Lusternik–Schnirelmann category and the connectivity of X

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We define and study a homotopy invariant called the connectivity weight to compute the weighted length between spaces X and Y. This is an invariant based on the connectivity of A_i, where A_i is a space attached in a mapping cone sequence from X to Y. We use the Lusternik–Schnirelmann category to prove a theorem concerning the connectivity of all spaces attached in any decomposition from X to Y. This theorem is used to prove that for any positive rational number q, there is a space X such that $q = \mathrm{cl}^{\omega}(X)$, the connectivity weighted cone-length of X. We compute $\mathrm{cl}^{\omega}(X)$ and $\mathrm{kl}^{\omega}(X)$ for many spaces and give several examples.

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1 Introduction

In [9], we introduced a *weighted length* between spaces which generalized the notion of the cone-length. Let X and Y be well-pointed CW complexes and $\mathcal{A}$ a collection of spaces. Then we may consider the smallest integer n such that

$$X \equiv X_0 \xrightarrow{j_0} X_1 \xrightarrow{j_2} \cdots \xrightarrow{j_{n-1}} X_n \equiv Y$$

where each j_i is part of a mapping cone sequence

$$A_i \longrightarrow X_i \xrightarrow{j_i} X_{i+1}$$

with $A_i \in \mathcal{A}$. Furthermore, we assign a *weight* $\omega(A)$ to each $A \in \mathcal{A}$ to obtain a weighted length between X and Y (see Section 2.1). The idea of a weight is to measure the complexity of a space so that $\omega(A)$ should be larger for "more complicated" spaces and smaller for "less complicated" spaces.

What ω should be chosen? Recall that a CW complex A is contractible if and only if $\pi_i(A) = 0$ for all i. Hence A is "further from being contractible" when A has smaller connectivity and A is "closer to being contractible" when it has larger connectivity. Thus we choose $\omega(A) = \omega_C(A) = 1/(1 + \mathrm{conn}(A))$ where $\mathrm{conn}(A)$ denotes the connectivity of A. An important invariant that we use to study ω_C is the Lusternik–Schnirelmann (LS) category. There is a wide variety of research in this area; see Cornea,

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Lupton, Oprea and Tanré [3], Oprea and Strom [6] and Stanley and Rodríguez Ordóñez [11]. Let X^n be a space with the homotopy type of the n–skeleton of X and define $\mathrm{cat}(X^n)$ to be the category of X^n in X (see Definition 2 and Proposition 3.) The *categorical sequence* of a CW complex X is the sequence $\sigma_X \colon \mathbb{N} \to \mathbb{N} \cup \{\infty\}$ defined by $\sigma_X(k) = \inf\{n \mid \mathrm{cat}_X(X^n) \geq k\}$. For $\omega_C(A) = 1/(1 + \mathrm{conn}(A))$, we are able to utilize categorical sequences to compute the weighted cone length (see Definition 1) for many spaces. This is seen in the following Corollary.

Corollary 12 *Let X be a space with $\mathrm{cat}(X) = n$ and let $\sigma_X = (m_1, m_2, m_3, \ldots, m_n)$. If $m_1 > 1$, then*

$$\sum_{k=1}^{n} \frac{1}{m_k - 1} \leq \mathrm{cl}^{\omega}(X).$$

If $m_1 = 1$, then

$$2 + \sum_{k=2}^{n} \frac{1}{m_k - 1} \leq \mathrm{cl}^{\omega}(X).$$

We use this Corollary to compute the weighted cone length of a finite product of spheres in Example 13. Finally we use Egyptian fractions in Lemma 14 to show that given a positive rational number q, one can choose a finite product of spheres whose ω_C–weighted cone length sums to q. This yields our main result.

Theorem 15 *Let $a \geq 1$ be an integer and $q \in \mathbb{Q}^{\geq 0}$ a rational number such that $q \geq \frac{1}{a}$. Then there exists a space $X(q)$ with $\mathrm{conn}(X(q)) = a$ and $\mathrm{cl}^{\omega}(X(q)) = q$.*

In addition, we devote Section 4 to computing $\mathrm{kl}^{\omega}(X)$, the weighted killing length of X (see Definition 1), for all X with abelian fundamental group, and we give several examples and computations throughout Section 5. In particular, we compute the weighted cone length of a sphere, real and complex projective spaces and $\mathrm{Sp}(3)$.

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2 Preliminaries

In this section we establish the basic notation and concepts that will be used in the paper. We use $*$ to denote a contractible space.

2.1 Weighted length

We recall the definitions introduced in [9]. Let $\mathcal{A}$ be any collection of spaces. A *weight function* $\omega\colon \mathcal{A} \to \mathbb{R}^{\geq 0}$ is any function such that

(a) $\omega(*) = 0$.

(b) $\omega(A_1 \vee A_2) \leq \omega(A_1) + \omega(A_2)$ for all spaces A_1, A_2.

