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# Locally symmetric spaces and $K$–theory of number fields

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For a closed locally symmetric space $M = \Gamma\backslash G/K$ and a representation $\rho\colon G \to \mathrm{GL}(N,\mathbb{C})$ we consider the pushforward of the fundamental class in $H_*(B\mathrm{GL}(\bar{\mathbb{Q}}))$ and a related invariant in $K_*(\bar{\mathbb{Q}}) \otimes \mathbb{Q}$. We discuss the nontriviality of this invariant and we generalize the construction to cusped locally symmetric spaces of $\mathbb{R}$–rank one.

57R19, 53C35, 57M50; 11R70, 22E46

## 1 Introduction

While elements in topological $K$–theory $K^{-*}(X)$ are, by definition, represented by (virtual) vector bundles over the space $X$, it is less evident what the topological meaning of elements in algebraic $K$–theory $K_*(A)$ for a commutative ring $A$ may be. An approach, which can be found eg in the appendix of Karoubi [18], is to consider elements in $K_d(A)$ associated to a flat $\mathrm{GL}(A)$–bundle over a $d$–dimensional homology sphere $M$. Namely, let

$$\rho\colon \pi_1 M \to \mathrm{GL}(A)$$

be the monodromy representation of the flat bundle, then its plusification

$$(B\rho)^+ : M^+ \to B\mathrm{GL}^+(A)$$

can, in view of $M^+ \simeq \mathbb{S}^d$, be considered as an element in algebraic $K$–theory

$$K_d(A) := \pi_d B\mathrm{GL}^+(A).$$

It was proved by Hausmann and Vogel [16] that for a finitely generated, commutative, unital ring $A$ and $d \geq 5$ or $d = 3$, all elements in $K_d(A)$ arise from such a construction.

If the manifold $M$ is not a homology sphere, but still possesses a fundamental class $[M] \in H_d(M;\mathbb{Q})$, one can still consider

$$(B\rho)_*[M] \in H_d(B\mathrm{GL}(A);\mathbb{Q})$$

and can use a suitably defined projection (see Section 2.4) to the primitive part of the homology to obtain

$$\gamma(M) \in PH_d(B\mathrm{GL}(A);\mathbb{Q}) \cong K_d(A) \otimes \mathbb{Q}.$$

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An interesting special case, which has been studied by Dupont and Sah and others, is $K_3(\mathbb{C})$. By a theorem of Suslin [27], $K_3^{\mathrm{ind}}(\mathbb{C})$ is, up to torsion, isomorphic to the Bloch group $B(\mathbb{C})$. On the other hand, each ideally triangulated hyperbolic 3–manifold yields, in a very natural way, an element in $B(\mathbb{C})$, the Bloch invariant. By Neumann and Yang [24], this element does not depend on the chosen ideal triangulation.

A generalization to higher-dimensional hyperbolic manifolds was provided by Goncharov [15]. To an odd-dimensional hyperbolic manifold $M^{2n-1}$ and the flat bundle coming from a half-spinor representation he associates an element $\gamma(M) \in K_{2n-1}(\overline{\mathbb{Q}}) \otimes \mathbb{Q}$ and proves its nontriviality by showing that evaluation of the Borel class yields (a fixed multiple of) the volume.

It thus arises as a natural question, whether other locally symmetric spaces and different flat bundles give nontrivial elements in the $K$–theory of number fields (and eventually how much of algebraic $K$–theory in odd degrees can be represented by locally symmetric spaces and representations of their fundamental groups).

In Section 2, we generalize the argument from [15] to the extent that, for a compact locally symmetric space $M^{2n-1} = \Gamma\backslash G/K$ of noncompact type and a representation $\rho\colon G \to \mathrm{GL}(N,\mathbb{C})$, nontriviality of the associated element $\gamma(M) \in K_{2n-1}(\overline{\mathbb{Q}}) \otimes \mathbb{Q}$ is (independently of $\Gamma$) equivalent to nontriviality of the Borel class $\rho^* b_{2n-1}$.

It does not in general work to associate elements in algebraic $K$–theory to flat bundles over manifolds with boundary. Nonetheless we succeed in Section 4 to associate an element $\gamma(M) \in K_{2n-1}(\overline{\mathbb{Q}}) \otimes \mathbb{Q}$ to flat bundles over locally rank one symmetric spaces of finite volume. (Goncharov [15] did this for hyperbolic manifolds and half-spinor representations, but implicitly assumed that $\partial M$ be connected.)

Nontriviality of classes in $K_{2n-1}(\overline{\mathbb{Q}}) \otimes \mathbb{Q} \cong PH_{2n-1}(\mathrm{GL}(N,\overline{\mathbb{Q}});\mathbb{Q})$ will be checked by pairing with the Borel classes $b_{2n-1} \in H_c^{2n-1}(\mathrm{GL}(N,\mathbb{C});\mathbb{R})$. The results of Section 2 (for closed manifolds) and Section 4 (for cusped manifolds) are subsumed as follows.

**Theorem** *For each symmetric space $G/K$ of noncompact type and odd dimension $d = 2n-1$, and to each representation $\rho\colon G \to \mathrm{GL}(N,\mathbb{C})$ with $\rho^* b_{2n-1} \neq 0$, there exists a constant $c_\rho \neq 0$ such that the following holds.*

*If $M = \Gamma\backslash G/K$ is a finite-volume, orientable, locally symmetric space and either $M$ is compact or $rk(G/K) = 1$, then there is an element*

$$\gamma(M) \in K_{2n-1}(\overline{\mathbb{Q}}) \otimes \mathbb{Q}$$

*such that application of the Borel class $b_{2n-1}$ yields*

$$\langle b_{2n-1}, \gamma(M) \rangle = c_\rho \operatorname{vol}(M).$$

In particular, if $\rho^* b_{2n-1} \neq 0$, then locally symmetric spaces $\Gamma\backslash G/K$ of $\mathbb{Q}$–independent volume give $\mathbb{Q}$–independent elements in $K_{2n-1}(\overline{\mathbb{Q}}) \otimes \mathbb{Q}$.

(In many cases one actually associates an element in $K_{2n-1}(\mathbb{F}) \otimes \mathbb{Q}$, for some number field $\mathbb{F}$; see Theorem 2 in Section 2.6.)

In Section 3, we work out the list of fundamental representations $\rho\colon G \to \mathrm{GL}(N,\mathbb{C})$ for which $\rho^* b_{2n-1} \neq 0$ holds true. It is easy to prove that $\rho^* b_{2n-1} \neq 0$ is always true if $2n-1 \equiv 3 \mod 4$ (and $\rho$ is not the trivial representation). We work out for which fundamental representations $\rho^* b_{2n-1} \neq 0$ holds if $2n-1 \equiv 1 \mod 4$.

**Theorem** *The following is a complete list of irreducible symmetric spaces $G/K$ of noncompact type and fundamental representations $\rho\colon G \to \mathrm{GL}(N,\mathbb{C})$ with $\rho^* b_{2n-1} \neq 0$ for $2n-1 := \dim(G/K)$.*

| Symmetric Space | Representation |
|---|---|
| $\mathrm{SL}_l(\mathbb{R})/\mathrm{SO}_l,\ l \equiv 0,3,4,7 \mod 8$ | any fundamental representation |
| $\mathrm{SL}_l(\mathbb{C})/\mathrm{SU}_l,\ l \equiv 0 \mod 2$ | any fundamental representation |
| $\mathrm{SL}_{2l}(\mathbb{H})/\mathrm{Sp}_l,\ l \equiv 0 \mod 2$ | any fundamental representation |
| $\mathrm{Spin}_{p,q}/(\mathrm{Spin}_p \times \mathrm{Spin}_q)$, $p,q \equiv 1 \mod 2,\ p \not\equiv q \mod 4$ | any fundamental representation |
| $\mathrm{Spin}_{p,q}/(\mathrm{Spin}_p \times \mathrm{Spin}_q)$, $p,q \equiv 1 \mod 2,\ p \equiv q \mod 4$ | positive and negative half-spinor representation |
| $\mathrm{SO}_l(\mathbb{C})/\mathrm{SO}_l,\ l \equiv 3 \mod 4$ | any fundamental representation |
| $\mathrm{Sp}_l(\mathbb{C})/\mathrm{Sp}_l,\ l \equiv 1 \mod 4$ | any fundamental representation |
| $E_7(\mathbb{C})/E_7$ | any fundamental representation |

In this list, the only examples with $\mathrm{rank}(G/K) = 1$ are the hyperbolic spaces $H^d = \mathrm{Spin}(d,1)/\mathrm{Spin}(d)$ with $d$ odd. Thus in the noncompact case we only get invariants for hyperbolic manifolds. (In forthcoming work with Inkang Kim we will generalize the construction to $\mathbb{Q}$–rank 1 lattices in symmetric spaces of higher rank.)

For hyperbolic manifolds and half-spinor representations, the construction of $\gamma(M)$ is due to Goncharov [15] (though the proof implicitly assumes that $\partial M$ be connected). For hyperbolic 3–manifolds, another construction is due to Cisneros-Molina and Jones [8]. (It was related in [8] to the construction of Neumann and Yang [24].) The latter has the advantage that the number of boundary components does not impose technical problems, contrary to the group-homological approach in [15].

Our construction for closed locally symmetric spaces in Section 2 is a straightforward generalization of [15]. In the case of cusped locally symmetric spaces (with possibly more than one cusp) it would have seemed more natural to stick to the approach of [8], and in fact this approach generalizes to locally symmetric spaces in a completely straightforward way. However, we did not succeed in evaluating the Borel class (in order to discuss nontriviality of the obtained invariants) in this approach. On the other hand, Goncharov's approach, even in the case of only one cusp, uses very special properties of the spinor representation, which can not be generalized to other representations.

Therefore, our argument is sort of a mixture of both approaches. On the one hand it is closer in spirit to the arguments of [15] (but with the cuspidal completion in Section 4.2 memorizing the geometry of *distinct* cusps). On the other hand the argument in Section 4.3 uses arguments from [8] to circumvent the very special group-homological arguments that were applied in [15] in the special setting of the half-spinor representations.

Of course, it should be interesting to relate the different constructions more directly.

# 2 The closed case

The results of this section are fairly straightforward generalizations of the results in [15] from hyperbolic manifolds to locally symmetric spaces of noncompact type. (Similar constructions have also appeared in work of other authors, mainly for hyperbolic 3–manifolds.) We define a notion of representations with nontrivial Borel class and, mimicking the arguments in [15], show that representations with nontrivial Borel class give rise to nontrivial elements in algebraic $K$–theory of number fields. The problem of constructing representations with nontrivial Borel class will be tackled in Section 3.

## 2.1 Preparations

**Classifying space** For a group G, its classifying space $BG$ (with respect to the discrete topology on $G$) is the simplicial set $BG$ defined as follows:

- the $k$–simplices of $BG$ are the $k$–tuples $(g_1,\dots,g_k)$ with $g_1,\dots,g_k\in G$,
- the operator $\partial$: $S_k(BG)\to S_{k-1}(BG)$ is defined by

$$\partial(g_1,\dots,g_k)$$
$$=(g_2,\dots,g_k)+\sum_{i=1}^{k-1}(-1)^i(g_1,\dots,g_ig_{i+1},\dots,g_k)+(-1)^k(g_1,\dots,g_{k-1}),$$

- the degeneracy maps are defined by

$$s_j(g_1,\dots,g_k)=(g_1,\dots,g_j,1,g_{j+1},\dots,g_k).$$

The simplicial chain complex of $BG$ will be denoted $C_*^{\mathrm{simp}}(BG)$. The group homology with coefficients in a ring $R$ is

$$H_*(G;R)=H_*^{\mathrm{simp}}(BG;R):=H_*(C_*^{\mathrm{simp}}(BG)\otimes_{\mathbb{Z}}R,\partial\otimes 1).$$

A homomorphism $\rho\colon \Gamma\to G$ induces $(B\rho)_*\colon H_*(\Gamma;R)\to H_*(G;R)$.

**Straight simplices** Let $M$ be a Riemannian manifold of nonpositive sectional curvature, $C_*(M)$ the chain complex of singular simplices.

Let $\pi\colon \widetilde{M}\to M$ be the universal covering. Fix a point $x_0\in M$ and a lift $\widetilde{x}_0\in\widetilde{M}$. Then $\Gamma:=\pi_1(M,x_0)$ acts isometrically by deck transformations on $\widetilde{M}$.

In a simply connected space of nonpositive sectional curvature each *ordered* $(k+1)$–tuple of vertices $(y_0,\dots,y_k)$ determines a unique straight $k$–simplex $\mathrm{str}(y_0,\dots,y_k)$. In particular, for $g_0,g_1,\dots,g_k\in\Gamma=\pi_1(M,x_0)$ there is a unique straight simplex

$$\mathrm{str}(g_0\widetilde{x}_0,g_1\widetilde{x}_0,\dots,g_k\widetilde{x}_0)$$

in $\widetilde{M}$. A simplex $\sigma\in C_*(M)$ is said to be straight if some (hence any) lift $\widetilde{\sigma}\in C_*(\widetilde{M})$ with $\pi\widetilde{\sigma}=\sigma$ is a straight simplex.

Let $C_*^{\mathrm{str},x_0}(M)$ be the chain complex of straight simplices with all vertices in $x_0$. There is a canonical chain map

$$\Psi\colon C_*^{\mathrm{simp}}(B\Gamma)\to C_*^{\mathrm{str},x_0}(M)$$

given by

$$\Psi(g_1,\dots,g_k):=\pi(\mathrm{str}(\widetilde{x}_0,g_1\widetilde{x}_0,g_1g_2\widetilde{x}_0,\dots,g_1\dots g_k\widetilde{x}_0)).$$

Let $w_0,\dots,w_k$ be the vertices of the standard simplex $\Delta^k$. For $j=0,\dots,k$ let $\gamma_j\subset\Delta^k$ be the sub–1–simplex with $\partial\gamma_j=w_j-w_{j-1}$ for $j=1,\dots,k$. Then there is a homomorphism

$$\Phi\colon C_*^{\mathrm{str},x_0}(M)\to C_*^{\mathrm{simp}}(B\Gamma)$$

defined by $\Phi(\sigma)=(g_1,\dots,g_k)$, where $g_j\in\Gamma=\pi_1(M,x_0)$ is the homotopy class (rel. vertices) of $\sigma|_{\gamma_j}$ for $j=1,\dots,k$.

Clearly $\Phi(\pi(\mathrm{str}(\widetilde{x}_0,g_1\widetilde{x}_0,g_1g_2\widetilde{x}_0,\dots,g_1\dots g_k\widetilde{x}_0)))=(g_1,\dots,g_k)$, thus $\Phi\Psi=\mathrm{id}$. On the other hand, a straight simplex $\sigma\colon\Delta^k\to M$ with all vertices in $x_0$ is uniquely determined by the homotopy classes (rel. vertices) of $g_j=[\sigma|_{\gamma_j}]$ for $j=1,\dots,k$, because its lift to $\widetilde{M}$ must be in the $\Gamma$–orbit of $\mathrm{str}(\widetilde{x}_0,g_1\widetilde{x}_0,g_1g_2\widetilde{x}_0,\dots,g_1\dots g_k\widetilde{x}_0)$. Thus $\Psi\Phi=\mathrm{id}$. This shows that $\Psi$ and $\Phi$ are chain isomorphisms, inverse to each other.

**Eilenberg–Mac Lane map** Let $C_*^{x_0}(M) \subset C_*(M)$ be the subcomplex generated by singular simplices with all vertices in $x_0$. The inclusions

$$C_*^{\mathrm{str},x_0}(M) \subset C_*^{x_0}(M) \subset C_*(M)$$

are chain homotopy equivalences[1]. For the first inclusion this is proved (for arbitrary aspherical manifolds, but with an isomorphic image of $C_*^{\mathrm{simp}}(B\Gamma)$ instead of the in this generality not defined $C_*^{\mathrm{str},x_0}(M)$) by Eilenberg and Mac Lane [13, Theorem 1a]. For the second inclusion it is proved by Eilenberg [12, Paragraph 31].

The composition of the chain isomorphism $\Psi\colon C_*^{\mathrm{simp}}(B\Gamma) \to C_*^{\mathrm{str},x_0}(M)$ with the inclusion $C_*^{\mathrm{str},x_0}(M) \to C_*(M)$ is thus a chain homotopy equivalence

$$C_*^{\mathrm{simp}}(B\Gamma) \to C_*(M),$$

the induced isomorphism

$$\mathrm{EM}\colon H_*^{\mathrm{simp}}(B\Gamma;\mathbb{Z}) \to H_*(M;\mathbb{Z})$$

will be called the Eilenberg–Mac Lane map. The chain homotopy inverse is given by the composition of str with the chain isomorphism $\Phi$, thus

$$\mathrm{EM}^{-1} = \Phi_* \circ \mathrm{str}_* .$$

The geometric realization $|B\Gamma|$ of $B\Gamma$ in the sense of Milnor [22] is aspherical by May [21, page 128]. Given a manifold $M$ and an isomorphism $I\colon \pi_1 M \cong \Gamma$, there is an up to homotopy unique continuous mapping $h^M\colon M \to |B\Gamma|$ which induces $I$ on $\pi_1$; see [21, page 177]. The map $h^M$ (rather its homotopy class) is called the classifying map for $\pi_1 M$. If $M$ is aspherical and has the homotopy type of a CW–complex then $h^M$ is a homotopy equivalence, and $h_*^M\colon H_*(M;\mathbb{Z}) \to H_*(|B\Gamma|;\mathbb{Z})$ is the composition of $\mathrm{EM}^{-1}$ with the isomorphism $i_*\colon H_*^{\mathrm{simp}}(B\Gamma;\mathbb{Z}) \to H_*(|B\Gamma|;\mathbb{Z})$ that is induced by the inclusion $i$ of the simplicial into the singular chain complex.

## 2.2 Construction of elements in algebraic $K$–theory

Throughout this paper, a *ring* $A$ will mean a *commutative ring with unit*. In all applications $A$ will be a subring (with unit) of the ring of complex numbers: $A \subset \mathbb{C}$.

One defines $\mathrm{GL}(A)$ as the increasing union $\mathrm{GL}(A) = \bigcup_{N\in\mathbb{N}} \mathrm{GL}(N,A)$, where $\mathrm{GL}(N,A)$ is considered as a subgroup of $\mathrm{GL}(N+1,A)$ via the canonical embedding

[1] Pictorially the chain homotopy inverse $\mathrm{str}\colon C_*(M) \to C_*^{\mathrm{str},x_0}(M)$ of the inclusion $C_*^{\mathrm{str},x_0}(M) \subset C_*(M)$ first homotopes all vertices of a given cycle into $x_0$ and then straightens the so-obtained cycle (by induction on dimension of subsimplices, depending on the given order of vertices) as in Benedetti and Petronio [2, Lemma C.4.3].

as $(N \times N)$–block matrices with complementary $(1 \times 1)$–block having entry $1$. We consider the simplicial set $B\mathrm{GL}(A)$ as defined in Section 2.1, and $|B\mathrm{GL}(A)|$ its geometrical realisation.

A representation $\rho\colon \Gamma \to \mathrm{GL}(A)$ induces a continuous map

$$|B\rho|\colon |B\Gamma| \to |B\mathrm{GL}(A)|.$$

**Definition 1** Let $M$ be a topological space with $\Gamma := \pi_1(M, x_0), x_0 \in M$, let $A$ be a commutative ring with unit and let $\rho\colon \Gamma \to \mathrm{GL}(A)$ be a homomorphism. Then we define

$$(H\rho)_*\colon H_d(M;\mathbb{Q}) \to H_d^{\mathrm{simp}}(B\mathrm{GL}(A);\mathbb{Q})$$

as the composition of $|B\rho|$ with the classifying map $h^M\colon M \to |B\Gamma|$:

$$H_d(M;\mathbb{Q}) \xrightarrow{h_*^M} H_d(|B\Gamma|;\mathbb{Q}) \xrightarrow{|B\rho|_*} H_d(|B\mathrm{GL}(A)|;\mathbb{Q}) \cong H_d^{\mathrm{simp}}(B\mathrm{GL}(A);\mathbb{Q}) \ .$$

(We will use without mention that inclusion $i\colon C_*^{\mathrm{simp}}(B\mathrm{GL}(A)) \to C_*(|B\mathrm{GL}(A)|)$ induces an isomorphism $i_*\colon H_*^{\mathrm{simp}}(B\mathrm{GL}(A)) \to H_*(|B\mathrm{GL}(A)|)$; see [22, Lemma 5].)

If $M$ is a closed, orientable, connected $d$–manifold, and the ring $A$ satisfies mild assumptions (see Section 2.5), eg for $A = \overline{\mathbb{Q}}$, then we will now explain how to construct an element in $K_d\,(A) \otimes \mathbb{Q}$.

Let $[M] \in H_d(M;\mathbb{Q})$ be the fundamental class. Consider

$$(H\rho)_*[M] \in H_d(|B\mathrm{GL}(A)|;\mathbb{Q}) \cong H_d(|B\mathrm{GL}(A)|^+;\mathbb{Q}).$$

By the Milnor–Moore Theorem, the Hurewicz homomorphism

$$K_d(A) := \pi_d(|B\mathrm{GL}(A)|^+) \to H_d|(B\mathrm{GL}(A)|^+;\mathbb{Z})$$

gives, after tensoring with $\mathbb{Q}$, an injective homomorphism

$$K_d(A) \otimes \mathbb{Q} = \pi_d(|B\mathrm{GL}(A)|^+) \otimes \mathbb{Q} \to H_d(|B\mathrm{GL}(A)|^+;\mathbb{Q}).$$

Its image, again by the Milnor–Moore theorem, is the subgroup $PH_d(|B\mathrm{GL}(A)|^+;\mathbb{Q})$ of primitive elements, which we will henceforth identify with $K_d(A) \otimes \mathbb{Q}$.

By Quillen (compare Burgos Gil [6, Section 9.1]), inclusion induces an isomorphism

$$Q_*\colon PH_*(|B\mathrm{GL}(A)|;\mathbb{Q}) \to PH_*(|B\mathrm{GL}(A)|^+;\mathbb{Q}) \cong K_*(A) \otimes \mathbb{Q}.$$

(For$d$ even and $A$ a ring of integers in any number field, $PH_d^{\mathrm{simp}}(B\mathrm{GL}(A);\mathbb{Q}) = 0$; cf [6, Theorem 9.9]. Therefore one is only interested in the case $d = 2n - 1$.)

Whenever we have a fixed projection

$$\mathrm{pr}_*\colon H_*^{\mathrm{simp}}(B\mathrm{GL}(A);\mathbb{Q}) \to PH_*^{\mathrm{simp}}(B\mathrm{GL}(A);\mathbb{Q}) \cong PH_*(|B\mathrm{GL}(A)|;\mathbb{Q}),$$

we can define an element $\gamma(M) \in K_d(A) \otimes \mathbb{Q}$ as

$$\gamma(M) := Q_*\, \mathrm{pr}_*\, (H\rho)_*\, [M].$$

In Section 2.5 we are going to show that eg for $A = \bar{\mathbb{Q}}$ (and also for many other rings) the projection $\mathrm{pr}_*$ can be chosen such that the evaluations of the Borel class on $h$ and $\mathrm{pr}_*(h)$ agree for all $h \in H_*^{\mathrm{simp}}(B\mathrm{GL}(\bar{\mathbb{Q}});\mathbb{Q})$. In particular, to check nontriviality of $\gamma(M)$ it will then suffice to apply the Borel class to $(H\rho)_*[M]$.

If $M$ is a (compact, orientable) manifold with *nonempty* boundary, then there is no general construction of an element in algebraic $K$–theory. However, we will show in Section 4 that for finite-volume locally rank one symmetric spaces one can generalize the above construction and again construct an invariant $\gamma(M) \in K_d(\bar{\mathbb{Q}}) \otimes \mathbb{Q}$.

## 2.3 The volume class in $H_c^d(\mathrm{Isom}(\widetilde{M}))$

**Volume class** For a Lie group $G$, let $C_c\left(G^{*+1},\mathbb{R}\right)$ be the *continuous* $G$–invariant mappings from $G^{*+1}$ to $\mathbb{R}$, $\delta$ the usual coboundary operator and $H_c^*(G;\mathbb{R})$ the cohomology of $(C_c(G^{*+1},\mathbb{R})^G,\delta)$. There is a comparison map comp: $H_c^*(G;\mathbb{R}) \to H^*(G;\mathbb{R})$ defined by the cochain map

$$\mathrm{comp}(f)(g_1,\ldots,g_k) := f(1,g_1,g_1g_2,\ldots,g_1g_2\cdots g_k).$$

**Remark 1** For $f \in H_c^*(G;\mathbb{R})$ and $c \in H_*(G;\mathbb{R})$, we will denote

$$\langle f,c\rangle = \mathrm{comp}(f)(c).$$

Let $\widetilde{M} = G/K$ be a symmetric space of noncompact type. The Riemannian metric and in particular the volume form are given via the Killing form and are thus canonical. It is well-known (see Helgason [17, Chapter V, Theorem 3.1]) that $\widetilde{M}$ has nonpositive sectional curvature. One can assume that $G$ is semisimple and acts by orientation-preserving isometries on $\widetilde{M}$.

