# Estimating the higher symmetric topological complexity of spheres

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We study questions of the following type: Can one assign continuously and $\Sigma_m$–equivariantly to any $m$–tuple of distinct points on the sphere $S^n$ a multipath in $S^n$ spanning these points? A *multipath* is a continuous map of the wedge of $m$ segments to the sphere. This question is connected with the *higher symmetric topological complexity* of spheres, introduced and studied by I Basabe, J González, Yu B Rudyak, and D Tamaki. In all cases we can handle, the answer is negative. Our arguments are in the spirit of the definition of the Hopf invariant of a map $f\colon S^{2n-1} \to S^n$ by means of the mapping cone and the cup product.

## 1 Introduction

Let us begin with some definitions following Basabe, González, Rudyak and Tamaki [1]. For $m \geq 2$, denote by $J_m$ the space obtained by gluing $m$ segments $I_i = [0, 1]_i$ ($i = 1, \ldots, m$) together at $t = 0$. Denote the set of $m$ right ends (for $t = 1$) of these segments by $\partial J_m$.

**Definition 1.1** For a topological space $X$ let $P_m(X)$ be the space of all continuous maps $J_m \to X$ with the compact-open topology.

**Definition 1.2** Consider the fibration

$$e_{m,X}\colon P_m(X) \to X^{\times m},$$

which assigns to a map $f\colon J_m \to X$ the $m$–tuple of points $f(1_i)$ comprising $f(\partial J_m)$. This map is $\Sigma_m$–equivariant with respect to the natural actions of the permutation group $\Sigma_m$ on $P_m(X)$ (by right multiplication) and on $X^{\times m}$.

The Schwarz genus (see Schwarz [14]) of the fibration $e_{m,X}\colon P_m(X) \to X^{\times m}$ is called the *$m$th topological complexity $TC_m(X)$* of $X$ in [1]. (We always use the "reduced" Schwarz genus, so a fibration has a continuous global cross section if and only if its Schwarz genus is 0.) Making use of the actions of $\Sigma_m$ and the equivariance of the projection map, the equivariant Schwarz genus [14] of the fibration $e_{m,X}\colon P_m(X) \to X^{\times m}$ is called the *$m$th symmetric topological complexity $TC_m^{\Sigma}(X)$* of $X$ in [1].

The study of topological complexity was begun by M Farber in [7], and much of its development is summarized in Farber's book [8, Chapter 4]. The number $TC_2(X)$ coincides with the Schwarz genus of the fibration $P_2(X) \to X \times X$. In this case $P_2(X)$ agrees with the space of paths in $X$ because $J_2$ is just a segment. A section of $e_{2,X}$ over an open subset of $X \times X$ may be viewed as a partial motion planning algorithm, which is a continuous assignment of paths joining certain pairs of points of $X$. The minimal number minus one of open subsets of $X \times X$ which cover $X \times X$ and over which sections of $e_{2,X}$ exist is the reduced Schwarz genus of the fibration $e_{2,X}$; hence the minimal topological complexity of a motion planning algorithm coincides with $TC_2(X)$. If one requires a motion planning algorithm to be symmetric (the considered open subsets of $X \times X$ contain $(y, x)$ when they contain $(x, y)$, and the path assigned to a pair $(y, x)$ is the reverse of the path assigned to $(x, y)$), then the corresponding topological complexity becomes the symmetric topological complexity $TC_2^{\Sigma}(X)$.

The higher topological complexity $TC_m(X)$ is obtained from motion planning algorithms spanning $m$–tuples of points by a path; it does not really matter whether we span points by a path or by an image of $J_m$ (see Rudyak [13]). The intuitive meaning of $TC_m^{\Sigma}(X)$ for motion planning algorithms is less clear, since in this case it is essential that $J_m$ is acted on by the symmetric group, and any "model" chosen as a replacement for $J_m$ would also need an action by the symmetric group.

**Definition 1.3** Define the configuration space $F_m(X)$ to be

$$F_m(X) = \{(x_1, \ldots, x_m) \in X^{\times m} : x_i \neq x_j \text{ for } i \neq j\}.$$

In [1, Theorem 4.12], Basabe, González, Rudyak and Tamaki show that lower and upper estimates of higher symmetric topological complexity involve the equivariant Schwarz genus of the restricted fibrations $\varepsilon_{m,X} := e_{m,X}\big|_{e_{m,X}^{-1}(F_m(X))}$; it is important that $\Sigma_m$ acts freely on $F_m(X)$ and its preimage under $e_{m,X}$. The first step is to distinguish between (reduced) equivariant Schwarz genus zero and positive equivariant Schwarz genus. This leads to the following question, raised and answered in some particular cases in [1], and studied further in this paper:

**Question 1.4** Does the fibration $\varepsilon_{m,S^n}$ have a $\Sigma_m$–equivariant section?

Here we are going to prove:

**Theorem 1.5** *If $m \geq 3$ and $n \equiv 2 \pmod 4$, then $\varepsilon_{m,S^n}$ cannot have a $\Sigma_m$–equivariant section.*

**Theorem 1.6** *If $n \equiv 0 \pmod 4$, $n > 4$, and $n/4$ is not a power of $3$, then $\varepsilon_{3,S^n}$ cannot have a $\Sigma_3$–equivariant section.*

We state a more general result for all odd primes (including $3$) as well as their multiples:

**Theorem 1.7** *If $m$ is divisible by an odd prime $p$ and $n = 4n' p^s$ where $s \geq 0$, $p \nmid n'$, and $n' > \frac{p-1}{2}$, then $\varepsilon_{m,S^n}$ cannot have a $\Sigma_m$–equivariant section.*

**Remark 1.8** Assuming that the fibration $\varepsilon_{m,S^n}$ has an equivariant section for $m$ divisible by two odd primes $p < q$ and $n = 4n' p^s q^t$ with $n'$ coprime with $p$ and $q$, we obtain the inequalities: $n' p^s \leq \frac{q-1}{2}$ and $n' q^t \leq \frac{p-1}{2}$. From the latter inequality it follows that $t = 0$, $n = 4n' p^s$, and the former inequality now becomes $n \leq 2(q-1)$. Hence for any $m$ with two distinct odd prime divisors the fibration $\varepsilon_{m,S^n}$ does not have an equivariant section for all $n$ apart from a finite number of possible exceptions.

The proofs of the above theorems rely on actually proving nonexistence of a $D_{2m}$–equivariant section over the space of regular $m$–gons; see Section 3 for the definitions. But the proofs of the following theorems require the whole configuration space:

**Theorem 1.9** *If $m \geq 5$ is a prime and $n \equiv 0 \pmod 4$, then $\varepsilon_{m,S^n}$ cannot have a $\Sigma_m$–equivariant section.*

**Theorem 1.10** *If $m \geq 4$ is even and $n \geq 2$, then $\varepsilon_{m,S^n}$ cannot have a $\Sigma_m$–equivariant section.*

**Remark 1.11** In [1, Proposition 5.5] it is proved that $\varepsilon_{m,S^n}$ has no $\Sigma_m$–equivariant section for odd $n$ and any $m \geq 2$. In the case of odd $n$ there is a simple cohomological obstruction based on the degree of a map of a sphere to itself, while in the above theorems the obstruction uses an analogue of the Hopf invariant. In addition, by using the mod 2 degree, it is proved in [1] prior to the proof of Proposition 5.5 that $\varepsilon_{2,S^n}$ has no $\Sigma_2$–equivariant section for all $n \geq 1$.