(c) $\omega(A_1) = \omega(A_2)$ whenever $A_1 \equiv A_2$.

In addition, if ω satisfies $\omega(\Sigma A) \leq \omega(A)$ for all spaces A, we say that ω is a Σ*–weight function*. If $\omega(A) \leq C$ for some constant C, then we say that ω is a *bounded* weight function. Let $f\colon X \to Y$. If f is a homotopy equivalence, set $\ell^{\omega}(f) = 0$. Otherwise, an $\mathcal{A}$*–decomposition of* f *of stepsize* $m < \infty$ is a homotopy commutative diagram D

$$\begin{tikzcd}
A_0 \arrow[d] & A_1 \arrow[d] & & A_{m-1} \arrow[d] & \\
X_0 \arrow[r] \arrow[d, equal] & X_1 \arrow[r] & \cdots \arrow[r] & X_{m-1} \arrow[r] & X_m \arrow[d, equal] \\
X \arrow[rrrr, "f"] & & & & Y
\end{tikzcd}$$

where each $A_i \longrightarrow X_i \longrightarrow X_{i+1}$ is a mapping cone sequence with $A_i \in \mathcal{A}$. Set $\ell^{\omega}(f) = \sum_{i=0}^{m-1} \omega(A_i)$. The ω*–length of* f is the number $\widetilde{\ell}^{\omega}(f) = \inf_D \{\ell^{\omega}_D(f)\}$ where the inf is taken over all such decompositions D of finite stepsize. If no such diagram D exists, we say that $\widetilde{\ell}^{\omega}(f) = \infty$. The weighted length is then defined as follows:

Definition 1 Let X and Y be spaces and ω a weight function. Define $\ell^{\omega}(X, Y) = \inf_f \{\widetilde{\ell}^{\omega}(f)\}$. We define the *$\omega$–weighted killing length* by $\mathrm{kl}^{\omega}(X) = \ell^{\omega}(X, *)$ and *ω–weighted cone length* by $\mathrm{cl}^{\omega}(X) = \ell^{\omega}(*, X)$.

When ω is a bounded weight function, there is an alternative characterization of $\widetilde{\ell}^{\omega}(f)$. We say that (i, j) is a *homotopy equivalence* from f to f' (and (r, s) is a homotopy equivalence from f' to f) if there is a homotopy commutative diagram

$$\begin{tikzcd}
X \arrow[r, "i"] \arrow[d, "f"] & X' \arrow[r, "r"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
Y \arrow[r, "j"] & Y' \arrow[r, "s"] & Y,
\end{tikzcd}$$

where $ri \simeq \mathrm{id}, sj \simeq \mathrm{id}, ir \simeq \mathrm{id}$ and $js \simeq \mathrm{id}$ and write $f \equiv f'$.

Now let L^ω be a function such that for every $f\colon X \to Y$, $L^\omega(f) \in [0,\infty]$ satisfies

(a) $L^\omega(f) = 0$ whenever f is a homotopy equivalence.

(b) If $A \longrightarrow X \xrightarrow{f} Y$ is a mapping cone sequence, then $L^\omega(f) \le \omega(A)$.

(c) $L^\omega(fg) \le L^\omega(f) + L^\omega(g)$.

(d) If $f \equiv g$, then $L^\omega(f) = L^\omega(g)$.

Define $\mathcal{L}^\omega(f) = \sup\{L^\omega(f) \mid L^\omega \text{ satisfies the above properties }\}$. It was shown in [9] that if ω is a bounded weight function, then $\widetilde{\ell}^\omega(f) = \mathcal{L}^\omega(f)$.

2.2 Lusternik–Schnirelmann category

Definition 2 The *Lusternik–Schnirelmann category* of a map $f\colon X \to Y$ is the least integer k for which X has a cover by open sets

$$X = X_0 \cup X_1 \cup \cdots \cup X_k$$

such that $f|_{X_i} \simeq *$ for each i. When $f = \mathrm{id}_X$, we write $\mathrm{cat}(X) = \mathrm{cat}(\mathrm{id}_X)$ and when $i\colon A \hookrightarrow X$ is the inclusion, we write $\mathrm{cat}_X(A) = \mathrm{cat}(i)$. In light of Proposition 3, when A has the homotopy type of the n–skeleton $X^n \subseteq X$, we write $\mathrm{cat}_X(X^n) = \mathrm{cat}(X^n)$ since X is clear from the context.

Proposition 3 (Nendorf–Scoville–Strom [5]) *Let $n > \mathrm{conn}(X)$ (see Definition 5) be a fixed integer. Then $\mathrm{cat}(X^n)$ depends only on the homotopy type of X, and not on the choice of n–skeleton.*

We recall the notion of *categorical sequences*, first introduced and studied in [5].