Fix an arbitrary point $\tilde{x} \in \widetilde{M} = G/K$. The volume class

$$v_d \in H_c^d(G;\mathbb{R})$$

is defined as follows. We define a simplicial cochain $cv_d \in C^d_{\mathrm{simp}}(BG)$ by

$$cv_d(g_1,\ldots,g_d) = \mathrm{algvol}(\mathrm{str}(\tilde{x},g_1\tilde{x},\ldots,g_1\cdots g_d\tilde{x})) := \mathrm{int}_{\mathrm{str}(\tilde{x},g_1\tilde{x},\ldots,g_d\tilde{x})}\, d\mathrm{vol},$$

that is the signed volume algvol [2, page 107] of the straight simplex with vertices $\widetilde{x}, g_1\widetilde{x}, \dots, g_1 \dots g_d\widetilde{x}$. From Stokes' Theorem and

$$\operatorname{algvol}(\operatorname{str}(g_1\widetilde{x}, g_1g_2\widetilde{x}, \dots, g_1 \dots g_d\widetilde{x})) = \operatorname{algvol}(\operatorname{str}(\widetilde{x}, g_2\widetilde{x}, \dots, g_2 \dots, g_d\widetilde{x}))$$

one can conclude

$$\begin{aligned}\delta c\nu_d(g_1, \dots, g_{d+1}) &= \operatorname{int}_{\partial \operatorname{str}(\widetilde{x}, g_1\widetilde{x}, \dots, g_1 \dots g_{d+1}\widetilde{x})}\, d\mathrm{vol}\\ &= \operatorname{int}_{\operatorname{str}(\widetilde{x}, g_1\widetilde{x}, \dots, g_{d+1}\widetilde{x})}\, d(d\mathrm{vol}) = 0.\end{aligned}$$

Thus $c\nu_d$ is a simplicial cocycle on $BG$.

Consider the cocycle $\nu_d \in C_c^d(G; \mathbb{R})$ given by the (clearly continuous) mapping

$$\begin{aligned}\nu_d(g_0, \dots, g_d) &= c\nu_d(g_0^{-1}g_1, \dots, g_{d-1}^{-1}g_d)\\ &= \operatorname{int}_{\operatorname{str}(\widetilde{x}, g_0^{-1}g_1\widetilde{x}, \dots, g_{d-1}^{-1}g_d\widetilde{x})}\, d\mathrm{vol} = \operatorname{int}_{\operatorname{str}(g_0\widetilde{x}, g_1\widetilde{x}, \dots, g_d\widetilde{x})}\, d\mathrm{vol}.\end{aligned}$$

It defines a cohomology class $v_d := [\nu_d] \in H_c^d(G; \mathbb{R})$ such that $\operatorname{comp}(v_d) \in H^d(BG; \mathbb{R})$ is represented by $c\nu_d$. The *volume class* $v_d$ does not depend on the chosen $\widetilde{x} \in G/K$.

**Theorem 1** *Let $M = \Gamma\backslash G/K$ be a closed, oriented, connected, $d$–dimensional locally symmetric space of noncompact type, let $j\colon \Gamma \to G$ be the inclusion of $\Gamma = \pi_1 M$ and $Bj_*\colon H_*^{\mathrm{simp}}(B\Gamma; \mathbb{Z}) \to H_*^{\mathrm{simp}}(BG; \mathbb{Z})$ the induced homomorphism. Let $[M] \in H_d(M; \mathbb{Z})$ be the fundamental class of $M$. Then*

$$\operatorname{vol}(M) = \langle v_d, Bj_* \operatorname{EM}^{-1}[M]\rangle.$$

**Proof** Let $\sum_{i=1}^r a_i\sigma_i$ represent $[M]$. Fix $\widetilde{x}_0 \in \widetilde{M}$ and $x_0 := \pi(x_0) \in M$. Then also $\sum_{i=1}^r a_i\tau_i := \sum_{i=1}^r a_i \operatorname{str}(\sigma_i) \in C_*^{\mathrm{str}, x_0}(M)$ represents $[M]$, and $\operatorname{vol}(M) = \sum_{i=1}^r a_i \operatorname{algvol}(\tau_i)$ from Stokes' Theorem. Let $\gamma_j^i \in \Gamma$ be the homotopy class (rel. vertices) of the closed edge from $\tau_i(w_{j-1})$ to $\tau_i(w_j)$. Then

$$\tau_j = \pi(\operatorname{str}(\widetilde{x}_0, \gamma_1^i\widetilde{x}_0, \dots, \gamma_1^i \cdots \gamma_d^i\widetilde{x}_0)).$$

Thus from $\operatorname{EM}^{-1} = \Phi_* \operatorname{str}_*$ we have that $Bj_* \operatorname{EM}^{-1}[M] \in H_d(G; \mathbb{Z})$ is represented by

$$\sum_{i=1}^r a_i(1, \gamma_1^i, \dots, \gamma_d^i) \in C_d^{\mathrm{simp}}(BG).$$

But

$$c\nu_d(\gamma_1^i, \dots, \gamma_d^i) = \operatorname{int}_{\operatorname{str}(\widetilde{x}_0, \gamma_1^i\widetilde{x}_0, \dots, \gamma_1^i \cdots \gamma_d^i\widetilde{x}_0)}\, d\mathrm{vol} = \operatorname{int}_{\tau_i}\, d\mathrm{vol} = \operatorname{algvol}(\tau_i),$$

which implies

$$\langle v_d, Bj_* \,\mathrm{EM}^{-1}[M]\rangle = cv_d\left(\sum_{i=1}^{r} a_i(\gamma_1^i,\dots,\gamma_d^i)\right) = \sum_{i=1}^{r} a_i \operatorname{algvol}(\tau_i) = \operatorname{vol}(M). \quad \square$$

## 2.4 Borel classes

**2.4.1 Dual symmetric space and dual representations** Let $\widetilde{M} = G/K$ be a symmetric space of noncompact type. Then $G$ is a semisimple, connected Lie group and $K$ is a maximal compact subgroup; see Helgason [17, Chapter VI.1].

Let $\underline{g}$ be the Lie algebra of $G$ and $\underline{g} = \underline{k} \oplus \underline{p}$ its Cartan decomposition. The Killing form $B(X,Y) = \mathrm{Tr}\,(\mathrm{ad}(X)\circ \mathrm{ad}(Y))$ is negatively definite on $\underline{k}$, positively definite on $\underline{p}$.

The dual symmetric space to $G/K$ is $G_u/K$, where $G_u$ is the simply connected Lie group with Lie algebra $\underline{g}_u = \underline{k} \oplus i\underline{p} \subset \underline{g} \otimes \mathbb{C}$; cf [17, Chapter V.2.]. The Killing form on $\underline{g}_u$ is negatively definite, thus $G_u/K$ is a compact symmetric space.

The Lie algebra cohomology $H^*(\underline{g})$ is the cohomology of the complex $(\Lambda^*\underline{g}, d)$ with $d\phi(X_0,\dots,X_n) = \sum_{i<j}(-1)^{i+j}\phi([X_i,X_j],X_0,\dots,\widehat{X}_i,\dots,\widehat{X}_j,\dots,X_n)$.

The relative Lie algebra cohomology $H^*(\underline{g},\underline{k})$ is the cohomology of the subcomplex $(C^*(\underline{g},\underline{k}),d)\subset(\Lambda^*\underline{g},d)$ with $C^*(\underline{g},\underline{k})=\{\phi\in\Lambda^*\underline{g} : i(X)\phi=0, \mathrm{ad}(X)\phi=\phi\ \forall X\in\underline{k}\}$, where $i(X)$ means insertion as first variable; cf [6, Section 5.5].

If $G/K$ is a symmetric space of noncompact type, and $G_u/K$ its compact dual, then there is an obvious isomorphism $H^*(\underline{g},\underline{k}) \to H^*(\underline{g}_u,\underline{k})$, dual to the obvious $\mathbb{R}$–linear map $\underline{k}\oplus i\underline{p} \to \underline{k}\oplus\underline{p}$.

Moreover, $H^*(\underline{g},\underline{k})$ is the cohomology of the complex of $G$–invariant differential forms on $G/K$; cf [6, Example 5.39]. Since $G_u$ is compact and connected, there is an isomorphism $H^*(G_u/K;\mathbb{R}) \to H^*(\underline{g}_u,\underline{k})$, defined by averaging over $G_u$.

**Definition 2** Let $\widetilde{M} = G/K$ be a symmetric space of noncompact type, and let $\rho\colon (G,K)\to(\mathrm{GL}(N,\mathbb{C}),U(N))$ be a smooth representation. We denote

$$D_e\rho\colon (\underline{g},\underline{k}) \to (\mathfrak{gl}(N,\mathbb{C}), u(N))$$

the associated Lie-algebra homomorphism, and, with $\underline{g} = \underline{k}\oplus\underline{p}$, $\underline{g}_u := \underline{k}\oplus i\underline{p}$,

$$D_e\rho_u\colon (\underline{g}_u,\underline{k}) \to (u(N)\oplus u(N), u(N))$$

the induced homomorphism on $\underline{k}\oplus i\underline{p}$. The corresponding Lie group homomorphism $\rho_u\colon (G_u,K)\to(U(N)\times U(N),U(N))$ will be called the dual homomorphism to $\rho$. Denote $\overline{\rho}_u\colon G_u/K \to U(N)$ the induced smooth map.

Here $\underline{g}_u$, $\underline{k}$ and $i\underline{p}$ are to be understood as subsets of the complexification $\underline{g} \otimes \mathbb{C}$. In particular, the complexification of $\mathfrak{gl}_N \mathbb{C}$ is isomorphic to $\mathfrak{gl}_N \mathbb{C} \oplus \mathfrak{gl}_N \mathbb{C}$, and $i\underline{p} \simeq u(N)$ in this case. We emphasize that $\rho_u$ *sends* $K$ *to the first factor of* $U(N) \times U(N)$, and not to the diagonal subgroup as has been claimed in [15, page 586].

**2.4.2 Van Est Theorem** The van Est Theorem (see Burgos Gil [6, Theorem 6.9]) states that there is a natural isomorphism

$$\nu_G\colon H_c^*(G;\mathbb{R}) \to H^*(\underline{g},\underline{k}).$$

If $\rho\colon (G,K) \to (\mathrm{GL}(N,\mathbb{C}), U(N))$ is a representation then we obtain the following commutative diagram, where all vertical arrows are isomorphisms

$$\begin{array}{ccc}
H_c^*(\mathrm{GL}(N,\mathbb{C});\mathbb{R}) & \xrightarrow{\ \rho^*\ } & H_c^*(G;\mathbb{R}) \\
{\scriptstyle \nu^{-1}_{\mathrm{GL}(N,\mathbb{C})}}\uparrow\cong & & {\scriptstyle \nu_G^{-1}}\uparrow\cong \\
H^*(\mathfrak{gl}(N,\mathbb{C}), u(N)) & \xrightarrow{\ D_e\rho^*\ } & H^*(\underline{g},\underline{k}) \\
\uparrow\cong & & \uparrow\cong \\
H^*(u(N)\oplus u(N), u(N)) & \xrightarrow{\ D_e\rho_u^*\ } & H^*(\underline{g}_u,\underline{k}) \\
\uparrow\cong & & \uparrow\cong \\
H^*(U(N);\mathbb{R}) & \xrightarrow{\ \overline{\rho}_u^*\ } & H^*(G_u/K;\mathbb{R})
\end{array}$$

**Corollary 1** *Let* $G$ *be a connected, semisimple Lie group,* $K$ *a maximal compact subgroup,* $d = \dim(G/K)$, $v_d \in H_c^d(G;\mathbb{R})$ *the volume class,* $[d\mathrm{vol}] \in H^d(G_u/K;\mathbb{R})$ *the de Rham cohomology class of the volume form on* $G_u/K$ *and*

$$D_G\colon H^*(G_u/K;\mathbb{R}) \to H_c^*(G;\mathbb{R})$$

*the isomorphism given by the right-hand column of the above diagram. Then*

$$D_G([d\mathrm{vol}]) = v_d.$$

**Proof** By [9, Proposition 1.5] $\nu_G$ maps $v_d$ to the class of the volume form in $H_d(G/K;\mathbb{R}) \cong H^d(\underline{g},\underline{k})$. The Riemannian metrics on $G/K$ and $G_u/K$ are defined by the negative of the Killing form. The $\mathbb{R}$–linear map $\underline{k} \oplus i\underline{p} \to \underline{k} \oplus \underline{p}$ clearly preserves the Killing form, thus the isomorphism $H^d(\underline{g},\underline{k}) \simeq H^d(\underline{g}_u,\underline{k})$ maps the volume form of $G/K$ to that of $G_u/K$. □

**2.4.3 Chern classes and Borel classes** Let $H$ be a *compact* connected Lie group. Let $I_S^n(H)$ resp. $I_A^n(H)$ be the ad–invariant symmetric resp. antisymmetric multilinear $n$–forms on its Lie algebra $\underline{h}$. By [6, Proposition 5.2] we have the isomorphism $\Phi_A\colon I_A^n(H) \to H^n(H;\mathbb{R})$. Moreover, we remind that there is the Chern–Weil isomorphism $\Phi_S\colon I_S^n(H) \to H^{2n}(BH;\mathbb{R})$ [6, Theorem 5.23], where *in this section (contrary to the remainder of the paper) $BH$ means the classifying space for $H$ with its Lie group topology.*

For $H = U(N)$ we consider the symmetric polynomial $\mathrm{Tr}_n \in I_S^n(U(N))$ defined by

$$\mathrm{Tr}_n(A_1,\dots,A_n) = \frac{1}{(2\pi i)^n}\frac{1}{n!}\sum_{\sigma\in S_n}\mathrm{Tr}(A_{\sigma(1)}\dots A_{\sigma(n)}) \in I_S^n(U(N)).$$

The $2n$–th component of the *universal Chern character*

$$\mathrm{ch}_n := \Phi_S(\mathrm{Tr}_n) \in H^{2n}(BU(N);\mathbb{Q}).$$

(We consider the rational valued Chern character whose $2n$–th component is obtained by multiplication with $1/(2\pi i)^n$ from that of the twisted Chern character considered in [6, Proposition 5.27].)

There is a "transgression map" $\tau$ which maps a subspace of $H^{2n-1}(H;\mathbb{Z})$ (whose elements are the so-called transgressive elements) to $H^{2n}(BH;\mathbb{Z})$; cf [6, Example 4.16]. By [7] there is a homomorphism

$$R\colon I_S^n(H) \to I_A^{2n-1}(H),$$

such that the image of $\Phi_A \circ R$ in $H^{2n-1}(H;\mathbb{R})$) consists precisely of the transgressive elements and such that $\tau\circ\Phi_A\circ R = \Phi_S$.

For $H = U(N)$, [6, Example 5.37] gives an explicit representative for the *Borel classes*

$$b_{2n-1} := \Phi_A(R(\mathrm{Tr}_n)) \in H^{2n-1}(U(N);\mathbb{R}) \cong H^{2n-1}(u(N))$$

by the Lie algebra cocycle whose value on $X_1,\dots,X_{2n-1} \in u(N)$ is

$$\frac{1}{(2\pi i)^n}\frac{(-1)^{n-1}(n-1)!}{(2n-1)!}\sum_{\sigma\in S_{2n-1}}(-1)^\sigma\,\mathrm{Tr}(X_{\sigma(1)}[X_{\sigma(2)},X_{\sigma(3)}]\cdots[X_{\sigma(2n-2)},X_{\sigma(2n-1)}]).$$

From $\tau\circ\Phi_A\circ R = \Phi_S$ we conclude that $\tau(b_{2n-1}) = \mathrm{ch}_n$.

**Lemma 1** *Let $G/K$ be a symmetric space of noncompact type, of odd dimension $d = 2n-1$, $\rho\colon G \to \mathrm{GL}(N,\mathbb{C})$ a representation. Then*

$$\rho_* b_{2n-1} \neq 0 \in H_c^{2n-1}(G;\mathbb{R}) \iff \overline{\rho}_u^* b_{2n-1} \neq 0 \in H^{2n-1}(G_u/K;\mathbb{R}) \iff \langle b_{2n-1}, (\rho_u)_*[G_u/K]\rangle \neq 0.$$

**Proof** This follows from naturality of $D_G$ and from $H_d\left(G_u/K;\mathbb{R}\right)\cong\mathbb{R}$. □

It will be clear from the context whether we consider the Borel classes as elements of $H^*\left(u(N)\right)\simeq H^*\left(U(N);\mathbb{R}\right)$ or as the (under the van Est isomorphism) corresponding elements of $H_c^*\left(\mathrm{GL}\left(N,\mathbb{C}\right);\mathbb{R}\right)$.

Stabilization $H^*\left(U\left(N+1\right);\mathbb{R}\right)\to H^*\left(U(N);\mathbb{R}\right)$ preserves $b_{2n-1}$, thus $b_{2n-1}$ may also be considered as an element of $H^{2n-1}\left(U;\mathbb{R}\right)\cong H_c^{2n-1}\left(\mathrm{GL}(\mathbb{C});\mathbb{R}\right)$.

In Theorem 2 and Theorem 3 we will for subrings $A\subset\mathbb{C}$ consider the homomorphism

$$K_{2n-1}(A)\otimes\mathbb{Q}\to\mathbb{R}$$

defined by application of the isomorphism

$$K_{2n-1}(A)\otimes\mathbb{Q}\cong PH_{2n-1}(|B\mathrm{GL}(A)|^+;\mathbb{Q})\cong PH_{2n-1}(B\mathrm{GL}(A);\mathbb{Q})$$

from Section 2.2 and pairing with the Borel class

$$b_{2n-1}\in H_c^{2n-1}\left(\mathrm{GL}(\mathbb{C});\mathbb{R}\right).$$

## 2.5 Projection $H_*^{\mathrm{simp}}(B\,\mathrm{GL}(\bar{\mathbb{Q}});\mathbb{Q})\to K_*(\bar{\mathbb{Q}})\otimes\mathbb{Q}$

Let $A\subset\mathbb{C}$ be a subring and $G=\mathrm{GL}(A)$. Let $I=H^{*\geq 1}_{\mathrm{simp}}(BG;\mathbb{Q})$ be the augmentation ideal of $H^*_{\mathrm{simp}}(BG;\mathbb{Q})$ and $D=I^2$ the subspace of decomposable cohomology classes.

Let $PH_*^{\mathrm{simp}}(BG;\mathbb{Q})$ be the subspace of primitive elements in homology. It is easy to check that $c(h)=0$ for all $c\in D, h\in PH_*^{\mathrm{simp}}(BG;\mathbb{Q})$. By [23, Proposition 3.10] $I/D$ is the dual of $PH_*^{\mathrm{simp}}(BG;\mathbb{Q})$, which implies

$$D=\{c\in I: c(h)=0\ \forall\ h\in PH_*^{\mathrm{simp}}(BG;\mathbb{Q})\}$$
$$PH_*^{\mathrm{simp}}(BG;\mathbb{Q})=\{h\in H_*^{\mathrm{simp}}(BG;\mathbb{Q}): c(h)=0\ \forall\ c\in D\}.$$

**Lemma 2** *Let $A\subset\mathbb{C}$ be a subring. Assume that* $\mathrm{comp}(b_{2n-1})\in H^{2n-1}_{\mathrm{simp}}(B\mathrm{GL}(A);\mathbb{R})$ *is not decomposable:* $\mathrm{comp}(b_{2n-1})\notin D$. *Then there exists a projection*

$$\mathrm{pr}_{2n-1}\colon H^{\mathrm{simp}}_{2n-1}(B\mathrm{GL}(A);\mathbb{Q})\to PH^{\mathrm{simp}}_{2n-1}(B\mathrm{GL}(A);\mathbb{Q})$$

*such that for all $h\in H^{\mathrm{simp}}_{2n-1}(B\mathrm{GL}(A);\mathbb{Q})$,*

$$\mathrm{comp}(b_{2n-1})(\mathrm{pr}_{2n-1}(h))=\mathrm{comp}(b_{2n-1})(h).$$

**Proof** Denote $G = \mathrm{GL}(A)$. We consider $\mathrm{comp}(b_{2n-1}) \in H^{2n-1}_{\mathrm{simp}}(BG;\mathbb{Q})$ as a linear map $\mathrm{comp}(b_{2n-1})\colon H^{\mathrm{simp}}_{2n-1}(BG;\mathbb{Q}) \to \mathbb{Q}$. We have

$$PH^{\mathrm{simp}}_{2n-1}(BG;\mathbb{Q}) = \left\{h \in H^{\mathrm{simp}}_{2n-1}(BG;\mathbb{Q}) : c(h) = 0 \ \forall c \in D\right\}.$$

Since $\mathrm{comp}(b_{2n-1}) \notin D$ there exists some $e_0 \in PH^{\mathrm{simp}}_{2n-1}(BG;\mathbb{Q})$ with

$$\mathrm{comp}(b_{2n-1})(e_0) \neq 0.$$

We extend $\{e_0\}$ to a basis $\{e_j : j \in J_P\}$ of $PH^{\mathrm{simp}}_{2n-1}(BG;\mathbb{Q})$ and then to a basis $\{e_j : j \in J\}$ of $H^{\mathrm{simp}}_{2n-1}(BG;\mathbb{Q})$, for some index sets $\{0\} \subset J_P \subset J$.

Since $\mathrm{comp}(b_{2n+1})(e_0) \neq 0$, we have that $\{e'_j : j \in J\}$ defined by

$$e'_0 := e_0, e'_j := \mathrm{comp}(b_{2n-1})(e_0)e_j - \mathrm{comp}(b_{2n-1})(e_j)e_0 \text{ for } j \in J - \{0\}$$

is another basis of $H^{\mathrm{simp}}_{2n-1}(BG;\mathbb{Q})$, and

$$\{e'_j : j \in J_P\}$$

is another basis of $PH^{\mathrm{simp}}_{2n-1}(BG;\mathbb{Q})$.

Let $S \subset H^{\mathrm{simp}}_{2n-1}(BG;\mathbb{Q})$ be the subspace spanned by $\{e'_j : j \notin J_P\}$, then $S \subset \ker(\mathrm{comp}(b_{2n-1}))$ and we have a decomposition

$$H^{\mathrm{simp}}_{2n-1}(BG;\mathbb{Q}) = PH^{\mathrm{simp}}_{2n-1}(BG;\mathbb{Q}) \oplus S.$$

We use this decomposition to define the projection

$$\mathrm{pr}_{2n-1}\colon H^{\mathrm{simp}}_{2n-1}(BG;\mathbb{Q}) \to PH^{\mathrm{simp}}_{2n-1}(BG;\mathbb{Q})$$

by $\mathrm{pr}_{2n-1}(p+s) = p$ for $p \in PH^{\mathrm{simp}}_{2n-1}(BG;\mathbb{Q})$ and $s \in S$. $S \subset \ker(\mathrm{comp}(b_{2n-1}))$ implies $\mathrm{comp}(b_{2n-1})(\mathrm{pr}_{2n-1}(p+s)) = \mathrm{comp}(b_{2n-1})(p+s)$. □

To decide whether the Borel class is indecomposable we apply[2] Borel's computation of $K$–theory of integer rings in number fields in [4].

[2]We remark that in the already interesting case $A = \mathbb{C}$ one can prove indecomposability of the Borel class without using Borel's $K$–theory computation.

First, $H^*_c(\mathrm{GL}(N,\mathbb{C});\mathbb{Q}) = \Lambda_{\mathbb{Q}}(b_1, b_3, b_5, \dots, b_{2N-1})$ implies that $b_{2n-1}$ is not decomposable in $H^*_c(\mathrm{GL}(N,\mathbb{C});\mathbb{Q})$ for any $N$. Next, by homology stability of the linear group [6, page 77], inclusion induces an isomorphism $H^{2n-1}(BG;\mathbb{Q}) = H^{2n-1}(B\mathrm{GL}(N,\mathbb{C});\mathbb{Q})$ if $N \geq 4n+3$.

By Borel's Theorem [4, Theorem 9.6], for each arithmetic subgroup $\Gamma \subset \mathrm{SL}(N,\mathbb{C})$ we have an isomorphism $j^*\colon H^{2n-1}_{\mathrm{simp}}(B\Gamma;\mathbb{Q}) \to H^{2n-1}_c(B\mathrm{SL}(N,\mathbb{C});\mathbb{Q})$ whenever $N \geq 8n+4$. This isomorphism is constructed via the van Est isomorphism, that is by integration of forms over simplices. In particular, if $h \in H^*_{\mathrm{simp}}(B\mathrm{SL}(N,\mathbb{C});\mathbb{Q})$ and $i\colon \Gamma \to \mathrm{SL}(N,\mathbb{C})$ is the inclusion, then $\mathrm{comp}(j^*i^*h) = h$.

Now if $\mathrm{comp}(b_{2n-1}) = xy$, then $b_{2n-1} = j^*i^*\,\mathrm{comp}(b_{2n-1}) = (j^*i^*x)(j^*i^*y)$ is decomposable in $H^*_c(\mathrm{GL}(N,\mathbb{C});\mathbb{Q})$ for all $h \in H^{\mathrm{simp}}_{2n-1}(B\mathrm{GL}(\overline{\mathbb{Q}});\mathbb{Q})$, giving a contradiction.

