n} V_\alpha \boxtimes \mathbb{Q} \hookrightarrow \operatorname{Ind}_{S_l\times S_{n+1-l}}^{S_{n+1}} V_\alpha \boxtimes \mathbb{Q}.$$

This sequence is monotone and uniform representation stable as proved in [6, Theorem 2.11]:

Lemma 6.4 *Let V be a finite dimensional S_l–representation, then the sequence of induced representations $\{\operatorname{Ind}_{S_l\times S_{n-l}}^{S_n} V \boxtimes \mathbb{Q}\}_{n=1}^\infty$ is monotone and uniformly representation stable for $n \geq 2l$.*

This lemma and the Künneth formula give us the following result.

Proposition 6.5 *Let G be any group of type FP_∞ and $q \geq 0$. The consistent sequence of S_n–representations $\{H^q(G^n;\mathbb{Q}), \phi_n\}_{n=1}^\infty$ is monotone and uniformly representation stable for $n \geq 2q$.*

Proof By the Künneth formula we have

$$H^q(G^n;\mathbb{Q}) \approx \bigoplus_{\mathfrak{a}} H^{\mathfrak{a}}(G^n)$$

where the sum is over all tuples $\mathfrak{a} = (a_1,\dots,a_n)$ such that $a_j \geq 0$ and $\sum a_j = q$ and $H^{\mathfrak{a}}(G^n)$ denotes $H^{a_1}(G;\mathbb{Q}) \otimes \cdots \otimes H^{a_n}(G;\mathbb{Q})$.

Let $\overline{\mathfrak{a}} = \alpha$ where $\alpha = (\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_l)$ is a partition of q and the α_j are the positive values of $\mathfrak{a}$ arranged in decreasing order. We define $\operatorname{supp}(\mathfrak{a})$ as the subset of $\{1, 2, \ldots, n\}$ for which $a_i \neq 0$. Observe that the length of α is $l = |\operatorname{supp}(\mathfrak{a})| \leq q$. Therefore we have

$$H^q(G^n; \mathbb{Q}) = \bigoplus_{\alpha} H^{\alpha}(G^n)$$

where now the sum is over all partitions α of q of length $l \leq q$ and $H^{\alpha}(G^n) = \bigoplus_{\overline{\mathfrak{a}}=\alpha} H^{\mathfrak{a}}(G^n)$.

The natural S_n–action on G^n induces an S_n–action on $H^q(G^n; \mathbb{Q})$. More precisely, the group S_n acts on n–tuples $\mathfrak{a}$ by permuting the coordinates. This induces an action on $\bigoplus_{\overline{\mathfrak{a}}=\alpha} H^{\mathfrak{a}}(G^n)$ by permuting the summands accordingly (with a sign, since cohomology is graded commutative). Hence, under this action, each $H^{\alpha}(G^n)$ is S_n–invariant. We now describe $H^{\alpha}(G^n)$ as an induced representation.

For a given α, take $\mathfrak{b} = (\alpha_1, \ldots, \alpha_l, 0, \cdots, 0)$. Observe that we can identify the S_n–translates of $H^{\mathfrak{b}}(G^n)$ with the cosets $S_n / \operatorname{Stab}(\mathfrak{b})$ by an orbit-stabilizer argument. Thus

$$H^{\alpha}(G^n) = \operatorname{Ind}_{\operatorname{Stab}(\mathfrak{b})}^{S_n} H^{\mathfrak{b}}(G^n).$$

Moreover, $S_{n-l} < \operatorname{Stab}(\mathfrak{b}) < S_l \times S_{n-l}$, where S_l permutes coordinates $\{1, \ldots, l\}$ and S_{n-l} permutes coordinates $\{l+1, \ldots, n\}$. Therefore $\operatorname{Stab}(\mathfrak{b}) = H \times S_{n-l}$, for some subgroup $H < S_l$.