Definition 4 The *categorical sequence* of a CW complex X is the sequence $\sigma_X\colon \mathbb{N} \to \mathbb{N} \cup \{\infty\}$ defined by

$$\sigma_X(k) = \inf\{n \mid \mathrm{cat}_X(X^n) \ge k\}.$$

This is well-defined by Proposition 3.

The idea behind a categorical sequence of a space X is simply to keep track of the dimensions in which the category increases by 1. For example, let $X = \mathbb{C}\mathrm{P}^n$. Then $\sigma_X = (0, 2, 4, 6, \ldots, 2n-2, 2n, \infty, \infty, \ldots)$. For notational simplicity, we will suppress the infinities unless it is of relevance.

2.3 Connectivity

It is well known that a CW complex A is contractible if and only if $\pi_i(A) = 0$ for all i. This leads to the idea that we can measure the complexity of A by considering the dimension of its first nontrivial homotopy group.

Definition 5 For a CW complex A, we define the *connectivity* of A, denoted $\mathrm{conn}(A)$, to be the largest integer n (or ∞) such that $\pi_i(A) = 0$ for $i < n+1$. If A is not path-connected, we say that $\mathrm{conn}(A) = -1$.

We will view $\mathrm{conn}(A)$ as one less than the dimension of the first reduced homology group. This follows from the Hurewicz Theorem; see Arkowitz [2, page 219].

We now define the connectivity weight, the main focus of this paper.

Definition 6 Let X, Y be path-connected CW complexes, and $\mathcal{A}$ the collection of all CW complexes with abelian fundamental group. Define

$$\omega_C(A) = \begin{cases} 0 & \text{if } A \equiv *, \\ 2 & \text{if } A \text{ is not path-connected}, \\ 1/(\mathrm{conn}(A)+1) & \text{otherwise.} \end{cases}$$

We say that ω_C is the *connectivity weight* and that $\ell^{\omega_C}(X, Y)$ is the *connectivity weighted length between X and Y*. Throughout the rest of this paper, let $\omega = \omega_C$.

Remark 7 A remark concerning our choice to define $\omega_C(A) = 2$ for A non-path-connected is in order. Let A_i be a space with $\mathrm{conn}(A_i) = i$, and write $\omega_C(A_{-1}) = \frac{1}{x}$. Since $\omega_C(A_i) > \omega(A_j)$ whenever $i < j$, it should be the case that $\omega_C(A_{-1}) > \omega_C(A_j)$ for all $j \neq -1$. Now $\dots, \omega_C(A_2), \omega_C(A_1), \omega_C(A_0), \omega_C(A_{-1}) = \dots, \frac{1}{3}, \frac{1}{2}, \frac{1}{1}, \frac{1}{x}$, and a choice of $x = \frac{1}{2}$ provides a nice symmetry in the sequence. Since $1/(1/2) = 2$, we choose $\omega_C(A) = 2$ for X non-path-connected. Furthermore, while we will allow *attachments* of spaces which are not necessarily path-connected, we will *not* consider the lengths between non-path-connected spaces. Hence, it is always assumed that when we consider $\ell^{\omega}(X, Y)$, both X and Y are path-connected, but the A_i which we attach are not necessarily path-connected. Again, each A_i has abelian fundamental group.

The following Proposition is easily verified.

Proposition 8 *The function ω_C is a bounded Σ–weight function.*

3 Connectivity weight

This section is devoted to proving our main results. We first state a technical lemma which is needed to ensure that given a mapping cone sequence of CW complexes, we may pass to a mapping cone sequence on the skeleta. Let $A \to B$ be a map of CW complexes and replace it with a cellular map. Then the cofiber C inherits a natural CW structure.

Lemma 9 *With the above setup, $A^{n-1} \to B^n \to C^n$ is a cofiber sequence.*

Proof See Stanley [10, Lemma 7.3]. □

The decompositions below will be helpful in following the proofs of Lemma 10 and Theorem 11. Let

$$\begin{tikzcd} A_0 \arrow[d] & A_1 \arrow[d] & & A_{n-1} \arrow[d] & \\ Z \equiv X_0 \arrow[r] & X_1 \arrow[r] & \cdots \arrow[r] & X_{n-1} \arrow[r] & X_n \equiv X \end{tikzcd}$$

be any ω–decomposition of Z into X. We keep track of the m–skeleta in the above diagram by considering the following diagram:

$$\begin{tikzcd} (A_0)^{m-1} \arrow[d] & (A_1)^{m-1} \arrow[d] & & (A_{n-1})^{m-1} \arrow[d] & \\ (X_0)^m \arrow[r] & (X_1)^m \arrow[r] & \cdots \arrow[r] & (X_{n-1})^m \arrow[r] & (X_n)^m \end{tikzcd} \tag{1}$$

By Lemma 9, each sequence $(A_i)^{m-1} \to (X_i)^m \to (X_{i+1})^m$ is also a mapping cone sequence, $0 \le i \le n-1$.