Let $O_F$ be the ring of integers in a number field $F$, which has $r_1$ real and $2r_2$ complex embeddings. Borel proves that the Borel regulator, applied to the different embeddings of $\mathrm{SL}(O_F)$, yields an isomorphism between $PH_{2n-1}^{\mathrm{simp}}(B\mathrm{SL}(O_F);\mathbb{Z})$ and $\mathbb{Z}^{r_1+r_2}$ resp. $\mathbb{Z}^{r_2}$ if $n$ is even resp. odd. Since decomposable cohomology classes vanish on primitive homology classes, this implies in particular:

*If $A = O_F$ for a number field F, then $b_{2n-1}$ is not decomposable for even $n$.*

*If moreover $F$ is not totally real, then $b_{2n-1}$ is not decomposable for all $n$.*

In particular, we can apply Lemma 2 to $A = O_F$, and therefore also to all rings $A$ with $O_F \subset A \subset \mathbb{C}$, in particular to $A = \overline{\mathbb{Q}}$:

**Corollary 2** *For all $n$, there exists a projection*

$$\mathrm{pr}_{2n-1}\colon H_{2n-1}^{\mathrm{simp}}(B\mathrm{GL}(\overline{\mathbb{Q}});\mathbb{Q}) \to PH_{2n-1}^{\mathrm{simp}}(B\mathrm{GL}(\overline{\mathbb{Q}});\mathbb{Q}) = K_{2n-1}(\overline{\mathbb{Q}}) \otimes \mathbb{Q}$$

*such that for all $h \in H_{2n-1}^{\mathrm{simp}}(B\mathrm{GL}(\overline{\mathbb{Q}});\mathbb{Q})$,*

$$\mathrm{comp}(b_{2n-1})(\mathrm{pr}_{2n-1}(h)) = \mathrm{comp}(b_{2n-1})(h).$$

## 2.6 Compact locally symmetric spaces and $K$–theory

In this subsection, we finally show that to each representation of nontrivial Borel class, and each compact, oriented, locally symmetric space of noncompact type $M$ we can find a nontrivial element $\gamma(M) \in K_*(\overline{\mathbb{Q}}) \otimes \mathbb{Q}$.

**Theorem 2** *For each symmetric space $G/K$ of noncompact type and odd dimension $d$, and each representation $\rho\colon G \to \mathrm{GL}(N,\mathbb{C})$ with $\rho^* b_d \neq 0$, there is some $c_\rho \neq 0$ such that to each compact, oriented, locally symmetric space $M = \Gamma\backslash G/K$, with $\rho(\Gamma) \subset \mathrm{GL}(N, A)$ for a subring $A \subset \mathbb{C}$ satisfying the conclusion of Lemma 2, there exists an element*

$$\gamma(M) \in K_d(A) \otimes \mathbb{Q}$$

*with $\langle b_d, \gamma(M)\rangle = c_\rho \operatorname{vol}(M)$.*

**Proof** Using the projection $\mathrm{pr}_d$ from Lemma 2, we obtain as in Section 2.2

$$\gamma(M) := Q_* \,\mathrm{pr}_*\, B\,(\rho j)_*\, \mathrm{EM}^{-1}[M] \in K_d(A) \otimes \mathbb{Q}.$$

$\rho_* b_d \neq 0$ together with $H_c^d(G;\mathbb{R}) = H^d(G_u/K;\mathbb{R}) = \mathbb{R}$ implies $\rho^* b_d = c_\rho v_d$ for some $c_\rho \neq 0$. Using Lemma 2 we get

$$\begin{aligned}\langle b_d, \gamma(M)\rangle &= \mathrm{comp}(b_d)(\mathrm{pr}_* \, B(\rho j)_* \, \mathrm{EM}^{-1}[M]) \\ &= \mathrm{comp}(b_d)(B(\rho j)_* \, \mathrm{EM}^{-1}[M]) \\ &= \mathrm{comp}(\rho^* b_d)(Bj_* \, \mathrm{EM}^{-1}[M]) \\ &= c_\rho \langle v_d, Bj_* \, \mathrm{EM}^{-1}[M]\rangle = c_\rho \, \mathrm{vol}(M),\end{aligned}$$

where the last equality is true by Theorem 1. □

**Corollary 3** *For each symmetric space $G/K$ of noncompact type and odd dimension $d = 2n-1$, and to each representation $\rho\colon G \to \mathrm{GL}(N,\mathbb{C})$ with $\rho^* b_{2n-1} \neq 0$, there exists a constant $c_\rho \neq 0$, such that the following holds: to each compact, oriented, locally symmetric space $M = \Gamma\backslash G/K$ there exists an element*

$$\gamma(M) \in K_{2n-1}(\overline{\mathbb{Q}}) \otimes \mathbb{Q}$$

*with $\langle b_{2n-1}, \gamma(M)\rangle = c_\rho \, \mathrm{vol}(M)$.*

**Proof** $\dim(G/K) = 2n-1$ implies that $G$ is not locally isomorphic to $\mathrm{SL}(2,\mathbb{R})$, thus we can apply Weil's rigidity theorem, which yields a $g \in G$ with $g\Gamma g^{-1} \in G(\overline{\mathbb{Q}})$. Hence $M$ is of the form $M = \Gamma\backslash G/K$ with $\Gamma \subset G(\overline{\mathbb{Q}})$.

Each representation $\rho\colon G \to \mathrm{GL}(N,\mathbb{C})$ is isomorphic to a representation $\rho'$ such that $G(\overline{\mathbb{Q}})$ is mapped to $\mathrm{GL}(N,\overline{\mathbb{Q}})$. This follows from the classification of irreducible representations of Lie groups; see Fulton and Harris [14].

By Corollary 2 we can then apply Theorem 2 to $A = \overline{\mathbb{Q}}$. □

**Corollary 4** *Let $G/K$ be a symmetric space of noncompact type and $\rho\colon G \to \mathrm{GL}(N,\mathbb{C})$ a representation with $\rho^* b_{2n-1} \neq 0$, for $2n-1 = \dim(G/K)$. Then compact, oriented, locally symmetric spaces $\Gamma\backslash G/K$ of rationally independent volumes yield rationally independent elements in $K_{2n-1}(\overline{\mathbb{Q}}) \otimes \mathbb{Q}$.*

**Remark** In [15] it was claimed that for $(2n-1)$–dimensional compact hyperbolic manifolds one can construct an element $\gamma(M) \in K_{2n-1}(\overline{\mathbb{Q}}) \otimes \mathbb{Q}$ such that $\langle b_{2n-1}, \gamma(M)\rangle = \mathrm{vol}(M)$. However, since $\rho^* b_{2n-1}$ is a rational cohomology class, $c_\rho$ is rational if and only if $v_{2n-1}$ is a rational cohomology class, and this is equivalent to $\mathrm{vol}(M) = \langle v_{2n-1}, [M]\rangle \in \mathbb{Q}$. Since, conjecturally, all hyperbolic manifolds have irrational volumes, one can probably not get rid of the factor $c_\rho$ in Theorem 2.

In conclusion, we are left with the problem of finding representations of nontrivial Borel class, which will be solved in Section 3.

*Compact examples* can eg be obtained by Borel's construction of locally symmetric spaces in [3]. A very special case is the construction of arithmetic hyperbolic manifolds using quadratic forms (cf [2, Chapter E.3]).

Let $u\in\mathbb{R}$ be an algebraic integer such that all roots of its minimal polynomial have multiplicity 1 and are real and negative (except possibly $u$). Assume moreover that $(0,\dots,0)$ is the only integer solution of $x_1^2+\cdots+x_{2n-1}^2-ux_{2n}^2=0$. Let $\widehat{\Gamma}\subset \mathrm{GL}(2n,\mathbb{Z}[u])$ be the group of maps preserving $x_1^2+\cdots+x_{2n-1}^2-ux_{2n}^2$. It is isomorphic to a discrete cocompact subgroup of $\mathrm{SO}(2n-1,1;\mathbb{Z}[u])\subset\mathrm{SO}(2n-1,1;\mathbb{R})$. By Selberg's lemma, it contains a torsion-free cocompact subgroup $\Gamma\subset\mathrm{SO}(2n-1,1;\mathbb{Z}[u])$. With the computations in Section 3 below one concludes: If $n$ is even, then the compact manifold $M:=\Gamma\backslash\mathbb{H}^{2n-1}$ (and, for example, a half-spinor representation) gives a nontrivial element $\gamma(M)\in K_{2n-1}(\mathbb{Z}[u])\otimes\mathbb{Q}$. If $n$ is odd, then Corollary 2 can not be applied to $\mathbb{Z}[u]$ but to $\overline{\mathbb{Q}}$, one gets at least a nontrivial element $\gamma(M)\in K_{2n-1}(\overline{\mathbb{Q}})\otimes\mathbb{Q}$.

Matthey, Pitsch and Scherer [20] constructed a somewhat stronger invariant for stably parallelisable manifolds: given an embedding $M^d\to\mathbb{R}^n$ with trivial normal bundle $\nu M$ and a regular neighborhood $U$ they consider the composition

$$\mathbb{S}^n\to\bar{U}/\partial U\to\bar{U}/\partial U\wedge M_+=Th(\nu M)\wedge M_+=\Sigma^{n-d}M_+\wedge M_+\to\mathbb{S}^{n-d}\wedge M_+$$

giving an element $\gamma(M)\in\pi_d^s(M)$.

# 3 Existence of representations of nontrivial Borel class

## 3.1 Trace criterion

**Lemma 3** *Let $G/K$ be a symmetric space of noncompact type, of dimension $2n-1$. Let $\underline{t}\subset\underline{p}$ be a Cartan subalgebra of $\underline{g}$.*

*Then for a representation $\rho\colon G\to\mathrm{GL}(N,\mathbb{C})$ and its dual $\rho_u\colon G_u\to U(N)\times U(N)$ the following are equivalent:*

(i) *$\rho$ has nonvanishing Borel class $\rho^*b_{2n-1}\neq 0\in H_c^{2n-1}(G;\mathbb{R})$.*

(ii) *$\mathrm{Tr}((D_e\rho_u(it))^n)\neq 0$ for some $t\in\underline{t}$.*

(iii) *$\mathrm{Tr}((D_e\rho(t))^n)\neq 0$ for some $t\in\underline{t}$.*

**Proof** As in Definition 2 we consider the dual representation

$$\rho_u\colon (G_u, K) \to (U(N) \times U(N), U(N))$$

and the smooth map $\overline{\rho_u}\colon G_u/K \to U(N)\times U(N)/U\,(N) \simeq U(N)$. Since $\rho_u$ sends $K$ to the first factor of $U(N)\times U(N)$ we have $\pi_2\rho_u = \overline{\rho_u}\,p$, where $\pi_2\colon U(N)\times U(N) \to U(N)$ is the projection to the second factor and $p\colon G_u \to G_u/K$ projection to the quotient.

By Lemma 1 we have that $\rho^* b_{2n-1} \neq 0 \in H_c^{2n-1}(G;\mathbb{R})$ if and only if

$$\overline{\rho_u}^* b_{2n-1} \neq 0 \in H^{2n-1}(G_u/K).$$

Averaging of differential forms over the compact group $K$ shows $p^*\colon H^*(G_u/K) \to H^*(G_u)$ is injective. Hence, $\overline{\rho_u}^* b_{2n-1} \neq 0$ if and only if its image in $H^{2n-1}(G_u)$ does not vanish. The latter equals

$$(\pi_2\rho_u)^* b_{2n-1},$$

because $\pi_2\rho_u = \overline{\rho_u}\,p$.

Using the notation and facts from Section 2.4.3 we have

$$(\pi_2\rho_u)^* b_{2n-1} = (\pi_2\rho_u)^*\Phi_A(R(\mathrm{Tr}_n)) = \Phi_A(R((\pi_2\rho_u)^*\,\mathrm{Tr}_n)),$$
$$(\pi_2\rho_u)^*\,\mathrm{ch}_n = (\pi_2\rho_u)^*\Phi_S(\mathrm{Tr}_n) = \Phi_S((\pi_2\rho_u)^*\,\mathrm{Tr}_n).$$

Now $\Phi_A$ and $\Phi_S$ are isomorphisms, moreover $\tau \circ \Phi_A \circ R = \Phi_S$ implies injectivity of $R$. Hence $(\pi_2\rho_u)^*\,\mathrm{ch}_n \neq 0$ if and only if $(\pi_2\rho_u)^*\, b_{2n-1} \neq 0$.

From the definition of $\mathrm{Tr}_n$ we see that

$$(\pi_2\rho_u)^*\,\mathrm{Tr}_n(A_1,\dots,A_n) = \frac{1}{(2\pi i)^n}\frac{1}{n!}\sum_{\sigma\in S_n}\mathrm{Tr}(D_e(\pi_2\rho_u)A_{\sigma(1)}\dots D_e(\pi_2\rho_u)A_{\sigma(n)}),$$

An easy exercise in multilinear algebra shows that a *symmetric* polynomial $P(x_1,\dots,x_n)$ is nontrivial if and only if there is some $x$ with $P(x,x,\dots,x)\neq 0$. Hence it is sufficient to check that the invariant polynomial

$$\mathrm{Tr}((D_e(\pi_2\rho_u)(\cdot))^n)$$

is not trivial on $\underline{g}_u$.

Let $\underline{t}_u$ be the Cartan subalgebra of $\underline{g}_u$, which corresponds to $\underline{t}$ under the canonical bijection $\underline{k} \oplus \underline{p} \simeq \underline{k} \oplus i\underline{p}$. There is an action of the Weyl group $W$ on $\underline{t}_u$, we denote its space of invariant polynomials by $S_*^W(\underline{t}_u)$. By a theorem of Chevalley (see

Bourbaki [5]), restriction induces an isomorphism

$$S_*^{G_u}(\underline{g}_u) \cong S_*^{W}(\underline{t}_u).$$

In particular, it suffices to check that $\mathrm{Tr}((D_e(\pi_2\rho_u)(\cdot))^n)$ is not trivial on $\underline{t}_u$.

By assumption the Cartan algebra $\underline{t}$ is contained in $\underline{p}$. (This can actually always be achieved by a suitable conjugation.) Thus $\underline{t}_u \subset i\underline{p}$. This implies that, for $t \in \underline{t}_u$, $D_e\rho_u(t)$ belongs to the second factor of $u(N) \oplus u(N)$, and thus $D_e(\pi_2\rho_u)(t) = D_e\rho_u(t)$ for $t \in \underline{t}_u$, which proves the equivalence of (i) and (ii). Finally we note that, for $t \in \underline{p}$, $\mathrm{Tr}((D_e\rho(t))^n)$ and $\mathrm{Tr}((D_e\rho_u(it))^n)$ coincide up to a power of $i$. The equivalence of (ii) and (iii) follows. □

**Corollary 5** *Let $G/K$ be a symmetric space of noncompact type. If $d := \dim(G/K) \equiv 3 \mod 4$, then every nontrivial representation $\rho\colon G \to \mathrm{GL}(N,\mathbb{C})$ has nonvanishing Borel class $\rho^* b_d \neq 0 \in H_c^d(G;\mathbb{R})$.*

**Proof** We apply Lemma 3 with $d = 2n-1$, that is $n$ is even.

For each $t \in \underline{t}$ we have that

$$D_e\rho_u(it) \in u(N) \oplus u(N)$$

has purely imaginary eigenvalues, since matrices in $u(N) \oplus u(N)$ are skew-symmetric. Hence, if $\rho$ is nontrivial (and thus $D_e\rho_u \not\equiv 0$), the eigenvalues of $(D_e\rho_u(it))^n$ are either all positive (if $n \equiv 0 \mod 4$) or all negative (if $n \equiv 2 \mod 4$). In either case $\mathrm{Tr}((D_e\rho_u(it))^n) \neq 0$. □

## 3.2 Borel class of Lie algebra representations

**3.2.1 Preliminaries** Let $\underline{g}$ be a semisimple Lie algebra and $R(\underline{g})$ its (real) representation ring, with addition $\oplus$ and multiplication $\otimes$. Let $\underline{t}$ be a Cartan subalgebra of $\underline{g}$.

In this section we consider, for $n \in \mathbb{N}$, the map $\beta_{2n-1}\colon R(\underline{g}) \to \mathbb{C}[\underline{t}]$ given by

$$\beta_{2n-1}(\pi)(t) = \mathrm{Tr}(\pi(t)^n).$$

For representations $\pi_1, \pi_2$, one has $\beta_{2n-1}(\pi_1 \oplus \pi_2) = \beta_{2n-1}(\pi_1) + \beta_{2n-1}(\pi_2)$ and $\beta_{2n-1}(\pi_1 \otimes \pi_2) = \beta_{2n-1}(\pi_1)\beta_{2n-1}(\pi_2)$.

By Lemma 3 a representation $\rho\colon G \to \mathrm{GL}(N,\mathbb{C})$ has *nontrivial Borel class* $\rho^* b_{2n-1} \neq 0$ in $H_c^{2n-1}(G;\mathbb{R})$ if and only if $\mathrm{Tr}(D_e\rho(A)^n) \neq 0$ for some $A \in \underline{t}$, in other words if and only if the associated Lie algebra representation $\pi = D_e\rho\colon \underline{g} \to \mathfrak{gl}(N,\mathbb{C})$ satisfies

$$\beta_{2n+1}(\pi) \neq 0 \in \mathbb{C}[\underline{t}].$$

In this section we will investigate for which fundamental representations of Lie algebras the latter condition is satisfied.

In the following subsections we will consider complex simple Lie algebras $\underline{g}$ and the ring $R_{\mathbb{C}}(\underline{g}) \subset R(\underline{g})$ of their $\mathbb{C}$–linear representations. The general picture can be reduced to that of $\mathbb{C}$–linear representations in view of the following observations.

**Noncomplex Lie algebras** Let $\pi\colon \underline{g} \to \mathfrak{gl}(N,\mathbb{C})$ be an $\mathbb{R}$–linear representation of a simple Lie-algebra $\underline{g}$ which is not a complex Lie algebra. Then $\underline{g}\otimes\mathbb{C}$ is a simple complex Lie algebra and $\pi$ is the restriction of some $\mathbb{C}$–linear representation $\underline{g}\otimes\mathbb{C} \to \mathfrak{gl}(N,\mathbb{C})$. Let $\underline{t}$ be a Cartan subalgebra of $\underline{g}$. Then it is obvious that an element $t \in \underline{t}\otimes\mathbb{C}$ with

$$\mathrm{Tr}(\pi(t)^n) \neq 0$$

exists if and only if such an element exists in $\underline{t}$. Thus $\pi$ has nontrivial Borel class if and only if the $\mathbb{C}$–linear representation $\pi\otimes\mathbb{C}$ has nontrivial Borel class.

**$\mathbb{R}$–linear representations of complex Lie algebras** If $\underline{g}$ is a simple complex Lie algebra, then each $\mathbb{R}$–linear representation $\pi\colon \underline{g} \to \mathfrak{gl}(N,\mathbb{C})$ is of the form $\pi = \pi_1\otimes\overline{\pi_2}$ for $\mathbb{C}$–linear representations $\pi_1, \pi_2$. The equality

$$\mathrm{Tr}(\pi(t)^n) = \mathrm{Tr}(\pi_1(t)^n)\,\mathrm{Tr}(\overline{\pi_2}(t)^n).$$

implies that $\mathbb{R}$–linear representations with $b_{2n-1}(\pi)$ exist only if there are $\mathbb{C}$–linear ones.

In the sequel we will go through the fundamental representations of simple Lie algebras and discuss whether their Borel class is nontrivial. The results will be subsumed in Section 3.3 in the proof of Theorem 3. For faster reading we are going to highlight the exceptional cases that will occur in the proof of Theorem 3.

**3.2.2** $\underline{g} = \mathfrak{sl}(l+1,\mathbb{C})$ Let $V = \mathbb{C}^{l+1}$ be the standard representation, with basis $e_1,\dots,e_{l+1}$. Then

$$R_{\mathbb{C}}(\underline{g}) = \mathbb{Z}[A_1,\dots,A_l]$$

with $A_k$ the induced representation on $\Lambda^k V$; cf [14, page 377]. In particular, irreducible representations occur as representations of dominant weight in tensor products of the fundamental representations $A_1,\dots,A_l$. We compute $\beta_{2n-1}$ on the fundamental representations $A_k, k = 1,\dots,l$.

A basis of $\Lambda^k V$ is given by

$$\{e_{i_1}\wedge\cdots\wedge e_{i_k} : 1 \le i_1 < \cdots < i_k \le l+1\}.$$

As Cartan subalgebra we may choose the diagonal matrices

$$\underline{t} = \{\mathrm{diag}(h_1, \dots, h_l, h_{l+1}) : h_1 + \dots + h_{l+1} = 0\}.$$

$\mathrm{diag}(h_1, \dots, h_l, h_{l+1})$ acts on $e_{i_1} \wedge \dots \wedge e_{i_k}$ by multiplication with $h_{i_1} + \dots + h_{i_k}$. Hence

$$\beta_{2n-1}(A_k)\begin{pmatrix} h_1 & 0 & \cdots & 0 \\ 0 & h_2 & \cdots & 0 \\ . & . & \cdots & . \\ 0 & 0 & \cdots & h_{l+1} \end{pmatrix} = \sum_{1 \le i_1 < \dots < i_k \le l+1} (h_{i_1} + \dots + h_{i_k})^n.$$

If $k = l = 1$ then $h_1^n + h_2^n$ is a multiple of $h_1 + h_2 = 0$ if and only if $n$ is odd. Thus for $l = 1$ we have $\beta_{2n-1}(A_1) \neq 0$ if $n$ is even and $\beta_{2n-1}(A_1) = 0$ *if n is odd.*

If $k = 1$ and $l \ge 2$, then $\sum_{i=1}^{l+1} h_i^n$ does not vanish for example for $h_1 = 2$, $h_2 = -1$, $h_3 = \dots = h_l = 0, h_{l+1} = -1$.

If $2 \le k \le l$ *and* $n = 1$, then

$$\sum_{1 \le i_1 < \dots < i_k \le l+1} (h_{i_1} + \dots + h_{i_k}) = \binom{l}{k-1}(h_1 + \dots + h_{l-1}) = 0,$$

thus $\beta_1(A_k) = 0$.

If $2 \le k \le l$ and $n > 1$, then $\beta_{2n-1}(A_k) \neq 0$. Indeed, nontriviality can be seen for example by considering again the diagonal matrix $(2, -1, 0, \dots, 0, -1) \in \underline{t}$, for which we obtain

$$\sum_{1 \le i_1 < \dots < i_k \le l+1} (h_{i_1} + \dots + h_{i_k})^n = (2^n - 1)\left(\binom{l-2}{k-1} - \binom{l-1}{k-1}\right) < 0.$$

**Conclusion** *The exceptional cases with* $\beta_{2n-1}(A_k) = 0$ *occur for*

- $k = l = 1$, *n odd,*
- $2 \le k \le l$, $n = 1$.

**3.2.3** $\underline{g} = \mathfrak{spin}(2l, \mathbb{C})$ Let $V = \mathbb{C}^{2l}$ with $\mathbb{C}$–basis $e_1, \dots, e_l, f_1, \dots, f_l$. Let $Q$ be the quadratic form given by $Q(e_i, f_i) = Q(f_i, e_i) = 1$ for $i = 1, \dots, l$, $Q(e_i, f_j) = Q(f_i, e_j) = 0$ for $i \neq j$ and $Q(e_i, e_j) = Q(f_i, f_j) = 0$ for all $i, j = 1, \dots, l$.

Following [14, page 268 ff] we consider $\mathfrak{spin}(2l, \mathbb{C})$ as the skew-symmetric matrices with respect to the quadratic form $Q\colon V \times V \to \mathbb{C}$. (All quadratic forms are equivalent over $\mathbb{C}$ under a suitable change of base, the corresponding Lie groups $\mathrm{SO}(Q) \subset$

$\mathrm{GL}(N,\mathbb{C})$ are conjugate, thus it is sufficient to consider the Lie algebra $\mathfrak{spin}(Q)$ with respect to this quadratic form $Q$.)

Let $D_1\colon \mathfrak{spin}(2l,\mathbb{C}) \to \mathfrak{gl}(V)$ be the standard representation.

Then

$$R_{\mathbb{C}}(\underline{g}) = \mathbb{Z}[D_1,\dots,D_{l-2},S^+,S^-]$$

with $D_k\colon \mathfrak{spin}(2l,\mathbb{C}) \to \mathfrak{gl}(\Lambda^k V)$ the representation induced from $D_1$ on $\Lambda^k V$, and $S^\pm$ the half-spinor representations.

As a Cartan subalgebra we may choose the diagonal matrices

$$\underline{t} = \{\mathrm{diag}(h_1,\dots,h_l,-h_1,\dots,-h_l) : h_1,\dots,h_l \in \mathbb{C}\}.$$

First we look at $\beta_{2n-1}(D_k)$ for the fundamental representations $D_k$.

A basis of $\Lambda^k V$ is given by

$$\{e_{i_1}\wedge\cdots\wedge e_{i_p}\wedge f_{j_1}\wedge\cdots\wedge f_{j_{k-p}} : 0\le p\le k, 1\le i_1<\cdots<i_p\le l, 1\le j_1<\cdots<j_{k-p}\le l\}.$$

$\mathrm{diag}(h_1,\dots,h_l,-h_1,\dots,-h_l)$ acts on $e_{i_1}\wedge\cdots\wedge e_{i_p}\wedge f_{j_1}\wedge\cdots\wedge f_{j_{k-p}}$ by multiplication with $h_{i_1}+\cdots+h_{i_p}-h_{j_1}-\cdots-h_{j_{k-p}}$. Hence

$$\beta_{2n-1}(D_k)\begin{pmatrix} h_1 & 0 & \cdots & 0 & 0 & \cdots \\ 0 & h_2 & \cdots & 0 & 0 & \cdots \\ \cdots & & & \cdots & & \\ 0 & 0 & \cdots & -h_1 & 0 & \cdots \\ 0 & 0 & \cdots & 0 & -h_2 & \cdots \\ \cdots & & & \cdots & & \end{pmatrix}$$
$$= \sum_{\substack{1\le i_1<\cdots<i_p\le l,\\ 1\le j_1<\cdots<j_{k-p}\le l}} (h_{i_1}+\cdots+h_{i_p}-h_{j_1}-\cdots-h_{j_{k-p}})^n.$$

If $n$ is even, then we get a nonvanishing polynomial. This follows from Corollary 5 or more explicitly for example from

$$\beta_{2n-1}(D_k)(\mathrm{diag}(1,0,\dots,0,-1,0,\dots,0)) > 0.$$

If $n$ is odd, then the permutation, which transposes $i_r$ and $j_r$ simultaneously for all $r$, multiplies the sum by $-1$, but on the other hand preserves the sum. Thus $\beta_{2n-1}(D_k)=0$ *if $n$ is odd.*

Next we look at $\beta_{2n-1}(S^\pm)$ for the half-spinor representations $S^\pm$.