**Remark 1.12** The first open case of Question 1.4 is therefore $m = 3$, $n = 4$; see the discussion of this case in the final section. Nonetheless, the theorems proved here (and Remark 1.8), together with the lack of any approach that seems likely to produce equivariant sections, provide evidence for the conjecture that there is no case with $m \geq 2$ in which $\varepsilon_{m,S^n}$ has a $\Sigma_m$–equivariant section.

**Remark 1.13**  In [1, Corollary 5.4] it is shown that the Schwarz genus of $\varepsilon_{3,S^n}$ is at most 1. Hence the above theorems imply that for $n$ not divisible by 4 and $n$ not of the form $4 \cdot 3^s$ with $s \geq 0$ the Schwarz genus of $\varepsilon_{3,S^n}$ is equal to 1.

The above theorems are negative results, but they lead to a positive consequence:

**Corollary 1.14**  *Assume $m \geq 2$ and $n \geq 2$ are as in any of the above theorems, or $n$ is odd. Then for any $\Sigma_m$–invariant map $f \colon F_m(S^n) \to S^n$, i.e. a map constant on orbits, there exists a configuration $(x_1, \ldots, x_m) \in F_m(S^n)$ such that $f(x_1, \ldots, x_m)$ coincides with one of the points $x_1, \ldots, x_m$.*

**Remark 1.15**  From the proofs of Theorems 1.5–1.7 and [1, Proposition 5.5] it follows that in the corresponding cases of this corollary the configuration may be chosen to be a regular $m$–gon centered at the origin. Moreover, in these cases it is sufficient to assume that $f$ is defined only for such regular $m$–gons and is invariant with respect to the action of the dihedral group $D_{2m}$. If we rely on Theorem 1.9 then the configuration still may be chosen to be a regular $m$–gon, but the map $f$ has to be defined and $D_{2m}$–invariant on the whole configuration space. In Theorem 1.10 the domain is necessarily larger; see the details in Section 10.

**Proof**  Assume the contrary and put

$$h(x_1, \ldots, x_m) = -f(x_1, \ldots, x_m).$$

The point $h(x_1, \ldots, x_m)$ is *not* antipodal to any of the points $x_1, \ldots, x_m$. So the point $h(x_1, \ldots, x_m)$ can be connected with every point $x_1, \ldots, x_m$ by a unique shortest path. Therefore we obtain a continuous $\Sigma_m$–equivariant section of $\varepsilon_{m,S^n}$, which cannot exist by the corresponding theorem of this paper or [1, Proposition 5.5].          $\square$

The paper is organized as follows. In Section 2 we reduce Question 1.4 to a question about existence of certain $\Sigma_m$–equivariant maps from configuration spaces to a sphere. In Sections 3 and 4 we describe the spaces of regular $m$–gons. In Section 5 we list the needed facts about the cohomology of dihedral groups. Then in Sections 6–10 we prove the main results. In Section 11 we discuss the first open case $m = 3$, $n = 4$ and explain why our approach fails in this case.

## 2  Reduction of Question 1.4

Fix the numbers $m$ and $n$ and denote $\varepsilon_{m,S^n}$ simply by $\varepsilon$. A $\Sigma_m$–equivariant section of $\varepsilon\colon e^{-1}(F_m(S^n)) \to F_m(S^n)$ assigns to an $m$–tuple $(x_1, \ldots, x_m) \in F_m(S^n)$ a map

$$g_{(x_1,\ldots,x_m)}\colon J_m \to S^n$$

so that the restriction $g_{(x_1,\ldots,x_m)}|_{\partial J_m}$ gives these $m$ points $x_1, \ldots, x_m$ in the prescribed order, and if we permute $(x_1, \ldots, x_m)$ the map $g_{(x_1,\ldots,x_m)}$ is right-multiplied by the corresponding permutation of branches of $J_m$.

Obviously, this gives an adjoint map $g\colon F_m(S^n) \times J_m \to S^n$. If we consider $J_m$ as $m$ segments $I_1, \ldots, I_m$ glued together, we see that $g$ gives $m$ maps

$$g_i\colon F_m(S^n) \times I_i \to S^n.$$

Define the map $\widetilde{g}\colon F_m(S^n) \times I \to (S^n)^{\times m}$ (here $I = [0,1]$) by

$$\widetilde{g}((x_1, \ldots, x_m), t) = (g_1((x_1, \ldots, x_m), t), \ldots, g_m((x_1, \ldots, x_m), t)).$$

The conditions on the section $g$ are equivalent to the following: $\widetilde{g}(\cdot, t)$ is $\Sigma_m$–equivariant for all $t$, $\widetilde{g}(\cdot, 1)$ coincides with the standard inclusion

$$\iota\colon F_m(S^n) \subseteq (S^n)^{\times m},$$

and $\widetilde{g}(\cdot, 0)$ assigns to any $m$–tuple $(x_1, \ldots, x_m)$ an $m$–tuple

$$(g_1((x_1, \ldots, x_m), 0), \ldots, g_m((x_1, \ldots, x_m), 0))$$

of *equal* points of $S^n$ (because $I_1, \ldots, I_m$ are glued at $t = 0$ to form $J_m$).

Putting $\widetilde{g}(\cdot, 0) = (h(\cdot), \ldots, h(\cdot))$ we have proved:

**Lemma 2.1** *Question 1.4 is equivalent to the following: Is there a $\Sigma_m$–equivariant homotopy between the natural inclusion $\iota\colon F_m(S^n) \to (S^n)^{\times m}$ and the map $\Delta \circ h$, where*

$$\Delta\colon S^n \to (S^n)^{\times m}$$

*is the inclusion of the thin diagonal, and $h\colon F_m(S^n) \to S^n$ is some $\Sigma_m$–equivariant map?*

**Remark 2.2**   Here and below we assume that the sphere $S^n$ has trivial $\Sigma_m$–action.

In the following lemma the action of $\Sigma_{m-1}$ on $F_m(S^n)$ is the action by permutations of $n$–tuples $(x_1, x_2, \ldots, x_n)$ which fix $x_1$.

**Lemma 2.3**   *The fibration $\varepsilon_{m,S^n}$ has a $\Sigma_m$–equivariant section over $F_m(S^n)$ if and only if there exists a $\Sigma_m$–equivariant map $h\colon F_m(S^n) \to S^n$, $\Sigma_{m-1}$–equivariantly homotopic to the composition $\pi_1 \circ \iota$, where $\pi_1\colon (S^n)^{\times m} \to S^n$ is the natural projection to the first factor.*

**Proof**   Assume that a $\Sigma_m$–equivariant section of $\varepsilon_{m,S^n}$ exists. By Lemma 2.1 we have a $\Sigma_m$–equivariant map $h\colon F_m(S^n) \to S^n$ such that $\Delta \circ h$ is equivariantly homotopic to $\iota\colon F_m(S^n) \to (S^n)^{\times m}$. Then left-multiplying by the $\Sigma_{m-1}$–equivariant map $\pi_1$ we obtain

$$\pi_1 \circ \Delta \circ h \simeq_{\Sigma_{m-1}} \pi_1 \circ \iota.$$

It remains to note that $\pi_1 \circ \Delta = \mathrm{id}_{S^n}$.

