Motivic twisted K–theory

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This paper sets out basic properties of motivic twisted K–theory with respect to degree three motivic cohomology classes of weight one. Motivic twisted K–theory is defined in terms of such motivic cohomology classes by taking pullbacks along the universal principal $\mathsf{B}\mathbf{G}_{\mathrm{m}}$–bundle for the classifying space of the multiplicative group scheme $\mathbf{G}_{\mathrm{m}}$. We show a Künneth isomorphism for homological motivic twisted K–groups computing the latter as a tensor product of K–groups over the K–theory of $\mathsf{B}\mathbf{G}_{\mathrm{m}}$. The proof employs an Adams Hopf algebroid and a trigraded Tor-spectral sequence for motivic twisted K–theory. By adapting the notion of an E_∞–ring spectrum to the motivic homotopy theoretic setting, we construct spectral sequences relating motivic (co)homology groups to twisted K–groups. It generalizes various spectral sequences computing the algebraic K–groups of schemes over fields. Moreover, we construct a Chern character between motivic twisted K–theory and twisted periodized rational motivic cohomology, and show that it is a rational isomorphism. The paper includes a discussion of some open problems.

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1 Motivation

Topological K–theory has many variants which have all been developed and exploited for geometric purposes. Twisted K–theory or "K–theory with coefficients" was introduced by Donovan and Karoubi in [11] using Wall's graded Brauer group. More general twistings of K–theory arise from automorphisms of its classifying space of Fredholm operators on an infinite dimensional separable complex Hilbert space. Of particular geometric interest are twistings given by integral 3–dimensional cohomology classes. The subject was further developed in the direction of analysis by Rosenberg in [41].

Twisted K–theory resurfaced in the late 1990's with Witten's work on classification of D-brane charges in type II string theory [54]. Fruitful interactions between algebraic topology and physics afforded by twisted K–theory continues today; see eg Oberwolfach Rep. 3, no. 4 [9], Atiyah and Segal [4; 5], Bouwknegt et al [8] and Tu, Xu

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and Laurent-Gengoux [47]. The work of Freed, Hopkins and Teleman [15] relates the twisted equivariant K–theory of a compact Lie group G to the "Verlinde ring" of its loop group. For a recent survey of twisted K–theory, refer to Karoubi [26].

We are interested in twistings of the motivic K–theory spectrum KGL in the algebro-geometric context of motivic homotopy theory; see Dundas et al [13] and Voevodsky [48]. It is not evident precisely how our homotopical approach relates to the more algebraic work of Walker [53] of twisted K–theory with respect to central simple algebras over fields. Twisting K–theory with respect to $\mathbf{G}_{\mathfrak{m}}$–gerbes in the étale topology along the lines of this paper seems to produce a theory – twisted étale K–theory – which compares more transparently with [53]. It would be interesting to have a precise comparison result between these versions of twisted K–theory. Over the complex numbers $\mathbf{C}$, or more generally any field with a complex embedding, our motivic twisted K–theory specializes to twisted K–theory under realization of complex points. The idea of twisting (co)homology theories has been used to great effect in topology. A classical example is cohomology with local coefficients, which can be used to formulate Poincaré duality and the Thom isomorphism for nonorientable manifolds. An analogous motivic version of twisted Poincaré duality is subject to future work.

Ando, Blumberg and Gepner [3] use the formalism of ∞–categories in order to construct twisted forms of multiplicative generalized (co)homology theories, and May and Sigurdsson [30] employ parametrized spectra to the same end. Their setups suggest the existence of a deep theory of "motivic twisted cohomology theories" which goes beyond the scope of this paper. Twisted K–theory is the first natural example of such a twisted cohomology theory and we set out its basic properties in this paper.

2 Main results

The isomorphism classes of principal $\mathsf{B}S^1$–bundles over a topological space X identifies canonically with the homotopy classes of maps from X to the Eilenberg–Mac Lane space $K(\mathbf{Z},3)$. The second delooping $\mathsf{BB}S^1$ of the circle gives a concrete model for $K(\mathbf{Z},3)$. We begin the paper by considering the analogous setup in motivic homotopy theory.

Let $\mathscr{X}$ be a motivic space and $\mathbf{G}_{\mathfrak{m}}$ the multiplicative group scheme over a noetherian base scheme S of finite dimension (usually implicit in the notation).

For any map $\tau\colon \mathcal{X} \to \mathsf{BB}\mathbf{G}_{\mathfrak{m}}$ define $\mathcal{X}^{\tau}$ as the homotopy pullback along $* \to \mathsf{BB}\mathbf{G}_{\mathfrak{m}}$ (which can be thought of as a universal principal $\mathsf{B}\mathbf{G}_{\mathfrak{m}}$–bundle):

$$\begin{tikzcd} \mathcal{X}^{\tau} \arrow[r] \arrow[d] & * \arrow[d] \\ \mathcal{X} \arrow[r, "\tau"] & \mathsf{BB}\mathbf{G}_{\mathfrak{m}} \end{tikzcd}$$

With this definition there is a naturally induced action

$$\mathbf{P}^{\infty} \times \mathcal{X}^{\tau} \to \mathcal{X}^{\tau}.$$

Here we use implicitly the motivic weak equivalence between $\mathsf{B}\mathbf{G}_{\mathfrak{m}}$ and the infinite projective space $\mathbf{P}^{\infty}$. By passing to motivic suspension spectra we get a naturally induced map

$$\Sigma^{\infty}\mathbf{P}^{\infty}_{+} \wedge \Sigma^{\infty}\mathcal{X}^{\tau}_{+} \to \Sigma^{\infty}\mathcal{X}^{\tau}_{+}$$

displaying $\Sigma^{\infty}\mathcal{X}^{\tau}_{+}$ as a module over the motivic ring spectrum $\Sigma^{\infty}\mathbf{P}^{\infty}_{+}$. (We defer the somewhat technical definition of this module structure to Section 3.)

When S is a smooth scheme over a field we can identify the homotopy classes of maps from $\mathcal{X}$ to $\mathsf{BB}\mathbf{G}_{\mathfrak{m}}$ with the third integral motivic cohomology group $\mathsf{M}\mathbf{Z}^{3,1}(\mathcal{X})$ of weight one. This group is in fact trivial for smooth schemes of finite type over S by Suslin and Voevodsky [46, Corollary 3.2.1]. On the other hand, it is nontrivial for motivic spheres, eg $S^{3,1}$.

Denote by $\mathsf{B}\mathbf{GL}$ the classifying space of the infinite Grassmannian over S. Suppose $f_{\zeta}\colon \mathcal{X} \to \mathbf{P}^{\infty}$ and $f_{\xi}\colon \mathcal{X} \to \mathbf{Z} \times \mathsf{B}\mathbf{GL}$ represent $\zeta \in \mathsf{M}\mathbf{Z}^{2,1}(\mathcal{X})$ and $\xi \in \mathsf{KGL}_{*}(\mathcal{X})$, respectively. Then the composite map

$$\mathcal{X} \xrightarrow{\Delta} \mathcal{X} \times \mathcal{X} \xrightarrow{f_{\zeta} \times f_{\xi}} \mathbf{P}^{\infty} \times (\mathbf{Z} \times \mathsf{B}\mathbf{GL}) \to \mathbf{Z} \times \mathsf{B}\mathbf{GL}$$

represents an element $\xi \otimes \zeta \in \mathsf{KGL}_{*}(\mathcal{X})$. The above defines the action of the Picard group on the K–theory ring of $\mathcal{X}$. On the level of motivic spectra there exists a corresponding composite map

$$\Sigma^{\infty}\mathbf{P}^{\infty}_{+} \wedge \mathsf{KGL} \to \mathsf{KGL} \wedge \mathsf{KGL} \to \mathsf{KGL},$$

where the second map is the ring multiplication on KGL. The first map is obtained via adjointness from the multiplicative map $\mathsf{B}\mathbf{G}_{\mathfrak{m}} \to \{1\} \times \mathsf{B}\mathbf{GL}$ that sends a line bundle represented by a map into $\mathbf{P}^{\infty}$ to its class in the Grothendieck group of all vector bundles by Spitzweck and Østvær [44, (3)]. Thus the motivic K–theory spectrum KGL is a module over $\Sigma^{\infty}\mathbf{P}^{\infty}_{+}$.

We are now ready to define our main objects of study in this paper. The distinction between homological and cohomological versions of motivic twisted K–theory is rooted in standard nomenclature for twisted K–theory.