Let $TV = \bigoplus_{k=0}^{m} V^{\otimes k}$ be the tensor algebra of $V$ and let $\mathrm{Cl}(Q) = TV/I(Q)$ be the Clifford algebra of $Q$, where $I(Q)$ is the ideal generated by all $v \otimes v + Q(v,v)1$ with $v \in V$. The grading of $\bigoplus_{k=0}^{m} V^{\otimes k}$ induces a well-defined $\mathbb{Z}/2\mathbb{Z}$–grading $\mathrm{Cl}(Q) = \mathrm{Cl}(Q)^{\mathrm{even}} \oplus \mathrm{Cl}(Q)^{\mathrm{odd}}$ on the Clifford algebra.

Denote by $E_{ij}$ the elementary matrix with entry 1 at position $(i,j)$ and entries 0 else. Then

$$\{A_i := E_{i,i} - E_{l+i,l+i}, i = 1, \dots, l\}$$

is a basis of $\underline{t}$.

By [14, pages 303–305], there is an injective homomorphism

$$\iota\colon \mathfrak{spin}(Q) \to \mathrm{Cl}(Q)^{\mathrm{even}}$$

which maps, in particular, $A_i$ to $\frac{1}{2}(e_i \otimes f_i - 1)$.

Let $W$ be the $\mathbb{C}$–subspace of $V$ spanned by $e_1, \dots, e_l$.

From the proof of [14, Lemma 20.9] we have a homomorphism

$$\Phi\colon \mathrm{Cl}(Q) \to \mathfrak{gl}(\Lambda^* W)$$

with $\quad \Phi(e_i)(v_1 \wedge \dots \wedge v_k) = e_i \wedge v_1 \wedge \dots \wedge v_k$

$$\Phi(f_i)(v_1 \wedge \dots \wedge v_k) = \sum_{j=1}^{k} (-1)^{j-1} 2Q(v_j, f_i) v_1 \wedge \dots \widehat{v_j} \dots \wedge v_k$$

for all $v_1 \wedge \dots \wedge v_k \in \Lambda^* W$ and $i = 1, \dots, l$, which implies

$$\Phi\left(\frac{1}{2}(e_i \otimes f_i - 1)\right)(e_{i_1} \wedge \dots \wedge e_{i_k}) = \frac{1}{2} e_{i_1} \wedge \dots \wedge e_{i_k}$$

if $i \in \{i_1, \dots, i_k\}$ and

$$\Phi\left(\frac{1}{2}(e_i \otimes f_i - 1)\right)(e_{i_1} \wedge \dots \wedge e_{i_k}) = -\frac{1}{2} e_{i_1} \wedge \dots \wedge e_{i_k}$$

if $i \notin \{i_1, \dots, i_k\}$.

By [14, page 305], restriction of $\Phi$ to $\mathrm{Cl}(Q)^{\mathrm{even}}$ gives rise to an isomorphism

$$\Phi^{\mathrm{even}}\colon \mathrm{Cl}(Q)^{\mathrm{even}} \to \mathrm{End}(\Lambda^{\mathrm{even}} W) \oplus \mathrm{End}(\Lambda^{\mathrm{odd}} W).$$

Let $\pi_1, \pi_2$ be the projections from $\mathrm{End}(\Lambda^{\mathrm{even}} W) \oplus \mathrm{End}(\Lambda^{\mathrm{odd}} W)$ to the first resp. second summand. The induced homomorphisms

$$S^+ := \pi_1 \Phi^{\mathrm{even}} \iota\colon \mathfrak{spin}(Q) \to \mathrm{End}(\Lambda^{\mathrm{even}} W)$$
$$S^- := \pi_2 \Phi^{\mathrm{even}} \iota\colon \mathfrak{spin}(Q) \to \mathrm{End}(\Lambda^{\mathrm{odd}} W)$$

give the positive resp. negative half-spinor representations that we are going to consider.

Thus

$$S^{\pm}(A_i)(e_{i_1}\wedge\cdots\wedge e_{i_k})=\frac{1}{2}e_{i_1}\wedge\cdots\wedge e_{i_k}$$

if $i\in\{i_1,\dots,i_k\}$ and

$$S^{\pm}(A_i)(e_{i_1}\wedge\cdots\wedge e_{i_k})=-\frac{1}{2}e_{i_1}\wedge\cdots\wedge e_{i_k}$$

if $i\notin\{i_1,\dots,i_k\}$.

For the positive half-spinor representation $S^+$ and any $n\in\mathbb{N}$ we obtain

$$\beta_{2n-1}(S^+)\begin{pmatrix} h_1 & 0 & \cdots & 0 & 0 & \cdots\\ 0 & h_2 & \cdots & 0 & 0 & \cdots\\ \cdots & & & \cdots & & \\ 0 & 0 & \cdots & -h_1 & 0 & \cdots\\ 0 & 0 & \cdots & 0 & -h_2 & \cdots\\ \cdots & & & \cdots & & \end{pmatrix}=\frac{1}{2^n}\sum_{\substack{0\le k\le l\\ k\text{ even}}}\sum_{|I|=k}\Big(\sum_{i\in I}h_i-\sum_{j\notin I}h_j\Big)^n.$$

If $n$ is even, then $\beta_{2n-1}(S^+)\neq 0$ follows from Corollary 5.

If $n$ *is odd and* $l$ *is even*, then for each $I$ with $k=|I|$ even we have $I':=\{1,\dots,l\}-I$ with $k'=|I'|$ even and $(\sum_{i\in I}h_i-\sum_{j\notin I}h_j)^n$ cancels against $(\sum_{i\in I'}h_i-\sum_{j\notin I'}h_j)^n$. Thus all summands cancel and $\beta_{2n-1}(S^+)=0$.

We prove that the polynomial is nontrivial for all $n\geq l$ with $n\equiv l \bmod 2$, in particular if $n$ and $l$ are both odd. It suffices to show that for example the coefficient of $h_1^{n-l+1}h_2\dots h_n$ is not zero. First we observe that the coefficient of $h_1^{n-l+1}h_2\dots h_n$ in $(\sum_{i\in I}h_i-\sum_{j\notin I}h_j)^n$ is

$$\frac{n!}{(n-l+1)!}(-1)^{n-k}\quad\text{if } 1\in I$$

$$\text{resp.}\quad \frac{n!}{(n-l+1)!}(-1)^{l-k}\quad\text{if } 1\notin I.$$

Thus the coefficient of $h_1^{n-l+1}h_2\dots h_n$ in $\sum_{|I|=k}(\sum_{i\in I}h_i-\sum_{j\notin I}h_j)^n$ is

$$\frac{n!}{(n-l+1)!}\bigg(\binom{l-1}{k-1}(-1)^{n-k}+\binom{l-1}{k}(-1)^{l-k}\bigg).$$

All summands have the same sign because of $n\equiv l \bmod 2$. Thus $\beta_{2n-1}(S^+)\neq 0$.

For the negative half-spinor representation $S^-$ and any $n \in \mathbb{N}$ we obtain

$$\beta_{2n-1}(S^-)\begin{pmatrix} h_1 & 0 & \cdots & 0 & 0 & \cdots \\ 0 & h_2 & \cdots & 0 & 0 & \cdots \\ \cdots & & & \cdots & & \\ 0 & 0 & \cdots & -h_1 & 0 & \cdots \\ 0 & 0 & \cdots & 0 & -h_2 & \cdots \\ \cdots & & & \cdots & & \end{pmatrix} = \frac{1}{2^n} \sum_{\substack{0\le k\le l \\ k \text{ odd}}} \sum_{|I|=k} \Big( \sum_{i\in I} h_i - \sum_{j\notin I} h_j \Big)^n .$$

If $n$ is even, then $\beta_{2n-1}\,(S^-) \neq 0$ by Corollary 5.

If $n$ is odd, then the same argument as in the computation of $\beta_{2n-1}(D_k) = 0$ shows

$$\beta_{2n-1}(S^+) + \beta_{2n-1}(S^-) = 0,$$

thus $\beta_{2n-1}(S^+) \neq 0$ implies $\beta_{2n-1}(S^-) \neq 0$ if $n \geq l$ and $n \equiv l \bmod 2$.

Conversely, if $n$ *is odd and* $l$ *is even*, then $\beta_{2n-1}\,(S^-) = 0$.

**Conclusion** *The cases with $\beta_{2n-1}(\pi) = 0$ are precisely*

- $\pi = D_k$ *($1 \leq k \leq l-2$), $n$ odd,*
- $\pi = S^{\pm}$*, $l$ even, $n$ odd.*

**3.2.4** $\underline{g} = \mathfrak{spin}(2l+1, \mathbb{C})$ Let $V = \mathbb{C}^{2l+1}$ with $\mathbb{C}$–basis $e_1, \dots, e_l, f_1, \dots, f_l, g$, and $Q$ the quadratic form given by $Q(g,g) = 1, Q(e_i, f_i) = Q(f_i, e_i) = 1$ for $i = 1, \dots, l$, and $Q(\cdot,\cdot) = 0$ for all other pairs of basis vectors.

Following [14, page 268 ff] we consider $\mathfrak{spin}(2l+1, \mathbb{C})$ as the skew-symmetric matrices with respect to the quadratic form $Q\colon V \times V \to \mathbb{C}$. Let $C_1\colon \mathfrak{spin}(2l+1,\mathbb{C}) \to \mathfrak{gl}(V)$ be the standard representation. Then

$$R_{\mathbb{C}}(\underline{g}) = \mathbb{Z}[C_1, \dots, C_{l-1}, S]$$

with $C_k\colon \mathfrak{spin}(2l+1,\mathbb{C}) \to \mathfrak{gl}(\Lambda^k V)$ the representation induced from $C_1$ on $\Lambda^k V$, and $S$ the spinor representation.

As a Cartan subalgebra we may choose the diagonal matrices

$$\underline{t} = \{\mathrm{diag}(h_1, \dots, h_l, -h_1, \dots, -h_l, 0) : h_1, \dots, h_l \in \mathbb{C}\}.$$

Then the computation of $\beta_{2n-1}$ on $C_k$ is exactly the same as for $\mathfrak{spin}(2l, \mathbb{C})$ and $D_k$, in particular $\beta_{2n-1}(C_k) \neq 0$ for $n$ even and $\beta_{2n-1}(C_k) = 0$ *for $n$ odd.*

We look at $\beta_{2n-1}(S)$ for the spinor representation $S$. As in the case of $\mathfrak{spin}(2l, \mathbb{C})$, we have $\iota\colon \mathfrak{spin}(Q) \to \mathrm{Cl}(Q)^{\mathrm{even}}$ with $\iota(E_{i,i} - E_{l+i,l+i}) = \frac{1}{2}(e_i \otimes f_i - 1)$.

Let $W$ be the $\mathbb{C}$–subspace of $V$ spanned by $e_1, \ldots, e_l$. It follows from the proof of [14, Lemma 20.16] that $\mathrm{Cl}(Q)$ acts on $\Lambda^* W$ as follows: the action of $e_i$ resp. $f_i$, for $i = 1, \ldots, l$ is defined as in the case of $\mathfrak{spin}(2l, \mathbb{C})$, and $g$ acts as multiplication by 1 on $\Lambda^{\mathrm{even}} W$ and as multiplication by -1 on $\Lambda^{\mathrm{odd}} W$. In particular, we have again that $\frac{1}{2}(e_i \otimes f_i - 1)$ acts by sending $e_{i_1} \wedge \cdots \wedge e_{i_k}$ to $\frac{1}{2} e_{i_1} \wedge \cdots \wedge e_{i_k}$ if $i \in \{i_1, \ldots, i_k\}$ resp. to $-\frac{1}{2} e_{i_1} \wedge \cdots \wedge e_{i_k}$ if $i \notin \{i_1, \ldots, i_k\}$.

This action gives rise to an isomorphism $\mathrm{Cl}(Q)^{\mathrm{even}} \cong \mathrm{End}(\Lambda W)$ (see [14, page 306]). The induced action of $\mathfrak{spin}(Q)$ on $\Lambda W$ is the spinor representation $S$.

Let $\{A_i : i = 1, \ldots, l\}$ be a basis of $\underline{t}$, where

$$A_i = E_{i,i} - E_{l+i,l+i}.$$

The element $A_i$ acts on $e_{i_1} \wedge \cdots \wedge e_{i_k}$ by multiplication with $\frac{1}{2}$ if $i \in \{i_1, \ldots, i_k\}$ and by multiplication with $-\frac{1}{2}$ if $i \notin \{i_1, \ldots, i_k\}$. Thus we obtain for any $n \in \mathbb{N}$,

$$\beta_{2n-1}(S) \begin{pmatrix} h_1 & 0 & \cdots & 0 & 0 & \cdots \\ 0 & h_2 & \cdots & 0 & 0 & \cdots \\ \cdots & & & \cdots & & \\ 0 & 0 & \cdots & -h_1 & 0 & \cdots \\ 0 & 0 & \cdots & 0 & -h_2 & \cdots \\ \cdots & & & \cdots & & \end{pmatrix} = \frac{1}{2^n} \sum_{0 \le k \le l} \sum_{|I|=k} \Big( \sum_{i \in I} h_i - \sum_{j \notin I} h_j \Big)^n.$$

Thus, by the same argument as for $D_k$ and $C_k$, $\beta_{2n-1}(S) = 0$ *for* $n$ *odd* and $\beta_{2n-1}(S) \neq 0$ for $n$ even.

**Conclusion** *The cases with $\beta_{2n-1}(\pi) = 0$ are precisely*

- $\pi = C_k$ *($1 \le k \le l-1$), $n$ odd,*
- $\pi = S$, *$n$ odd.*

**3.2.5** $\underline{g} = \mathfrak{sp}(l, \mathbb{C})$ Let $V = \mathbb{C}^{2l}$ with basis $\{e_1, \ldots, e_l, f_1, \ldots, f_l\}$. Consider the symplectic form $Q\colon V \times V \to \mathbb{R}$ given by $Q(e_i, f_i) = 1 = -Q(f_i, e_i)$ for $i = 1, \ldots, l$, and $Q(\cdot, \cdot) = 0$ for each other pair of basis vectors. Let $\mathrm{Sp}(l, \mathbb{C})$ be the Lie group of linear maps preserving this symplectic form. Then its lie algebra $\mathfrak{sp}(l, \mathbb{C})$ consists of matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

such that the $(l \times l)$–blocks $A, B, C, D$ satisfy $B^T = B, C^T = C, A^T = -D$. As a Cartan subalgebra we may choose the diagonal matrices

$$\underline{t} = \{\mathrm{diag}(h_1, \ldots, h_l, -h_1, \ldots, -h_l) : h_1, \ldots, h_l \in \mathbb{C}\}.$$

Then

$$R_{\mathbb{C}}(\underline{g}) = \mathbb{Z}[B_1, \dots, B_l]$$

where by [14, page 377] the fundamental representations $B_k$ are the induced representations of $\mathfrak{sp}(l, \mathbb{C})$ on $\ker(\phi_k\colon \Lambda^k V \to \Lambda^{k-2} V)$ for $k = 1, \dots, l$, where $\phi_k$ is the contraction using $Q$ defined in [14, page 260] by

$$\phi_k(v_1 \wedge \cdots \wedge v_k) = \sum_{i<j} Q(v_i, v_j)(-1)^{i+j-1} v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_k.$$

We consider $\beta_{2n-1}$ for the fundamental representations $B_k, k = 1, \dots, l$.

If $n$ is even, then $\beta_{2n-1}(B_k) \neq 0$ follows from Corollary 5.

We claim that $\beta_{2n-1}(B_k) = 0$ *if $n$ is odd*. This can be seen as follows. Consider the involution $B \in \mathfrak{gl}(2l, \mathbb{C})$ given by $B(e_i) = f_i, B(f_i) = -e_i$ for $i = 1, \dots, l$. It induces an involution on $\Lambda^k V$. $B$ preserves the symplectic form $Q$, thus we have

$$\begin{aligned}
&\phi_k(Bv_1 \wedge \cdots \wedge Bv_k)\\
&= \sum_{i<j} Q(Bv_i, B_j)(-1)^{i+j-1} Bv_1 \wedge \cdots \wedge \widehat{Bv}_i \wedge \cdots \wedge \widehat{Bv}_j \wedge \cdots \wedge Bv_k\\
&= \sum_{i<j} Q(v_i, v_j)(-1)^{i+j-1} Bv_1 \wedge \cdots \wedge \widehat{Bv}_i \wedge \cdots \wedge \widehat{Bv}_j \wedge \cdots \wedge Bv_k = B(\phi_k(v_1 \wedge \cdots \wedge v_k)).
\end{aligned}$$

In particular $B$ maps $\ker(\phi_k)$ to itself. If $\{b_1, \dots, b_{\dim(\ker(\phi_k))}\}$ is a basis of $\dim(\ker(\phi_k))$, then $\{Bb_1, \dots, Bb_{\dim(\ker(\phi_k))}\}$ is a basis of $\dim(\ker(\phi_k))$.

Let $\langle \cdot, \cdot \rangle$ be the standard scalar product on $\mathbb{C}^{2l}$ such that $\{e_1, \dots, e_l, f_1, \dots, f_l\}$ is an orthonormal basis. Note $B$ preserves this scalar product. Thus, if $\{b_1, \dots, b_{\dim(\ker(\phi_k))}\}$ is an orthonormal basis of $\ker(\phi_k) \subset \mathbb{C}^{2l}$, then $\{Bb_1, \dots, Bb_{\dim(\ker(\phi_k))}\}$ is an orthonormal basis of $\ker(\phi_k)$ as well and we have

$$\mathrm{Tr}(B_k(H)^n) = \sum_{i=1}^{\dim(\ker(\phi_k))} \langle B_k(H)^n b_i, b_i \rangle = \sum_{i=1}^{\dim(\ker(\phi_k))} \langle B_k(H)^n Bb_i, Bb_i \rangle$$

for each $H \in \mathfrak{sp}(l, \mathbb{C})$.

On the other hand, for $H = \mathrm{diag}(h_1, \dots, h_l, -h_1, \dots, -h_l) \in \underline{t} \subset \mathfrak{sp}(l, \mathbb{C})$ and $n$ odd we have

$$\langle B_k(H)^n e_i, e_i \rangle = h_i^n, \langle B_k(H)^n f_i, f_i \rangle = -h_i^n$$

for $i = 1, \dots, n$, which implies

$$\langle B_k(H)^n e_i, e_i \rangle = -\langle B_k(H)^n Be_i, Be_i \rangle, \langle B_k(H)^n f_i, f_i \rangle = -\langle B_k(H)^n Bf_i, Bf_i \rangle.$$

From bilinearity of the scalar product we conclude

$$\langle B_k(H)^n v, v\rangle = -\langle B_k(H)^n Bv, Bv\rangle$$

for all $v \in \mathbb{C}^{2l}$, in particular for $v = b_1, \dots, b_{\dim(\ker(\phi_k))} \in \ker(\phi_k)$. Thus

$$\sum_{i=1}^{\dim(\ker(\phi_k))} \langle B_k(H)^n b_i, b_i\rangle = -\sum_{i=1}^{\dim(\ker(\phi_k))} \langle B_k(H)^n Bb_i, Bb_i\rangle,$$

which implies

$$\mathrm{Tr}(B_k(H)^n) = \sum_{i=1}^{\dim(\ker(\phi_k))} \langle B_k(H)^n b_i, b_i\rangle = 0.$$

**Conclusion** *The cases with $\beta_{2n-1}(B_k) = 0$ are precisely*

- $1 \le k \le l$, $n$ *odd.*

**3.2.6 Exceptional Lie groups** For the applications of Theorem 2 and Theorem 3 we will have to consider only odd-dimensional manifolds and therefore we are only interested in Lie groups which admit a symmetric space of odd dimension. The only exceptional Lie group admitting an odd-dimensional symmetric space is $E_7$ with $\dim(E_7/E_7(\mathbb{R})) = 163$. The fact that $163 \equiv 3 \bmod 4$ implies by Corollary 5 that $\rho^* b_{163} \neq 0$ holds for each irreducible representation $\rho$.

For completeness we also show, at least for a specific representation, that $\rho^* b_{2n-1} \neq 0$ holds for each $n \geq 6$. Namely, we consider the representation $\rho\colon E_7 \to \mathrm{GL}(56, \mathbb{C})$, which has been constructed by Adams [1, Corollary 8.2], and we are going to show that this representation satisfies $\rho^* b_{2n-1} \neq 0$ for each $n \geq 6$, in particular for $n = 82$.

By [1, Chapter 7–8] there is a monomorphism $\mathrm{Spin}(12) \times \mathrm{SU}(2)/\mathbb{Z}_2 \to E_7$ and the Cartan subalgebra of the Lie algebra $e_7$ coincides with the Cartan subalgebra $t$ of $\mathfrak{spin}(12) \oplus \mathfrak{spin}(2)$. By [1, Corollary 8.2], the restriction of $\rho$ to $\mathrm{Spin}(12) \times \mathrm{SU}(2)$ is $\lambda_{12}^1 \otimes \lambda_1 \oplus S^- \otimes 1$, where $\lambda_{12}^1$ resp. $\lambda_1$ are the standard representations and $S^-$ is the negative spinor representation.

For even $n$, we know that $\rho^* b_{2n-1} \neq 0$. If $n$ is odd then, for the derivative $\pi_1$ of the standard representation $\lambda_1$ of SU(2) we have $\mathrm{Tr}(\pi_1(h)^n) = 0$, whenever $h \in \underline{t} \cap \mathfrak{spin}(2)$ belongs to the Cartan subalgebra of $\mathfrak{spin}(2)$, because the latter are the diagonal $(2x2)$–matrices of trace 0. Thus the first direct summand $\lambda_{12}^1 \otimes \lambda_1$ does not contribute to $\mathrm{Tr}(\pi(h)^n)$. Hence, for $h = (h_{\mathfrak{spin}}, h_{\mathfrak{spin}}) \in \underline{t} \subset \mathfrak{spin}(12) \oplus \mathfrak{spin}(2)$, we have $\mathrm{Tr}(\pi(h)^n) = \mathrm{Tr}(S^-(h_{\mathfrak{spin}})^n)$. But the nontriviality of the latter has already been shown in Section 3.2.3.

## 3.3 Conclusion

In this section, we discuss, for which symmetric spaces $G/K$ (irreducible, of non-compact type, of dimension $2n-1$) and which representations $\rho\colon G \to \mathrm{GL}(N,\mathbb{C})$ the inequality $\rho^* b_{2n-1} \neq 0$ holds.

**Theorem** *The following is a complete list of irreducible symmetric spaces $G/K$ of noncompact type and fundamental representations $\rho\colon G \to \mathrm{GL}(N,\mathbb{C})$ with $\rho^* b_{2n-1} \neq 0$ for $2n-1 := \dim(G/K)$.*

| Symmetric Space | Representation |
|---|---|
| $\mathrm{SL}_l(\mathbb{R})/\mathrm{SO}_l,\ l \equiv 0,3,4,7 \bmod 8$ | any fundamental representation |
| $\mathrm{SL}_l(\mathbb{C})/\mathrm{SU}_l,\ l \equiv 0 \bmod 2$ | any fundamental representation |
| $\mathrm{SL}_{2l}(\mathbb{H})/\mathrm{Sp}_l,\ l \equiv 0 \bmod 2$ | any fundamental representation |
| $\mathrm{Spin}_{p,q}/(\mathrm{Spin}_p \times \mathrm{Spin}_q)$, $p,q \equiv 1 \bmod 2,\ p \not\equiv q \bmod 4$ | any fundamental representation |
| $\mathrm{Spin}_{p,q}/(\mathrm{Spin}_p \times \mathrm{Spin}_q)$, $p,q \equiv 1 \bmod 2,\ p \equiv q \bmod 4$ | positive and negative half-spinor representation |
| $\mathrm{SO}_l(\mathbb{C})/\mathrm{SO}_l,\ l \equiv 3 \bmod 4$ | any fundamental representation |
| $\mathrm{Sp}_l(\mathbb{C})/\mathrm{Sp}_l,\ l \equiv 1 \bmod 4$ | any fundamental representation |
| $E_7(\mathbb{C})/E_7$ | any fundamental representation |

**Proof** By Lemma 3 it suffices to check whether $\beta_{2n-1}(\pi) \not\equiv 0$, where $\pi$ is the Lie algebra representation induced by $\rho$. Thus we can use the results from Section 3.2.

We use the classification of symmetric spaces as it can be read off Onishchik and Vinberg [25, Table 4, page 229 ff]. Of course, we are only interested in symmetric spaces of odd dimension. A simple inspection shows that all odd-dimensional irreducible symmetric spaces of noncompact type are given in the first table on the next page.

If $2n-1 \equiv 3 \bmod 4$, then $n$ is even and by Corollary 5 all representations $\rho\colon G \to \mathrm{GL}(N,\mathbb{C})$ satisfy $\rho^* b_{2n-1} \neq 0$. This applies to the symmetric spaces in the second table on the next page.

Next we look at the irreducible locally symmetric spaces of dimension $\equiv 1 \bmod 4$.