Now assume that there exists a $\Sigma_{m-1}$–equivariant map

$$g_1\colon F_m(S^n) \times I \to S^n$$

such that $g_1(\cdot, 0)$ is $\Sigma_m$–equivariant and $g_1((x_1, \ldots, x_m), 1) = x_1$. Put

$$g_k((x_1, \ldots, x_m), t) = g_1((x_k, x_1, \ldots, \cancel{x_k}, \ldots, x_m), t)$$

for $k > 1$ and

$$\widetilde{g}((x_1, \ldots, x_m), t) = (g_1((x_1, \ldots, x_m), t), \ldots, g_m((x_1, \ldots, x_m), t)).$$

The map $\widetilde{g}$ is $\Sigma_m$–equivariant:

$$\begin{aligned}
\widetilde{g}((x_{\sigma(1)}, \ldots, x_{\sigma(m)}), t) &= \big(g_1((x_{\sigma(1)}, \ldots, x_{\sigma(m)}), t), \ldots, g_m((x_{\sigma(1)}, \ldots, x_{\sigma(m)}), t)\big) \\
&= \big(g_1((x_{\sigma(1)}, \ldots, x_{\sigma(m)}), t), \ldots, g_1((x_{\sigma(m)}, x_{\sigma(1)}, \ldots, x_{\sigma(m-1)}), t)\big) \\
&= \big(g_{\sigma(1)}((x_1, \ldots, x_m), t), \ldots, g_{\sigma(m)}((x_1, \ldots, x_m), t)\big),
\end{aligned}$$

and obviously the map $\widetilde{g}$ is a homotopy between $\iota$ (at $t = 1$) and $\Delta \circ h$, where $h = g_1(\cdot, 0)$. So we apply Lemma 2.1.                                                                                     $\square$

## 3   Regular polygons in $F_m(S^n)$

We assume $m \geq 3$. Let us describe a subspace of $F_m(S^n)$.

**Definition 3.1**  We denote by $D_m(S^n) \subset F_m(S^n)$ the space of regular planar $m$–gons $x_1, \ldots, x_m$ centered at the origin. The vertices $x_1, \ldots, x_m$ lie on the unit sphere $S^n$ and are considered to be labeled in one of the two cyclic orders.

This subset is not $\Sigma_m$–invariant unless $m = 3$, but it is always invariant with respect to the action of the dihedral group $D_{2m}$ of order $2m$, which we view as a subgroup of $\Sigma_m$. In order to prove Theorems 1.5 and 1.6 we are going to use Lemma 2.3 and replace the full space $F_m(S^n)$ by its subspace $D_m(S^n)$ and $\Sigma_m$–equivariance by $D_{2m}$–equivariance. We denote by $\mathbb{Z}_m$ the integers mod $m$, frequently viewed as the subgroup of $D_{2m}$ or the symmetric group $\Sigma_m$ consisting of the cyclic permutations.

The space $D_m(S^n)$ is easily seen to be homeomorphic to the Stiefel manifold $V_{n+1,2}$. Namely, to $(x, y) \in V_{n+1,2}$ we assign $(x_1, \ldots, x_m) \in D_m(S^n)$, where $x_1 = x$ together with the further points $x_2, \ldots, x_m$ form the regular $m$–gon in the plane spanned by $x$ and $y$ so that the order of the vertices proceeds in the same sense as the rotation from $x$ to $y$. In this way, *we shall identify these two spaces.* We need the following information about the map $\pi_1 \colon D_m(S^n) \to S^n$ $((x_1, \ldots, x_m) \mapsto x_1)$:

**Lemma 3.2**  *Let $n$ be even. The map $\pi_1 \colon D_m(S^n) \to S^n$ is the projection of the sphere bundle of the tangent bundle of $S^n$. Therefore $H^i(D_m(S^n)) = \mathbb{Z}$ in dimensions $i = 0$ and $i = 2n - 1$, $H^n(D_m(S^n)) = \mathbb{Z}_2$, all other cohomology groups are zero, and the map $\pi_1^* \colon H^n(S^n) \to H^n(D_m(S^n))$ is a surjection.*

**Proof**  Since $D_m(S^n)$ has been identified with $V_{n+1,2}$, the identification of $D_m(S^n)$ with the set of unit tangent vectors of $S^n$ is obvious. From the spectral sequence of this fibration with

$$E_2(\pi_1) = H^*(S^n) \otimes H^*(S^{n-1})$$

we see that the only nonzero differential is $d_n \colon E_n^{0,n-1} \to E_n^{n,0}$, which is multiplication by 2 (the Euler characteristic of $S^n$). This implies all the needed facts (which also follow from the Gysin sequence). □

Since $H^{2n-1}(D_m(S^n)) = \mathbb{Z}$, we conclude that $D_m(S^n)$ is an orientable manifold. The following lemma shows that when $n$ is even $D_m(S^n)/D_{2m}$ is also an orientable manifold.

**Lemma 3.3**  *For $m \geq 3$ and even $n$, the action of the dihedral group $D_{2m}$ on $D_m(S^n)$ preserves orientation.*

**Proof**  We first observe that the action of the cyclic subgroup $\mathbb{Z}_m$ preserves orientation. Indeed, the action of $\mathbb{Z}_m$ extends to an action of the circle group, in which $e^{i\vartheta}$ maps a pair $(x, y)$ in the corresponding Stiefel manifold to $(\cos(\vartheta)\, x + \sin(\vartheta)\, y, -\sin(\vartheta)\, x + \cos(\vartheta)\, y)$;, so that $(x, y)$ is rotated through the angle $\vartheta$ in the plane they span; hence the action of $\mathbb{Z}_m$ preserves orientation. It remains to show that one element of $D_{2m}$ not in its cyclic subgroup also preserves orientation, and for this we take the element which sends an element $(x_1, x_2, \dots, x_m) \in D_m(S^n)$ to the element $(x_1, x_m, \dots, x_2)$ listed in the reverse cyclic order. Then each element $(x, y)$ of the corresponding Stiefel manifold is sent to $(x, -y)$. Hence we obtain an involution of the Stiefel manifold which for the bundle projection $(x, y) \mapsto x$ can be described as an automorphism which fixes the base $S^n$ and acts as the antipodal involution on each fiber. The fibers are spheres of odd dimension, on which the antipodal map preserves orientation; hence the orientation of the Stiefel manifold is also preserved.                                       □

Denote by $\mathbb{Z}_2 \subset D_{2m}$ the subgroup generated by the flip fixing $x_1$ (which appeared in the previous proof). Lemma 2.3 yields the following reduction:

**Lemma 3.4**  *If the fibration $\varepsilon_{m, S^n}$ has a $D_{2m}$–equivariant section over $D_m(S^n)$ then there exists a $D_{2m}$–equivariant map $h\colon D_m(S^n) \to S^n$, $\mathbb{Z}_2$–equivariantly homotopic to the composition $\pi_1 \circ \iota$, where $\iota\colon D_m(S^n) \to (S^n)^{\times m}$ is the natural inclusion.*

## 4  A note on triangles

In the particular case $m = 3$ we know even more. Define the function $E\colon F_3(S^n) \to \mathbb{R}$ by

$$E(a, b, c) = \frac{1}{|a - b|} + \frac{1}{|b - c|} + \frac{1}{|c - a|}.$$

This function is smooth and proper. A configuration $(a, b, c) \in F_3(S^n)$ is a critical point for $E$ if and only if $abc$ is a regular triangle centered at the origin. This can be proved in two steps:

- Note that the affine 2–flat spanned by $a, b, c$ should contain the origin;
- Make an explicit calculation after expressing the side lengths of $\triangle abc$ by $2\sin\alpha$, $2\sin\beta$, and $2\sin\gamma$ with $\alpha + \beta + \gamma = \pi$.

Therefore the gradient flow of $E$ establishes a $\Sigma_3$–equivariant deformation retraction of $F_3(S^n)$ onto $D_3(S^n)$.