Definition 2.1 For $\tau\colon \mathscr{X}\to \mathsf{BB}\mathbf{G}_{\mathfrak{m}}$ define the motivic twisted

- homological K–theory of τ by $\mathsf{KGL}^{\tau}\equiv \Sigma^{\infty}\mathscr{X}^{\tau}_{+}\wedge_{\Sigma^{\infty}\mathbf{P}^{\infty}_{+}}\mathsf{KGL}$.
- cohomological K–theory of τ by $\mathsf{KGL}_{\tau}\equiv \underline{\mathrm{Hom}}_{\Sigma^{\infty}\mathbf{P}^{\infty}_{+}}(\Sigma^{\infty}\mathscr{X}^{\tau}_{+},\mathsf{KGL})$.

The smash product in the definition of KGL^{τ} is derived in the sense that it is formed in the homotopy category of highly structured modules over $\Sigma^{\infty}\mathbf{P}^{\infty}_{+}$. In order to make sense of the derived smash product, we implicitly use a closed symmetric monoidal model for the motivic stable homotopy category; see Jardine's work [25] on motivic symmetric spectra, for example. The internal hom in the definition of KGL_{τ} is also formed in the derived sense. Later in the paper we prove that homotopy equivalent maps from $\mathscr{X}$ to $\mathsf{BB}\mathbf{G}_{\mathfrak{m}}$ give rise to isomorphic motivic twisted K-theories. In addition, the derived style definition of KGL^{τ} requires a strict ring model for KGL as a $\Sigma^{\infty}\mathbf{P}^{\infty}_{+}$–ring spectrum. Such a model was only recently constructed by Röndigs, Spitzweck and Østvær [40] using the Bott tower

$$\Sigma^{\infty}\mathbf{P}^{\infty}_{+}\xrightarrow{\beta}\Sigma^{-2,-1}\Sigma^{\infty}\mathbf{P}^{\infty}_{+}\xrightarrow{\Sigma^{-2,-1}\beta}\cdots. \tag{1}$$

Similarly, in the cohomological setup, the hom-object appearing in the definition of KGL_{τ} is formed in the homotopy category of $\Sigma^{\infty}\mathbf{P}^{\infty}_{+}$–modules.

An alternate definition of KGL^{τ} can be made precise by simply inverting the $(2,1)$ self-map of $\Sigma^{\infty}\mathscr{X}^{\tau}_{+}$ obtained from the Bott map realizing K–theory KGL as the Bott inverted infinite projective space. (The Bott map β is indeed a $\Sigma^{\infty}\mathbf{P}^{\infty}_{+}$–module map by construction.) Independent proofs of the latter result have appeared in Gepner and Snaith [17] and Spitzweck and Østvær [44]. For other discussions of the Bott inverted model for K–theory we refer to Naumann, Spitzweck and Østvær [32; 33] and Röndigs, Spitzweck and Østvær [40]. Making use of the Bott map provides also an alternate definition of KGL_{τ}. We shall be using this viewpoint on a number of occasions in this paper.

Now suppose the twisting class τ for $\mathscr{X}$ is null and the base scheme S is regular. In Section 3 we show that the homotopy fiber $\mathscr{X}^{\tau}$ identifies with the product $\mathbf{P}^{\infty}\times\mathscr{X}$ and KGL^{τ} identifies with the smash product $\Sigma^{\infty}\mathscr{X}_{+}\wedge\mathsf{KGL}$. Accordingly, we may view motivic twisted K–theory as a generalization of K–theory. In the course of the paper we shall work out some of the differences and similarities arising from this generalization, and suggest some open problems.

For a general twist τ it turns out that the motivic twisted K–theory spectra KGL^{τ} and KGL_{τ} do not acquire ring structures in the motivic stable homotopy category. In particular, the motivic twisted K–groups KGL^{τ}_{*} do not form a ring in general. (Here we use the standard motivic grading with a topological degree and a weight.) The lack of a product structure tends to complicate computations. On the other hand we establish two far more powerful tools for performing computations. First we prove a Künneth isomorphism for homological motivic twisted K–groups and second we construct spectral sequences relating motivic (co)homology to motivic twisted K–groups.

By applying the Tor-spectral sequence in Dugger and Isaksen [12, Proposition 7.7] to the commutative motivic ring spectrum $\Sigma^{\infty}\mathbf{P}^{\infty}_{+} \wedge \mathsf{KGL}$ and its modules KGL and $\Sigma^{\infty}\mathscr{X}^{\tau}_{+} \wedge \mathsf{KGL}$ we arrive at the strongly convergent trigraded spectral sequence

$$\mathsf{Tor}^{\mathsf{KGL}_{*}(\mathbf{P}^{\infty})}_{a,(b,c)}(\mathsf{KGL}_{*}(\mathscr{X}^{\tau}), \mathsf{KGL}_{*}) \Rightarrow \mathsf{KGL}^{\tau}_{a+b,c}(\mathscr{X}). \tag{2}$$

Here we should infer that $\Sigma^{\infty}\mathbf{P}^{\infty}_{+} \wedge \mathsf{KGL}$ and KGL are stably cellular motivic ring spectra, also known as "Tate spectra" [32]. To begin with, the motivic K–theory spectrum KGL is stably cellular [12, Theorem 6.2]. The suspension spectrum of the pointed infinite projective space is also stably cellular by [12, Propositions 2.13, 2.17, Lemma 3.1] since any filtered colimit of stably cellular motivic spaces is stably cellular [12, Definition 2.1(3)]. Hence the smash product $\Sigma^{\infty}\mathbf{P}^{\infty}_{+} \wedge \mathsf{KGL}$ is cellular. (Although the case of fields is emphasized in [12] the results we employ from loc. cit. hold over arbitrary noetherian base schemes of finite dimension.)

Theorem 2.2 *The edge homomorphism in the Tor-spectral sequence* (2) *induces a natural isomorphism*

$$\mathsf{KGL}^{\tau}_{*}(\mathscr{X}) \cong \mathsf{KGL}_{*}(\mathscr{X}^{\tau}) \otimes_{\mathsf{KGL}_{*}(\mathbf{P}^{\infty})} \mathsf{KGL}_{*}.$$

This theorem is the motivic analogue of the corresponding topological result shown by Khorami [27]. Theorem 2.2 follows from Equation (2) by proving the Tor-group $\mathsf{Tor}^{\mathsf{KGL}_{*}(\mathbf{P}^{\infty})}_{a,(b,c)}(\mathsf{KGL}_{*}, \mathsf{KGL}_{*}(\mathscr{X}^{\tau}))$ is trivial for $a > 0$. It is worthwhile to point out that KGL_{*} is not a flat $\mathsf{KGL}_{*}(\mathbf{P}^{\infty})$–module, that is, the vanishing result for the Tor-groups does not hold for an "obvious" reason. Our proof of the vanishing employs flatness of the ring map $\mathsf{KGL}_{*}(\mathbf{P}^{\infty}) \to \mathsf{KGL}_{*}\mathsf{KGL}$ and the homotopy theory of Hopf algebroids. More precisely, it is shown that the composite map $\mathsf{KGL}_{*}(\mathbf{P}^{\infty}) \to \mathsf{KGL}_{*}\mathsf{KGL} \to \mathsf{KGL}_{*}$ satisfies the Landweber exactness criterion relative to the Hopf algebroid $(\mathsf{KGL}_{*}(\mathbf{P}^{\infty}), \mathsf{KGL}_{*}\mathsf{KGL} \otimes_{\mathsf{KGL}_{*}} \mathsf{KGL}_{*}(\mathbf{P}^{\infty}))$. Furthermore, $\mathsf{KGL}_{*}(\mathscr{X}^{\tau})$ is a comodule over the same Hopf algebroid, and the Tor-groups are computed by a cofibrant replacement in the projective model structure on the category of unbounded chain complexes of such comodules. The Hopf algebroid in question is an "Adams Hopf algebroid." This notion is recalled in Section 7 together with some background from stable homotopy theory. By combining these facts we show the desired vanishing of the Tor-groups in positive degrees. The model structure allows us to circumvent an explicit construction employed in the topological situation [27].

Algebraic K–theory is closely related to motivic cohomology and more classically to higher Chow groups via Chern characters; see Bloch [6]. We shall briefly examine a Chern character for motivic twisted K–theory with target twisted periodized rational motivic cohomology

$$\mathsf{Ch}^\tau\colon \mathsf{KGL}^\tau \to \mathsf{PM}^\tau\mathbf{Q}.$$

The construction of Ch^τ follows the setup for the Chern character for KGL in [33]. As it turns out, the rationalization of Ch^τ is an isomorphism under a mild assumption on the base scheme originating in the work of Cisinski and Déglise [10]. (We leave the formulation of the corresponding result for KGL_τ to the main body of the paper.)