Lemma 10 *Let X and Z be spaces and let m be the first dimension such that $\mathrm{cat}(X^m) - \mathrm{cat}(Z^m) = 1$. Then there exists an attachment of a space with connectivity at most $m-2$ in any ω–decomposition of Z into X.*

Proof Suppose that $\mathrm{cat}(X^m) - \mathrm{cat}(Z^m) = 1$ for the first time in dimension m. If $(A_i)^{m-1} = *$ for all i in (1), then $X^m \equiv Z^m$, which is impossible since X and Z have different categories in dimension m. Hence, there must be at least one $(A_i)^{m-1} \neq *$ which implies that $\mathrm{conn}(A_i)$ is at most $m-2$ for some space A_i. □

We translate the preceding Lemma into the language of the connectivity weight to obtain the following Theorem.

Theorem 11 *Let X and Z be spaces with $m_1 \le m_2 \le \cdots \le m_N < \infty$ the first dimension of X such that $\mathrm{cat}(X^{m_i}) - \mathrm{cat}(Z^{m_i}) = i > 0$ for $1 \le i \le N$. If $\mathrm{cat}(X^1) - \mathrm{cat}(Z^1) = 1$, then*

$$2 + \sum_{i=2}^{N} \frac{1}{m_i - 1} \le \ell^{\omega}(Z, X).$$

Otherwise,

$$\sum_{i=1}^{N} \frac{1}{m_i - 1} \le \ell^{\omega}(Z, X).$$

Proof Let D be any ω–decomposition of Z into X. We will apply Lemma 10 for each value of i, $1 \le i \le N$, to obtain a lower bound.

Consider the first case where $\mathrm{cat}(X^1) - \mathrm{cat}(Z^1) = 1 = m_1$. For $i = 1$, by Lemma 10 there is 1 attachment in D with connectivity at most $1-2=-1$ ie there is an attachment of a non-path-connected space, say A_{j_0}. By definition of ω_C, this attachment contributes a value of $\omega(A_{j_0}) = 2$ to the lower bound estimate for $\ell^{\omega}(Z, X)$. If m_2 does not exist (and since category can increase by at most 1 per attachment, consequently $m_3, m_4, \dots$ also do not exist), we finish with an estimate of $2 \le \ell^{\omega}(Z, X)$.

We proceed by induction on the i of m_i. If m_2 exists, it is defined as the first dimension such that $\mathrm{cat}(X^{m_2}) - \mathrm{cat}(Z^{m_2}) = 2$. Now $\mathrm{cat}(X^{m_2}) - \mathrm{cat}(X^{m_1}) = 1$, so by Lemma 10, there is an attachment in D, say A_{j_1}, with connectivity at most $m_2 - 2$. Clearly A_{j_1} must be a different attachment than A_{j_0} since otherwise this would imply that a single attachment can increase the category by 2 which is impossible. This yields the estimate $2 + 1/(m_2 - 1) \le 2 + 1/(\mathrm{conn}(A_{j_1}) + 1) = \omega(A_{j_0}) + \omega(A_{j_1}) \le \ell^{\omega}(Z, X)$. If m_3 does not exist, we are done.

Assume the inductive hypothesis that we have found $A_{j_0}, A_{j_1}, \dots, A_{j_k}$ satisfying $1/(m_i - 1) \le \omega(A_{j_{i-1}})$ for $1 \le i \le k$ so that $2 + \sum_{i=2}^{k} 1/(m_i - 1) \le \ell^{\omega}(Z, X)$. If m_{k+1} exists, m_{k+1} is by definition the first dimension such that $\mathrm{cat}(X^{m_{k+1}}) - \mathrm{cat}(Z^{m_{k+1}}) = k + 1$. Now $\mathrm{cat}(X^{m_{k+1}}) - \mathrm{cat}(X^{m_k}) = 1$ and so by Lemma 10, there are is an attachment in D, say $A_{j_{k+1}}$, such that $\mathrm{conn}(A_{j_{k+1}}) \le m_{k+1} - 2$. For the same reason as above, $A_{j_{k+1}}$ must be a different attachment than the other $A_{j_0}, A_{j_1}, \dots, A_{j_k}$. Therefore, $2 + \sum_{i=2}^{k+1} 1/(m_i - 1) \le \ell^{\omega}(Z, X)$.

We thus obtain the estimate $2 + \sum_{i=2}^{N} 1/(m_i - 1) \le \ell^{\omega}(Z, X)$. The case where $\mathrm{cat}(X^1) - \mathrm{cat}(Z^1) \ne 1$ is almost identical. □

By taking $Z = *$ in Theorem 11, we obtain the following useful lower bound for the weighted cone length of any space.