Another way to establish an equivariant homotopy equivalence in the case $n = 2$ is to first identify $F_3(S^2)$ with the group of Möbius transformations $\mathrm{PSL}(2, \mathbb{C})$ of the

Riemann sphere (it is classical that triples of points in $\mathbb{C}P^1 = S^2$ admit a simply transitive action by $\mathrm{PSL}(2, \mathbb{C})$), and to then notice that $\mathrm{SO}(3)$ is a maximal compact subgroup of $\mathrm{PSL}(2, \mathbb{C})$. A non-equivariant homotopy equivalence $F_3(S^n) \simeq D_3(S^n)$ is established by Fadell and Husseini in [6, Chapter III, Proposition 1.1, page 32].

Finally we obtain the following:

**Lemma 4.1** *The fibration $\varepsilon_{3, S^n}$ has a $\Sigma_3$–equivariant section over $F_3(S^n)$ if and only if there exists a $\Sigma_3$–equivariant map $h\colon D_3(S^n) \to S^n$, $\mathbb{Z}_2$–equivariantly homotopic to the composition $\pi_1 \circ \iota$, where $\iota\colon D_3(S^n) \to (S^n)^{\times 3}$ is the natural inclusion.*

# 5 The cohomology of dihedral groups

Before proving Theorems 1.5 and 1.6 we need some facts about the cohomology of dihedral groups and the spaces $D_m(S^n)/D_{2m}$. The first lemma follows from the explicit description of the additive structure of $H^*(D_{2m})$ given in the proofs of Theorems 5.2 and 5.3 in Handel [9]:

**Lemma 5.1** *For $i > 0$, if the cohomology group $H^i(D_{2m})$ fails to be 2–primary then $i \equiv 0 \pmod 4$.*

*More precisely, suppose that $p$ is an odd prime and that $p^r \| m$, i.e. $m = p^r k$ with $(p, k) = 1$. If $i > 0$ and $i \equiv 0 \pmod 4$, then the $p$–primary component of $H^i(D_{2m})$ is $\mathbb{Z}_{p^r}$, while if $j \not\equiv 0 \pmod 4$ then $H^j(D_{2m})$ has no $p$–torsion.*

**Lemma 5.2** *If the cohomology group $H^n(D_m(S^n)/D_{2m})$ fails to be 2–primary then $n \equiv 0 \pmod 4$. If the group $H^{n-1}(D_m(S^n)/D_{2m})$ fails to be 2–primary then $n \equiv 1 \pmod 4$.*

**Proof** Consider the Cartan–Leray spectral sequence [2; 3] with

$$E_2^{p,q} = H^p(D_{2m}; H^q(D_m(S^n))),$$

converging to $H^*(D_m(S^n)/D_{2m})$. By Lemma 3.3 the group $D_{2m}$ acts trivially on the cohomology of $D_m(S^n)$. From Lemma 3.2 it follows that the only nonzero rows of this spectral sequence occur for $q = 0, n, 2n - 1$. In the $n$th row we have $H^p(D_{2m}; \mathbb{Z}_2)$, which has no odd torsion. Hence all of the odd torsion in $H^n(D_m(S^n)/D_{2m})$ and $H^{n-1}(D_m(S^n)/D_{2m})$ comes from the group cohomology and we may apply Lemma 5.1. □

**Lemma 5.3** *If $m$ is even and $n \equiv 2$ (mod 4), then the group $H^n(D_m(S^n)/D_{2m})$ is a 4–torsion group. In the particular case $m = 4$ the group $H^n(D_4(S^n)/D_8)$ is a 2–torsion group.*

Here an abelian group is called a *$k$–torsion group* for a positive integer $k$ if each of its elements $a$ satisfies $ka = 0$.

**Proof** It follows from Lemma 5.1 that the group cohomology $H^n(D_{2m})$ is a 2–torsion group. Consider again the spectral sequence with

$$E_2^{p,q} = H^p(D_{2m}; H^q(D_m(S^n)))$$

with blank rows in the range $1 \leq q \leq n - 1$. If $d_{n+1}$ is nonzero on

$$E_{n+1}^{0,n} = H^0(D_{2m}; \mathbb{Z}_2) = \mathbb{Z}_2$$

then $H^n(D_m(S^n)/D_{2m}) = H^n(D_{2m})$ and the proof is complete.

Otherwise we obtain the exact sequence

$$0 \to H^n(D_{2m}) \to H^n(D_m(S^n)/D_{2m}) \to H^n(D_m(S^n)) \to 0$$

and note that $H^n(D_m(S^n)/D_{2m})$ is a 4–torsion group since $H^n(D_m(S^n)) = \mathbb{Z}_2$.

If $m = 4$ we note that $D_4(S^n)/D_8 = V_{n+1,2}/D_8 \simeq F_2(\mathbb{R}P^n)/\Sigma_2 = B_2(\mathbb{R}P^n)$. The latter space is studied by Domínguez, González and the second author in [5], and [5, Theorem 2.2] implies $H^n(B_2(\mathbb{R}P^n))$ is a 2–torsion group when $n \equiv 2$ (mod 4). $\quad\square$

# 6  Proof of Theorem 1.5

We are going to use reasoning in the spirit of the definition of the Hopf invariant of a map $f: S^{2n-1} \to S^n$ by means of the mapping cone and the cup product.

Using Lemma 3.4, we assume that a continuous map $h: D_m(S^n) \to S^n$ is $D_{2m}$–equivariant and $\mathbb{Z}_2$–equivariantly homotopic to the map $\pi_1$ described in Lemma 3.2.

There results a map $h': D_m(S^n)/D_{2m} \to S^n$ so that $h$ admits the factorization $h = h' \circ q$, where $q: D_m(S^n) \to D_m(S^n)/D_{2m}$ is the orbit projection. Consider the mapping cones $X = S^n \cup_h CD_m(S^n)$ and $X' = S^n \cup_{h'} C(D_m(S^n)/D_{2m})$ and the corresponding commutative diagram

$$
\begin{array}{ccccccc}
D_m(S^n) & \xrightarrow{\ h\ } & S^n & \xrightarrow{\ \lambda\ } & X & \longrightarrow & S(D_m(S^n)) \\
\downarrow & & \Big\| & & \downarrow{\scriptstyle\pi} & & \downarrow \\
D_m(S^n)/D_{2m} & \xrightarrow[\ h'\ ]{} & S^n & \xrightarrow[\ \mu\ ]{} & X' & \twoheadrightarrow & S(D_m(S^n)/D_{2m}).
\end{array}
$$

There results a commutative diagram of long exact sequences of cohomology groups (we observe that $H^n(S(D_m(S^n))) \cong H^{n-1}(D_m(S^n)) = 0$):

(6-1)
$$
\begin{array}{ccccccc}
0 & \longrightarrow & H^n(X) & \xrightarrow{\lambda^*} & H^n(S^n) & \xrightarrow{h^*} & H^n(D_m(S^n)) \\
& & \uparrow & & \uparrow{\scriptstyle \pi^*} & & \uparrow \\
H^{n-1}(D_m(S^n)/D_{2m}) & \xrightarrow[\nu^*]{} & H^n(X') & \xrightarrow[\mu^*]{} & H^n(S^n) & \xrightarrow[h'^*]{} & H^n(D_m(S^n)/D_{2m})
\end{array}
$$

The map $h$ is non-equivariantly homotopic to $\pi_1$. From Lemma 3.2 it follows that the space $X$ is homotopy equivalent to the Thom space of the tangent bundle of $S^n$, so its cohomology is isomorphic to $\mathbb{Z}$ in dimensions $0, n, 2n$ and zero in other dimensions. Denote generators by $x \in H^n(X)$ and $y \in H^{2n}(X)$. From the definition of the Euler class it follows that $x \smile x = 2y$ for a suitable choice of the generator $y$.