Theorem 2.3 *For geometrically unibranched excellent base schemes the rational Chern character*

$$\mathsf{Ch}^\tau_{\mathbf{Q}}\colon \mathsf{KGL}^\tau_{\mathbf{Q}} \to \mathsf{PM}^\tau\mathbf{Q}.$$

is an isomorphism in the homotopy category of modules over $\Sigma^\infty\mathbf{P}^\infty_+$.

Section 4 provides streamlined proofs of the results reviewed in the above. In the same section we work out explicit computations of the motivic twisted K–groups of the motivic $(3,1)$–sphere. Over finite fields the motivic twisted K–groups in positive degrees $2k-1$ and $2k$ turn out to be finite cyclic groups of the same order. This amusing computation is closely related to Quillen's computation of the K–groups of finite fields. Some of the basic facts concerning flat Adams Hopf algebroids required in Section 4 are deferred to Section 7.

In Section 5 we establish powerful integral relations between motivic (co)homology and motivic twisted K–theory in the form of spectral sequences

$$\mathsf{M}\mathbf{Z}_*(\Sigma^\infty\mathscr{X}_+) \Longrightarrow \mathsf{KGL}^\tau_*(\mathscr{X}),$$
$$\mathsf{M}\mathbf{Z}^*(\Sigma^\infty\mathscr{X}_+) \Longrightarrow \mathsf{KGL}^*_\tau(\mathscr{X}).$$

For closely related papers on spectral sequences computing (nontwisted) K–groups in terms of motivic cohomology groups, refer to Bloch and Lichtenbaum [7], Friedlander and Suslin [16], Grayson [18], Levine [28], Suslin [45] and Voevodsky [50]. The question of strong convergence of these spectral sequences is a tricky problem. Our approach involves the very effective motivic stable homotopy category $\mathbf{SH}(S)^{\mathbf{Veff}}$. We define it as the smallest full subcategory of the motivic stable homotopy category $\mathbf{SH}(S)$

that contains suspension spectra of smooth schemes of finite type over S and is closed under extensions and homotopy colimits. This is not a triangulated category; however, it is a subcategory of the effective motivic stable homotopy category $\mathbf{SH}(S)^{\mathbf{eff}}$. (In fact, it is the homologically positive part of a t–structure on $\mathbf{SH}(S)^{\mathbf{eff}}$.) The very effective motivic stable homotopy category is of independent interest. We show that the algebraic cobordism spectrum MGL lies in $\mathbf{SH}(S)^{\mathbf{Veff}}$. When S is a field of characteristic zero, we show that the connective K–theory spectrum kgl lies in $\mathbf{SH}(S)^{\mathbf{Veff}}$. This is a key input for showing strong convergence of the spectral sequences.

The main body of the paper ends in Section 6 with a discussion of open problems. In particular, we suggest extending the techniques in this paper to the settings of both equivariant K–theory and hermitian K–theory.

For legibility, bigraded motivic homology theories are written with a single grading. The precise meaning of the gradings should always be clear from the context.

3 Main definitions and first properties

In this section we first put the definitions of the motivic twisted K–theory spectra on rigorous grounds. This part deals with model structures and classifying spaces. The constructions are rigged so that smashing with the sphere spectrum over $\Sigma^\infty \mathbf{P}^\infty_+$ yields a useful "untwisting" result detailed in Lemma 3.6. For algebro-geometric reasons we shall explain below, some of the results require mild restrictions on the base scheme.

Let $\mathbf{Spc}$ be the category of motivic spaces on $\mathbf{Sm}$, ie simplicial presheaves on the Nisnevich site of smooth schemes of finite type over S, with the injective motivic model structure. This model structure satisfies the monoid axiom by Schwede and Shipley [42]. Hence for any monoid G in $\mathbf{Spc}$, the category $\mathbf{Mod}(G)$ of G–modules acquires a model structure. (In which the fibrations and weak equivalences of modules are just maps which are fibrations and weak equivalences in the underlying model structure.) For a map $G \to H$ of monoids there is an induced left Quillen functor $\mathbf{Mod}(G) \to \mathbf{Mod}(H)$. In particular, the pushforward of a G–module $\mathcal{X}$ along the canonical map $G \to *$ is the quotient $\mathcal{X}/G$. The homotopy quotient is defined similarly by first taking a cofibrant replacement of $\mathcal{X}$ in $\mathbf{Mod}(G)$.

We denote by $\mathbf{Mod}_{\mathcal{Y}}(G)$ the category of G–modules in the slice category $\mathbf{Spc}/\mathcal{Y}$ comprised of motivic spaces over a motivic space $\mathcal{Y}$. It should be noted that $\mathbf{Mod}_{\mathcal{Y}}(G)$ is a model category: To begin with, $\mathbf{Spc}/\mathcal{Y}$ inherits an evident model structure from the motivic model structure on $\mathbf{Spc}$ which turns the pairing $\mathbf{Spc} \times \mathbf{Spc}/\mathcal{Y} \to \mathbf{Spc}/\mathcal{Y}$ sending $(\mathcal{X}, \mathcal{X}' \to \mathcal{Y})$ to $(\mathcal{X} \times \mathcal{X}') \to \mathcal{X}' \to \mathcal{Y}$ into a Quillen bifunctor. Moreover, every object of **Spc** is cofibrant. Thus the existence of the model structure on $\mathbf{Mod}_{\mathcal{Y}}(G)$ follows from a relative version of [42, Theorem 3.1.1]. For a G–module $\mathcal{X}$ and a map of motivic spaces $\mathcal{Y}' \to \mathcal{Y}$, note that $\mathcal{X} \in \mathbf{Mod}_{\mathcal{X}/G}(G)$ and there is a pullback functor $\mathbf{Mod}_{\mathcal{Y}}(G) \to \mathbf{Mod}_{\mathcal{Y}'}(G)$.

In what follows we specialize to the commutative monoid $\mathsf{B}\mathbf{G}_{\mathfrak{m}} \simeq \mathbf{P}^{\infty}$. As a model for the classifying space $\mathsf{B}\mathbf{P}^{\infty}$ of $\mathbf{P}^{\infty}$ we may use the standard bar construction. Viewing $*$ as a $\mathbf{P}^{\infty}$–module we are entitled to a cofibrant replacement $\mathcal{Q} \to *$ in $\mathbf{Mod}(\mathbf{P}^{\infty})$. The homotopy quotient $\mathcal{Q}/\mathbf{P}^{\infty}$ gives an alternative model for the classifying space of $\mathbf{P}^{\infty}$. In the proof of Lemma 3.6 we find it convenient to use the latter.

If $\tau\colon \mathcal{X} \to \mathcal{Q}/\mathbf{P}^{\infty}$ is a fibration in **Spc**, then the homotopy pullback

$$\mathcal{X}^{\tau} \equiv \mathcal{X} \times_{\mathcal{Q}/\mathbf{P}^{\infty}} \mathcal{Q} \in \mathbf{Mod}_{\mathcal{X}}(\mathbf{P}^{\infty})$$

is a $\mathbf{P}^{\infty}$–module over $\mathcal{X}$, in particular a $\mathbf{P}^{\infty}$–module. Suppose $\tau\colon \mathcal{X} \to (\mathcal{Q}/\mathbf{P}^{\infty})^{\mathsf{f}}$ is a map in **Spc** with target some fibrant replacement of $\mathcal{Q}/\mathbf{P}^{\infty}$. The model structure ensures there exists a functorial fibrant replacement $(\mathcal{Q})^{\mathsf{f}}$ of $\mathcal{Q}$ in $\mathbf{Mod}_{(\mathcal{Q}/\mathbf{P}^{\infty})^{\mathsf{f}}}(\mathbf{P}^{\infty})$. Using these fibrant replacements we define $\mathcal{X}^{\tau}$ by the homotopy pullback

$$\mathcal{X}^{\tau} \equiv \mathcal{X} \times_{(\mathcal{Q}/\mathbf{P}^{\infty})^{\mathsf{f}}} (\mathcal{Q})^{\mathsf{f}} \in \mathbf{Mod}_{\mathcal{X}}(\mathbf{P}^{\infty}).$$

Working in motivic symmetric spectra we note that $\Sigma^{\infty}\mathcal{X}^{\tau}_{+}$ is a strict $\Sigma^{\infty}\mathbf{P}^{\infty}_{+}$–module. (Recall that Σ^{∞} is a left Quillen functor for the injective motivic model structure [25].)