For those $G/K$, whose Lie algebra $\underline{g}$ is *not* a complex Lie algebra (this concerns the first 3 cases), we can, as observed in Section 3.2.1, directly apply the results for the respective complexifications. Thus we have to check whether $\beta_{2n-1}(\rho_{\mathbb{C}}) \neq 0$.

| Symmetric Space | Dimension |
|---|---|
| $SL_l(\mathbb{R})/SO_l, l \equiv 0,3,4,7 \bmod 8$ | $\frac{1}{2}(l-1)(l+2)$ |
| $SL_l(\mathbb{C})/SU_l, l \equiv 0 \bmod 2$ | $l^2-1$ |
| $SL_{2l}(\mathbb{H})/Sp_l, l \equiv 0 \bmod 2$ | $(l-1)(2l+1)$ |
| $Spin_{p,q}/(Spin_p \times Spin_q),\ p,q \equiv 1 \bmod 2$ | $pq$ |
| $SO_l(\mathbb{C})/SO_l, l \equiv 2,3 \bmod 4$ | $\frac{1}{2}l(l-1)$ |
| $Sp_l(\mathbb{C})/Sp_l, l \equiv 1 \bmod 2$ | $l(2l+1)$ |
| $E_7(\mathbb{C})/E_7$ | 163 |

| Symmetric Space | Condition |
|---|---|
| $SL_l(\mathbb{R})/SO_l$ | $l \equiv 0,7 \bmod 8$ |
| $SL_l(\mathbb{C})/SU_l$ | $l \equiv 0 \bmod 2$ |
| $SL_{2l}(\mathbb{H})/Sp_l$ | $l \equiv 0 \bmod 4$ |
| $Spin_{p,q}/(Spin_p \times Spin_q)$ | $p,q \equiv 1 \bmod 2,\ p \not\equiv q \bmod 4$ |
| $SO_l(\mathbb{C})/SO_l$ | $l \equiv 3 \bmod 4$ |
| $Sp_l(\mathbb{C})/Sp_l$ | $l \equiv 1 \bmod 4$ |
| $E_7(\mathbb{C})/E_7$ | |

- For $SL_l(\mathbb{R})/SO_l, l \equiv 3,4 \bmod 8$, every fundamental representation $\rho$ satisfies $\rho^* b_{2n-1} \neq 0$. (Indeed $l \geq 3, n \geq 3$, thus we are not in one of the exceptional cases from Section 3.2.2.)

- For $SL_{2l}(\mathbb{H})/Sp_l, l \equiv 2 \bmod 4$, every fundamental representation $\rho$ satisfies $\rho^* b_{2n-1} \neq 0$. (Indeed the complexification of $\mathfrak{sl}_{2l}(\mathbb{H})$ is $\mathfrak{sl}_{4l}(\mathbb{C})$. We have $4l \geq 8$ and $n \geq 3$, thus we are not in one of the exceptional cases from Section 3.2.2.)

- For $Spin_{p,q}/(Spin_p \times Spin_q)$, $p,q \equiv 1 \bmod 2$, $p \equiv q \bmod 4$, the positive and negative half-spinor representations are the only fundamental representations $\rho$ satisfying $\rho^* b_{2n-1} \neq 0$. (The assumptions imply that the complexification is $\mathfrak{spin}(2l,\mathbb{C})$ with $l$ odd, because of $2l = p+q \equiv 2 \bmod 4$. In particular $n \equiv l \bmod 2$ and we are not in the exceptional case of Section 3.2.3.)

For those $G/K$ whose Lie algebra $\underline{g}$ is a complex Lie algebra, we use the fact that each $\mathbb{R}$–linear representation is of the form $\rho_1 \otimes \overline{\rho_2}$. We get:

- For $SO_l(\mathbb{C})/SO_l, l \equiv 3 \bmod 4$, we have $l \equiv n \bmod 2$ and by Section 3.2.4 no fundamental representation $\rho$ satisfies $\rho^* b_{2n-1} \neq 0$.

- For $Sp_l(\mathbb{C})/Sp_l, l \equiv 1 \bmod 4$, by Section 3.2.5 no fundamental representation $\rho$ satisfies $\rho^* b_{2n-1} \neq 0$. □

**Example** (Goncharov) Consider hyperbolic space $\mathbb{H}^d = \mathrm{Spin}_{d,1}/(\mathrm{Spin}_d \times \mathrm{Spin}_1)$. It was shown in [15] that the half-spinor representations have nontrivial Borel class if $d$ is odd. The above results show that for $d \equiv 3 \bmod 4$ each irreducible representation has nontrivial Borel class, but for $d \equiv 1 \bmod 4$ the positive and half-negative spinor representation are the only fundamental representations with this property.

For $d = 3$ we will however compute in Section 3.4 that the invariants coming from irreducible representations, albeit all distinct, are rational multiples of each other.

## 3.4 Some clues on computation

So far we have been using Lemma 3 to decide when $\rho^* b_{2n-1} \neq 0$. In this subsection we will give some clues to the actual computation of $\rho^* b_{2n-1}$.

Recall that $H^*(\underline{g}_u, \underline{k})$ is the cohomology of the complex of $G_u$–invariant forms on $G_u/K$. For a $d$–dimensional compact symmetric space $G_u/K$, there is an isomorphism $H^d(\underline{g}_u, \underline{k}) \simeq H^d(G_u/K; \mathbb{R}) \simeq \mathbb{R}$ given by integration over $[G_u/K]$. Moreover, a $G_u$–invariant $d$–form is uniquely determined by its value on an orthonormal (for the metric given by the negative of the Killing form) basis $X_1, \dots, X_d$ of $T_{[e]}G_u/K \simeq i\underline{p}$. By definition, the volume form takes the value 1 on each orthonormal basis. On the other hand, the volume form represents $\mathrm{vol}(G_u/K)[G_u/K]^v \in H^d(G_u/K)$, where $[G_u/K]^v$ means the dual of the fundamental class. Thus we have:

**Lemma 4** *Let $G_u/K$ be a compact symmetric space of dimension $d$, $\omega \in C^d(\underline{g}_u, \underline{k})$ a $G_u$–invariant $d$–form and $X_1, \dots, X_d$ an orthonormal basis of $i\underline{p}$. Then*

$$[\omega] = \omega(X_1, \dots, X_d)\, \mathrm{vol}(G_u/K)\, [G_u/K]^v \in H^d(G_u/K; \mathbb{R}).$$

The Borel class $b_{2n-1} \in H^{2n-1}(u(N))$ is represented by $b_{2n-1}(X_1, \dots, X_{2n-1}) =$

$$\frac{1}{(2\pi i)^n} \frac{(-1)^n (n-1)!}{(2n-1)!} \sum_{\sigma \in S_{2n-1}} (-1)^\sigma \operatorname{Tr}(X_{\sigma(1)}[X_{\sigma(2)}, X_{\sigma(3)}] \cdots [X_{\sigma(2n-2)}, X_{\sigma(2n-1)}]).$$

Under the identification $H^{2n-1}(u(N) \oplus u(N), u(N)) \simeq H^{2n-1}(u(N))$, the element $b_{2n-1}^{u \oplus u}$ is represented by $b_{2n-1}^{u \oplus u}(Y_1, \dots, Y_{2n-1}) = b_{2n-1}(X_1, \dots, X_{2n-1})$. Here $Y_1, \dots, Y_{2n-1} \in u(N) \oplus u(N)$ and $X_1 := \pi_2(Y_1), \dots, X_{2n-1} := \pi_2(Y_n) \in u(N)$, where $\pi_2$ is the projection to the second summand of $u(N) \oplus u(N)$.

After the identification $H^{2n-1}(\mathfrak{gl}(N, \mathbb{C}), u(N)) \simeq H^{2n-1}(u(N) \oplus u(N), u(N))$ this gives $b_{2n-1}^{\mathfrak{gl}}$ represented by $b_{2n-1}^{\mathfrak{gl}}(y_1, \dots, y_{2n-1}) = i^{2n-1} b_{2n-1}(x_1/i, \dots, x_{2n-1}/i)$. Here $x_1 = \pi(y_1), \dots, x_{2n-1} = \pi(y_{2n-1})$, where $\pi\colon \mathfrak{gl}(N, \mathbb{C}) \to iu(N)$ is the projection associated to $\mathfrak{gl}(N, \mathbb{C}) = u(N) \oplus iu(N)$.

**Borel element** In the notation of [6, Section 9.7], we have $b^{\mathfrak{gl}}_{2n-1}=(1/(2\pi i)^n)\Phi_{2n-1}$. The *Borel element* $\mathrm{Bo}_n\in C^*(\mathfrak{gl}(N,\mathbb{C}),u(N);\mathbb{R}(n-1))$ is given in [6, Section 9.7] by $\mathrm{Bo}_n(\bigwedge_{j=1}^{2n-1} y_j)=\Phi_{2n-1}(\bigwedge_{j=1}^{2n-1}(\overline{y}_j^t+y_j)))$ and this is then used to define the Borel regulator.

For $x_j\in iu(N)$ we have $\overline{x}_j^t+x_j=2x_j$, hence

$$\mathrm{Bo}_n(\textstyle\bigwedge_{j=1}^{2n-1} x_j)=(2\pi i)^n 2^{2n-1} b_{2n-1}(x_1,\dots,x_{2n-1}).$$

**Example** (Hyperbolic 3–manifolds) A Killing form-orthonormal basis of

$$T_{[e]}\,\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)=\{B\in\mathrm{Mat}(2,\mathbb{C}):\mathrm{Tr}(B)=0, B=\overline{B}^T\}$$

is $\{(1/2\sqrt{2})H,(1/2\sqrt{2})X,(1/2\sqrt{2})Y\}$, with

$$H=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad X=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad Y=\begin{pmatrix}0&i\\-i&0\end{pmatrix}.$$

For a representation $\rho\colon \mathrm{SL}(2,\mathbb{C})\to\mathrm{GL}(m+1,\mathbb{C})$ and $\pi=D_e\rho$ we have

$$\begin{aligned}
&\rho^* b_3^{\mathfrak{gl}}(H,X,Y)\\
&=\frac{i}{(2\pi i)^2}\frac{1}{6}\big(2\,\mathrm{Tr}(\pi iH[\pi iX,\pi iY])+2\,\mathrm{Tr}(\pi iX[\pi iY,\pi iH])+2\,\mathrm{Tr}(\pi iY[\pi iH,\pi iX])\big)\\
&=-\frac{1}{6\pi^2}\mathrm{Tr}((\pi iH)^2)-\frac{1}{6\pi^2}\mathrm{Tr}((\pi iX)^2)-\frac{1}{6\pi^2}\mathrm{Tr}((\pi iY)^2).
\end{aligned}$$

Each $(m+1)$–dimensional irreducible representation is equivalent to $\pi_m$ given by

$$\pi_m(iH)=\begin{pmatrix} im & 0 & 0 & \cdots & 0\\ 0 & i(m-2) & 0 & \cdots & 0\\ \cdot & \cdot & \cdot & \cdots & \cdot\\ 0 & 0 & \cdot & \cdots & -im\end{pmatrix},$$

$$\pi_m(iX)=\begin{pmatrix} 0 & -i & 0 & \cdots & 0\\ -im & 0 & -2i & \cdots & 0\\ 0 & -i(m-1) & 0 & \cdots & 0\\ \cdot & \cdot & \cdot & \cdots & -im\\ 0 & 0 & 0 & \cdots -i & 0\end{pmatrix},$$

$$\pi_m(iY)=\begin{pmatrix} 0 & 1 & 0 & \cdots & 0\\ -m & 0 & 2 & \cdots & 0\\ 0 & -(m-1) & 0 & \cdots & 0\\ \cdot & \cdot & \cdot & \cdots & m\\ 0 & 0 & 0 & \cdots -1 & 0\end{pmatrix}.$$

The diagonal entries of $\pi_m(iH)^2$ are

$$(-m^2, -(m-2)^2, \dots, 0, \dots, -(m-2)^2, -m^2),$$

and the diagonal entries of $\pi_m(iX)^2$ and $\pi_m(iY)^2$ are both equal to

$$(-m, -m-2(m-1), -2(m-1)-3(m-2), \dots).$$

In particular $\mathrm{Tr}(\pi_m(iX)^2) = \mathrm{Tr}(\pi_m(iY)^2)$. After scaling with $1/2\sqrt{2}$ we conclude

$$\rho_m^* b_3^{\mathfrak{gl}}\Big(\frac{1}{2\sqrt{2}}H, \frac{1}{2\sqrt{2}}X, \frac{1}{2\sqrt{2}}Y\Big) = \frac{i}{96\sqrt{2}\pi^2}\sum_{k=0}^{m}(m-2k)^2 + \frac{i}{48\sqrt{2}\pi^2}\sum_{k=0}^{m} k(m-k+1)+(k+1)(m-k).$$

**Comparison of Borel element and hyperbolic volume form** If $m=1$, that is for the inclusion $\rho_1 = j : \mathrm{SL}(2,\mathbb{C}) \subset \mathrm{GL}(2,\mathbb{C})$, we get

$$j^* b_3^{\mathfrak{gl}}\Big(\frac{1}{2\sqrt{2}}H, \frac{1}{2\sqrt{2}}X, \frac{1}{2\sqrt{2}}Y\Big) = \frac{i}{16\sqrt{2}\pi^2}, \; j^* b_3^{u\oplus u}\Big(\frac{i}{2\sqrt{2}}H, \frac{i}{2\sqrt{2}}X, \frac{i}{2\sqrt{2}}Y\Big) = \frac{1}{16\sqrt{2}\pi^2}.$$

Therefore by Lemma 4, the element $j^* b_3^{u\oplus u} \in H^3(\underline{g}_u, \underline{k}) \simeq H^3(S^3)$ represents $(1/(16\sqrt{2}\pi^2))\,\mathrm{vol}(S^3)[S^3]$.

Explicit computation shows the projection $\mathrm{SL}(2,\mathbb{C}) \to \mathbb{H}^3$ maps

$$\frac{1}{2\sqrt{2}}H, \quad \frac{1}{2\sqrt{2}}X, \quad \frac{1}{2\sqrt{2}}Y$$

to vectors of hyperbolic length $1/\sqrt{2}$. Thus *the hyperbolic metric is given by one half of the Killing form*: lengths are multiplied by $1/\sqrt{2}$, volumes by $1/2\sqrt{2}$. By Lemma 4 this means that the isomorphism $H^3(\mathfrak{sl}(2,\mathbb{C}), \mathfrak{spin}(2)) \simeq H^3(S^3)$ sends the class $[d\mathrm{vol}]$ of the hyperbolic volume form to $(1/2\sqrt{2})\,\mathrm{vol}(S^3)[S^3]$. In particular $j^* b_3^{\mathfrak{gl}} \in C^3(\mathfrak{sl}(2,\mathbb{C}), \mathfrak{spin}(2))$ is $1/8\pi^2$ times the class of the hyperbolic volume form.

For the Borel element we have $\mathrm{Bo}_2 = -32\pi^2 b_3^{\mathfrak{gl}}$, it follows that the Borel element is $-16$ times the hyperbolic volume. (Dupont and Sah [10] and Neumann and Yang [24] compute the imaginary part of the Borel regulator to be $1/2\pi^2$ times the hyperbolic volume, but they are using a different definition.)

**Example** $(\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3))$ Let $\rho\colon \mathrm{SL}(3,\mathbb{R}) \to \mathrm{GL}(3,\mathbb{C})$ be the inclusion. Since $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$ is 5–dimensional, we wish to compute $\rho^* b_5$. Let

$$H_1 = \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_1 = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We will use the convention that, for $A \in \{H, X, Y\}$ if $A_1$ is defined (in a given basis), then $A_2$ is obtained via the base change $e_1 \to e_2, e_2 \to e_3, e_3 \to e_1$ and $A_3$ is obtained via the base change $e_1 \to e_3, e_3 \to e_2, e_2 \to e_1$.

We have $[H_1, H_2] = 0$, $[H_1, X_1] = 2Y_1$, $[H_1, X_2] = -Y_2$, $[H_1, X_3] = -Y_3$, $[X_1, X_2] = iY_3$ and more relations are obtained out of these ones by base changes.

A basis of $i\underline{p}$ is given by $H_1, H_2, X_1, X_2, X_3$. There are 120 summands in the formula for $\rho^* b_5(H_1, H_2, X_1, X_2, X_3)$. (24 of them contain $[H_1, H_2] = 0$ or $[H_2, H_1] = 0$.)

Each summand appears four times because, for example, $H_1[H_2, X_1][X_2, X_3]$ also shows up as $-H_1[X_1, H_2][X_2, X_3], -H_1[H_2, X_1][X_3, X_2]$ and $H_1[X_1, H_2][X_3, X_2]$. Thus one has to add 30 summands (6 of them zero), and multiply their sum by 4.

We note that all summands of the form $H_1[H_2, \cdot][\cdot, \cdot]$ give after base change corresponding elements of the form $H_2[H_1, \cdot][\cdot, \cdot]$, which are summed with the opposite sign. Thus these terms cancel each other. The same cancellation occurs between summands of the form $X_2[\cdot, \cdot][\cdot, \cdot]$ and $X_3[\cdot, \cdot][\cdot, \cdot]$. Thus we only have to sum up summands of the form $X_1[\cdot, \cdot][\cdot, \cdot]$ and we get

$$\begin{aligned}
&(2\pi i)^3 5! \rho^* b_5(H_1, H_2, X_1, X_2, X_3) \\
&\quad = 4\,\mathrm{Tr}(X_1[H_1, H_2][X_2, X_3]) + 4\,\mathrm{Tr}(X_1[X_2, X_3][H_1, H_2]) \\
&\qquad\qquad + 4\,\mathrm{Tr}(X_1[H_1, X_2][X_3, H_2]) + 4\,\mathrm{Tr}(X_1[X_3, H_2][H_1, X_2]) \\
&\qquad\qquad + 4\,\mathrm{Tr}(X_1[H_1, X_3][H_2, X_2]) + 4\,\mathrm{Tr}(X_1[H_2, X_2][H_1, X_3]) \\
&\quad = 0 + 0 + 4\,\mathrm{Tr}(X_1 Y_2 Y_3) + 4\,\mathrm{Tr}(X_1 Y_3 Y_2) + 4\,\mathrm{Tr}(-2X_1 Y_3 Y_2) + 4\,\mathrm{Tr}(-2X_1 Y_2 Y_3) \\
&\quad = 0 + 0 + 4i + 4i - 8i - 8i = -8i.
\end{aligned}$$

Note that $H_1, H_2, X_1, X_2, X_3$ are pairwise orthogonal and have norm $2\sqrt{3}$. Dividing each of them by $2\sqrt{3}$ gives an orthonormal basis, on which evaluation of $\rho^* b_5$ gives

$$\begin{aligned}
\rho^* b_5\left(\frac{1}{2\sqrt{3}}H_1, \frac{1}{2\sqrt{3}}H_2, \frac{1}{2\sqrt{3}}X_1, \frac{1}{2\sqrt{3}}X_2, \frac{1}{2\sqrt{3}}X_3\right) &= \frac{1}{(2\sqrt{3})^5}\frac{1}{5!}\frac{1}{(2\pi i)^3}(-8i) \\
&= \frac{1}{34560\sqrt{3}\pi^3}.
\end{aligned}$$

The Borel element is $-256\pi^3 i b_5$, so its value on the orthonormal basis is $-i/(135\sqrt{3})$.

# 4 The cusped case

**Outline** In Section 2 we defined $\gamma(M)$ for closed manifolds $M = \Gamma\backslash G/K$, using the image of the fundamental class $[M] \in H_*(M) \cong H_*(B\Gamma)$ in $H_*(BG)$ for the construction, and the volume cocycle in $C^*_{\mathrm{simp}}(BG)$ for the proof of the desired nontriviality properties. In this section we would like to give an analogous construction for cusped manifolds.

For this we would like to map the fundamental class

$$[M, \partial M] \in H_*\Big(D\,\mathrm{Cone}\Big(\bigcup_i \partial_i M \to M\Big)\Big) \cong H_*\Big(D\,\mathrm{Cone}\Big(\bigcup_i B\Gamma_i \to B\Gamma\Big)\Big)$$

(all notions are defined in Section 4.2) to the homology of some completion $BG^{\mathrm{comp}}$ of $BG$, where the completion should be chosen such that the volume class extends to $BG^{\mathrm{comp}}$.

The completion $BG^{\mathrm{comp}}$ that we define in Section 4.2.2 will be chosen such that for each $c \in \partial_\infty G/K$ a cone over $BG$ is added. There is a natural extension of the volume class to this set and the addition of all points at infinity will leave us the flexibility to remember the geometry of cusps.

Now, if $G/K$ has rank one, then each path-component $\partial_i M$ corresponds to a cusp $c_i \in \partial_\infty G/K$ and this will allow us to define an image of $[M, \partial M]$ in $H_*(BG^{\mathrm{comp}})$.

Of course this does not apply to $\mathrm{SL}(N,\mathbb{C})/\mathrm{SU}(N)$, which has rank $N-1$, but if $\rho\colon (G,K) \to (\mathrm{SL}(N,\mathbb{C}), \mathrm{SU}(N))$ is a representation for a rank one space $G/K$, then we get a well-defined image of $c_i$ in $\partial_\infty \mathrm{SL}(N,\mathbb{C})/\mathrm{SU}(N)$ and can thus define the image of $[M, \partial M]$ in $H_*(B\mathrm{SL}(N,\mathbb{C})^{\mathrm{comp}})$.

In Section 4.4 we will show this homology class has a preimage in $H_*(B\mathrm{SL}(N,\mathbb{C}))$. This then finally allows to generalize the Goncharov construction (Theorem 3).

## 4.1 Preparations

Let $G$ be a connected, semisimple Lie group with maximal compact subgroup $K$. Thus $G/K$ is a symmetric space of noncompact type. Throughout Section 4 we will make the assumption $\mathrm{rank}(G/K) = 1$.

We will consider a manifold $M$ with boundary $\partial M$ such that $\mathrm{int}(M) = M - \partial M$ is a finite-volume locally symmetric space of noncompact type of rank one. This means

$$\mathrm{int}(M) = \Gamma\backslash G/K$$

for a (not necessarily cocompact) lattice $\Gamma \subset G$.

We note that connected, semisimple Lie groups are perfect, hence each representation $\rho\colon G \to \mathrm{GL}(N,\mathbb{C})$ has image in $\mathrm{SL}(N,\mathbb{C})$. Further we will assume that $\rho$ maps $K$ to $\mathrm{SU}(N)$, which can be achieved upon conjugation.

**4.1.1 Negative curvature and visibility manifolds** If $\mathrm{int}(M) = \Gamma\backslash G/K$ is a locally symmetric space of noncompact type of rank one, then its sectional curvature sec is bounded between two negative constants, after scaling with a constant factor one has

$$-4 \leq \sec \leq -1.$$

In particular, by [11, page 440], the universal covering $\widetilde{\mathrm{int}(M)} = G/K$ is a "visibility manifold" in the sense of [11].

The structure of finite-volume quotients of visibility manifolds has been described by Eberlein [11]. The following Lemma collects those results from the proof of [11, Theorem 3.1] that we will frequently use in this paper. (We denote by $\partial_\infty \widetilde{\mathrm{int}(M)} = \partial_\infty(G/K)$ the ideal boundary of $\widetilde{\mathrm{int}(M)} = G/K$, that is the set of equivalence classes of geodesic rays, where rays are equivalent if they are asymptotic; see [11, Section 1].)

**Lemma 5** *Let $\widetilde{N}$ be a simply connected, complete Riemannian manifold, $\Gamma$ be a discrete group of isometries of $\widetilde{N}$ and $N = \widetilde{N}/\Gamma$.*

*If $\widetilde{N}$ is a visibility manifold [11] of nonpositive sectional curvature and $N$ has finite volume, then each end of $N$ has a neighborhood $E$ homeomorphic to $U_c/P_c$, where $c \in \partial_\infty \widetilde{N}$, $U_c$ is a horoball centered at $c$ and $P_c \subset \Gamma$ is a discrete group of parabolic isometries fixing $c$.*

*In particular, if $N$ has finitely many ends, then there are end neighborhoods $E_1, \dots, E_s$ such that $K = N - \bigcup_{i=1}^s E_i$ is compact and for $i = 1, \dots, s$ there are homeomorphisms of pairs $(E_i, \partial \overline{E}_i) \to (U_{c_i}/Pc_i, L_{c_i}/P_{c_i})$, where $c_i \in \partial_\infty \widetilde{N}$ and $L_{c_i}$ is the horosphere centered at $c_i$ which bounds the horoball $U_{c_i}$.*

**Corollary 6** *If $M$ is a compact manifold with boundary, $\partial_1 M, \dots, \partial_s M$ are the connected components of $\partial M$, and $N := \mathrm{int}(M) = M - \partial M$ carries a Riemannian metric of finite volume such that $\widetilde{N}$ is a visibility manifold, then, with the notation of Lemma 5, we have a homeomorphism of tuples*

$$(M, \partial_1 M, \dots, \partial_s M) \to \left(\left(\widetilde{N} - \bigcup_{i=1}^{s} U_{c_i}\right)/\,\Gamma, L_{c_1}/P_{c_1}, \dots, L_{c_s}/P_{c_s}\right).$$

**Proof** By the proof of [11, Theorem 3.1], the neighborhood $E_i$ is *Riemannian collared*, which implies in particular the existence of a diffeomorphism $E_i \cong \partial \overline{E}_i \times (0,\infty)$. The claim follows. □

We will say that $\Gamma c_i \subset \partial_\infty \widetilde{N}$ is *the set of parabolic fixed points corresponding to* $\partial_i M$.

It is at this point where we need the assumption $\operatorname{rank}(G/K) = 1$. In the higher rank case it is not true that there is a unique $\Gamma$–orbit of parabolic fixed points $\Gamma c_i \subset \partial_\infty(G/K)$ associated to a boundary component $\partial_i M$. The isomorphism $\pi_1 M \cong \Gamma$ does *not* send $\pi_1 \partial_i M$ to a subgroup of some $Fix\,(c_i)$, if $\operatorname{rank}(G/K) \geq 2$.