Corollary 12 *Let X be a space with $\mathrm{cat}(X) = n$ and let $\sigma_X = (m_1, m_2, m_3, \dots, m_n)$. If $m_1 > 1$, then*

$$\sum_{k=1}^{n} \frac{1}{m_k - 1} \leq \mathrm{cl}^{\omega}(X).$$

If $m_1 = 1$, then

$$2 + \sum_{k=2}^{n} \frac{1}{m_k - 1} \leq \mathrm{cl}^{\omega}(X).$$

We will use this to compute the weighted cone length of a product of spheres.

Example 13 Let $X = S^{n_1} \times S^{n_2} \times \cdots \times S^{n_k}$ with $1 \leq n_1 \leq n_2 \leq \cdots \leq n_k$. The standard cone decomposition of X is given by

$$\begin{tikzcd}
A_0 \arrow[d] & A_1 \arrow[d] & A_2 \arrow[d] & & A_{k-1} \arrow[d] & \\
* \arrow[r] & X(1) \arrow[r] & X(2) \arrow[r] & \cdots \arrow[r] & X(k-1) \arrow[r] & X(k)
\end{tikzcd}$$

where $X(i) = \{(x_1, x_2, \dots) \mid \text{at most } i \text{ entries are not } *\} \subseteq X$, and each A_i is attached via a higher order Whitehead product [7] with $\mathrm{conn}(A_i) = n_1 + n_2 + \cdots + n_{i+1} - 2$. We thus obtain the upper bound of

$$\mathrm{cl}^{\omega}(S^{n_1} \times S^{n_2} \times \cdots \times S^{n_k}) \leq \frac{1}{n_1 - 1} + \frac{1}{n_1 + n_2 - 1} + \cdots + \frac{1}{n_1 + n_2 + \cdots + n_k - 1}$$

for $n_1 \neq 1$ and

$$\mathrm{cl}^{\omega}(S^{n_1} \times S^{n_2} \times \cdots \times S^{n_k}) \leq 2 + \frac{1}{n_1 + n_2 - 1} + \cdots + \frac{1}{n_1 + n_2 + \cdots + n_k - 1}$$

for $n_1 = 1$.

We now show the lower bound. By [5, Corollary 17], $\sigma_X(r) = n_1 + n_2 + \cdots + n_r$ for $r \leq k$ and ∞ otherwise. By Corollary 12 and the upper bound, we conclude that

$$\mathrm{cl}^{\omega}(S^{n_1} \times S^{n_2} \times \cdots \times S^{n_k})$$
$$= \begin{cases} \dfrac{1}{n_1 - 1} + \dfrac{1}{n_1 + n_2 - 1} + \cdots + \dfrac{1}{n_1 + n_2 + \cdots + n_k - 1} & \text{if } n_1 \neq 1, \\[2ex] 2 + \dfrac{1}{n_1 + n_2 - 1} + \cdots + \dfrac{1}{n_1 + n_2 + \cdots + n_k - 1} & \text{if } n_1 = 1. \end{cases}$$

The last step in proving Theorem 15 is to show that any rational number can be realized as a finite sum of the above form.

Lemma 14 *Let $a \geq 1$ be an integer and r a rational number such that $r \geq \frac{1}{a}$. Then there exists a finite sequence of positive integers $a < a_2 \leq a_3 \leq \cdots \leq a_n$ such that*

$$\frac{1}{a} + \frac{1}{a+a_2} + \frac{1}{a+a_2+a_3} + \cdots + \frac{1}{a+a_2+\cdots+a_n} = r.$$

Proof It suffices to show that any positive rational r can be written as $r = 1/A_1 + 1/A_2 + \cdots + 1/A_n$ where the difference $D_i = A_{i+1} - A_i$ satisfies $A_1 < D_1 \leq D_2 \leq D_3 \leq \cdots \leq D_{n-1}$. Let k be a positive integer such that $r \geq 1/k$. Find the value j that satisfies

$$S_0 := \frac{1}{k} + \frac{1}{k+(k+1)} + \frac{1}{k+2(k+1)} + \cdots + \frac{1}{(k+1)j-1} \leq r,$$
$$r < \frac{1}{k} + \frac{1}{k+(k+1)} + \frac{1}{k+2(k+1)} + \cdots + \frac{1}{(k+1)j-1} + \frac{1}{(k+1)(j+1)-1}.$$