The motivic twisted homological K–theory of $\tau\colon \mathcal{X} \to (\mathcal{Q}/\mathbf{P}^{\infty})^{\mathsf{f}}$ in **Spc** is defined as the derived smash product

$$\mathsf{KGL}^{\tau} \equiv \Sigma^{\infty}\mathcal{X}^{\tau}_{+} \wedge_{\Sigma^{\infty}\mathbf{P}^{\infty}_{+}} \mathsf{KGL}.$$

in the homotopy category of $\Sigma^{\infty}\mathbf{P}^{\infty}_{+}$–modules. For the strict module structure on KGL we use the Bott inverted model discussed in [40]. Likewise, by making the same fibrant replacements, the motivic twisted cohomological K–theory of $\tau\colon \mathcal{X} \to (\mathcal{Q}/\mathbf{P}^{\infty})^{\mathsf{f}}$ is defined as the derived internal hom

$$\mathsf{KGL}_{\tau} \equiv \underline{\mathrm{Hom}}_{\Sigma^{\infty}\mathbf{P}^{\infty}_{+}}(\Sigma^{\infty}\mathcal{X}^{\tau}_{+}, \mathsf{KGL}).$$

In the following we let $\mathsf{BB}\mathbf{G}_{\mathfrak{m}}$ denote the homotopy quotient $(\mathcal{Q}/\mathbf{P}^{\infty})^{\mathsf{f}}$. Here $\mathcal{Q}$ can be modeled as an E_{∞}–monoid over $\mathbf{P}^{\infty}$ and thus we may assume $\mathsf{BB}\mathbf{G}_{\mathfrak{m}}$ is an E_{∞}–monoid.

Proposition 3.1 *Suppose S is a regular scheme and $\tau\colon \mathcal{X} \to \mathsf{BB}\mathbf{G}_{\mathfrak{m}}$. If τ is null then KGL^{τ} is isomorphic to $\Sigma^{\infty}\mathcal{X}_{+} \wedge \mathsf{KGL}$ in the motivic stable homotopy category.*

Proof Corollary 3.4 identifies $\mathcal{X}^{\tau}_{+}$ with the smash product of motivic pointed spaces $\mathcal{X}_{+} \wedge \mathbf{P}^{\infty}_{+}$. This follows because of the assumption on τ the former is obtained by first taking the homotopy pullback of the diagram

$$* \to \mathsf{B}\mathsf{B}\mathbf{G}_{\mathrm{m}} \leftarrow *$$

and second forming the homotopy pullback along the canonical map $\mathcal{X} \to *$. Using this we obtain the isomorphisms

$$\Sigma^{\infty}\mathcal{X}^{\tau}_{+} \wedge_{\Sigma^{\infty}\mathbf{P}^{\infty}_{+}} \mathsf{KGL} \cong (\Sigma^{\infty}\mathcal{X}_{+} \wedge \Sigma^{\infty}\mathbf{P}^{\infty}_{+}) \wedge_{\Sigma^{\infty}\mathbf{P}^{\infty}_{+}} \mathsf{KGL} \cong \Sigma^{\infty}\mathcal{X}_{+} \wedge \mathsf{KGL}.$$

The regularity assumption on the base scheme S enters the proof of Corollary 3.4, which we discuss next. □

Recall from [32, Section 2] the definition of the simplicial Picard functor $\nu\mathbf{Pic}$ on $\mathbf{Sm}$: For a scheme X in $\mathbf{Sm}$, let $\underline{\mathbf{Pic}}(X)$ denote the associated Picard groupoid. Then the pseudo-functor $X \mapsto \underline{\mathbf{Pic}}(X)$ can be strictified to a presheaf on $\mathbf{Sm}$. Applying the nerve functor to any such strictification defines the simplicial presheaf $\nu\mathbf{Pic}$ on $\mathbf{Sm}$.

Lemma 3.2 *Suppose S is a normal scheme. Then the simplicial Picard functor $\nu\mathbf{Pic}$ is a Nisnevich local $\mathbf{A}^1$–invariant simplicial presheaf.*

Proof Nisnevich localness holds because the groupoid valued Picard functor $\mathbf{Pic}$ satisfies flat descent. And $\mathbf{A}^1$–invariance holds because $\mathbf{Pic}$ is $\mathbf{A}^1$–invariant by assumption. For more details, refer to [32, Section 2]. □

We remark that $\nu\mathbf{Pic}$ is a commutative monoid in $\mathbf{Spc}$ by strictification.

Lemma 3.3 *Suppose S is a regular scheme. Applying the classifying space functor sectionwise to $\nu\mathbf{Pic}$ in simplicial sets determines a Nisnevich local $\mathbf{A}^1$–invariant simplicial presheaf.*

Proof Since S is regular, it is well known that the cohomology group $H^2_{\mathbf{Nis}}(X,\mathbf{G}_{\mathrm{m}})$ is trivial for every X in $\mathbf{Sm}$. For an outline of a proof, we note that the sheaf $\mathcal{M}^*_X$ of meromorphic functions on X and the sheaf $\mathcal{Z}^1_X$ of codimension 1 cycles on X are flasque in the Nisnevich topology. Thus, using [19, Section 21.6], the vanishing of $H^2_{\mathbf{Nis}}(X,\mathbf{G}_{\mathrm{m}})$ follows from the exact sequence

$$0 \to \mathcal{O}^*_X \to \mathcal{M}^*_X \to \mathcal{Z}^1_X \to 0. \tag{3}$$

Let $B^s\nu\mathbf{Pic}$ be the sectionwise classifying space of $\nu\mathbf{Pic}$ and $\varphi\colon B^s\nu\mathbf{Pic} \to RB^s\nu\mathbf{Pic}$ a Nisnevich local replacement. Then $\pi_i((RB^s\nu\mathbf{Pic})(X)) = H^{2-i}_{\mathbf{Nis}}(X,\mathbf{G}_{\mathrm{m}})$ for $0 \le i \le 2$.

It follows that φ is a sectionwise equivalence. Thus the sectionwise classifying space construction is Nisnevich local (sectionwise equivalent to every Nisnevich local fibrant replacement) and $\mathbf{A}^1$–invariance is preserved. □

Corollary 3.4 *If S is a regular scheme then the homotopy pullback of the diagram*

$$* \to \mathsf{BB}\mathbf{G}_{\mathrm{m}} \leftarrow *$$

is isomorphic to $\mathbf{P}^\infty$ in the motivic homotopy category.

Proof Let $\mathbf{H}^s(S)$ denote the homotopy category of simplicial presheaves on **Sm** with the objectwise model structure, and let $\mathbf{H}(S)$ denote the motivic homotopy category. Then the inclusion $\mathbf{H}(S) \to \mathbf{H}^s(S)$ arises from a right Quillen functor. Therefore, in order to compute the homotopy pullback of $* \to \mathsf{BB}\mathbf{G}_{\mathrm{m}} \leftarrow *$ in the motivic homotopy category, it is sufficient to compute the homotopy pullback of its image in $\mathbf{H}^s(S)$.

By Lemma 3.3, $B^s\nu\mathbf{Pic}$ is an $\mathbf{A}^1$– and Nisnevich local replacement of $\mathsf{BB}\mathbf{G}_{\mathrm{m}}$ (for notation see the proof of Lemma 3.3). Thus $B^s\nu\mathbf{Pic}$ is a model for the image of $\mathsf{BB}\mathbf{G}_{\mathrm{m}}$ in $\mathbf{H}^s(S)$. The homotopy pullback of $* \to B^s\nu\mathbf{Pic} \leftarrow *$ in $\mathbf{H}^s(S)$ is clearly $\mathsf{B}\mathbf{G}_{\mathrm{m}}$. □

Remark 3.5 The previous corollary would have been trivially true provided the infinity category of motivic spaces had been an infinity topos in the sense of Lurie [29]. Alas, this is not true for motivic spaces: Recall that in any infinity topos the loop functor provides an equivalence between the connected 1–truncated pointed objects and discrete group objects. In motivic homotopy theory, the simplicial loop space of a 1–truncated pointed motivic space is strongly $\mathbf{A}^1$–invariant. Recall also that $\mathbf{A}^1$–invariant and strongly $\mathbf{A}^1$–invariant sheaves of groups are different notions; see Morel [31]. A discrete motivic group object is synonymous with an $\mathbf{A}^1$–invariant sheaf of groups. This shows that the equivalence does not hold in the motivic setting. We thank J Lurie for this remark. It is unclear if Corollary 3.4 extends to an interesting class of base schemes. We note that the sequence (3) is exact if and only if X is a locally factorial scheme. However, a smooth scheme of finite type over a locally factorial scheme need not be locally factorial in general. Thus we cannot expect that the group $H^2_{\mathbf{Nis}}(X, \mathbf{G}_{\mathrm{m}})$ is trivial over every locally factorial base scheme. We thank M Levine for clarifying this remark.