**$\pi_1$–injective boundary** In the proof of Proposition 1 and Theorem 3 we will use that $\pi_1 \partial_i M \to \pi_1 M$ is injective for each path-component $\partial_i M$ of $\partial M$. We are going to explain how this fact follows from well-known properties of visibility manifolds.

**Corollary 7** *Under the assumptions of Corollary 6 we have that $\pi_1 \partial_i M \to \pi_1 M$ is injective for each path-component $\partial_i M$ of $\partial M$.*

**Proof** From Corollary 6 we get a commutative diagram

$$\begin{array}{ccc} \partial_i M & \longrightarrow & M \\ \downarrow & & \downarrow \\ L_{c_i}/P_{c_i} & \longrightarrow & (\widetilde{N} - \bigcup_{i=1}^s U_{c_i})/\Gamma \end{array}$$

where the vertical arrows are homeomorphisms, thus inducing isomorphisms $\pi_1 \partial_i M \to P_{c_i}$ and $\pi_1 M \to \Gamma$, and the horizontal arrows are induced by inclusions. If $P_{c_i} \to \Gamma$ were not injective, then the lift of $\iota\colon L_{c_i}/P_{c_i} \to (\widetilde{N} - \bigcup_{i=1}^s U_{c_i})/\Gamma$ to the universal coverings would not be injective. However the lift of $\iota$ is the inclusion $\widetilde{\iota}\colon L_{c_i} \to \widetilde{N} - \bigcup_{i=1}^s U_{c_i}$. □

Moreover $M$ and all $\partial_i M$ are aspherical by the Cartan–Hadamard Theorem and by [11].

**Identification of $\pi_1 \partial_i M$ with a subgroup of $\pi_1 M$** If $\partial M$ is not connected, then we have to choose different basepoints $x, x_1, \ldots, x_s$ for the definition of $\pi_1(M, x)$, $\pi_1(\partial_1 M, x_1), \ldots, \pi_1(\partial_s M, x_s)$. We can obtain subgroups $\Gamma_1, \ldots, \Gamma_s \subset \pi_1(M, x)$ isomorphic to $\pi_1(\partial_1 M, x_1), \ldots, \pi_1(\partial_s M, x_s)$, respectively, as follows:

**Definition 3** Let $M$ be a manifold, $\partial_1 M, \ldots, \partial_s M$ the connected components of $\partial M$, $x \in M, x_1 \in \partial_1 M, \ldots, x_s \in \partial_s M, \Gamma = \pi_1(M, x)$.

Fix lifts $\widetilde{x}, \widetilde{x}_1, \ldots, \widetilde{x}_s$ of $x, x_1, \ldots, x_s$ to the universal covering $\pi\colon \widetilde{M} \to M$, for $i = 1, \ldots, s$ fix pathes $\widetilde{l}_i\colon [0,1] \to \widetilde{M}$ with $\widetilde{l}_i(0) = \widetilde{x}$ and $\widetilde{l}_i(1) = \widetilde{x}_i$, let $l_i = \pi \circ \widetilde{l}_i\colon [0,1] \to M$, denote $[l_i]$ its homotopy class rel. $\{0,1\}$ and define

$$\Gamma_i := \{[l_i]^{-1} * \gamma * [l_i] : \gamma \in \pi_1(\partial_i M, x_i)\} \subset \Gamma$$

to be the subgroup of $\Gamma$ which corresponds to $\pi_1(\partial_i M, x_i)$ after conjugation with $[l_i]$.

The subgroup $\Gamma_i$ depends on the chosen lift $\widetilde{x}_i$ but, for given $\widetilde{x}, \widetilde{x}_i$, not on $\widetilde{l}_i$.

With the homeomorphism from Corollary 6 we obtain $\Gamma_i = P_{c_i}$. We will say that $c_i$ is the *cusp associated to* $\Gamma_i$. In particular, $\Gamma_i \subset \mathrm{Fix}(c_i)$.

(The choice of $c_i$ in its $\Gamma$–orbit depends on the chosen lift $\widetilde{x}_i$ of $x_i$.)

**Compactification of universal covering by cusps** In the following Corollary we consider $\widetilde{\mathrm{int}(M)} \cup \bigcup_{i=1}^s \Gamma c_i$ as a subspace of $\widetilde{\mathrm{int}(M)} \cup \partial_\infty \widetilde{\mathrm{int}(M)}$, where the latter has the well-known topology defined for example in [11, Section 1]. The definition of the disjoint cone $D\,\mathrm{Cone}$ is given in Section 4.2.1 below.

**Corollary 8** *Let the assumptions of Lemma 5 hold and let a fixed homeomorphism* $f\colon \mathrm{int}(M) - \bigcup_{i=1}^s E_i \to M$ *be given. Then we have a projection*

$$\overline{\pi}\colon \widetilde{\mathrm{int}(M)} \cup \bigcup_{i=1}^s \Gamma c_i \to D\,\mathrm{Cone}\left(\bigcup_{i=1}^s \partial_i M \to M\right)$$

*such that*

$$\overline{\pi}|_{\widetilde{\mathrm{int}(M)} - \bigcup_{i=1}^s \Gamma U_{c_i}}\colon \widetilde{\mathrm{int}(M)} - \bigcup_{i=1}^s \Gamma U_{c_i} \to \mathrm{int}(M) - \bigcup_{i=1}^s E_i$$

*is the restriction of the universal covering* $\pi\colon \widetilde{\mathrm{int}(M)} \to \mathrm{int}(M)$, $\overline{\pi}|_{\Gamma U_{c_i}}\colon \Gamma U_{c_i} \to E_i \cup \mathrm{Cone}\,(\partial_i M) - C_i$ *is a covering with deck group* $\Gamma$ *and* $\overline{\pi}$ *maps* $\Gamma c_i$ *to* $C_i$ *for* $i = 1, \ldots, s$, *where* $C_i$ *is the cone point of* $\mathrm{Cone}\,(\partial_i M)$.

**Proof** Each boundary component $\partial_i M$ corresponds to an end (with neighborhood $E_i$) of $\mathrm{int}(M)$ and thus by Lemma 5 to a unique $\Gamma$–orbit $\Gamma c_i$ with $c_i \in \partial_\infty \widetilde{\mathrm{int}(M)}$ such that $E_i = U_{c_i} / P_{c_i}$. Let $\overline{E}_i$ be the one-point compactification of $\overline{E}_i$, denote $C_i^+$ be the compactifying point, and let $M_+$ be the compactification of $M$ obtained by adding $C_1^+, \ldots, C_s^+$ to $M$. (This is homeomorphic to the space $M_+$ which will be considered in Section 4.1.2.) Then we have homeomorphisms $f_0\colon \mathrm{int}(M) - \bigcup_{i=1}^s E_i \to M$ and $f_i\colon \overline{E}_i \to \mathrm{Cone}\,(\partial_i M)$ such that $f_0 = f_i$ on $\partial E_i$ for $i = 1, \ldots, s$, hence they yield a well-defined homeomorphism $f\colon M_+ \to D\,\mathrm{Cone}\left(\bigcup_{i=1}^s \partial_i M \to M\right)$ which sends $C_i^+$ to $C_i$, the cone point over $\partial_i M$.

Moreover, the universal covering $\pi\colon \widetilde{\mathrm{int}(M)} \to \mathrm{int}(M)$ sends $\gamma U_{c_i}$ to $E_i$ for each $\gamma \in \Gamma$, thus it can be continuously extended to $\Gamma c_i$ by $\pi\,(\gamma c_i) = C_i^+$ for $\gamma \in \Gamma$.

Composition of $\pi$ with the homeomorphism $f$ yields the desired projection $\overline{\pi}$. □

Again, by the remark after Corollary 6, also Corollary 8 requires the assumptions of Lemma 5 and would not work if $\widetilde{\mathrm{int}(M)} = G/K$ were a symmetric space with $\mathrm{rank}(G/K) \geq 2$. (However there is a version of Corollary 8 for locally symmetric spaces of $\mathbb{Q}$–rank 1, which we will exploit in forthcoming work with Inkang Kim.)

**4.1.2 Generalized Cisneros-Molina–Jones construction** The aim of Section 4 will be to associate a $K$–theoretic invariant to cusped locally symmetric spaces. We mention that, by an argument completely analogous to [8], one can define an element $\alpha(M)$ and can associate to each representation $\rho\colon (G, K) \to (\mathrm{SL}(N, \mathbb{C}), \mathrm{SU}(N))$ the pushforward

$$(B\rho)_d(\alpha(M)) \in H_d(B(\mathrm{SL}(N, \mathbb{C}), \mathcal{F}(B))),$$

where $B \subset \mathrm{SL}(N, \mathbb{C})$ is a maximal unipotent subgroup.

In the case of hyperbolic 3–manifolds, Cisneros-Molina and Jones lifted the invariant $\alpha(M)$ to $K_3(\mathbb{C}) \otimes \mathbb{Q}$, and proved its nontriviality by relating it to the Bloch invariant. We describe now how to do a very similar construction for arbitrary locally symmetric spaces of noncompact type with finite volume. Unfortunately we did not succeed to evaluate the Borel class on the constructed invariant. This is the reason why we will actually pursue another approach, using relative group homology and closer in spirit to [15], in the remainder of this section. The construction is however included at this point because its main step, Lemma 6, will be crucial for the proof of Proposition 1.

Let $M$ be an aspherical (compact, orientable, connected) $d$–manifold with aspherical boundary, $\mathbb{F} \subset \mathbb{C}$ a subring and $\rho\colon \pi_1 M \to \mathrm{SL}(\mathbb{F})$ a representation[3].

To push forward the fundamental class $[M_+] \in H_d(M_+; \mathbb{Q})$ one would like to have a map $R\colon M_+ \to B\mathrm{SL}(\mathbb{F})^+$ such that the following diagram commutes up to homotopy:

$$\begin{array}{ccc} M & \xrightarrow{\;q\;} & M_+ \\ \Big\downarrow{\scriptstyle |B\rho| h^M} & & \Big\downarrow{\scriptstyle R} \\ |B\mathrm{SL}(\mathbb{F})| & \xrightarrow{\;\mathrm{incl}\;} & |B\mathrm{SL}(\mathbb{F})|^+ \end{array}$$

Given this, one can consider $R_*[M_+]$ and use the isomorphism $H_d(|B\mathrm{SL}(\mathbb{F})|^+; \mathbb{Q}) \cong H_d(|B\mathrm{SL}(\mathbb{F})|; \mathbb{Q})$ to define an element in $H_d(|B\mathrm{SL}(\mathbb{F})|; \mathbb{Q})$.

[3] Notation: We will denote by $\mathbb{F} \subset \mathbb{C}$ an arbitrary subring (with 1), while $A \subset \mathbb{C}$ will denote a subring satisfying the assumptions of Lemma 2.

**Lemma 6** *Let $M$ be a manifold with boundary such that $M$ and the path-components $\partial_1 M, \dots, \partial_s M$ of $\partial M$ are aspherical. Let $q\colon M \to M_+$ be the canonical projection.*

*Let $\mathbb{F} \subset \mathbb{C}$ a subring and let $\rho\colon \pi_1 M \to \mathrm{SL}(N;\mathbb{F})$ be a representation such that $\rho(\pi_1 \partial_i M)$ is unipotent for $i = 1, \dots, s$.*

*Then there exists a continuous map $R\colon M_+ \to |B\mathrm{SL}(N;\mathbb{F})|^+$ such that*

$$R \circ q = \mathrm{incl} \circ |B\rho| \circ h^M,$$

*where $\mathrm{incl}\colon |B\mathrm{SL}(N;\mathbb{F})| \to |B\mathrm{SL}(N;\mathbb{F})|^+$ is the inclusion.*

**Proof** Let $F$ be the homotopy fiber of $|B\mathrm{SL}(N,\mathbb{F})| \to |B\mathrm{SL}(N;\mathbb{F})|^+$. It is well-known (eg [8, page 336]) that $\pi_1 F$ is isomorphic to the Steinberg group $\mathrm{St}(\mathbb{F})$. Let $\Phi\colon \mathrm{St}(N;\mathbb{F}) \to \mathrm{SL}(N;\mathbb{F})$ be the canonical homomorphism.

By assumption, $\rho$ maps $\pi_1\partial_1 M$ into some maximal unipotent subgroup $B \subset \mathrm{SL}(n,\mathbb{F})$ of parabolic elements. $B$ is conjugate to $B_0 \subset \mathrm{SL}(n,\mathbb{F})$, the group of upper triangular matrices with all diagonal entries equal to 1. By [26, Lemma 4.2.3] there exists a homomorphism $\Pi\colon B_0 \to \mathrm{St}(N;\mathbb{F})$ with $\Phi\Pi = \mathrm{id}$. Applying conjugations and composing with $\rho$, we get a homomorphism $\tau\colon \pi_1\partial_1 M \to \mathrm{St}(N;\mathbb{F})$ such that $\Phi\tau = \rho|_{\pi_1\partial_1 M}$.

The component $\partial_1 M$ is aspherical, hence $\tau$ is induced by some continuous mapping $g_1\colon \partial_1 M \to F$, and the diagram

$$\begin{array}{ccc} \partial_1 M & \xrightarrow{\;i_1\;} & M \\ \big\downarrow{\scriptstyle g_1} & & \big\downarrow{\scriptstyle |B\rho| h^M} \\ F & \xrightarrow{\;j\;} & |B\mathrm{SL}(N,\mathbb{F})| \end{array}$$

commutes up to some homotopy $H_t$.

This construction can be repeated for all connected components $\partial_1 M, \dots, \partial_s M$ of $\partial M$. For each $r = 1, \dots, s$ we get a continuous map $g_r\colon \partial_r M \to F$ such that $j g_r \sim |B\rho| h^M i_r$. Altogether, we get a continuous map $g\colon \partial M \to F$ such that $jg$ is homotopic to $|B\rho| B\rho| h^M i$.

By [8, Lemma 8.1] this implies the existence of the desired map $R$. □

Hence one obtains an element in $H_d(|B\mathrm{SL}(\mathbb{F})|;\mathbb{Q})$. Unfortunately we did not succeed to prove its nontriviality, ie to evaluate the Borel class. Therefore we will in the remainder of Section 4 pursue a different approach, closer in spirit to [15], but surrounding the problem that $\partial M$ may be disconnected.

We mention that another "basis-point independent" approach might use multicomplexes in the sense of Gromov, but also here we were able to evaluate the Borel class only in the case that there are 2 or less boundary components. Also, in the case of hyperbolic 3–manifolds, yet another approach is due to Neumann and Yang [24]. For hyperbolic 3–manifolds of finite volume, Zickert [28] has given a direct construction of a fundamental class $[M, \partial M] \in H_3\,(\mathrm{SL}(2,\mathbb{C}), B_0)$, even in the case of possibly disconnected boundary. It should be interesting to generalize and compare the different constructions.

## 4.2 Cuspidal completion

**4.2.1 Disjoint cone** We start with a *notational remark*: the notion of disjoint cone for topological spaces resp. simplicial sets. This notion will be useful for considering the homology of a group relative to possibly more than one subgroup.

**Disjoint cone of topological spaces** Let $X$ be a topological space and $A_1,\dots,A_s \subset X$ a set of (not necessarily disjoint) subspaces. There is a (not necessarily injective) continuous mapping

$$i\colon A_1 \dot\cup \cdots \dot\cup A_s \to X$$

from the *disjoint* union $A_1 \dot\cup \cdots \dot\cup A_s$ to $X$.

We define the *disjoint cone*

$$D\,\mathrm{Cone}\Big(\bigcup_{i=1}^{s} A_i \to X\Big)$$

to be the pushout of the diagram

$$\begin{array}{ccc} A_1 \dot\cup \cdots \dot\cup A_s & \xrightarrow{\quad i \quad} & X \\ \downarrow & & \downarrow \\ \mathrm{Cone}(A_1) \dot\cup \cdots \dot\cup \mathrm{Cone}(A_s) & \longrightarrow & D\,\mathrm{Cone}(\bigcup_{i=1}^{s} A_i \to X) \end{array}$$

If $X$ is a CW–complex and $A_1,\dots,A_s$ are disjoint sub–CW–complexes, then for $* \ge 2$,

$$H_*\Big(D\,\mathrm{Cone}\Big(\bigcup_{i=1}^{s} A_i \to X\Big)\Big) \cong H_*\Big(\mathrm{Cone}\Big(\bigcup_{i=1}^{s} A_i \to X\Big)\Big) = H_*\Big(X, \bigcup_{i=1}^{s} A_i\Big).$$

**Disjoint cone of simplicial sets** We will need the cuspidal completion of a classifying space, which fits into the setting of simplicial sets. (The point of the construction is that it may remember the geometry of the cusps of locally symmetric spaces. Thus it will serve as a technical device to handle the cusped case.)

For a simplicial set $(B, \partial_B, s_B)$ and a symbol $c$, *the cone over $B$ with cone point $c$* is the quasisimplicial set whose $k$–simplices are

- either $k$–simplices in $B$,
- or cones over $(k-1)$–simplices in $B$ with cone point $c$. (*By convention, the cone point is always the* last *vertex of the cone over a $(k-1)$–simplex.*)

The boundary operator $\partial$ in $\mathrm{Cone}(B)$ is defined by $\partial\sigma = \partial_B\sigma$ if $\sigma \in B$ and $\partial\,\mathrm{Cone}\,(\tau) = \mathrm{Cone}\,(\partial_B\tau) + (-1)^{\dim(\tau)+1}\tau$ if $\tau \in B$.

If $Y$ is a simplicial set and $\{B_i : i \in I\}$ a family of simplicial subsets indexed over a set $I$, then we define the quasisimplicial set $D\,\mathrm{Cone}\left(\bigcup_{i\in I} B_i \to Y\right)$ as the pushout

$$\begin{array}{ccc} \dot{\bigcup}_{i\in I} B_i & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \dot{\bigcup}_{i\in I} \mathrm{Cone}\,(B_i) & \longrightarrow & D\,\mathrm{Cone}\left(\bigcup_{i\in I} B_i \to Y\right). \end{array}$$

**4.2.2 Construction of $BG^{\mathrm{comp}}$ and $B\Gamma^{\mathrm{comp}}$** Recall from the beginning of Section 2.1 that $BG$ is the simplicial set realizing the bar construction. Thus its $k$–simplices are of the form $(g_1, \ldots, g_k)$ with $g_1, \ldots, g_k \in G$. We recall that $\partial_\infty(G/K)$ denotes the ideal boundary of $G/K$. The point of the following definition is that it allows to consider the geometry at each $c \in \partial_\infty(G/K)$ separately. (As in the remark after Corollary 6 the definition of $B\Gamma^{\mathrm{comp}}$ will assume $\mathrm{rank}(G/K) = 1$.)

**Definition 4** Let $G/K$ be a symmetric space of noncompact type. We define the cuspidal completion $BG^{\mathrm{comp}}$ of $BG$ to be

$$D\,\mathrm{Cone}\left(\dot{\bigcup_{c\in\partial_\infty(G/K)}} BG \to BG\right).$$

**Notation** The cone point of corresponding to $c \in \partial_\infty(G/K)$ will also be denoted by $c$.

**Definition 5** Let $M$ be a manifold with $\pi_1$–injective boundary $\partial M$, let $\partial_1 M, \dots, \partial_s M$ be the connected components of $\partial M$, fix $x_0 \in M$ and $x_i \in \partial_i M$ for $i = 1, \dots, s$, and let $\Gamma_i \subset \Gamma := \pi_1(M, x)$ be defined according to Definition 3.

Assume that $M$ satisfies the assumptions of Corollary 6 and let $c_i \in \partial_\infty \widetilde{\mathrm{int}(M)}$ be the cusp associated to $\Gamma_i$. Then we define

$$B\Gamma^{\mathrm{comp}} = D\,\mathrm{Cone}\left( \bigcup_{i=1}^{s} B\Gamma_i \to B\Gamma \right)$$

to be the quasisimplicial set whose $k$–simplices $\tau$ are either of the form

$$\tau = (\gamma_1, \dots, \gamma_k)$$

with $\gamma_1, \dots, \gamma_k \in \Gamma$ or for some $i \in \{1, \dots, s\}$ of the form

$$\tau = (p_1, \dots, p_{k-1}, c_i)$$

with $p_1, \dots, p_{k-1} \in \Gamma_i$.

**Notation** The cone point of Cone $(B\Gamma_i) \subset B\Gamma^{\mathrm{comp}}$ will be denoted by $c_i \in \partial_\infty(G/K)$, the cusp associated to $\Gamma_i$. This is justified by the following observation.

**Observation 1** Let $M$ be a compact manifold with boundary $\partial M = \partial_1 M \cup \dots \cup \partial_s M$ such that $\mathrm{int}(M) = \Gamma \backslash G/K$ is a locally symmetric space of noncompact type of rank one with finite volume. Then $B\Gamma^{\mathrm{comp}} \subset BG^{\mathrm{comp}}$, where the cone point $c_i$ of Cone $(B\Gamma_i)$ corresponds to $c_i \in \partial_\infty(G/K)$ as the cone point of the corresponding copy of Cone$(BG)$.

**Remark** $B\Gamma^{\mathrm{comp}}$, as a subset of $BG^{\mathrm{comp}}$, depends on the chosen identification of $\pi_1(\partial_i M, x_i)$ with a subgroup $\Gamma_i$ of $\Gamma$.

**4.2.3 Volume cocycle** In Section 2.3 we defined the volume cocycle $cv_d \in C^d_{\mathrm{simp}}(BG)$ for a symmetric space $G/K$ of noncompact type. In this subsection we will extend $cv_d$ to $\overline{cv}_d \in C^d_{\mathrm{simp}}(BG^{\mathrm{comp}})$.

For the remainder of this section we fix some $\tilde{x} \in G/K$. Let $d = \dim(G/K)$.

We define the *volume cocycle* $\overline{cv}_d \in C^d_{\mathrm{simp}}(BG^{\mathrm{comp}})$ as follows.

For $(g_1, \dots, g_d) \in BG$ we define

$$\overline{cv}_d(g_1, \dots, g_d) = \mathrm{algvol}(\mathrm{str}(\tilde{x}, g_1\tilde{x}, \dots, g_1 \dots g_d\tilde{x})) = \mathrm{int}_{\mathrm{str}(\tilde{x}, g_1\tilde{x}, \dots, g_1 \dots g_d\tilde{x})}\, d\mathrm{vol}$$

and for $(p_1,\dots,p_{d-1},c)\in \mathrm{Cone}(BG)$ with $c\in\partial_\infty(G/K)$ we define

$$\begin{aligned}\overline{cv}_d\,(p_1,\dots,p_{d-1},c) &= \mathrm{algvol}\,(\mathrm{str}\,(\tilde{x},\, p_1\tilde{x},\dots,\, p_1\dots p_{d-1}\tilde{x},c))\\ &= \mathrm{int}_{\mathrm{str}(\tilde{x},p_1\tilde{x},\dots,p_1\dots p_{d-1}\tilde{x},c)}\, d\mathrm{vol}\,.\end{aligned}$$

(This is defined because ideal d-simplices in a $d$–dimensional symmetric space $G/K$ of noncompact type have finite volume.)

The computation in Section 2.3 shows $\delta\overline{cv}_d\,(g_1,\dots,g_{d+1})=0$ for $(g_1,\dots,g_{d+1})\in BG$. Moreover, for $(p_1,\dots,p_d,c)\in\mathrm{Cone}(BG)$ with $c\in\partial_\infty(G/K)$ we have

$$\begin{aligned}&\delta\overline{cv}_d(p_1,\dots,p_d,c)\\ &=\overline{cv}_d\Big((p_2,\dots,p_d,c)+\sum_{i=1}^{d-1}(p_1,\dots,p_ip_{i+1},\dots,p_d,c)+(-1)^{d+1}(p_1,\dots,p_d)\Big)\\ &=\cdots=\mathrm{int}_{\partial\,\mathrm{str}(\tilde{x},p_1\tilde{x},\dots,p_1\dots p_d\tilde{x},c)}\,d\mathrm{vol}=\mathrm{int}_{\mathrm{str}(\tilde{x},p_1\tilde{x},\dots,p_1\dots p_d\tilde{x},c)}\,d(d\mathrm{vol})=0.\end{aligned}$$

This proves $\overline{cv}_d$ is a simplicial cocycle on $BG^{\mathrm{comp}}$. Let $\overline{cv}_d=[\overline{cv}_d]\in H^d_{\mathrm{simp}}(BG^{\mathrm{comp}})$.

By construction we have $\overline{cv}_d|_{BG}=cv_d$ and thus $\overline{cv}_d|_{BG}=\mathrm{comp}(v_d)$ for the volume class $v_d=[v_d]\in H^d_c(G;\mathbb{R})$ defined in Section 2.3.