Notice that

$$\begin{aligned} H^{\mathfrak{b}}(G^n) &= H^{b_1}(G; \mathbb{Q}) \otimes \cdots \otimes H^{b_l}(G; \mathbb{Q}) \otimes \cdots \otimes H^0(G; \mathbb{Q}) \\ &\approx H^{b_1}(G; \mathbb{Q}) \otimes \cdots \otimes H^{b_l}(G; \mathbb{Q}) \end{aligned}$$

can be regarded as an H–representation.

Let $V_\alpha := \operatorname{Ind}_H^{S_l} H^{\mathfrak{b}}(G^n)$ and let $V_\alpha \boxtimes \mathbb{Q}$ denote the corresponding $(S_l \times S_{n-l})$–representation. Then

$$\begin{aligned} H^{\alpha}(G^n) &= \operatorname{Ind}_{\operatorname{Stab}(\mathfrak{b})}^{S_n} H^{\mathfrak{b}}(G^n) \\ &= \operatorname{Ind}_{H \times S_{n-l}}^{S_n} (H^{\mathfrak{b}}(G^n) \boxtimes \mathbb{Q}) \\ &= \operatorname{Ind}_{S_l \times S_{n-l}}^{S_n} \big(\operatorname{Ind}_{H \times S_{n-l}}^{S_l \times S_{n-l}} (H^{\mathfrak{b}}(G^n) \boxtimes \mathbb{Q}) \big) \\ &= \operatorname{Ind}_{S_l \times S_{n-l}}^{S_n} \big(\big(\operatorname{Ind}_H^{S_l} H^{\mathfrak{b}}(G^n) \big) \boxtimes \mathbb{Q} \big) \\ &= \operatorname{Ind}_{S_l \times S_{n-l}}^{S_n} V_\alpha \boxtimes \mathbb{Q}. \end{aligned}$$

Moreover, we notice that the forgetful map ϕ_n restricted to the summand $H^\alpha(G^n)$ corresponds to the inclusion

$$\mathrm{Ind}_{S_l \times S_{n-l}}^{S_n} V_\alpha \boxtimes \mathbb{Q} \hookrightarrow \mathrm{Ind}_{S_l \times S_{n+1-l}}^{S_{n+1}} V_\alpha \boxtimes \mathbb{Q}.$$

Therefore, by Lemma 6.4, the consistent sequence $\{H^\alpha(G^n)\}$ is monotone and uniformly representation stable with stable range $n \geq 2l$, where l is the length of α and $l \leq q$. The result for $\{H^q(G^n;\mathbb{Q}), \phi_n\}$ then follows from Proposition 2.3. □

We illustrate the notation in the previous proof with the concrete case of $G = \mathbb{Z}$.

By the Künneth formula we have

$$H^q(\mathbb{Z}^n;\mathbb{Q}) \approx \bigoplus_{\sum a_i = q} H^{a_1}(\mathbb{Z};\mathbb{Q}) \otimes \cdots \otimes H^{a_n}(\mathbb{Z};\mathbb{Q}).$$

Following our previous notation we take the n–tuple $\mathfrak{b} = (1,\dots,1,0,\dots,0)$ with $|\operatorname{supp}(\mathfrak{b})| = q$ and $\alpha := \overline{\mathfrak{b}}$. Since $H^q(\mathbb{Z};\mathbb{Q}) = \mathbb{Q}$ for $q = 0, 1$ and zero otherwise, we have that

$$H^q(\mathbb{Z}^n;\mathbb{Q}) = \bigoplus_{\overline{\mathfrak{a}} = \alpha} H^{\mathfrak{a}}(\mathbb{Z}^n) = \mathrm{Ind}_{\mathrm{Stab}(\mathfrak{b})}^{S_n} H^{\mathfrak{b}}(\mathbb{Z}^n).$$