Lemma 3.6 *There is an isomorphism of $\Sigma^\infty\mathbf{P}^\infty_+$–modules $\Sigma^\infty\mathscr{X}^\tau_+ \wedge_{\Sigma^\infty\mathbf{P}^\infty_+} \mathbf{1} \cong \Sigma^\infty\mathscr{X}_+$ where $\Sigma^\infty\mathbf{P}^\infty_+ \to \mathbf{1}$ is induced by the canonical map $\mathbf{P}^\infty \to *$.*

Proof For a fibration $\mathscr{Y}' \to \mathscr{Y}$ in **Spc** the pullback functor $\mathbf{Spc}/\mathscr{Y} \to \mathbf{Spc}/\mathscr{Y}'$ is a left Quillen functor for the injective motivic model structure on **Spc**. Indeed, it has a right adjoint and it preserves monomorphisms and weak equivalences. (Recall that **Spc** is right proper.) Thus we obtain a left Quillen functor

$$\mathbf{Mod}_{\mathcal{Y}}(\mathbf{P}^\infty) \to \mathbf{Mod}_{\mathcal{Y}'}(\mathbf{P}^\infty).$$

This functor commutes with pushforward along the canonical map $\mathbf{P}^\infty \to *$ and thus it preserves homotopy quotients by $\mathbf{P}^\infty$. We deduce that the natural map $\mathcal{Q}\mathcal{X}^\tau/\mathbf{P}^\infty \to \mathcal{X}$ is a weak equivalence, where $\mathcal{Q}\mathcal{X}^\tau \to \mathcal{X}^\tau$ is a cofibrant replacement in $\mathbf{Mod}_{\mathcal{X}}(\mathbf{P}^\infty)$. On the other hand, the forgetful functor $\mathbf{Mod}_{\mathcal{X}}(\mathbf{P}^\infty) \to \mathbf{Mod}(\mathbf{P}^\infty)$ is also a left Quillen functor. Combining the above findings shows there is an isomorphism

$$\mathcal{X}^\tau \times^{\mathbf{L}}_{\mathbf{P}^\infty} * \cong \mathcal{X}.$$

Applying the motivic symmetric suspension spectrum functor yields the result. □

Next we consider some basic functorial properties of motivic twisted K–theory. First we note there exists a functor

$$\mathsf{KGL}^-\colon \mathbf{Ho}(\mathbf{Spc}/\mathsf{BB}\mathbf{G}_{\mathrm{m}}) \to \mathbf{SH}(S).$$

Note here that for a map from $\tau\colon \mathcal{X} \to \mathsf{BB}\mathbf{G}_{\mathrm{m}}$ to $\tau'\colon \mathcal{X}' \to \mathsf{BB}\mathbf{G}_{\mathrm{m}}$ there is an induced map between pullbacks $\mathcal{X}^\tau \to (\mathcal{X}')^{\tau'}$ of $\mathbf{P}^\infty$–modules, which induces a map of motivic symmetric spectra $\mathsf{KGL}^\tau \to \mathsf{KGL}^{\tau'}$. Clearly this factors through the homotopy category of motivic spaces over $\mathsf{BB}\mathbf{G}_{\mathrm{m}}$. In particular, if τ and τ' are $\mathbf{A}^1$–homotopy equivalent maps, then KGL^τ and $\mathsf{KGL}^{\tau'}$ are isomorphic. We also note that KGL^- can be enhanced to a functor from $\mathbf{Ho}(\mathbf{Spc}/\mathsf{BB}\mathbf{G}_{\mathrm{m}})$ taking values in the homotopy category of highly structured KGL–modules. Likewise, in the cohomological setup, there exists a functor

$$\mathsf{KGL}_-\colon \mathbf{Ho}(\mathbf{Spc}/\mathsf{BB}\mathbf{G}_{\mathrm{m}})^{\mathrm{op}} \to \mathbf{SH}(S).$$

Some parts of our discussion of motivic twisted K–theory rely on the notion of a "motivic E_∞–ring spectrum." For every operad $\mathcal{O}$ in motivic symmetric spectra, the category of $\mathcal{O}$–algebras acquires a combinatorial model structure. An account of this model structure has been written up by Hornbostel [21]. Motivic symmetric spectra are equipped with its stable flat positive model structure. In this setup there exists model structures for the commutative motivic operad. A motivic E_∞–ring spectrum is an algebra over a Σ–cofibrant replacement of the commutative motivic operad. Motivic E_∞–ring spectra and strict commutative motivic ring spectra are related by a Quillen equivalence [21]. For our purposes we may therefore use these two notions interchangeably.

4 A Künneth isomorphism for motivic twisted homological K–theory

An explicit computation furnishes a natural base change isomorphism expressing the KGL–homology of $\mathbf{P}^\infty$ in terms of KGL_* and unitary topological K–theory

$$\mathsf{KGL}_*(\mathbf{P}^\infty) \cong \mathsf{KGL}_* \otimes_{\mathsf{KU}_*} \mathsf{KU}_*(\mathbf{CP}^\infty). \tag{4}$$

The multiplicative structure on $\mathsf{KGL}_*(\mathbf{P}^\infty)$ induced from the H–space structure on $\mathbf{P}^\infty$ can be read off from this isomorphism by using the ring structure on $\mathsf{KU}_*(\mathbf{CP}^\infty)$ and the coefficient ring. We refer to the work of Ravenel and Wilson [37] for a description of the ring structure on $\mathsf{KU}_*(\mathbf{CP}^\infty)$ in terms of the multiplicative formal group law.

An application of motivic Landweber exactness [32, Proposition 9.1] shows there is a natural base change isomorphism

$$\mathsf{KGL}_*\mathsf{KGL} \cong \mathsf{KGL}_* \otimes_{\mathsf{KU}_*} \mathsf{KU}_*\mathsf{KU}. \tag{5}$$

The multiplicative structure on $\mathsf{KGL}_*\mathsf{KGL}$ induced from the ring structure on KGL can be read off from this isomorphism by using the ring structure on $\mathsf{KU}_*\mathsf{KU}$ and the coefficient ring. For a description of the Hopf algebra $\mathsf{KU}_*\mathsf{KU}$ we refer to the work of Adams and Harris [1, Part II, Section 13].

Lemma 4.1 *Under the naturally induced composite map*

$$\mathsf{KGL}_*(\mathbf{P}^\infty) \to \mathsf{KGL}_*\mathsf{KGL} \to \mathsf{KGL}_*,$$

the generator β_i maps to 1 if $i = 0, 1$ and to 0 if $i \neq 0, 1$.

Proof We note that $\mathsf{KGL}_*(\mathbf{P}^\infty)$ is a free KGL_*–module generated by elements β_i for $i \geq 0$ [44]. Thus the claim follows from the analogous result for unitary topological K–theory of $\mathbf{CP}^\infty$ (see [27] for example) by applying the functor $\mathsf{KGL}_* \otimes_{\mathsf{KU}_*} -$ and appealing to the base change isomorphisms (4) and (5). □

Lemma 4.1 verifies the previous assertion that KGL_* is not a flat $\mathsf{KGL}_*(\mathbf{P}^\infty)$–module. Next we establish a result which is pivotal for our proof of the vanishing of the Tor-groups discussed in Section 2.

Lemma 4.2 *The naturally induced map $\mathsf{KGL}_*(\mathbf{P}^\infty) \to \mathsf{KGL}_*\mathsf{KGL}$ is a flat ring map.*

Proof In [17; 44] it is shown that KGL is isomorphic to the Bott inverted motivic suspension spectrum of $\mathbf{P}^\infty_+$. Thus the map in question is a localization. In particular it is flat. For an alternate proof, combine the base change isomorphisms (4) and (5) with flatness of the naturally induced map $\mathsf{KU}_*(\mathbf{CP}^\infty) \to \mathsf{KU}_*\mathsf{KU}$. (This map is a localization according to the topological analogue of our first argument, which is well known and follows from the motivic result by taking complex points, or alternatively by [27].) □

In Proposition 7.3 we show that $(\mathsf{KGL}_*(\mathbf{P}^\infty), \mathsf{KGL}_*\mathsf{KGL} \otimes_{\mathsf{KGL}_*} \mathsf{KGL}_*(\mathbf{P}^\infty))$ has the structure of a flat graded Adams Hopf algebroid. We refer the reader to Section 7 for background on the notions and results appearing in the formulation and proof of the following key result.