Denote by $\alpha$ a generator of $H^n(S^n)$. By Lemma 3.2 in the upper row of (6-1) the map $h^*$ is a surjection and we have $\lambda^*(x) = \pm 2\alpha$.

Let us describe the lower row of (6-1). By Lemma 5.2 and the hypothesis that $n \equiv 2 \pmod 4$, the groups $H^n(D_m(S^n)/D_{2m})$ and $H^{n-1}(D_m(S^n)/D_{2m})$ are 2–primary. Hence $H^n(X')/\nu^*(H^{n-1}(D_m(S^n)/D_{2m}))$ is isomorphic to $\mathbb{Z}$; denote its generator by $x'$. It is clear that $\mu^*(x') = \pm s\alpha$ where $s$ is a power of 2. Since $\lambda^*(x) = \pm 2\alpha$ and $\mu^*(x') = \pm s\alpha$ it follows that $\pi^*(x') = \pm tx$, where $t$ is a power of 2, possibly equal to 1. (Observe that $s = 2t$.) From the diagram (another part of the map between the long exact sequences)

$$
\begin{array}{ccccccc}
0 & \xrightarrow{h^*} & H^{2n-1}(D_m(S^n)) & \longrightarrow & H^{2n}(X) & \xrightarrow{\lambda^*} & 0 \\
& & \uparrow & & \uparrow{\scriptstyle \pi^*} & & \\
0 & \xrightarrow{h'^*} & H^{2n-1}(D_m(S^n)/D_{2m}) & \xrightarrow{\nu^*} & H^{2n}(X') & \xrightarrow{\mu^*} & 0
\end{array}
$$

we conclude that the group $H^{2n}(X')$ is isomorphic to $\mathbb{Z}$ with a generator $y'$ such that $\pi^*(y') = 2my$, since the manifolds $D_m(S^n)$ and $D_m(S^n)/D_{2m}$ are orientable by Lemma 3.3.

For some integer $c$ we have $x' \smile x' = cy'$; this equality does not depend on adding an element of $\nu^*(H^{n-1}(D_m(S^n)/D_{2m}))$ to $x'$. Then applying $\pi^*$ we obtain $t^2x \smile x = 2mcy$, and therefore $t^2 = mc$. From the latter equation it is clear that $m$ must be a power of 2. In this theorem $m \geq 3$, so in particular $m$ is even. Now we apply Lemma 5.3 to conclude that $s \leq 4$, so $t = s/2 \leq 2$ and the equality $t^2 = mc$ is possible only for $m = 4$ and $t = 2$. But applying Lemma 5.3 again for $m = 4$ we obtain that $s \leq 2$, and therefore $t = 1$, a contradiction. This completes the proof of Theorem 1.5.

# 7 Proof of Theorem 1.6

We repeat the above reasoning literally and only have to show that the map $\mu^*$ in (6-1) is multiplication by a power of 2 on generators. In order to show this we have to prove that the image of the map $h'^*\colon H^n(S^n) \to H^n(D_3(S^n)/\Sigma_3)$ lies in the 2–primary component of $H^n(D_3(S^n)/\Sigma_3)$.

Besides the 2–primary component of $H^n(D_3(S^n)/\Sigma_3)$ there exists a 3–primary component, isomorphic to $\mathbb{Z}_3$ (see Lemma 5.1). So we pass to modulo 3 cohomology and to the cyclic subgroup $\mathbb{Z}_3 \subset \Sigma_3$. Assume the map $h$ factors through $h''\colon D_m(S^n)/\mathbb{Z}_3 \to S^n$. Now we have to prove that

$$h''^*\colon H^n(S^n;\mathbb{F}_3) \to H^n(D_3(S^n)/\mathbb{Z}_3;\mathbb{F}_3)$$

is the zero map. The space $D_3(S^n)$ is a mod 3 cohomology sphere having dimension $2n-1$, so the cohomology ring $H^*(D_3(S^n)/\mathbb{Z}_3;\mathbb{F}_3)$ is a truncation of the group cohomology, i.e. it has generators $u, w$ with $\dim u = 1$, $\dim w = 2$, satisfying the relations ($\beta$ denotes the Bockstein homomorphism)

$$u^2 = 0, \qquad \beta(u) = w, \qquad w^n = 0.$$

Note that a generator $\alpha \in H^n(S^n;\mathbb{F}_3)$ is annihilated by any cohomology operation of positive degree. But the Steenrod reduced powers of $w^{n/2} \in H^n(D_3(S^n)/\mathbb{Z}_3;\mathbb{F}_3)$ satisfy

$$\mathcal{P}^i(w^{n/2}) = \binom{n/2}{i}w^{n/2+2i}.$$

So if $\alpha$ maps to $cw^{n/2}$ with $c \in \mathbb{F}_3^*$, then $\binom{n/2}{i} \equiv 0 \pmod 3$ for $i = 1, \dots, n/4-1$. Equivalently, putting $N = n/4$, $\binom{2N}{i} \equiv 0 \pmod 3$ for $i = 1, \dots, N-1$. By hypothesis $N$ does not have the form $3^s$ with $s \geq 0$, so a contradiction follows from:

**Lemma 7.1** *For a positive integer $N$, all binomial coefficients $\binom{2N}{i}$ vanish mod 3 for $0 < i < N$ if and only if $N$ has the form $3^s$ for some $s \geq 0$.*

**Proof** Let $t$ be an indeterminate. Then the vanishing of the binomial coefficients in the statement of the lemma is equivalent to requiring that

(7-1) $$(1+t)^{2N} \equiv 1 + \binom{2N}{N}t^N + t^{2N} \pmod 3.$$

In case $N$ is a power of 3, possibly equal to 1, this last condition is easily verified (and one learns that the "middle" binomial coefficient is congruent to 2 mod 3). Of course, it is the converse that is really needed here. Suppose $N$ is not a power of 3. Then the formula of Lucas for binomial coefficients modulo a prime implies that there is a smallest integer $i$, $0 < i < N$, such that $\binom{N}{i} \not\equiv 0 \pmod 3$. Then expand

$(1 + t)^{2N} = ((1 + t)^N)^2$ to find that there is a nonzero term $2\binom{N}{i}t^i$ appearing in the expansion of $(1 + t)^{2N}$ mod 3, which violates the condition (7-1) at the start of the proof. $\qquad\square$

## 8  Proof of Theorem 1.7

First we note that both spaces $D_p(S^n)$ and $D_m(S^n)$ are identified with the same $V_{n+1,2}$. The corresponding homeomorphism $\phi\colon D_p(S^n) \to D_m(S^n)$ is obtained by adding $m/p - 1$ further vertices uniformly placed on every geodesic arc $[x_i x_{i+1}]$ between successive (mod $p$) vertices of a $p$–gon. It is clear that $\phi$ is $D_{2p}$–equivariant. If $h\colon D_m(S^n) \to S^n$ satisfies the hypothesis of Lemma 3.4 then so does $\phi \circ h\colon D_p(S^n) \to S^n$. So we have reduced the problem to the case of regular $p$–gons.

For odd primes $p$ the cohomology $H^*(\mathbb{Z}_p; \mathbb{F}_p)$ has structure similar to $H^*(\mathbb{Z}_3; \mathbb{F}_3)$ considered above, with a generator $w \in H^2(\mathbb{Z}_p; \mathbb{F}_p)$. The space $D_p(S^n)$ is a $(2n-1)$–dimensional mod $p$ homology sphere. As above, it suffices to find a cohomology operation on the cohomology $H^*(D_p(S^n)/\mathbb{Z}_p; \mathbb{F}_p)$ which acts nontrivially on $w^{n/2}$.