Consider $r - S_0 = r'$. Clearly $r' < 1/((k+1)(j+1)-1)$ and in particular, $r' < 1$. If $r' = 0$, then we are done. Otherwise, write $r' = 1/m_1 + 1/m_2 + \cdots + 1/m_t$ where each $m_{i+1} = m_i^2 - m_i + \epsilon_i$, ϵ_i a positive integer [8, Theorems 1 and 2]. Then

$$r = \frac{1}{k} + \frac{1}{k+(k+1)} + \frac{1}{k+2(k+1)} + \cdots + \frac{1}{(k+1)j-1} + \frac{1}{m_1} + \frac{1}{m_2} + \cdots + \frac{1}{m_t}$$

and $k < k+1 = D_1 = D_2 = \cdots = D_{j-1}$. It remains to show that $D_i \leq D_{i+1}$, for $j-1 \leq i \leq t-1$. We first show that $D_{j-1} \leq D_j$. Observe that $1/m_1 \leq r' < 1/((k+1)(j+1)-1)$ so that $D_j - D_{j-1} = m_1 - (k+1)(j+1) + 1 > 0$. We now show that $D_i \leq D_{i+1}$ for $j \leq i \leq t-1$. We have

$$\begin{aligned} D_{i+1} &= m_{i+1} - m_i \\ &= (m_i^2 - m_i + \epsilon_i) - m_i \\ &= m_i^2 - 2m_i + \epsilon \\ &\geq m_i^2 - 2m_i \\ &\geq m_i - 2 \\ &\geq m_i - m_{i-1} \\ &= D_i, \end{aligned}$$

which completes the proof. □

Our main result follows.

Theorem 15 *Let $a \geq 1$ be an integer and $q \in \mathbb{Q}^{\geq 0}$ such that $q \geq \frac{1}{a}$. Then there exists a space $X(q)$ with $\mathrm{conn}(X(q)) = a$ and $\mathrm{cl}^{\omega}(X(q)) = q$.*

Proof Let q and a be as above. By Lemma 14, there exists positive integers $a = n_1 < n_2 \leq \cdots \leq n_k$ such that

$$\frac{1}{n_1} + \frac{1}{n_1+n_2} + \cdots + \frac{1}{n_1+n_2+\cdots+n_k} = q.$$

Write $X = S^{n_1+1} \times S^{n_2} \times S^{n_3} \times \cdots \times S^{n_k}$. By Example 13,

$$\mathrm{cl}^{\omega}(X) = \frac{1}{n_1+1-1} + \frac{1}{n_1+1+n_2-1} + \frac{1}{n_1+1+n_2+n_3-1} + \cdots + \frac{1}{n_1+n_2+\cdots+n_k-1} = q.$$

It is clear that $\mathrm{conn}(X(q)) = a$. □

4 Killing and cone length

Lemma 16 *If $X \xrightarrow{f} Y \longrightarrow *$ is a mapping cone sequence and X and Y are simply connected CW complexes, then $X \equiv Y$.*

Proof This follows from Whitehead's first and second Theorems [2, pages 53, 220]. □

We show that $\mathrm{kl}^{\omega}(X)$ can easily be computed for all spaces X by first showing a lower bound.

Proposition 17 *Let X and Y be spaces with different homology groups in at least one dimension and $m \geq 1$ the first dimension with $H_m(X) \ncong H_m(Y)$. If $\omega = \omega_C$, then $\frac{1}{m} \leq \ell^{\omega}(X,Y)$.*

Proof Take any ω–decomposition

$$\begin{array}{ccccccccc} A_0 & & A_1 & & & & A_{n-1} & & \\ \downarrow & & \downarrow & & & & \downarrow & & \\ X \equiv X_0 & \longrightarrow & X_1 & \longrightarrow & \cdots & \longrightarrow & X_{n-1} & \longrightarrow & X_n \equiv Y \end{array}$$

of X into Y. Assume by way of contradiction that $\mathrm{conn}(A_i) > m-1$ for all $0 \leq i \leq n-1$. Consider any of the mapping cone sequences $A_j \to X_j \to X_{j+1}$ and the long exact homology sequence which it induces:

$$\cdots \longrightarrow H_m(A_j) \longrightarrow H_m(X_j) \longrightarrow H_m(X_{j+1}) \longrightarrow H_{m-1}(A_j) \longrightarrow \cdots$$

Since $\mathrm{conn}(A_j) > m-1$, we see that $H_m(X_j) \cong H_m(X_{j+1})$ for all j so that $H_m(X) \cong H_m(Y)$. Thus there is at least one A_i with $\mathrm{conn}(A_i) \leq m-1$ so that $\frac{1}{m} \leq \ell^{\omega}(X,Y)$. □

Corollary 18 *Let X and Y be spaces and $\omega = \omega_C$. If* $\mathrm{conn}(X) < \mathrm{conn}(Y)$*, then* $\omega(X) \le \ell^\omega(X, Y)$.

Proof Let $m - 1 = \mathrm{conn}(X)$. Since $\mathrm{conn}(X) < \mathrm{conn}(Y)$, m is the first dimension in which $H_m(X) \not\cong H_m(Y)$. By Proposition 17, $\frac{1}{m} = \omega(X) \le \ell^\omega(X, Y)$. □

We now compute $\mathrm{kl}^\omega(X)$ for all spaces X.