The Borel class $b_d\in H^d_c(\mathrm{GL}(\mathbb{C});\mathbb{R})$ defined in Section 2.4 may also be considered as a class $b_d\in H^d_c(\mathrm{SL}(\mathbb{C});\mathbb{R})$. For a representative $\beta_d\in C^d_c(\mathrm{SL}(\mathbb{C});\mathbb{R})$ of $b_d$ we define $c\beta_d\in C^d_{\mathrm{simp}}(B\mathrm{SL}(\mathbb{C});\mathbb{R})$ by

$$c\beta_d(g_1,\dots,g_d):=\beta_d(1,g_1,g_1g_2,\dots,g_1g_2\dots g_d).$$

Then $c\beta_d$ represents

$$\mathrm{comp}(b_d)\in H^d_{\mathrm{simp}}(B\mathrm{SL}(\mathbb{C});\mathbb{R}).$$

**Lemma 7** *Let $d,N\in\mathbb{N}$ with $d$ odd. Let $G/K$ be a $d$–dimensional symmetric space of noncompact type. If $\rho\colon (G,K)\to(\mathrm{SL}(N,\mathbb{C}),\mathrm{SU}(N))$ is a representation, then there exists a quasisimplicial set $B\mathrm{SL}(N,\mathbb{C})^{\mathrm{fb}}$ with*

$$B\mathrm{SL}(N,\mathbb{C})\subset B\mathrm{SL}(N,\mathbb{C})^{\mathrm{fb}}\subset B\mathrm{SL}(N,\mathbb{C})^{\mathrm{comp}}$$

*and a homomorphism*

$$\overline{c\beta}_d\colon C_d^{\mathrm{simp}}(B\mathrm{SL}(N,\mathbb{C})^{\mathrm{fb}};\mathbb{R})\to\mathbb{R},$$

*such that*

(i) $\overline{c\beta}_d|_{C_d^{\mathrm{simp}}(B\mathrm{SL}(N,\mathbb{C});\mathbb{R})}$ *is a cocycle representing* $\mathrm{comp}(b_d)$,

(ii) *we have*

$$(B\rho)_d(C_d^{\mathrm{simp}}(BG^{\mathrm{comp}};\mathbb{R})) \subset C_d^{\mathrm{simp}}(B\mathrm{SL}(N,\mathbb{C})^{\mathrm{fb}};\mathbb{R})$$

*and* $\rho^*\overline{c\beta}_d$ *represents* $c_\rho\overline{cv}_d$.
*(In particular,* $\overline{c\beta}_d$ *is well-defined on* $(B\rho)_d H_d(BG^{\mathrm{comp}};\mathbb{R})$*.)*

Here, $c_\rho \in \mathbb{R}$ is defined by the equality $\rho^* b_d = c_\rho v_d \in H_c^d(G;\mathbb{R})$ from Theorem 2.

**Proof** Let $d\mathrm{bol}$ be[4] an $\mathrm{SL}(N,\mathbb{C})$–invariant differential form on $\mathrm{SL}(N,\mathbb{C})/\mathrm{SU}(N)$ representing $\nu_{\mathrm{SL}(N,\mathbb{C})}(b_d)$, where $\nu_{\mathrm{SL}(N,\mathbb{C})}$ is the map from $H_c^d(\mathrm{SL}(N,\mathbb{C});\mathbb{R})$ to $H^d(\mathfrak{sl}(N,\mathbb{C}),\mathfrak{spin}(N))$ from Section 2.4.2. Then a representative $\beta_d$ of $b_d$ is given by

$$\beta_d(g_0,g_1,\dots,g_d) := \mathrm{int}_{\mathrm{str}(g_0\tilde{x},g_1\tilde{x},\dots,g_d\tilde{x})}\, d\mathrm{bol}$$

for each $(g_0,g_1,\dots,g_d)\in(\mathrm{SL}(N,\mathbb{C}))^{d+1}$. (This follows from the explicit description of the van Est isomorphism in [9, Theorem 1.1].)

The van Est isomorphism is functorial and $\rho^* b_d = c_\rho v_d$, so $\rho^* d\mathrm{bol} - c_\rho d\mathrm{vol}$ is an exact differential form. Moreover, $\rho^* d\mathrm{bol}$ and $d\mathrm{vol}$ are $G$–invariant differential forms on $G/K$. Hence they are harmonic and $\rho^* d\mathrm{bol} - c_\rho d\mathrm{vol}$ is an exact harmonic form, thus zero and we conclude

$$\rho^* d\mathrm{bol} = c_\rho d\mathrm{vol}.$$

Let $\widetilde{X} = \mathrm{SL}(N,\mathbb{C})/\mathrm{SU}(N)$. Define

$$B\mathrm{SL}(N,\mathbb{C})_d^{\mathrm{fb}} := B\mathrm{SL}(N,\mathbb{C})_d \cup \bigcup_{c\in\partial_\infty\widetilde{X}} \Big\{(p_1,\dots,p_{d-1},c)\in \mathrm{Cone}(B\mathrm{SL}(N,\mathbb{C})) : \mathrm{int}_{\mathrm{str}(\tilde{x},p_1\tilde{x},\dots,p_1\dots p_{d-1}\tilde{x},c)}\, d\mathrm{bol} < \infty\Big\}.$$

This defines the d-simplices of $B\mathrm{SL}(N,\mathbb{C})^{\mathrm{fb}}$ and we define $B\mathrm{SL}(N,\mathbb{C})^{\mathrm{fb}}$ to be the quasisimplicial set generated by $B\mathrm{SL}(N,\mathbb{C})_d^{\mathrm{fb}}$ under face maps.

Define $\overline{c\beta}_d\colon B\mathrm{SL}(N,\mathbb{C})_d^{\mathrm{fb}} \to \mathbb{R}$ by

$$\overline{c\beta}_d(g_1,\dots,g_d) = \mathrm{int}_{\mathrm{str}(\tilde{x},g_1\tilde{x},\dots,g_1\dots g_d\tilde{x})}\, d\mathrm{bol}$$

if $(g_1,\dots,g_d)\in B\mathrm{SL}(N,\mathbb{C})_d$, and

$$\overline{c\beta}_d(p_1,\dots,p_{d-1}) = \mathrm{int}_{\mathrm{str}(\tilde{x},\dots,p_1\tilde{x},p_1\dots p_{d-1}\tilde{x},c)}\, d\mathrm{bol}$$

if $(p_1,\dots,p_{d-1})\in B\mathrm{SL}(N,\mathbb{C})_{d-1}$, $c\in\partial_\infty\widetilde{X}$ and $(p_1,\dots,p_{d-1},c)\in B\mathrm{SL}(N,\mathbb{C})_d^{\mathrm{fb}}$.

By construction, $\overline{c\beta}_d|_{C_d^{\mathrm{simp}}(B\mathrm{SL}(N,\mathbb{C});\mathbb{R})}$ is a cocycle representing $\mathrm{comp}(b_d)$.

[4] The reason for the notation "$d\mathrm{bol}$" is that $d\mathrm{bol}$ relates to the Borel class $b_n$ as $d\mathrm{vol}$ relates to $v_n$. The superscript "$fb$" in $B\mathrm{SL}(N,\mathbb{C})^{\mathrm{fb}}$ stands for "finite Borel class".

The homomorphism $\rho\colon G \to \mathrm{SL}(N,\mathbb{C})$ extends to a well-defined map $\partial_\infty G/K \to \partial_\infty \widetilde{X}$, thus we obtain a well-defined simplicial map $B\rho\colon BG^{\mathrm{comp}} \to B\mathrm{SL}(N,\mathbb{C})^{\mathrm{comp}}$. Now $\rho^* d\mathrm{bol} = c_\rho d\mathrm{vol}$ so $(B\rho)_d(C_d^{\mathrm{simp}}(BG^{\mathrm{comp}};\mathbb{R})) \subset C_d^{\mathrm{simp}}(B\mathrm{SL}(N,\mathbb{C})^{\mathrm{fb}};\mathbb{R})$ and $\rho^*\overline{c\beta}_d$ represents $c_\rho \overline{cv}_d$. □

**Definition 6** Let $\mathbb{F} \subset \mathbb{C}$ be a subring (with 1) and $G/K$ a symmetric space of noncompact type. Then we define

$$BG(\mathbb{F})^{\mathrm{comp}} = D\,\mathrm{Cone}\Big(\dot{\bigcup}_{c \in \partial_\infty(G/K)} BG(\mathbb{F}) \to BG(\mathbb{F})\Big) \subset BG^{\mathrm{comp}}.$$

For $G = \mathrm{SL}(N,\mathbb{C})$ we define

$$B\mathrm{SL}\,(N,\mathbb{F})^{\mathrm{fb}} = B\mathrm{SL}(N,\mathbb{C})^{\mathrm{fb}} \cap B\mathrm{SL}\,(N,\mathbb{F})^{\mathrm{comp}}.$$

## 4.3 Straightening of interior and ideal simplices

The purpose of this section is to describe an explicit realization of the isomorphism

$$H_*\Big(D\,\mathrm{Cone}\Big(\bigcup_{i=1}^{s} \partial_i M \to M\Big)\Big) \cong H_*^{\mathrm{simp}}\Big(D\,\mathrm{Cone}\Big(\bigcup_{i=1}^{s} B\Gamma_i \to B\Gamma\Big)\Big)$$

for $\Gamma = \pi_1 M, \Gamma_1 = \pi_1\partial_1 M, \ldots, \Gamma_s = \pi_1 \partial_s M$, under the assumptions of Lemma 5, that is if $M$ is a finite-volume quotient of a nonpositively curved visibility manifold.

In Section 2.1 we used straightening to define the Eilenberg–Mac Lane map on genuine simplices. In this section we will extend the Eilenberg–Mac Lane map to ideal simplices.

**Definition 7** Let $M$ be a compact manifold with boundary, let $\partial_1 M, \ldots, \partial_s M$ be the connected components of $\partial M$. Let $x_0, x_i, \Gamma, \Gamma_i$ be defined according to Definition 3. We denote

$$\widehat{C}_*(M) := C_*\Big(D\,\mathrm{Cone}\Big(\bigcup_{i=1}^{s} \partial_i M \to M\Big)\Big).$$

For $i = 1, \ldots, s$ let $C_i$ be the cone point of $\mathrm{Cone}(C_*(\partial_i M))$. A vertex of a simplex in $\widehat{C}_*(M)$ is an ideal vertex, if it is in one of the cone points $C_1, \ldots, C_s$, and an interior vertex else. Then we define

$$\widehat{C}_*^{x_0}(M) \subset \widehat{C}_*(M)$$

to be the subcomplex freely generated by those simplices for which

- either all vertices are in $x_0$,
- or the last vertex is an ideal vertex $C_i$, all other vertices are in $x_0$, and the homotopy classes (rel. $\{0,1\}$) of all edges between interior vertices belong to $\Gamma_i \subset \pi_1(M, x_0)$.

By construction, $\widehat{C}_*(M)$ and $\widehat{C}_*^{x_0}(M)$ are chain complexes.

From now on we assume that the assumptions of Corollary 8 (and thus the assumptions of Lemma 5) hold for $N = \mathrm{int}(M) = M - \partial M$. In particular we have the projection $\overline{\pi} : \widetilde{\mathrm{int}(M)} \cup \bigcup_{i=1}^s \Gamma c_i \to D\,\mathrm{Cone}\left(\bigcup_{i=1}^s \partial_i M \to M\right)$ from Corollary 8.

**Definition 8** Let the assumptions of Corollary 8 hold. We say that a simplex in $D\,\mathrm{Cone}\left(\bigcup_{i=1}^s \partial_i M \to M\right)$ is *straight* if some (hence any) lift to $\widetilde{\mathrm{int}(M)} \cup \bigcup_{i=1}^s \Gamma c_i \subset \widetilde{\mathrm{int}(M)} \cup \partial_\infty \widetilde{\mathrm{int}(M)}$ is straight.

In particular a $k$–simplex $\sigma \in \widehat{C}^*(M)$ is straight if it is either of the form

$$\sigma = \pi\left(\mathrm{str}\left(\widetilde{x}_0, \widetilde{x}_1, \ldots, \widetilde{x}_k\right)\right)$$

with $\widetilde{x}_0, \widetilde{x}_1, \ldots, \widetilde{x}_k \in \widetilde{\mathrm{int}(M)}$ or of the form

$$\sigma = \pi\left(\mathrm{str}\left(\widetilde{x}_0, \widetilde{x}_1, \ldots, \widetilde{x}_{k-1}, \gamma c_i\right)\right)$$

with $\widetilde{x}_0, \widetilde{x}_1, \ldots, \widetilde{x}_{k-1} \in \widetilde{\mathrm{int}(M)}, \gamma \in \Gamma, i \in \{1, \ldots, s\}$.

**Definition 9** Let $M$ be a manifold satisfying the assumptions of Definition 8. Let $x_0 \in M$. Then we define the chain complex

$$\widehat{C}_*^{\mathrm{str},x_0}(M) := \mathbb{Z}[\{\sigma \in \widehat{C}_*^{x_0}(M) : \sigma \text{ straight}\}].$$

**Lemma 8** *Let $M$ be a compact manifold with boundary, let $\partial_1 M, \ldots, \partial_s M$ be the connected components of $\partial M$. Let $x_0, x_i, \Gamma, \Gamma_i$ be defined according to Definition 3. Moreover let the assumptions of Corollary 8 hold.*

(a) *Then there is an isomorphism of chain complexes*

$$\Phi\colon \widehat{C}_*^{\mathrm{str},x_0}(M) \to C_*^{\mathrm{simp}}\left(B\Gamma^{\mathrm{comp}}\right).$$

(b) *The inclusion*

$$\widehat{C}_*^{\mathrm{str},x_0}(M) \to \widehat{C}_*(M)$$

*is a chain homotopy equivalence.*

(c) *The composition of $\Psi := \Phi^{-1}$ with the inclusion*

$$\widehat{C}_*^{\mathrm{str},x_0}(M) \to C_*\left(D\,\mathrm{Cone}\left(\bigcup_{i=1}^s \partial_i M \to M\right)\right)$$

*induces an isomorphism*

$$\mathrm{EM}_*\colon H_*^{\mathrm{simp}}(B\Gamma^{\mathrm{comp}}) \to H_*\left(D\,\mathrm{Cone}\left(\bigcup_{i=1}^s \partial_i M \to M\right)\right).$$

**Proof** (a) In Section 2.1 we defined a chain isomorphism

$$\Phi\colon C_*^{\mathrm{str},x_0}(M)\to C_*^{\mathrm{simp}}(B\Gamma)$$

by $\Phi(\sigma)=(g_1,\dots,g_k)$, where $\sigma\in C_k^{\mathrm{str},x_0}(M)$ is a continuous map $\sigma\colon\Delta^k\to M$ with $\sigma(w_j)=x_0$ for $j=0,\dots,k$, and $g_j\in\Gamma=\pi_1(M,x_0)$ is the homotopy class (rel. vertices) of $\sigma|_{\gamma_j}$ for $j=1,\dots,k$. Moreover, we defined a chain isomorphism

$$\Psi\colon C_*^{\mathrm{simp}}(B\Gamma)\to C_*^{\mathrm{str},x_0}(M)$$

by $\Psi(g_1,\dots,g_k):=\pi(\mathrm{str}(\widetilde{x}_0,g_1\widetilde{x}_0,g_1g_2\widetilde{x}_0,\dots,g_1\dots g_k\widetilde{x}_0))$ and we proved $\Phi\Psi=\mathrm{id}$ and $\Psi\Phi=\mathrm{id}$. We will now extend $\Phi$ and $\Psi$ to chain isomorphisms

$$\Phi\colon \widehat{C}_*^{\mathrm{str},x_0}(M)\to C_*^{\mathrm{simp}}(B\Gamma^{\mathrm{comp}}),$$
$$\Psi\colon C_*^{\mathrm{simp}}(B\Gamma^{\mathrm{comp}})\to \widehat{C}_*^{\mathrm{str},x_0}(M)$$

and will prove that the extensions are inverse to each other.

Let $\sigma\in\widehat{C}_k^{\mathrm{str},x_0}(M)$ be a straight $k$–simplex which is not in $C_k^{\mathrm{str},x_0}(M)$. This means that the lift $\widetilde{\sigma}$ of $\sigma$ to

$$\widetilde{\mathrm{int}(M)}\cup\bigcup_{i=1}^{s}\Gamma c_i\subset\widetilde{\mathrm{int}(M)}\cup\partial_\infty\widetilde{\mathrm{int}(M)}$$

is of the form

$$\widetilde{\sigma}=\pi(\mathrm{str}(\gamma_0\widetilde{x}_0,\gamma_1\widetilde{x}_0,\dots,\gamma_{k-1}\widetilde{x}_0,\gamma c_i))$$

for some $i\in\{1,\dots,s\}$ and some $\gamma_0,\dots,\gamma_{k-1},\gamma\in\Gamma_i$. We define

$$\Phi(\sigma)=(\gamma_1\gamma_0^{-1},\dots,\gamma_{k-1}\gamma_{k-2}^{-1},c_i),$$

where $c_i$ is the cone point of $\mathrm{Cone}(B\Gamma_i)$.

Conversely, if a simplex $\tau\in C_*^{\mathrm{simp}}(B\Gamma^{\mathrm{comp}})$ does not belong to $C_*^{\mathrm{simp}}(B\Gamma)$ then $\tau\in\mathrm{Cone}(B\Gamma_i)$ for some $i\in\{1,\dots,s\}$, but $\tau\notin B\Gamma_i$, thus $\tau$ is of the form

$$\tau=(p_1,\dots,p_{k-1},c_i)\in C_*^{\mathrm{simp}}\Big(D\,\mathrm{Cone}\Big(\bigcup_{i=1}^{s}B\Gamma_i\to B\Gamma\Big)\Big)$$

for some $i\in\{1,\dots,s\}$, with $p_1,\dots,p_{k-1}\in\Gamma_i$ and $c_i$ the cone point of $\mathrm{Cone}(B\Gamma_i)$. Then we define

$$\Psi(\tau)=\pi(\mathrm{str}(\widetilde{x}_0,p_1\widetilde{x}_0,\dots,p_1\dots p_{k-1}\widetilde{x}_0,c_i))\in\widehat{C}_*^{\mathrm{str},x_0}(M).$$

From Section 2.1 we have $\Psi(\partial\tau)=\partial\Psi(\tau)$ for $\tau\in C_*^{\mathrm{simp}}(B\Gamma)$. On the other hand, if $\tau=(p_1,\dots,p_{k-1},c_i)\in C_*^{\mathrm{simp}}(B\Gamma^{\mathrm{comp}})$, then a straightforward computation shows

$$\Psi(\partial\tau)-\partial\Psi(\tau)=\pi(\mathrm{str}(\widetilde{x}_0,p_2\widetilde{x}_0,\dots,p_2\dots p_{k-1}\widetilde{x}_0,c_i)) \\ -\pi(\mathrm{str}(p_1\widetilde{x}_0,p_1p_2\widetilde{x}_0,\dots,p_1p_2\dots p_{k-1}\widetilde{x}_0,c_i)).$$

Thus $p_1\in\Gamma_i\subset\mathrm{Fix}(c_i)$ and $\Gamma$–invariance of $\pi(\mathrm{str}(\cdot))$ implies $\Psi(\partial\tau)=\partial\Psi(\tau)$, that is $\Psi$ is a chain map.

Clearly $\Phi(\pi(\mathrm{str}(\widetilde{x}_0,p_1\widetilde{x}_0,\dots,p_1\dots p_{k-1}\widetilde{x}_0,c_i)))=(p_1,\dots,p_{d-1},c_i)$, therefore $\Phi\Psi=\mathrm{id}$. On the other hand, a straight simplex $\sigma\colon\Delta^k\to M$ with the first $k$ vertices in $x_0$ and the last vertex in $\Gamma c_i$ is uniquely determined by the homotopy classes (rel. vertices) of $p_j=[\sigma|_{\gamma_j}]$ for $j=1,\dots,k-1$, because its lift to $\widetilde{M}$ must be in the $\Gamma$–orbit of $\mathrm{str}(\widetilde{x}_0,p_1\widetilde{x}_0,\dots,p_1\dots p_{k-1}\widetilde{x}_0,c_i)$. Thus $\Psi\Phi=\mathrm{id}$. This shows that $\Psi$ and $\Phi$ are inverse to each other, in particular both are chain isomorphisms.

(b) We define a chain homotopy $\widehat{C}_*(M)\to\widehat{C}_*^{x_0}(M)$, left-inverse to the inclusion, by induction on the dimension of simplices. First, for each $v\in C_0\,(\partial_iM)$ we fix a chain homotopy from $v$ to $x_i$ *inside* $\partial_iM$. The fixed path $l_i$ from Definition 3 provides us with a chain homotopy from $x_i$ to $x_0$. Composition of these two chain homotopies yields a chain homotopy from $v$ to $x_0\in\widehat{C}_0^{x_0}(M)$. If $v\in C_0(M)-C_0\,(\partial M)$, then we fix an arbitrary chain homotopy from $v$ to $x_0$. For the cone points fix the constant chain homotopy. Now for each 1–simplex $e$ we have a chain homotopy of its vertices into either $x_0$ or one of the cone points. This chain homotopy of $\partial e$ can be extended to a chain homotopy of $e$. If $e$ had vertices in $\partial_iM$, then we observe that the chain homotopy of the vertices consisted of two steps. In the first step the vertices were homotoped *inside* $\partial_iM$ into $x_i$. Thus $e$ can be homotoped inside $\partial_iM$ into a loop with vertices in $x_i$, which then represents an element of $\pi_1\,(\partial_iM,x_i)$. In the second step the vertices were homotoped along the $l_i$, thus $e$ can be homotoped into a loop representing an element of $\Gamma_i$ as defined in Definition 3. Thus we have a chain homotopy from $\widehat{C}_1(M)$ to $\widehat{C}_1^{x_0}(M)$. A standard argument shows that this chain homotopy can be recursively extended to the $\widehat{C}_k(M)$ for all $k\in\mathbb{N}$.

We then apply the usual straightening procedure [2, Lemma C.4.3] to construct a chain homotopy $\widehat{C}_*^{x_0}(M)\to\widehat{C}_*^{\mathrm{str},x_0}(M)$, left-inverse to the inclusion.

(c) This follows from (a) and (b). □

Thus for $d$–dimensional compact, orientable Riemannian manifold $M$ of nonpositive sectional curvature, $\mathrm{EM}_d^{-1}[M,\partial M]\in H_d^{\mathrm{simp}}\left(D\,\mathrm{Cone}\left(\bigcup_{i=1}^s B\Gamma_i\to B\Gamma\right)\right)$ is well-defined.

## 4.4 Construction of $\bar{\gamma}(M)$

The results of the previous sections allow us to consider the image of the fundamental class $[M, \partial M]$ in $H_d^{\mathrm{simp}}\left(D\,\mathrm{Cone}\left(\bigcup_{i=1}^s B\Gamma_i \to B\Gamma\right)\right)$ and its pushforward, for representations satisfying suitable assumptions, in $H_d^{\mathrm{simp}}(B\mathrm{SL}(N,\mathbb{F})^{\mathrm{fb}})$. The aim of this subsection will be to show that this element has a (uniquely defined) preimage in $H_d^{\mathrm{simp}}(B\mathrm{SL}(N,\mathbb{F}))$.

**Proposition 1** *Let $M$ be a compact, oriented, connected manifold with boundary components $\partial_1 M, \dots, \partial_s M$ such that $\mathrm{Int}(M) = \Gamma\backslash G/K$ is a locally symmetric space of noncompact type of rank one with finite volume.*

*Fix $x_0 \in M$ and $x_i \in \partial_i M$ for $i = 1, \dots, s$, and fix the isomorphisms of $\pi_1(\partial_i M, x_i)$ with subgroups $\Gamma_i$ of $\Gamma = \pi_1(M, x_0)$ given by Definition 3. Assume that, for some subring $\mathbb{F} \subset \mathbb{C}$, we have an inclusion*

$$j\colon (\Gamma, \Gamma_i) \to (G(\mathbb{F}), \Gamma_i).$$

*Let*

$$\rho\colon G(\mathbb{F}) \to \mathrm{SL}(N, \mathbb{F})$$

*be a representation. Denote by*

$$[M, \partial M] \in H_d\left(D\,\mathrm{Cone}\left(\bigcup_{i=1}^s \partial_i M \to M\right); \mathbb{Q}\right)$$

*the fundamental class of $M$. Then*

$$B(\rho j)_* \,\mathrm{EM}^{-1}[M, \partial M] \in H_d^{\mathrm{simp}}(B\mathrm{SL}(N,\mathbb{F})^{\mathrm{fb}}; \mathbb{Q})$$

*has a preimage*

$$\bar{\gamma}(M) \in H_d^{\mathrm{simp}}(B\mathrm{SL}(N,\mathbb{F}); \mathbb{Q}).$$

*The preimage does not depend on the chosen identification of $\pi_1 \partial_i M$ with a subgroup $\Gamma_i \subset \Gamma$.*

**Proof** Let $\Gamma_i' := \rho(\Gamma_i)$ for $i = 1, \dots, s$. In a first step we will prove that the desired preimage exists if

$$B(\rho j)_* \,\mathrm{EM}^{-1}[\partial_i M] = 0 \in H_{d-1}^{\mathrm{simp}}(B\Gamma_i'; \mathbb{Q})$$

for $i = 1, \dots, s$. In the second step we will then prove this equality.