We examine the mod $p$ Steenrod reduced powers, which satisfy

$$\mathcal{P}^i(w^{n/2}) = \binom{n/2}{i}w^{n/2 + (p-1)i}.$$

For an indeterminate $t$, notice the following congruence of polynomials mod $p$:

$$(1 + t)^{n/2} = (1 + t)^{2n'p^s} \equiv (1 + t^{p^s})^{2n'} = 1 + 2n't^{p^s} + \text{ higher powers of } t.$$

Assuming that all the Steenrod powers $\mathcal{P}^i(w^{n/2})$ with $n/2 + (p-1)i < n$ vanish, we obtain the following (equivalent) inequalities:

$$2n'p^s + (p-1)p^s \geq n = 4n'p^s \;\Rightarrow\; (p-1) \geq 2n' \;\Rightarrow\; n' \leq \tfrac{p-1}{2},$$

which contradicts the hypothesis of the theorem.

## 9  Proof of Theorem 1.9

In this theorem we use the squaring relation $(w^{n/2})^2 = w^n$ instead of the Steenrod operations. In the mod $m$ cohomology of $D_m(S^n)/\mathbb{Z}_m$ we have $w^n = 0$. (Recall that $m$ is assumed to be a prime, $m \geq 5$.) So we have to consider the whole configuration space and use its cohomology.

We need to prove that the map $h''^*\colon H^n(S^n; \mathbb{F}_m) \to H^n(D_m(S^n)/\mathbb{Z}_m; \mathbb{F}_m)$ is trivial. The map $h''$ is a composition $h'' = \iota \circ h_0''$, where $\iota\colon D_m(S^n)/\mathbb{Z}_m \to F_m(S^n)/\mathbb{Z}_m$

is the natural inclusion and $h_0''\colon F_m(S^n)/\mathbb{Z}_m \to S^n$ comes from the original map $h_0\colon F_m(S^n) \to S^n$ satisfying the hypothesis of the theorem and Lemma 2.1.

Again let $w \in H^2(\mathbb{Z}_m; \mathbb{F}_m)$ be a generator. We need the following facts about the cohomology of $F_m(\mathbb{R}^n)/\mathbb{Z}_m$:

**Lemma 9.1**  *Let $m \geq 5$ be a prime. The cohomology $H^n(F_m(\mathbb{R}^n)/\mathbb{Z}_m; \mathbb{F}_m)$ is generated by the power $w^{n/2}$, and its square $w^n$ is nonzero.*

**Remark 9.2**  For $m = 3$ the cohomology $H^n(F_3(\mathbb{R}^n)/\mathbb{Z}_3; \mathbb{F}_3)$ is also generated by the power $w^{n/2}$, but $w^n = 0$ because the group $H^{2n}(F_3(\mathbb{R}^n)/\mathbb{Z}_3; \mathbb{F}_3)$ vanishes. Moreover, the group $H^{2n}(F_3(\mathbb{S}^n)/\mathbb{Z}_3; \mathbb{F}_3) = H^{2n}(D_3(\mathbb{S}^n)/\mathbb{Z}_3; \mathbb{F}_3)$ also vanishes and the approach with squaring is not applicable to the case $m = 3$.

**Proof**  The cohomology $H^*(F_m(\mathbb{R}^n); \mathbb{F}_m)$ is nonzero only in dimensions $0, n - 1, \ldots, (m-1)(n-1)$; and it is composed of free $\mathbb{F}_m[\mathbb{Z}_m]$–modules except for dimensions 0 and $(m-1)(n-1)$, see Cohen and Taylor [4, Theorem 3.4, Corollaries 3.5 and 3.6]. Therefore the only nonzero entries in the $E_2$–term of the Cartan–Leray spectral sequence for $F_m(\mathbb{R}^n)$ are $H^*(\mathbb{Z}_m; \mathbb{F}_m)$ in the zeroth row, something nonzero in dimensions $i(n-1)$ $(0 \leq i \leq m-1)$ in the zeroth column, and something nonzero in the $(m-1)(n-1)$th row. Because of the $H^*(\mathbb{Z}_m; \mathbb{F}_m)$–module structure of this spectral sequence the only nonzero differential in this spectral sequence can act from the $(m-1)(n-1)$th row to the zeroth row, thus preserving $w^{n/2}$ and $w^n$ of dimensions $n, 2n \leq (m-1)(n-1)$ (we use $m \geq 5$ and $n \geq 4$ here!) in the term $E_\infty$.  □

Evidently, there is a $\Sigma_m$–equivariant inclusion $F_m(\mathbb{R}^n) \to F_m(S^n)$ and $w^n$ is also nonzero in the group $H^{2n}(F_m(S^n)/\mathbb{Z}_m; \mathbb{F}_m)$. Thus $w^{n/2}$ cannot be a multiple of $h_0''^*(\alpha)$ because $\alpha^2 = 0$. But this does not exclude the possibility $h''^*(\alpha) \neq 0$ because the group $H^n(F_m(S^n))/\mathbb{Z}_m; \mathbb{F}_m)$ may not be generated by $w^{n/2}$.

The inclusion $F_m(\mathbb{R}^n) \to F_m(S^n)$ is obtained by choosing a base point $x_0 \in S^n$ and identifying $\mathbb{R}^n$ with $S^n \setminus \{x_0\}$. Let us describe the difference set $F_m(S^n) \setminus F_m(\mathbb{R}^n)$. This difference corresponds to configurations $(x_1, \ldots, x_m)$ such that for one index $i$ we have $x_i = x_0$. The other points $\{x_j\}_{j \neq i}$ form a configuration in $F_{m-1}(\mathbb{R}^d)$. Hence $F_m(S^n) \setminus F_m(\mathbb{R}^n)$ is a union of $m$ smooth manifolds $X_1 \cup \cdots \cup X_m$; each $X_i$ is diffeomorphic to $F_{m-1}(\mathbb{R}^n)$ and they are permuted freely by $\mathbb{Z}_m$. Moreover we note that $X_1$ is the preimage of zero under the natural projection map $\pi_1\colon F_m(S^n) \to S^n$, so it has a normal framing in $F_m(S^n)$. It follows now that the pair

$$(F_m(S^n)/\mathbb{Z}_m, F_m(\mathbb{R}^n)/\mathbb{Z}_m) = (F_m(S^n), F_m(S^n) \setminus X_1)$$

is homotopy equivalent to $(D^n \times X_1, S^n \times X_1)$, where $(D^n, S^n)$ is the $n$–ball and its boundary. Hence the cohomology $H^i(F_m(S^n)/\mathbb{Z}_m, F_m(\mathbb{R}^n)/\mathbb{Z}_m; \mathbb{F}_m)$ is trivial for $i < n$ and generated by one element $x$ for $i = n$. Now from the long exact sequence of the pair $(F_m(S^n)/\mathbb{Z}_m, F_m(\mathbb{R}^n)/\mathbb{Z}_m)$ we obtain that the group $H^n(F_m(S^n)/\mathbb{Z}_m; \mathbb{F}_m)$ is spanned by two elements: $w^{n/2}$ and the image of $x$, which we denote again by $x$. If this image is zero then we are done, so we may assume that $x \neq 0 \in H^n(F_m(S^n)/\mathbb{Z}_m; \mathbb{F}_m)$. The multiplication is determined by the relations $x^2 = 0$ (from the cohomology of $(D^n, S^n)$) and $wx = 0$, because the pairs $(F_m(S^n), X_i)$ are permuted freely by $\mathbb{Z}_m$ and the entire cohomology $H^*(F_m(S^n)/\mathbb{Z}_m, F_m(\mathbb{R}^n)/\mathbb{Z}_m; \mathbb{F}_m)$ is therefore annihilated by $w$.