Corollary 19 *Let X be a space and $\omega_C = \omega$. Then* $\mathrm{kl}^\omega(X) = \omega(X)$*. If X is simply connected, the decomposition is $X \longrightarrow X \longrightarrow *$. Furthermore,* $\mathrm{kl}^\omega(X) \le \ell^\omega(X, Y)$ *for all spaces Y.*

Proof Clearly $\mathrm{kl}^\omega(X) \le \omega(X)$. Let $Y = *$ and apply Corollary 18 for the reverse direction. For X simply connected, the only way to obtain this is with the decomposition $X \longrightarrow X \longrightarrow *$ by Lemma 16. The last inequality follows from Corollary 18. □

Though we are not able to compute $\mathrm{cl}^\omega(X)$ for all spaces, we can compute it for many spaces. We first compute $\mathrm{cl}^\omega(X)$ whenever X is a suspension. We then give examples of classes of spaces whose weighted cone length may be computed.

Corollary 20 *Let $\omega = \omega_C$ and A a noncontractible space. If $X = \Sigma A \not\equiv *$, then* $\mathrm{cl}^\omega(X) = \omega(A)$.

Proof Observe that the diagram

$$
\begin{tikzcd}
A \arrow[d] & \\
* \arrow[r] & \Sigma A \equiv X
\end{tikzcd}
$$

shows that $\ell^\omega(*, X) \le \omega(A)$.

We apply Corollary 12. Since by definition m_1 is the first dimension in which $\mathrm{cat}(X^{m_1}) - \mathrm{cat}(*) = \mathrm{cat}(X^{m_1}) = 1$, it follows that $m_1 = \mathrm{conn}(X) + 1$. We have $\omega(A) = 1/(1 + \mathrm{conn}(A)) = 1/(m_1 - 1) \le \mathrm{cl}^\omega(X)$ by Corollary 12 which completes the proof. □

5 Computations and examples

Example 21 By Corollary 19 and Corollary 20, $\ell^\omega(*, S^n) = \frac{1}{n-1}$ and $\ell^\omega(S^n, *) = \frac{1}{n}$ for $n \ge 2$.

Example 22 The converse of Corollary 20 is not true. That is, if $\mathrm{cl}^\omega(X) = \omega(A)$ for some A, X is not necessarily a suspension. Indeed, Theorem 15 allows us to construct many such examples. We will restrict our attention to products of only two spheres. To do this, we seek positive integers a, b, c such that $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$ if and only if $(a+b)|ab$. For example, if $a = 5$ and $b = 20$, we choose $n_1 = 6$ and $n_2 = 15$ so that $\mathrm{cl}^\omega(S^6 \times S^{15}) = \frac{1}{6-1} + \frac{1}{6+15-1} = \frac{1}{4} = \omega(A)$ for all 3–connected spaces A but $S^6 \times S^{15} \not\equiv \Sigma A$ for any A.

Example 23 Let $X = \mathbb{C}\mathrm{P}^n$. As noted above, $\sigma_{\mathbb{C}\mathrm{P}^n} = (0, 2, 4, 6, \dots, 2n-2, 2n)$. By Corollary 12, $\sum_{i=1}^n (1/(2i-1)) \le \mathrm{cl}^\omega(X)$. The standard CW decomposition of $\mathbb{C}\mathrm{P}^n$

$$\begin{tikzcd} S^1 \arrow[d] & S^3 \arrow[d] & & S^{2n-1} \arrow[d] & \\ * \arrow[r] & \mathbb{C}\mathrm{P}^1 \arrow[r] & \cdots \arrow[r] & \mathbb{C}\mathrm{P}^{n-1} \arrow[r] & \mathbb{C}\mathrm{P}^n \end{tikzcd}$$

yields the estimate $\mathrm{cl}^\omega(\mathbb{C}\mathrm{P}^n) \le \sum_{i=1}^n (1/(2i-1))$ so $\mathrm{cl}^\omega(\mathbb{C}\mathrm{P}^n) = \sum_{i=1}^n (1/(2i-1))$. The exact value of the sum can be computed using the digamma function [1, 6.3.4].

Example 24 Using the same technique as in Example 23, we can compute $\mathrm{cl}^\omega(\mathbb{R}\mathrm{P}^n) = 2 + \sum_{i=1}^n (1/i)$, 2 plus the i–th partial sum of the harmonic series. In particular, this shows that cl^ω can take on arbitrarily large values.