Theorem 4.3 *The naturally induced composite map*

$$\mathsf{KGL}_*(\mathbf{P}^\infty) \to \mathsf{KGL}_*\mathsf{KGL} \to \mathsf{KGL}_*$$

is Landweber exact for the flat graded Adams Hopf algebroid

$$(\mathsf{KGL}_*(\mathbf{P}^\infty), \mathsf{KGL}_*\mathsf{KGL} \otimes_{\mathsf{KGL}_*} \mathsf{KGL}_*(\mathbf{P}^\infty)).$$

Proof By Lemma 7.6 it suffices to show that the left unit map

$$\begin{aligned}\eta_{\mathsf{KGL}_*(\mathbf{P}^\infty)}\colon \mathsf{KGL}_*(\mathbf{P}^\infty) &\to \mathsf{KGL}_*\mathsf{KGL} \otimes_{\mathsf{KGL}_*} \mathsf{KGL}_*(\mathbf{P}^\infty)\\ &\cong (\mathsf{KGL}_*\mathsf{KGL} \otimes_{\mathsf{KGL}_*} \mathsf{KGL}_*(\mathbf{P}^\infty)) \otimes_{\mathsf{KGL}_*(\mathbf{P}^\infty)} \mathsf{KGL}_*(\mathbf{P}^\infty)\\ &\to (\mathsf{KGL}_*\mathsf{KGL} \otimes_{\mathsf{KGL}_*} \mathsf{KGL}_*(\mathbf{P}^\infty)) \otimes_{\mathsf{KGL}_*(\mathbf{P}^\infty)} \mathsf{KGL}_*\end{aligned}$$

for the displayed Hopf algebroid is flat (the target is canonically isomorphic to $\mathsf{KGL}_*\mathsf{KGL}$). Remark 7.4 provides more details on the Hopf algebroid structure.

The left unit map $\eta_{\mathsf{KGL}_*(\mathbf{P}^\infty)}$ and its topological analogue $\eta_{\mathsf{KU}_*(\mathbf{CP}^\infty)}$ determines a commutative diagram where the horizontal maps are the base change isomorphisms given in (4) and (5):

$$\begin{tikzcd}
\mathsf{KGL}_*(\mathbf{P}^\infty) \arrow[r, "\cong"] \arrow[d, "\eta_{\mathsf{KGL}_*(\mathbf{P}^\infty)}"'] & \mathsf{KGL}_* \otimes_{\mathsf{KU}_*} \mathsf{KU}_*(\mathbf{CP}^\infty) \arrow[d, "\mathsf{KGL}_*\otimes\eta_{\mathsf{KU}_*(\mathbf{CP}^\infty)}"] \\
\mathsf{KGL}_*\mathsf{KGL} \arrow[r, "\cong"] & \mathsf{KGL}_* \otimes_{\mathsf{KU}_*} \mathsf{KU}_*\mathsf{KU}
\end{tikzcd}$$

Khorami [27] has shown that $\eta_{\mathsf{KU}_*(\mathbf{CP}^\infty)}$ coincides with the naturally induced map from $\mathsf{KU}_*(\mathbf{CP}^\infty)$ to $\mathsf{KU}_*\mathsf{KU}$. By motivic Landweber exactness [32] we deduce that $\eta_{\mathsf{KGL}_*(\mathbf{P}^\infty)}$ coincides with the naturally induced flat map in Lemma 4.2. This finishes the proof. □

By Landweber exactness the functor from comodules over $\mathsf{KGL}_*\mathsf{KGL}\otimes_{\mathsf{KGL}_*}\mathsf{KGL}_*(\mathbf{P}^\infty)$ to KGL_*–algebras is exact (where KGL_* is viewed with its $\mathsf{KGL}_*(\mathbf{P}^\infty)$–algebra structure induced by the projection $\mathbf{P}^\infty \to *$). This observation is a crux input in the proof of the next result.

Corollary 4.4 *Suppose E is a comodule over $\mathsf{KGL}_*\mathsf{KGL}\otimes_{\mathsf{KGL}_*}\mathsf{KGL}_*(\mathbf{P}^\infty)$. The group*

$$\mathsf{Tor}_*^{\mathsf{KGL}_*(\mathbf{P}^\infty)}(E, \mathsf{KGL}_*)$$

is trivial in positive degrees.

Proof Proposition 7.3 implies there exists a projective model structure on the category of nonconnective chain complexes of $\mathsf{KGL}_*\mathsf{KGL}\otimes_{\mathsf{KGL}_*}\mathsf{KGL}_*(\mathbf{P}^\infty)$–comodules for the set of dualizable comodules [22, Theorem 2.1.3]. The projective model structure is proper, finitely generated, stable symmetric monoidal and satisfies the monoid axiom. Moreover, a map is a cofibration if and only if it is a degreewise split monomorphism with cofibrant cokernel. The cofibrant objects are retracts of certain sequential cell-complexes described in detail in [22, Theorem 2.1.3]. These results are easily transferred to the graded setting.

Due to the existence of the projective model structure we are entitled to a cofibrant replacement $\mathfrak{Q}E \to E$ (recall this is a projective weak equivalence and $\mathfrak{Q}E$ is cofibrant). Proposition 7.3 shows that $(\mathsf{KGL}_*(\mathbf{P}^\infty), \mathsf{KGL}_*\mathsf{KGL}\otimes_{\mathsf{KGL}_*}\mathsf{KGL}_*(\mathbf{P}^\infty))$ is a graded Adams Hopf algebroid. This additional structure guarantees that every weak equivalence in the projective model structure is a quasi-isomorphism [22, Proposition 3.3.1].

We claim that the Tor-groups in question are computed by the homology of the chain complex

$$\mathfrak{Q}E \otimes_{\mathsf{KGL}_*(\mathbf{P}^\infty)} \mathsf{KGL}_*.$$

The proof proceeds by comparing chain complexes of comodules over the tensor product $\mathsf{KGL}_*\mathsf{KGL}\otimes_{\mathsf{KGL}_*}\mathsf{KGL}_*(\mathbf{P}^\infty)$ with chain complexes of $\mathsf{KGL}_*(\mathbf{P}^\infty)$–modules. Indeed, by [22, Proposition 1.3.4] (cf the proof of [22, Theorem 2.1.3]), the generating cofibrations are of such a form that $\mathfrak{Q}E$ is even cofibrant as a complex of $\mathsf{KGL}_*(\mathbf{P}^\infty)$–modules for the usual projective model structure. (Note that the tensor factor KGL_* need not be cofibrantly replaced because the monoid axiom holds in the projective model structure.)

As noted earlier there exists an exact functor from the category of comodules over $\mathsf{KGL}_*\mathsf{KGL}\otimes_{\mathsf{KGL}_*}\mathsf{KGL}_*(\mathbf{P}^\infty)$ to KGL_*–algebras. We note that any such functor preserves quasi-isomorphisms. In particular there is a quasi-isomorphism

$$\mathfrak{Q}E \otimes_{\mathsf{KGL}_*(\mathbf{P}^\infty)} \mathsf{KGL}_* \simeq E \otimes_{\mathsf{KGL}_*(\mathbf{P}^\infty)} \mathsf{KGL}_*.$$

By combining the above we conclude that the Tor-groups vanish in positive degrees. □

By strong convergence of the spectral sequence (2) we are almost ready to conclude the proof of the Künneth isomorphism in Theorem 2.2. It only remains to observe that $\mathsf{KGL}_*(\mathscr{X}^\tau)$ is a comodule over the Hopf algebroid

$$(\mathsf{KGL}_*(\mathbf{P}^\infty), \mathsf{KGL}_*\mathsf{KGL} \otimes_{\mathsf{KGL}_*} \mathsf{KGL}_*(\mathbf{P}^\infty)).$$

To begin with, the naturally induced action of $\mathbf{P}^\infty$ on $\mathscr{X}^\tau$ yields a map

$$\mathsf{KGL}_*(\mathbf{P}^\infty \times \mathscr{X}^\tau) \to \mathsf{KGL}_*(\mathscr{X}^\tau).$$