Notice that $\mathrm{Stab}(\mathfrak{b}) = S_q \times S_{n-q}$. The corresponding $(S_q \times S_{n-q})$–representation is

$$H^{\mathfrak{b}}(\mathbb{Z}^n) = H^1(\mathbb{Z};\mathbb{Q}) \otimes \cdots \otimes H^1(\mathbb{Z};\mathbb{Q}) \otimes \cdots \otimes H^0(\mathbb{Z};\mathbb{Q}) \approx V_\alpha \boxtimes \mathbb{Q}$$

where

$$V_\alpha := H^1(\mathbb{Z};\mathbb{Q}) \otimes \cdots \otimes H^1(\mathbb{Z};\mathbb{Q}) \approx H^{\mathfrak{b}}(\mathbb{Z}^n)$$

is regarded as an S_q–representation. Then, as an induced representation,

$$H^q(\mathbb{Z}^n;\mathbb{Q}) = \mathrm{Ind}_{S_q \times S_{n-q}}^{S_n} V_\alpha \boxtimes \mathbb{Q}.$$

Moreover, if $\mathbb{Q}^n$ denotes the permutation S_n–representation, then

$$\begin{aligned}
\mathrm{Ind}_{S_q \times S_{n-q}}^{S_n} V_\alpha \boxtimes \mathbb{Q} &= \textstyle\bigwedge^q(\mathbb{Q}^n) = \bigwedge^q\big(V(0)_n \oplus V(1)_n\big) \\
&= \Big(\textstyle\bigwedge^q V(1)_n\Big) \oplus \Big(\bigwedge^{q-1} V(1)_n\Big) \\
&= V(\underbrace{1,\dots,1}_{q})_n \oplus V(\underbrace{1,\dots,1}_{q-1})_n
\end{aligned}$$

Hence, we see explicitly how uniform multiplicity stability holds for this particular case.

7 Classifying spaces for diffeomorphism groups

In this last section we see how the same ideas also imply representation stability for the cohomology of classifying spaces for diffeomorphism groups.

Let M be a connected and compact smooth manifold of dimension $d \geq 3$. We denote by $\mathcal{E}(M, \mathbb{R}^\infty)$ the space of smooth embeddings $M \to \mathbb{R}^\infty$. It is a contractible space and $\mathrm{Diff}(M \operatorname{rel} \partial M)$ acts freely by pre-composition. The quotient space $\mathcal{E}(M, \mathbb{R}^\infty)/\operatorname{Diff}(M \operatorname{rel} \partial M)$ is a classifying space $B\operatorname{Diff}(M \operatorname{rel} \partial M))$ for the group $\mathrm{Diff}(M \operatorname{rel} \partial M))$. Similarly we can consider the action of the subgroup $\mathrm{PDiff}^n(M)$ of $\mathrm{Diff}(M \operatorname{rel} \partial M)$ (defined in the Introduction) on $\mathcal{E}(M, \mathbb{R}^\infty)$. The quotient space is a classifying space $B\,\mathrm{PDiff}^n(M)$ for $\mathrm{PDiff}^n(M)$ and we have a fiber bundle

$$B\,\mathrm{PDiff}^n(M) \to B\,\mathrm{Diff}(M \operatorname{rel} \partial M) \tag{9}$$

where the fiber is given by $\mathrm{Diff}(M \operatorname{rel} \partial M)/\operatorname{PDiff}^n(M) \approx C_n(M)$, the configuration space of n ordered points in M.

On the other hand we can consider the forgetful homomorphism $\mathrm{PDiff}^{n+1}(M) \to \mathrm{PDiff}^n(M)$, which induces a corresponding map between classifying spaces

$$f_n\colon B\,\mathrm{PDiff}^{n+1}(M) \to B\,\mathrm{PDiff}^n(M).$$

There is a Leray–Serre spectral sequence associated to the fiber bundle (9) that converges to the cohomology $H^*(B\,\mathrm{PDiff}^n(M); \mathbb{Q})$ with E_2–page given by

$$E_2^{p,q}(n) = H^p(B\,\mathrm{Diff}(M \operatorname{rel} \partial M); H^q(C_n(M); \mathbb{Q})). \tag{10}$$