Example 25 Let $X = \mathrm{Sp}(3)$. The following cone decomposition was explicitly shown in [4]:

$$\begin{tikzcd} S^2 \arrow[d] & C_6 \arrow[d] & C_9 \arrow[d] & S^{17} \arrow[d] & S^{20} \arrow[d] & \\ * \arrow[r] & S^3 \arrow[r] & X_2 \arrow[r] & X_3 \arrow[r] & X_4 \arrow[r] & X_5 \equiv \mathrm{Sp}(3), \end{tikzcd}$$

where $C_n = S^n \cup_{\nu_n} D^{n+4}$ (here ν_n is the generator of the 2–primary component of $\pi_{n+3}(S^n)$ [12]). This yields an upper bound. On the other hand, $\mathrm{Sp}(3)$ has categorical sequence $(3, 7, 10, 18, 21)$. By Corollary 12, we then obtain the same value as the lower bound. Thus $\mathrm{cl}^\omega(\mathrm{Sp}(3)) = \frac{1}{2} + \frac{1}{6} + \frac{1}{9} + \frac{1}{17} + \frac{1}{20} \approx .8866$.

Example 26 We find spheres whose product has ω–cone length 3.141, the first few digits of π. The following decomposition can be found using an elementary number theory computer program such as PARI:

$$3.141 = 2 + 1 + \frac{1}{8} + \frac{1}{63} + \frac{1}{7875}.$$

This yields the sequence $1, 1, 7, 56, 7813$ so we choose $X = S^1 \times S^1 \times S^7 \times S^{56} \times S^{7875}$, hence $\mathrm{cl}^{\omega}(X) = 3.141$.

6 Open questions

Question 27 In the examples we have seen, $\mathrm{cl}^{\omega}(X)$ is realized using the "standard" decomposition of X. In particular, if $\mathrm{cl}(X) = n$, the classical cone-length of X, we have found the connectivity weighted cone length of X in exactly n attachments. Is there a space X such that $\mathrm{cl}(X) = n$ but $\mathrm{cl}^{\omega}(X)$ is realized in more than n attachments?

Question 28 Theorem 11 provides a good lower bound for $\ell^{\omega}(X, Y)$ whenever $\mathrm{cat}(X^n) \le \mathrm{cat}(Y^n)$ for all n. However, this lower bound is clearly less helpful if there are integers i such that $\mathrm{cat}(X^i) > \mathrm{cat}(Y^i)$, and the theorem tells us nothing when $\mathrm{cat}(X^n) \ge \mathrm{cat}(Y^n)$ for all n. In particular, let $A \to B \to C$ be a mapping cone sequence such that $\mathrm{cat}(B) + 1 = \mathrm{cat}(C)$. Is there a good lower bound for $\ell^{\omega}(C, B)$? What about the special case of $S^n \to \mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^{n+1}$?

Question 29 Suppose that $\mathrm{cat}(X) = n$, $\dim(X) = d$, and $\mathrm{conn}(X) = c$; what can be said about $\mathrm{cl}^{\omega}(X)$?

Question 30 Is it possible to define ω so that for finite complexes, $\mathrm{cl}^{\omega}(X) = \mathrm{cl}^{\omega}(Y)$ if and only if $X \equiv Y$?

References

[1] **M Abramowitz**, **I A Stegun** (editors), *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, Dover, New York (1992) MR1225604

[2] **M Arkowitz**, *Introduction to homotopy theory*, Universitext, Springer, New York (2011) MR2814476

[3] **O Cornea**, **G Lupton**, **J Oprea**, **D Tanré**, *Lusternik–Schnirelmann category*, Math. Surveys and Monogr. 103, Amer. Math. Soc. (2003) MR1990857

[4] **L Fernández-Suárez**, **A Gómez-Tato**, **J Strom**, **D Tanré**, *The Lusternik–Schnirelmann category of* $\mathrm{Sp}(3)$, Proc. Amer. Math. Soc. 132 (2004) 587–595 MR2022385

[5] **R Nendorf**, **N Scoville**, **J Strom**, *Categorical sequences*, Algebr. Geom. Topol. 6 (2006) 809–838 MR2240916

[6] **J Oprea**, **J Strom**, *Mixing categories*, Proc. Amer. Math. Soc. 139 (2011) 3383–3392 MR2811292

[7] **G J Porter**, *Higher-order Whitehead products*, Topology 3 (1965) 123–135 MR0174054

[8] **H E Salzer**, *The approximation of numbers as sums of reciprocals*, Amer. Math. Monthly 54 (1947) 135–142 MR0020339

[9] **N A Scoville**, *Mapping cone sequences and a generalized notion of cone length*, JP J. Geom. Topol. 11 (2011) 209–233

[10] **D Stanley**, *Spaces and Lusternik–Schnirelmann category n and cone length $n+1$*, Topology 39 (2000) 985–1019 MR1763960

[11] **D Stanley**, **H Rodríguez Ordóñez**, *A minimum dimensional counterexample to Ganea's conjecture*, Topology Appl. 157 (2010) 2304–2315 MR2670507

[12] **H Toda**, *Composition methods in homotopy groups of spheres*, Annals of Math. Studies 49, Princeton Univ. Press (1962) MR0143217

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