Since $\mathsf{KGL}_*(\mathbf{P}^\infty)$ is free over the coefficient ring KGL_* there is an isomorphism

$$\mathsf{KGL}_*(\mathbf{P}^\infty \times \mathscr{X}^\tau) \cong \mathsf{KGL}_*(\mathbf{P}^\infty) \otimes_{\mathsf{KGL}_*} \mathsf{KGL}_*(\mathscr{X}^\tau).$$

It follows that $\mathsf{KGL}_*(\mathscr{X}^\tau)$ is a module over $\mathsf{KGL}_*(\mathbf{P}^\infty)$. Using the unit map from the motivic sphere spectrum $\mathbf{1}$ to KGL we get a map between motivic spectra

$$\mathsf{KGL} \wedge \Sigma^\infty \mathscr{X}^\tau_+ \cong \mathsf{KGL} \wedge \mathbf{1} \wedge \Sigma^\infty \mathscr{X}^\tau_+ \to \mathsf{KGL} \wedge \mathsf{KGL} \wedge \Sigma^\infty \mathscr{X}^\tau_+.$$

From this we immediately obtain the desired comodule map

$$\begin{aligned} \mathsf{KGL}_*(\mathscr{X}^\tau) &\to \mathsf{KGL}_*\mathsf{KGL} \otimes_{\mathsf{KGL}_*} \mathsf{KGL}_*(\mathscr{X}^\tau) \\ &\cong (\mathsf{KGL}_*\mathsf{KGL} \otimes_{\mathsf{KGL}_*} \mathsf{KGL}_*(\mathbf{P}^\infty)) \otimes_{\mathsf{KGL}_*(\mathbf{P}^\infty)} \mathsf{KGL}_*(\mathscr{X}^\tau). \end{aligned}$$

This is clearly a coassociative and unital map between $\mathsf{KGL}_*(\mathbf{P}^\infty)$–modules.

Remark 4.5 As noted earlier the KGL_*–module $\mathsf{KGL}_*(\mathbf{P}^\infty)$ is free on the generators β_i for $i \geq 0$. Its multiplicative structure can be described in terms of power series. Modulo the problem of computing the coefficient ring KGL_* this leaves us with investigating the K–theory of the homotopy fiber $\mathscr{X}^\tau$. On the other hand, an inspection of the module structures in Theorem 2.2, cf Lemma 4.1, reveals there is an isomorphism

$$\mathsf{KGL}^\tau_*(\mathscr{X}) \cong \mathsf{KGL}_*(\mathscr{X}^\tau)/(\beta_0 - 1, \beta_1 - 1, \beta_i)_{i \geq 2}.$$

In case τ is the identity map on $\mathsf{B}\mathsf{B}\mathbf{G}_\mathrm{m}$ then $\mathsf{KGL}_*(\mathsf{B}\mathsf{B}\mathbf{G}^\tau_\mathrm{m}) \cong \mathsf{KGL}_*$. We claim that $\mathsf{KGL}^\tau_*(\mathsf{B}\mathsf{B}\mathbf{G}_\mathrm{m})$ is the trivial group. This follows by comparing the images of β_0 or β_1 in the respective tensor factors. For example, the class β_1 maps to the unit in KGL_* and to zero in $\mathsf{KGL}_*(\mathsf{B}\mathsf{B}\mathbf{G}^\tau_\mathrm{m})$.

Next we turn to the constructions of the twisted Chern characters. The proof of our main result Theorem 2.3 relies on results in [10; 33]. Let $\mathsf{M}\mathbf{Q}$ denote the motivic Eilenberg–Mac Lane spectrum introduced by Voevodsky [48]. (Refer to Dundas, Röndigs and Østvær [14] for a definition of $\mathsf{M}\mathbf{Q}$ viewed as a motivic functor.) It has the structure of a commutative monoid in the category of motivic symmetric spectra [38; 39]. The periodization $\mathsf{PM}\mathbf{Q}$ of $\mathsf{M}\mathbf{Q}$ is also highly structured: Form the free commutative $\mathsf{M}\mathbf{Q}$–algebra $\mathsf{PM}\mathbf{Q}_{\geq 0}$ on one generator in degree $(2,1)$ (perform this in $\mathbf{P}^1$–spectra of simplicial presheaves of $\mathbf{Q}$–vector spaces and then transfer the spectrum back to obtain a strictly commutative ring spectrum). Inverting the same generator by following the method in [40] produces the commutative monoid $\mathsf{PM}\mathbf{Q}$ whose underlying spectrum is the infinite wedge sum $\bigvee_{i\in\mathbf{Z}} \Sigma^{2i,i}\mathsf{M}\mathbf{Q}$.

Lemma 4.6 *There is an isomorphism of E_∞–algebras between $\mathsf{PM}\mathbf{Q}$ and $\mathsf{KGL}\wedge\mathsf{M}\mathbf{Q}$.*

Proof By the universal property of $\mathsf{PM}\mathbf{Q}_{\geq 0}$ there is a commutative diagram of $\mathsf{M}\mathbf{Q}$–algebras:

$$\begin{tikzcd} & \Sigma^\infty\mathbf{P}^\infty_+\wedge\mathsf{M}\mathbf{Q} \arrow[d] \\ \mathsf{PM}\mathbf{Q}_{\geq 0} \arrow[ur] \arrow[r] & \mathsf{KGL}\wedge\mathsf{M}\mathbf{Q} \end{tikzcd}$$

Here the $\mathsf{M}\mathbf{Q}$–algebra structure on $\Sigma^\infty\mathbf{P}^\infty_+\wedge\mathsf{M}\mathbf{Q}$ is obtained from the unit map of $\Sigma^\infty\mathbf{P}^\infty_+$ and the identity map of $\mathsf{M}\mathbf{Q}$. The generator in degree $(2,1)$ maps to the canonical element in $\Sigma^\infty\mathbf{P}^\infty_+\wedge\mathsf{M}\mathbf{Q}$ determined by the Bott element of $\Sigma^\infty\mathbf{P}^\infty_+$ [44]. The diagonal map is an isomorphism. Inverting the generator and the Bott element gives the desired isomorphism. □

Lemma 4.6 furnishes an $\Sigma^\infty\mathbf{P}^\infty_+$–algebra structure on $\mathsf{PM}\mathbf{Q}$ via the map

$$\Sigma^\infty\mathbf{P}^\infty_+ \to \mathsf{KGL} \to \mathsf{KGL}\wedge\mathsf{M}\mathbf{Q} \cong \mathsf{PM}\mathbf{Q}.$$

Combining Lemma 4.6 and the canonical ring map $\mathsf{KGL}\to\mathsf{KGL}\wedge\mathsf{M}\mathbf{Q}$ we arrive at the Chern character

$$\mathsf{Ch}\colon \mathsf{KGL}\to\mathsf{PM}\mathbf{Q} \tag{6}$$

from algebraic K–theory to the periodized rational motivic Eilenberg–Mac Lane spectrum. (This is a map of motivic ring spectra.) For any twist τ, smashing (6) with $\mathscr{X}^\tau$ in the homotopy category of $\Sigma^\infty\mathbf{P}^\infty_+$–modules defines the twisted Chern character

$$\mathsf{Ch}^\tau\colon \mathsf{KGL}^\tau\to\mathsf{PM}^\tau\mathbf{Q}. \tag{7}$$

As asserted in Theorem 2.3, the rationalization of (7) is an isomorphism for geometrically unibranched excellent base schemes. This follows by combining [10, Corollary 15.1.6; 33, Theorem 10.1, Corollary 10.3].

Similarly we define the cohomological Chern character

$$\mathsf{Ch}_\tau\colon \mathsf{KGL}_\tau \to \mathsf{PM}_\tau\mathbf{Q} \tag{8}$$

by taking internal hom-objects from $\Sigma^\infty\mathscr{X}^\tau_+$ into the untwisted Chern character Ch. We note that the rationalization of (8) is an isomorphism over geometrically unibranched excellent base schemes provided $\Sigma^\infty\mathscr{X}^\tau_+$ is strongly dualizable in $\mathbf{Ho}(\Sigma^\infty\mathbf{P}^\infty_+-\mathbf{Mod})$. Indeed, this follows immediately by smashing the rational isomorphism in (6) with the dual of $\Sigma^\infty\mathscr{X}^\tau_+$.

Remark 4.7 In the topological setup, Atiyah and Segal [5] employed a different method in order to construct a Chern character for twisted K–theory and a corresponding theory of Chern classes. We leave the comparison of the two constructions as an open question.