Here, we regard (10) as the pth cohomology group of $B\,\mathrm{Diff}(M \operatorname{rel} \partial M)$ with local coefficients in the G–module $H^q(C_n(M); \mathbb{Q})$, where $G = \pi_1(B\,\mathrm{Diff}(M \operatorname{rel} \partial M))$ (see Hatcher [16, Section 3.H]). Notice that the actions of S_n and G on $H^q(C_n(M); \mathbb{Q})$ commute. Therefore $\{H^q(C_n(M); \mathbb{Q})\}_{n=1}^\infty$ is a consistent sequence compatible with G–actions. Moreover, by Theorem 1.8, it is monotone and uniformly representation stable, with stable range $n \geq 2q$. Monotonicity and uniform representation stability for the terms in the E_2–page will be a consequence of the following result, which is essentially Theorem 1.7 from before.

Theorem 7.1 (Representation stability with changing coefficients 2) *Let G be the fundamental group of a connected CW complex X with finitely many cells in each dimension. Consider a consistent sequence $\{V_n, \phi_n\}_{n=1}^\infty$ of finite dimensional rational representations of S_n compatible with G–actions. If the sequence $\{V_n, \phi_n\}_{n=1}^\infty$ is monotone and uniformly representation stable with stable range $n \geq N$, then for any non-negative integer p, the sequence of cohomology groups with local coefficients*

$\{H^p(X;V_n),\phi_n^*\}_{n=1}^{\infty}$ *is monotone and uniformly representation stable with the same stable range.*

Proof Since $G=\pi_1(X)$, the universal cover $\widetilde{X}$ of X has a G–equivariant cellular chain complex. Given that X has finitely many cells in each dimension, for each p the group $C_p(\widetilde{X})$ is a free G–module of finite rank, where a preferred G–basis can be provided by selecting a p–cell in $\widetilde{X}$ over each p–cell in X. Hence, the proof of Theorem 7.1 is the same as the one for Theorem 1.7, by replacing the notions of cohomology of groups by cohomology of a space with local coefficients. □

Hence when $B\,\mathrm{Diff}(M \;\mathrm{rel}\; \partial M)$ has the homotopy type of a CW-complex with finitely many cells in each dimension, we can apply the inductive argument from Section 5.2 on the successive pages of the Leray–Serre spectral sequence from above and obtain the following result.

Lemma 7.2 *For every $i\geq 0$ and every $n\geq 2$, the consistent sequence of rational S_n–representations*

$$\left\{E_2^{i-q,q}(n)=H^{i-q}\big(B\,\mathrm{Diff}(M \;\mathrm{rel}\; \partial M);H^q(C_n(M);\mathbb{Q})\big)\right\}_{n=1}^{\infty}$$

is monotone and uniformly representation stable with stable range $n\geq 2q$. Furthermore $E_\infty^{i-q,q}(n)=E_{i+2}^{i-q,q}(n)$, which is monotone and uniformly representation stable with stable range

$$n\geq 2q+2(i+1)(i).$$

As a consequence we get Theorem 1.6 for the cohomology of the classifying space of a group of diffeomorphisms.

Since the manifold M is orientable, we can replace $\mathrm{Diff}(M \;\mathrm{rel}\; \partial M)$ by the group of orientation-preserving diffeomorphims $\mathrm{Diff}^+(M \;\mathrm{rel}\; \partial M)$ in the above argument. In particular, Hatcher and McCullough proved in [17] that if M is an irreducible, compact connected orientable 3–manifold with nonempty boundary, then $B\,\mathrm{Diff}^+(M \;\mathrm{rel}\; \partial M)$ is a finite $\mathrm{K}(\pi,1)$–space for the mapping class group $\mathrm{Mod}(M)$. Therefore, Theorem 1.6 is true for this type of manifold. Moreover, if M satisfies conditions (i)–(iv) in [17, Section 3], then $\pi_1(M)$ is centerless and we can apply Theorem 1.4 to get uniform representation stability for the cohomology of $\mathrm{PMod}^n(M)$.

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