Now assume that $h_0''^*(\alpha) = ax + bw^{n/2}$. By taking squares we obtain $(ax + bw^{n/2})^2 = b^2 w^n = 0$ and therefore $b = 0$. Hence we have to calculate $y = \iota^*(x) \in H^n(D_m(S^n)/\mathbb{Z}_m; \mathbb{F}_m)$. Consider the $(n-1)$–sphere $Y_1 \subset D_m(S^n)$ formed by $m$–gons with the first vertex coinciding with $x_0$. It is clear that $y$ is Poincaré dual to the cycle

$$c_y = \left( \bigcup_{g \in \mathbb{Z}_m} g(Y_1) \right) \Big/ \mathbb{Z}_m.$$

It is clear that the cycle $c_y$ of $D_m(S^n)/\mathbb{Z}_m$ is the natural image of the cycle $Y_1$ of $D_m(S^n)$. But the latter cycle represents the generator of $H_{n-1}(D_m(S^n); \mathbb{Z}) = \mathbb{Z}_2$. Hence modulo $m$ we have $[S_1] = 0$, $[c_y] = 0$, and therefore $y = 0$.

Now we have established that $h''^*(\alpha) = 0 \in H^n(D_m(S^n); \mathbb{F}_m)$; the rest of the proof proceeds as in the previous arguments.

## 10 Proof of Theorem 1.10

Let us start with the case $m = 4$. Consider the subspace $Q_4(S^n) \subset F_4(S^n)$ consisting of 4–tuples $(x_1, x_2, x_3, x_4)$ satisfying the equations (the distance is measured in the ambient Euclidean $\mathbb{R}^{n+1}$):

(10-1) $$\|x_1 - x_3\| = \|x_2 - x_4\| = 2\delta$$

(10-2) $$x_1 + x_3 + x_2 + x_4 = 0$$

(10-3) $$\|x_1 + x_3 - x_2 - x_4\| = 4\sqrt{1 - \delta^2}.$$

Here $\delta$ is a fixed number satisfying $0 < \delta < 1$. Informally, we consider configurations of four points such that the midpoints of the geodesic segments $[x_1, x_3]$ and $[x_2, x_4]$ are antipodal and the lengths of these segments are fixed. This subspace is obviously

$D_8$–invariant. In the particular case $\delta = 1/\sqrt{2}$ this space includes squares inscribed in $S^n$.

Consider another configuration space $Q_4(\mathbb{R}^{n+1})$ given by the same equations (10-1)–(10-3) but without the assumption that $x_1, \dots, x_4 \in S^n$. If $\delta < 1/\sqrt{2}$ we have the inclusion $Q_4(\mathbb{R}^{n+1}) \subset F_4(\mathbb{R}^{n+1})$. Informally, $Q_4(\mathbb{R}^{n+1})$ consists of 4–tuples $(x_1, x_2, x_3, x_4)$ such that the distances $\|x_1 - x_3\|$ and $\|x_2 - x_4\|$ are fixed, the midpoints of $[x_1, x_3]$ and $[x_2, x_4]$ are antipodal, and the distance between those midpoints is also fixed. Obviously $Q_4(\mathbb{R}^{n+1})$ can be identified with $S^n \times S^n \times S^n$ because the directions of $x_1 + x_3 - x_2 - x_4$, $x_1 - x_3$, and $x_2 - x_4$ define uniquely the configuration $(x_1, x_2, x_3, x_4) \in Q_4(\mathbb{R}^{n+1})$ and can be prescribed arbitrarily. Thus $Q_4(\mathbb{R}^{n+1})$ is $(n-1)$–connected.

The space $Q_4(S^n)$ is not so simple as $Q_4(\mathbb{R}^{n+1})$. The map $\pi \colon Q_4(S^n) \to S^n$ defined by

$$(10\text{-}4) \qquad\qquad \pi \colon (x_1, x_2, x_3, x_4) \mapsto \frac{x_1 + x_3}{\|x_1 + x_3\|}$$

makes $Q_4(S^n)$ a bundle over $S^n$ with fiber $S^{n-1} \times S^{n-1}$. More precisely, if we denote by $\tau$ the tangent bundle of $S^n$ with its corresponding sphere bundle $S(\tau)$, then $Q_4(S^n) = S(\tau) \times_{S^n} S(\tau)$. Hence the space $Q_4(S^n)$ is $(n-2)$–connected.

If $n$ grows both spaces $Q_4(S^n)/D_8$ and $Q_4(\mathbb{R}^{n+1})/D_8$ approach the classifying space $BD_8$. In fact, $Q_4(\mathbb{R}^{n+1})/D_8$ is a standard approximation to $BD_8$ which is useful in the study of the cohomology $H^*(D_8; \mathbb{F}_2)$, see the definition of the wreath product of projective spaces ($\widetilde{M}(q, 4) = Q_4(\mathbb{R}^{q+1})$ and $M(q, 4) = Q_4(\mathbb{R}^{q+1})/D_8$ in our notation) in Hung [10, Section 1, page 254]. We need the following fact about the cohomology of $Q_4(S^n)/D_8$:

**Lemma 10.1** *The natural map $\kappa_8^* \colon H^*(D_8; \mathbb{F}_2) \to H^*(Q_4(S^n)/D_8; \mathbb{F}_2)$ is surjective in dimensions $\leq n$.*

Until the end of this section we assume mod 2 cohomology and omit the coefficients $\mathbb{F}_2$ from the notation.

**Proof** Note that $BD_8 = Q_4(\mathbb{R}^\infty)/D_8$ is a bundle over $\mathbb{R}P^\infty$ with fiber $\mathbb{R}P^\infty \times \mathbb{R}P^\infty$. Its corresponding spectral sequence starts with

$$E_2 = H^*(\mathbb{Z}_2; H^*(\mathbb{Z}_2) \otimes H^*(\mathbb{Z}_2)),$$

where we identify $H^*(\mathbb{R}P^\infty)$ with $H^*(\mathbb{Z}_2)$ and assume that $\mathbb{Z}_2$ acts on $H^*(\mathbb{Z}_2) \otimes H^*(\mathbb{Z}_2)$ by permuting the factors. A lemma of Nakaoka [12] (see also Leary [11,

Theorem 2.1] and the accompanying remark about the graded algebra structure) asserts that this spectral sequence collapses at $E_2$ and there is an isomorphism of graded algebras

$$H^*(D_8) = H^*(\mathbb{Z}_2 \wr \mathbb{Z}_2) \cong H^*(\mathbb{Z}_2; H^*(\mathbb{Z}_2) \otimes H^*(\mathbb{Z}_2)).$$