**Step 1** Assume that $B(\rho j)_* \mathrm{EM}^{-1}[\partial_i M] = 0 \in H_{d-1}^{\mathrm{simp}}(B\Gamma_i'; \mathbb{Q})$ for $i = 1, \dots, s$ and consider the commutative diagram

$$\begin{array}{ccccccccc}
Z_d(M,\partial M) & \xrightarrow{c} & \widehat{Z}_d(M) & \xrightarrow{\Phi\circ \mathrm{str}} & Z_d^{\mathrm{simp}}(B\Gamma^{\mathrm{comp}}) & \xrightarrow{Bj_*} & Z_d^{\mathrm{simp}}(BG(\mathbb{F})^{\mathrm{comp}}) & \xrightarrow{B\rho_*} & Z_d^{\mathrm{simp}}(B\mathrm{SL}(N,\mathbb{F})^{\mathrm{fb}}) \\
\Big\downarrow{\scriptstyle \partial_i} & & {\scriptstyle \mathrm{Cone}}\Big\uparrow & & {\scriptstyle \mathrm{Cone}}\Big\uparrow & & {\scriptstyle \mathrm{Cone}}\Big\uparrow & & {\scriptstyle \mathrm{Cone}}\Big\uparrow \\
Z_{d-1}(\partial_i M) & \xrightarrow{=} & Z_{d-1}(\partial_i M) & \xrightarrow{\mathrm{EM}^{-1}} & Z_{d-1}^{\mathrm{simp}}(B\Gamma_i) & \xrightarrow{=} & Z_{d-1}^{\mathrm{simp}}(B\Gamma_i) & \xrightarrow{B(\rho j)_*} & Z_{d-1}^{\mathrm{simp}}(B\Gamma_i')
\end{array}$$

where $Z_d(M, \partial M) \subset C_d(M, \partial M)$ is the subgroup of relative cycles, and for a relative cycle $z$ we define $c(z) = z + \mathrm{Cone}(\partial z) \in \widehat{C}_d(M)$ and $\partial_i z$ to be the image of $\partial z \in C_{d-1}(\partial M)$ under the projection from $C_{d-1}(\partial M)$ to its direct summand $C_{d-1}(\partial_i M)$.

If $z \in C_d(M, \partial M)$ is a relative cycle that represents $[M, \partial M]$, then $\partial_i z$ represents $[\partial_i M]$. By the assumption of Step 1 we have chains $z_i' \in C_d^{\mathrm{simp}}(B\Gamma_i')$ with

$$\partial z_i' = B(\rho j)_* \mathrm{EM}^{-1}(\partial_i z),$$

then

$$B(\rho j)_* \Phi(\mathrm{str}(c(z))) - \sum_{i=1}^{s} \mathrm{Cone}(z_i') \in Z_d^{\mathrm{simp}}(B\mathrm{SL}(N, \mathbb{F})^{\mathrm{fb}})$$

is a genuine cycle in $Z_d^{\mathrm{simp}}(B\mathrm{SL}(N, \mathbb{F}))$, whose image in $Z_d(B\mathrm{SL}(N, \mathbb{F})^{\mathrm{fb}})$ again represents $B(\rho j)_* \mathrm{EM}^{-1}[M, \partial M]$. Therefore its homology class gives the desired $\overline{\gamma}(M)$.

**Step 2** It remains to prove $B(\rho j)_* \mathrm{EM}^{-1}[\partial_i M] = 0$. Let $f_i\colon \partial_i M \to M$ be the inclusion, $q\colon M \to M_+$ the projection. Thus $q f_i$ is constant. Recall that $\Gamma_i \subset G$ consists of parabolic isometries with the same fixed point in $\partial_\infty G/K$ (see [11, Theorem 3.1]), thus $\Gamma_i$ and hence $\Gamma_i' := \rho(\Gamma_i)$ are unipotent and we can apply Lemma 6 and obtain a continuous map

$$R\colon M_+ \to |B\mathrm{SL}(N, \mathbb{F})|^+$$

such that

$$R \circ q \circ f_i = \mathrm{incl} \circ |B(\rho j)| \circ h^M \circ f_i.$$

In particular, $\mathrm{incl} \circ |B(\rho j)| \circ h^{\partial_i M} = \mathrm{incl} \circ |B(\rho j)| \circ h^M \circ f_i\colon \partial_i M \to |B\mathrm{SL}(N, \mathbb{F})|^+$ is constant.

Since $\mathrm{incl} \circ |B(\rho j)|\colon |B\Gamma_i| \to |B\mathrm{SL}(N; \mathbb{F})|^+$ factors over $|B\Gamma_i'|^+$ and since

$$|B\Gamma_i'| \subset |B\mathrm{SL}(N; \mathbb{F})| \subset |B\mathrm{SL}(N; \mathbb{F})|^+$$

are inclusions (the first by $\Gamma_i' \subset \mathrm{SL}(N; \mathbb{F})$, the second by the definition of the plus construction via attaching cells to $|B\mathrm{SL}(N; \mathbb{F})|$), this implies that

$$\mathrm{incl} \circ |B(\rho j)| \circ h^{\partial_i M}\colon \partial_i M \to |B\Gamma_i'|^+$$

is constant. Since $\mathrm{incl}_*\colon H_*(|B\Gamma_i'|;\mathbb{Q}) \to H_*(|B\Gamma_i'|^+;\mathbb{Q})$ is an isomorphism,

$$|B(\rho j)|_* h_*^{\partial_i M} = 0,$$

in particular

$$|B(\rho j)|_* h_*^{\partial_i M}[\partial_i M] = 0 \in H_{d-1}(|B\Gamma_i'|;\mathbb{Q})$$

for $i = 1, \dots, s$.

But $h_*^{\partial_i M}[\partial_i M]$ is the image of $\mathrm{EM}^{-1}[\partial_i M]$ under the isomorphism $H_{d-1}^{\mathrm{simp}}(B\Gamma_i;\mathbb{Q}) \to H_{d-1}(|B\Gamma_i|;\mathbb{Q})$ (see Section 2.1), hence $|B(\rho j)|_* h_*^{\partial_i M}[\partial_i M]$ is the image of

$$B(\rho j)_* \,\mathrm{EM}^{-1}[\partial_i M]$$

under the isomorphism $H_{d-1}^{\mathrm{simp}}(B\Gamma_i';\mathbb{Q}) \to H_{d-1}(|B\Gamma_i'|;\mathbb{Q})$. Thus

$$B(\rho j)_* \,\mathrm{EM}^{-1}[\partial_i M] = 0.$$

**Welldefinedness** The construction of $\overline{\gamma}(M)$ as the homology class represented by $B(\rho j)_* \Phi(\mathrm{str}(c(z))) - \sum_{i=1}^s \mathrm{Cone}(z_i')$ involves a choice of chains $z_i' \in C_d^{\mathrm{simp}}(B\Gamma_i')$ with $\partial z_i' = B(\rho j)_* \,\mathrm{EM}^{-1}(\partial_i z)$.

If $z_i'$ and $z_i''$ are two such choices, then $\partial z_i' = \partial z_i''$ implies $z_i' - z_i'' \in Z_d^{\mathrm{simp}}(B\Gamma_i')$. Now we have $\Gamma_i = \pi_1 \partial_i M$ and $\partial_i M$ is an aspherical $(d-1)$–manifold, hence $\mathrm{cd}_{\mathbb{Q}}(\Gamma_i) = d-1$.

We claim that this implies $\mathrm{cd}_{\mathbb{Q}}(\Gamma_i') \le d-1$. By a theorem of Gruenberg one has $\mathrm{cd}_{\mathbb{Q}}(\Gamma_i) = h(\Gamma_i)$ for a finitely generated torsion-free nilpotent group $\Gamma_i$, where $h$ denotes the Hirsch length and $\mathrm{cd}_{\mathbb{Q}}$ the rational cohomological dimension. Also $\Gamma_i' = \rho(\Gamma_i)$ is nilpotent, finitely generated and obviously $h(\Gamma_i') \le h(\Gamma_i)$. Let $N \subset G$ be the maximal nilpotent (in the sense of the Iwasawa decomposition) group containing $\Gamma_i$, then $\rho(N)$ is conjugate into the group of upper triangular matrices, in particular it is torsion free. Thus $\Gamma_i'$ is a finitely generated torsion-free nilpotent group, to which Gruenberg's Theorem applies, and we obtain

$$\mathrm{cd}_{\mathbb{Q}}(\Gamma_i') = h(\Gamma_i') \le h(\Gamma_i) = d-1.$$

Hence the $d$–cycle $z_i' - z_i''$ must be 0–homologous in $C_*^{\mathrm{simp}}(B\Gamma_i')$. This implies that $\mathrm{Cone}(z_i') - \mathrm{Cone}(z_i'')$ is 0–homologous in $C_*^{\mathrm{simp}}(B\mathrm{SL}(N,\mathbb{F})^{\mathrm{fb}})$ and the homology class of $\overline{\gamma}(M)$ does not depend on the choice of $z_i'$.

Also, the construction of $\overline{\gamma}(M)$ involves a choice of a relative cycle $z \in Z_d(M, \partial M)$.

If $z$ and $z'$ are two relative cycles representing $[M, \partial M]$, then $z - z' = \partial w + u$ for some $w \in C_{d-1}(M), u \in Z_d(\partial M)$. Again $H_d(\Gamma_i';\mathbb{Q}) = 0$ implies that $B\,(\rho j)_* \,\mathrm{EM}^{-1}(u)$ is a boundary. Hence $B\,(\rho j)_*\, \Phi\,(\mathrm{str}\,(c\,(z - z')))$ is a boundary and thus the homology class $\overline{\gamma}(M)$ does not depend on the choice of $z$.

**Independence of $\pi_1\partial_i M \cong \Gamma_i$** The identification of $\pi_1\partial_i M$ with a subgroup of $\Gamma$ depends on a path $\widetilde{l_i}\colon [0,1]\to\widetilde{M}$. For two different paths one obtains subgroups which are conjugate in $\Gamma$. Conjugation in $\Gamma$ induces the identity homomorphism in group homology, thus the image of $H_*(B\Gamma_i)$ in $H_*(B\Gamma)$ and the image of $H_*(B\Gamma_i')$ in $H_*(B\mathrm{SL}(N,\mathbb{F})^{\mathrm{fb}})$ do not depend on the chosen identification. In particular $\overline{\gamma}(M)$ does not depend on this identification. □

Proposition 1 was proved in [15, Theorem 2.12] for the special case of hyperbolic manifolds and half-spinor representations. The proof in [15] uses very special properties of the half-spinor representations and seems not to generalize to other representations.

## 4.5 Evaluation of Borel classes

In Theorem 2 we proved for closed manifolds the equality $\langle b_{2n-1},\gamma(M)\rangle = c_\rho \operatorname{vol}(M)$. The following theorem will prove the analogous result for the cusped case.

**Theorem 3** (a) *Let $M$ be a compact, oriented, connected $(2n-1)$–manifold with boundary components $\partial_1 M,\dots,\partial_s M$ such that $\mathrm{Int}(M)$ is a locally symmetric space of noncompact type $\mathrm{Int}(M)=\Gamma\backslash G/K$ of rank one with finite volume. Let $\rho\colon (G,K)\to(\mathrm{SL}(N,\mathbb{C}),\mathrm{SU}(N))$ be a representation and let $c_\rho$ be defined by Theorem 2. Let*

$$\overline{\gamma}(M)\in H_{2n-1}(B\mathrm{SL}(N,\overline{\mathbb{Q}}),\mathbb{Q})$$

*be defined by Proposition 1, denote the image of $\overline{\gamma}(M)$ in $H_{2n-1}(B\mathrm{GL}(\overline{\mathbb{Q}}),\mathbb{Q})$ by $\overline{\overline{\gamma}}(M)$, and define*

$$\gamma(M) := \mathrm{pr}_{2n-1}(\overline{\overline{\gamma}}(M)) \in PH_{2n-1}(B\mathrm{GL}(\overline{\mathbb{Q}}),\mathbb{Q}) \cong K_{2n-1}(\overline{\mathbb{Q}})\otimes\mathbb{Q},$$

*where $\mathrm{pr}_{2n-1}$ is defined in Corollary 2. Then*

$$\langle b_{2n-1},\gamma(M)\rangle = c_\rho \operatorname{vol}(M).$$

(b) *If $\Gamma\subset G(A)$ for a subring $A\subset\mathbb{C}$ that satisfies the assumption of Lemma 2 and if $\rho$ maps $G(A)$ to $\mathrm{SL}(N,A)$* [5]*, and if*

$$\gamma(M) := \mathrm{pr}_{2n-1}(\overline{\overline{\gamma}}(M)) \in PH_{2n-1}(B\mathrm{GL}(A),\mathbb{Q}) \cong K_{2n-1}(A)\otimes\mathbb{Q},$$

*where $\overline{\overline{\gamma}}(M)$ is the image of $\overline{\gamma}(M)\in H_{2n-1}(B\mathrm{SL}(N,A),\mathbb{Q})$ (defined by Proposition 1) in $H_{2n-1}(B\mathrm{GL}(A),\mathbb{Q})$, and $\mathrm{pr}_{2n-1}$ is given by Lemma 2, then*

$$\langle b_{2n-1},\gamma(M)\rangle = c_\rho \operatorname{vol}(M).$$

[5] For a semisimple Lie group $G$, each representation $\rho\colon G\to\mathrm{SL}(N,\mathbb{C})$ is isomorphic to a representation which maps $G(A)$ to $\mathrm{SL}(N,A)$. (This can be read off the classification of representations of semisimple Lie groups; see [14].)

**Proof** Denote $d = 2n-1$.

The group $G$ is a linear semisimple Lie group without compact factors, not locally isomorphic to $\mathrm{SL}(2,\mathbb{R})$. By Weil rigidity we can assume (upon conjugation) that $\Gamma \subset G(\overline{\mathbb{Q}})$. By Corollary 2, $A = \overline{\mathbb{Q}}$ satisfies the assumptions of Lemma 2. Thus (a) is a consequence of (b). We are going to prove (b).

Let $z \in C_d(M, \partial M)$ represent $[M, \partial M]$. Then $\partial z \in C_{d-1}(\partial M)$ and

$$z + \mathrm{Cone}(\partial z) \in \widehat{C}_d(M) \subset C_d\left(D\,\mathrm{Cone}\left(\bigcup_{i=1}^{s} \partial_i M \to M\right)\right)$$

represents the fundamental class.

From Corollary 8 we get a homeomorphism

$$D\,\mathrm{Cone}\left(\bigcup_{l=1}^{s} \partial_i M \to M\right) \cong \Gamma\backslash G/K \cup \{c_1, \dots, c_s\},$$

where $c_l$ corresponds to the cone point of $\mathrm{Cone}\,(\partial_l M)$ for $l = 1, \dots, s$. Thus we can define $\mathrm{algvol}\,(\sigma) = \mathrm{int}_\sigma\, d\mathrm{vol}$ for $\sigma \in C_*\left(D\,\mathrm{Cone}\left(\bigcup_{i=1}^{s} \partial_i M \to M\right)\right)$, where $d\mathrm{vol}$ is the volume form for the locally symmetric metric and the cusps $c_l$ are declared to have measure zero.

By Stokes' Theorem, evaluation of the volume form on $z + \mathrm{Cone}\,(\partial z)$ does not depend on the chosen representative $z$ of $[M, \partial M]$. In particular we can, by Whitehead's Theorem, assume that $z$ is given by a triangulation of $(M, \partial M)$. Then $z + \mathrm{Cone}\,(\partial z)$ is an ideal triangulation of $M$ and evaluation of the volume form gives the sum of the signed volumes of simplices in that triangulation, that is $\mathrm{vol}(M)$. This shows that

$$\mathrm{algvol}\,(z + \mathrm{Cone}\,(\partial z)) = \mathrm{vol}(M).$$

Let $x_0, x_i, \Gamma, \Gamma_i$ be defined according to Definition 3. Let

$$\mathrm{str}\colon \widehat{C}_*(M) \to \widehat{C}_*^{\mathrm{str},x_0}(M)$$

be the chain homotopy inverse of the inclusion given by part (b) of Lemma 8.

Then $\mathrm{str}(z + \mathrm{Cone}(\partial z))$ is homologous to $z + \mathrm{Cone}(\partial z)$, thus Stokes' Theorem implies

$$\mathrm{algvol}(\mathrm{str}(z + \mathrm{Cone}(\partial z))) = \mathrm{algvol}(z + \mathrm{Cone}(\partial z)) = \mathrm{vol}(M).$$

Let

$$z + \mathrm{Cone}(\partial z) = \sum_{i=1}^{r} a_i \tau_i + \sum_{j=1}^{p} b_j \kappa_j$$

with $\tau_i \in C_*(M)$ and $\kappa_j \in \bigcup_{l=1}^{s} \mathrm{Cone}(C_*(\partial_l M))$ for $i = 1, \dots, r$, $j = 1, \dots, p$.

Let $w_0, \dots, w_d$ be the vertices of the standard simplex $\Delta^d$. By the proof of Lemma 8, the isomorphism

$$\Phi\colon \widehat{C}_*^{\mathrm{str},x_0}(M) \to C_*^{\mathrm{simp}}(B\Gamma^{\mathrm{comp}})$$

maps the interior simplex $\mathrm{str}(\tau_i)$ to

$$(\gamma_1^i, \dots, \gamma_d^i) \in B\Gamma,$$

where $\gamma_k^i \in \Gamma$ is the homotopy class of the (closed) edge from $\tau_i(w_{k-1})$ to $\tau_i(w_k)$, and the ideal simplex $\mathrm{str}(\kappa_j)$ to

$$(p_1^j, \dots, p_{d-1}^j, c_{l_j}) \in \mathrm{Cone}(B\Gamma_{l_j} \to B\Gamma),$$

where $\kappa_j \in \mathrm{Cone}(C_*(\partial_{l_j} M))$ and $c_{l_j} \in \partial_\infty G/K$ is the cusp associated to $\Gamma_{l_j}$ (cf the remark after Definition 3) and $p_k^j$ is the homotopy class of the (closed) edge from $\kappa_j(w_{k-1})$ to $\kappa_j(w_k)$. Thus, in the setting of Proposition 1, we have that

$$(Bj)_*\,\mathrm{EM}^{-1}[M, \partial M] \in H_d(BG(A)^{\mathrm{comp}}; \mathbb{Q})$$

is represented by

$$\sum_{i=1}^{r} (\gamma_1^i, \dots, \gamma_d^i) + \sum_{j=1}^{p} (p_1^j, \dots, p_{d-1}^j, c_{l_j}).$$

Let $\mathrm{str}(\tilde{x}, \gamma_1^i \tilde{x}, \dots, \gamma_1^i \cdots \gamma_d^i \tilde{x})$ be the unique straight simplex with vertices $\tilde{x}, \gamma_1^i \tilde{x}, \dots, \gamma_1^i \cdots \gamma_d^i \tilde{x}$, and $\mathrm{str}(\tilde{x}, p_1^j \tilde{x}, \dots, p_1^j \cdots p_{d-1}^j \tilde{x}, c_{l_j})$ the unique ideal straight simplex with interior vertices $\tilde{x}, p_1^j \tilde{x}, \dots, p_1^j \cdots p_{d-1}^j \tilde{x}$ and ideal vertex $c_{l_j}$.

By construction we have

$$\bar{\pi}(\mathrm{str}(\tilde{x}, \gamma_1^i \tilde{x}, \dots, \gamma_1^i \cdots \gamma_d^i \tilde{x})) = \mathrm{str}(\tau_i),$$
$$\bar{\pi}(\mathrm{str}(\tilde{x}, p_1^j \tilde{x}, \dots, p_1^j \cdots p_{d-1}^j \tilde{x}, c_{l_j})) = \mathrm{str}(\kappa_j).$$

Hence

$$\begin{aligned}
\mathrm{int}_{\mathrm{str}(\tilde{x}, \gamma_1^i \tilde{x}, \dots, \gamma_1^i \cdots \gamma_d^i \tilde{x})}\, d\mathrm{vol}_{G/K} &= \mathrm{int}_{\mathrm{str}(\tilde{x}, \gamma_1^i \tilde{x}, \dots, \gamma_1^i \cdots \gamma_d^i \tilde{x})}\, \bar{\pi}^*\, d\mathrm{vol}_M \\
&= \mathrm{int}_{\mathrm{str}(\tau_i)}\, d\mathrm{vol}_M = \mathrm{algvol}(\mathrm{str}(\tau_i)), \\
\mathrm{int}_{\mathrm{str}(\tilde{x}, p_1^j \tilde{x}, \dots, p_1^j \cdots p_{d-1}^j \tilde{x}, c_{l_j})}\, d\mathrm{vol}_{G/K} &= \mathrm{int}_{\mathrm{str}(\tilde{x}, p_1^j \tilde{x}, \dots, p_1^j \cdots p_{d-1}^j \tilde{x}, c_{l_j})}\, \bar{\pi}^*\, d\mathrm{vol}_M \\
&= \mathrm{int}_{\mathrm{str}(\kappa_j)}\, d\mathrm{vol}_M = \mathrm{algvol}(\mathrm{str}(\kappa_j)).
\end{aligned}$$

By the construction of the volume cocycle $\overline{cv}_d$ in Section 4.2.3 this implies

$$\overline{cv}_d\left(\sum_{i=1}^{r} a_i(1,\gamma_1^i,\dots,\gamma_d^i)+\sum_{j=1}^{p} b_j(1,p_1^j,\dots,p_{d-1}^j,c_{l_j})\right)$$
$$=\sum_{i=1}^{r} a_i\,\mathrm{algvol}(\mathrm{str}(\tau_i))+\sum_{j=1}^{p} b_j\,\mathrm{algvol}(\mathrm{str}(\kappa_j))=\mathrm{algvol}(z+\mathrm{Cone}(\partial z))=\mathrm{vol}(M).$$

By Lemma 7(b) and Definition 6, $B(\rho j)_*\,\mathrm{EM}^{-1}[M,\partial M]\in H_*^{\mathrm{simp}}(B\mathrm{SL}(N,A)^{\mathrm{fb}};\mathbb{Q})$.

By Lemma 7, there is $\overline{c\beta}_d\colon C_d^{\mathrm{simp}}(B\mathrm{SL}(N,\mathbb{C})^{\mathrm{fb}};\mathbb{R})\to\mathbb{R}$ such that $\overline{c\beta}_d$ restricted to $C_d^{\mathrm{simp}}(B\mathrm{SL}(N,\mathbb{C});\mathbb{R})$ represents $\mathrm{comp}(b_d)$ and $\rho^*\overline{c\beta}_d$ represents $c_\rho\overline{cv}_d$. (In particular, $\overline{c\beta}_d$ is well-defined on $(B\rho)_*H_d^{\mathrm{simp}}(BG(A)^{\mathrm{comp}};\mathbb{Q})$.) Then we have

$$\begin{aligned}
[\overline{c\beta}_d](B(\rho j)_*\,\mathrm{EM}^{-1}[M,\partial M])&\\
&=\rho^*[\overline{c\beta}_d]((Bj)_*\,\mathrm{EM}^{-1}[M,\partial M])\\
&=c_\rho\overline{cv}_d(Bj_*\,\mathrm{EM}^{-1}[M,\partial M])\\
&=c_\rho\overline{cv}_d\left(\sum_{i=1}^{r}(1,\gamma_1^i,\dots,\gamma_d^i)+\sum_{j=1}^{p}(1,p_1^j,\dots,p_{d-1}^j,c^j)\right)\\
&=c_\rho\,\mathrm{vol}(M).
\end{aligned}$$

Let $i\colon B\mathrm{SL}(N,A)\to B\mathrm{SL}(N,A)^{\mathrm{fb}}$ be the inclusion, then Proposition 1 gives

$$i_*\overline{\gamma}(M)=B\,(\rho j)_*\,\mathrm{EM}^{-1}\,[M,\partial M].$$

Applying Lemma 7(a) to $C_d^{\mathrm{simp}}\,(B\mathrm{SL}(N,A);\mathbb{R})\subset C_d^{\mathrm{simp}}\,(B\mathrm{SL}(N,\mathbb{C});\mathbb{R})$ we obtain

$$i^*\overline{c\beta}_d=\mathrm{comp}\,(b_d)\,.$$

Thus, confusing $\overline{\gamma}(M)$ with its image in $H_d(B\mathrm{SL}(N,\mathbb{C});\mathbb{R}))$ we have

$$\begin{aligned}
\langle b_d,\overline{\gamma}(M)\rangle&=\mathrm{comp}(b_d)(\overline{\gamma}(M))\\
&=[i^*\overline{c\beta}_d](\overline{\gamma}(M))=[\overline{c\beta}_d](i_*\overline{\gamma}(M))\\
&=[\overline{c\beta}_d](B(\rho j)_*\,\mathrm{EM}^{-1}[M,\partial M])=c_\rho\,\mathrm{vol}(M).
\end{aligned}$$

By Lemma 2 this implies $\langle b_d,\gamma(M)\rangle=c_\rho\,\mathrm{vol}(M)$. □

**Examples** Cusped hyperbolic 3–manifolds were discussed to some extent in [24].

If $M$ is any hyperbolic 3–manifold of finite volume, then $\pi_1 M$ can be conjugated to a subgroup of $\mathrm{SL}\,(2,\mathbb{F})$, where $\mathbb{F}$ is an at most quadratic extension of the trace field [19], thus one gets an element in $K_3(\mathbb{F}) \otimes \mathbb{Q}$. In [24, Section 9] some examples of this construction are given. (The discussion in [24] is about elements $\beta(M) \in B(\mathbb{F}) \otimes \mathbb{Q}$ for the Bloch group $B(\mathbb{F})$ but of course, using Suslin's isomorphism $B(\mathbb{F}) \otimes \mathbb{Q} \cong K_3^{\mathrm{ind}}(\mathbb{F}) \otimes \mathbb{Q}$ from [27] and the isomorphism $K_3^{\mathrm{ind}}(\mathbb{F}) \otimes \mathbb{Q} \cong K_3(\mathbb{F}) \otimes \mathbb{Q}$ for number fields, this construction yields elements in $K_3(\mathbb{F}) \otimes \mathbb{Q}$ associated to the respective manifolds and it can actually be shown that $\gamma(M)$ corresponds to $\beta(M)$ under this isomorphism.)

For example (see [24, Section 9.4]) for any number field $\mathbb{F}$ with just one complex place there exists a hyperbolic 3–manifold of finite volume, such that its invariant trace field equals $\mathbb{F}$. The associated $\gamma(M)$ gives a nontrivial element, and actually *a generator*, in $K_3(\mathbb{F}) \otimes \mathbb{Q}$.

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