Now we consider $Q_4(S^n)/D_8$ as a bundle over $\mathbb{R}P^n$ with fiber $\mathbb{R}P^{n-1} \times \mathbb{R}P^{n-1}$. The homomorphism of $E_2$–terms

$$\kappa_8^*\colon H^*(\mathbb{Z}_2; H^*(\mathbb{Z}_2) \otimes H^*(\mathbb{Z}_2)) \to H^*(\mathbb{R}P^n; H^*(\mathbb{R}P^{n-1}) \otimes H^*(\mathbb{R}P^{n-1}))$$

is surjective in dimensions $\leq n$. Indeed, it is an isomorphism on $E_2^{p,q}$ for $p + q \leq n$ and $q \leq n - 1$, while in $E_2^{0,n}$ it annihilates the element

$$c_1^n \otimes 1 + 1 \otimes c_2^n \in H^0(\mathbb{Z}_2, H^*(\mathbb{R}P^\infty) \otimes H^*(\mathbb{R}P^\infty)),$$

where we denote the generators of $H^1(\mathbb{R}P^\infty)$ in the respective factors by $c_1$ and $c_2$. Therefore the resulting homomorphism

$$\kappa_8^*\colon H^*(D_8) \to H^*(Q_4(S^n)/D_8)$$

is surjective in dimensions $\leq n$. $\qquad\square$

Now we are going to use Lemma 2.3. The subgroup of $D_8$ that fixes $x_1$ is the copy of $\mathbb{Z}_2$ that exchanges $x_2$ and $x_4$. The map $\pi_1\colon Q_4(S^n) \to S^n$ of Lemma 2.3 is $\mathbb{Z}_2$–equivariantly homotopic to the map $\pi$ defined in (10-4) by connecting $x_1$ and $\frac{x_1+x_3}{\|x_1+x_3\|}$. By Lemma 2.3 we assume that $\pi$ is $\mathbb{Z}_2$–equivariantly homotopic to some $D_8$–equivariant map $h\colon Q_4(S^n) \to S^n$.

Denote the corresponding maps by $h_2\colon Q_4(S^n)/\mathbb{Z}_2 \to S^n$, $h_8\colon Q_4(S^n)/D_8 \to S^n$, and $q\colon Q_4(S^n)/\mathbb{Z}_2 \to Q_4(S^n)/D_8$ so that $h_2 = h_8 \circ q$. From the commutative diagram

$$
\begin{array}{ccc}
H^n(Q_4(S^n)/\mathbb{Z}_2) & \xleftarrow{\;q^*\;} & H^n(Q_4(S^n)/D_8) \\
\kappa_2^* \Big\uparrow & & \Big\uparrow \kappa_8^* \\
H^n(\mathbb{Z}_2) & \xleftarrow{\hspace{3cm}} & H^n(D_8)
\end{array}
$$

and Lemma 10.1 we obtain that the image $q^*(H^n(Q_4(S^n)/D_8))$ (and therefore the image $h_2^*(H^n(S^n))$) is contained in the image $\kappa_2^*(H^n(\mathbb{Z}_2))$.

The space $Q_4(S^n)/\mathbb{Z}_2$ is a bundle $S(\tau) \times_{S^n} P(\tau)$ over $S^n$. Here $P(\tau)$ is the projective bundle corresponding to $\tau$ and the bundle projection is the map $\pi$ defined in (10-4). The spectral sequence of this bundle collapses and therefore

$$(10\text{-}5) \qquad H^*(Q_4(S^n)/\mathbb{Z}_2) \cong H^*(S^{n-1}) \otimes H^*(\mathbb{R}P^{n-1}) \otimes H^*(S^n)$$

additively. Since all the Stiefel–Whitney classes of $\tau$ vanish we in fact obtain an isomorphism of graded algebras. It follows that the image of $\kappa_2^*$ is multiplicatively generated by $H^1(\mathbb{R}P^{n-1})$ and does not contain the nontrivial element of $\pi^*(H^n(S^n))$. Thus the conditions of Lemma 2.3 cannot be satisfied. Another way to prove this without the multiplicative structure in (10-5) is to note that there is an $(n-1)$–sphere bundle $\varphi\colon Q_4(S^n)/\mathbb{Z}_2 \to P(\tau)$ and the maps $\kappa_2^*$ and $\pi^*$ pass through $H^*(P(\tau))$ (because the maps $\kappa_2$ and $\pi$ factor through $\varphi$). The vanishing of Stiefel–Whitney classes of $\tau$ ensures the isomorphism of graded algebras $H^*(P(\tau)) \cong H^*(\mathbb{R}P^{n-1}) \otimes H^*(S^n)$; hence the image of $\kappa_2^*$ is generated by $H^1(\mathbb{R}P^{n-1})$ and cannot contain the generator of $\pi^*(H^n(S^n))$.

Finally we treat the case of arbitrary even $m = 2\ell > 4$ by producing a $D_8$–equivariant embedding $Q_4(S^n) \to F_m(S^n)$. This embedding is constructed by replacing the pair $(x_1, x_3)$ with $\ell$ points (including the end points) distributed uniformly on the geodesic segment $[x_1, x_3]$, and the same for the segment $[x_2, x_4]$. A $\Sigma_m$–invariant map $h_m\colon F_m(S^n) \to S^n$ therefore induces a $D_8$–invariant map $h_4\colon Q_4(S^n) \to S^n$ which has already been shown to be impossible.

## 11 Remarks about the case $m = 3$ and $n = 4$

In the open case $m = 3$, $n = 4$ it would be sufficient (following the approach in Section 7) to find a cohomology operation of positive degree that takes $w^2 \in H^4(D_3(S^4)/\mathbb{Z}_3; \mathbb{F}_3)$ to a nonzero element of dimension $\leq 7$. The Steenrod reduced powers fail to do so (see Section 7) and moreover squaring does not work (see Remark 9.2).

**Remark 11.1** The rest of this section replaces $\Sigma_3$ by its subgroup $\mathbb{Z}_3$, so we try to solve positively a relaxed version of Question 1.4 without any success.

A relevant example is the original Hopf map $S^7 \to S^4$ (considered as the natural projection of unit vectors in $\mathbb{H}^2$ to $\mathbb{H}P^1 = S^4$), which is $\mathbb{Z}_3$–equivariant (we may assume $\mathbb{Z}_3 \subset \mathrm{Sp}(1)$) and maps the generator $\alpha \in H^4(S^4; \mathbb{F}_3)$ to a nonzero multiple of

$$w^2 \in H^4(S^7/\mathbb{Z}_3; \mathbb{F}_3) = H^4(\mathbb{Z}_3; \mathbb{F}_3).$$

So the class $w^2 \in H^4(S^7/\mathbb{Z}_3; \mathbb{F}_3)$ is annihilated by any cohomology operation of positive degree. The sphere $S^7$ has the same mod 3 cohomology (and $\mathbb{Z}_3$–equivariant cohomology mod 3) as $D_3(S^4)$. We note one difference, but do not know if it will prove helpful: The action of $\mathbb{Z}_3$ on $S^7$ cannot be extended to an action of $\Sigma_3 \supset \mathbb{Z}_3$.

In an attempt to give a positive answer for Question 1.4 for $m = 3$, $n = 4$ we may try to construct a $\mathbb{Z}_3$–equivariant map $D_3(S^4) \to S^4$ homotopic to $\pi_1 \circ \iota$ as in Lemma 4.1.

In the case $n = 2$ such a map $D_3(S^2) \to S^2$ is easily constructed by considering two "centers" of the triangle and selecting one of them according to the orientation (here $\mathbb{Z}_3$–equivariance is essential).

But the situation is more difficult in the case $n = 4$. For a triple $(a, b, c) \in D_3(S^4)$ denote its 2–dimensional linear span by $\alpha(a, b, c)$ and its 3–dimensional orthogonal complement by $\beta(a, b, c)$. It would be sufficient to find a $\mathbb{Z}_3$–equivariant map $h \colon D_3(S^4) \to S^4$ such that $h(a, b, c) \in \beta(a, b, c)$ for any triple. But in Whitehead [15] it is shown that we cannot assign continuously to any pair $(x_1, x_2) \in V_{n+1,2}$ another vector $x_3$ orthogonal to $x_1$ and $x_2$ except for the cases $n = 2, 6$, even without assuming any invariance under a group action on $V_{n+1,2}$.

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