We end this section by outlining computations of nontrivial twisted K–groups for the motivic $(3,1)$–sphere. To begin with we allow the base scheme to be an arbitrary field. In the interest of explicit computations in all degrees, we specialize to finite fields.

Recall the smash product decomposition $S^{3,1} = S^2\wedge\mathbf{G}_{\mathrm{m}}$ for the motivic $(3,1)$–sphere. Moreover, there is a homotopy pushout square of motivic spaces:

$$\begin{tikzcd} \mathbf{P}^1 \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & S^{3,1} \end{tikzcd}$$

We shall consider the twist $\tau_n\colon S^{3,1}\to \mathsf{B}\mathsf{B}\mathbf{G}_{\mathrm{m}}$ corresponding to n times the canonical map $S^{3,1}\to \mathsf{B}\mathsf{B}\mathbf{G}_{\mathrm{m}}$. Precomposing with the map $*\to S^{3,1}$ produce null homotopic twists on $\mathbf{P}^1$ and the point. In order to proceed we infer, leaving details to the interested reader, there is a homotopy pushout diagram:

$$\begin{tikzcd} \mathbf{P}^\infty\times\mathbf{P}^1 \arrow[r] \arrow[d] & \mathbf{P}^\infty\times(\mathbf{P}^\infty)^n \arrow[r] & \mathbf{P}^\infty \arrow[d] \\ \mathbf{P}^\infty \arrow[rr] & & (S^{3,1})^{\tau_n} \end{tikzcd}$$

The left vertical map is the projection on the first factor. The upper composite horizontal map arise from embedding $\mathbf{P}^1$ into $(\mathbf{P}^\infty)^n$ along the diagonal map $\mathbf{P}^\infty\subseteq(\mathbf{P}^\infty)^n$ and using the H–space structure on the infinite projective space. With this in hand we get an induced long exact sequence

$$\cdots\to\Sigma^{2,1}\mathsf{KGL}_*\oplus\mathsf{KGL}_*\to\mathsf{KGL}_*\oplus\mathsf{KGL}_*\to\mathsf{KGL}^{\tau_n}_*(S^{3,1})\to\cdots. \tag{9}$$

Next we infer that the map between the direct sums in (9) is uniformly given by

$$(a,b) \mapsto (an\beta + b, -b). \tag{10}$$

Again we leave the details to the interested reader. (Note that (10) is compatible with its evident topological counterpart.) From (9) we deduce the exact sequence

$$K_1 \oplus K_1 \to K_1 \oplus K_1 \to K_1^{\tau_n}(S^{3,1}) \to K_0 \oplus K_0 \to K_0 \oplus K_0 \to K_0^{\tau_n}(S^{3,1}) \to 0. \tag{11}$$

Using (11) and the fact that K_0 is infinite cyclic for any field we read off the isomorphism

$$K_0^{\tau_n}(S^{3,1}) \cong \mathbf{Z}/n,$$

where, in general, $K_i^{\tau}(\mathscr{X})$ is shorthand for $\mathsf{KGL}^{\tau}_{i,0}(\mathscr{X})$, $i \in \mathbf{Z}$. By specializing to a finite field $\mathbb{F}_q$ and an odd integer $i \geq 1$, we deduce the exact sequence

$$0 \to K_{i+1}^{\tau_n}(S^{3,1}_{\mathbb{F}_q}) \to K_i \oplus K_i \to K_i \oplus K_i \to K_i^{\tau_n}(S^{3,1}_{\mathbb{F}_q}) \to 0. \tag{12}$$

This follows from (9) since the K–groups for finite fields vanish in positive even degrees [35]. Combining (10) and (12) yields the isomorphisms

$$\begin{aligned} K_{2i}^{\tau_n}(S^{3,1}_{\mathbb{F}_q}) &\cong \ker(\mathbf{Z}/(q^i-1) \xrightarrow{\times n} \mathbf{Z}/(q^i-1)) \cong \mathbf{Z}/\gcd(n, q^i-1), \\ K_{2i-1}^{\tau_n}(S^{3,1}_{\mathbb{F}_q}) &\cong \mathbf{Z}/\gcd(n, q^i-1). \end{aligned}$$

5 Spectral sequences for motivic twisted K–theory

In this section we shall construct and show strong convergence of the spectral sequences relating motivic (co)homology to motivic twisted K–theory. The review of this material in Section 2 provides motivation and background from K–theory. Our approach employs the slice tower formalism introduced by Voevodsky [49] and further developed from the viewpoint of colored operads by Gutiérrez, Röndigs, Spitzweck and Østvær [20].

Let r_i denote the right adjoint of the natural inclusion functor $\Sigma^i_T\mathbf{SH}(S)^{\mathbf{eff}} \subseteq \mathbf{SH}(S)$. Define $f_i\colon \mathbf{SH}(S) \to \mathbf{SH}(S)$ as the composite functor

$$\mathbf{SH}(S) \xrightarrow{r_i} \Sigma^i_T\mathbf{SH}(S)^{\mathbf{eff}} \subseteq \mathbf{SH}(S).$$

In [49] the i–th slice $s_i(X)$ of X is defined as the cofiber of the canonical map

$$f_{i+1}(X) \to f_i(X).$$

In the companion paper [20] we show that f_0 and s_0 respect motivic E_∞–structures, and f_q and s_q respect module structures over E_∞–algebras. Recall that f_0 is reminiscent of the connective cover in topology. As a sample result we state the following key result [20, Theorem 5.14].

Theorem 5.1 *Suppose A is an A_∞– or an E_∞–algebra in $\mathsf{Spt}_T^\Sigma(S)$. Then $f_0(A)$ is naturally equipped with the structure of an A_∞– resp. E_∞–algebra. The canonical map $f_0 A \to A$ can be modelled as a map of A_∞– resp. E_∞–algebras.*

The corresponding statements dealing with s_0 and modules are formulated in [20]. In the interest of keeping this paper concise we refer to loc. cit. for further details.

We define the connective K–theory spectrum kgl to be $f_0\mathsf{KGL}$. With this definition, kgl is a $\Sigma^\infty\mathbf{P}^\infty_+$–module (even an $\Sigma^\infty\mathbf{P}^\infty_+$–algebra) because the E_∞–map $\Sigma^\infty\mathbf{P}^\infty_+ \to \mathsf{KGL}$ factors uniquely through the connective K–theory spectrum. Here we use that f_0 is a lax monoidal functor that respects E_∞–objects; see Theorem 5.1. More generally, $f_i\mathsf{KGL} = \Sigma^{2i,i}\mathsf{kgl}$ is a $\Sigma^\infty\mathbf{P}^\infty_+$–module. (The two possible module structures, using either the shift functor or the fact that f_i produces a module over f_0, coincide.) Moreover, $f_{i+1}\mathsf{KGL} \to f_i\mathsf{KGL}$ is a kgl–module map, hence a $\Sigma^\infty\mathbf{P}^\infty_+$–module map. By stitching these maps together we obtain a sequential filtration of KGL by shifted copies of the connective K–theory spectrum

$$\cdots \to \Sigma^{2i+2,i+1}\mathsf{kgl} \to \Sigma^{2i,i}\mathsf{kgl} \to \cdots \to \mathsf{KGL}. \tag{13}$$

The filtration (13) coincides with the slice filtration of KGL up to isomorphism. The maps in (13) are $\Sigma^\infty\mathbf{P}^\infty_+$–module maps. Hence for every twist $\tau\colon \mathcal{X} \to \mathsf{B}\mathsf{B}\mathbf{G}_{\mathrm{m}}$ there is an induced filtration of the motivic twisted K–theory spectrum

$$\begin{aligned}\cdots \to \Sigma^\infty\mathcal{X}^\tau_+ \wedge_{\Sigma^\infty\mathbf{P}^\infty_+} \Sigma^{2i+2,i+1}\mathsf{kgl} &\to \Sigma^\infty\mathcal{X}^\tau_+ \wedge_{\Sigma^\infty\mathbf{P}^\infty_+} \Sigma^{2i,i}\mathsf{kgl} \\ &\to \cdots \to \Sigma^\infty\mathcal{X}^\tau_+ \wedge_{\Sigma^\infty\mathbf{P}^\infty_+} \mathsf{KGL} = \mathsf{KGL}^\tau.\end{aligned} \tag{14}$$

Likewise, by applying the functor $\underline{\mathrm{Hom}}_{\Sigma^\infty\mathbf{P}^\infty_+}(\Sigma^\infty\mathcal{X}^\tau_+, -)$ to the filtration (13) of KGL we obtain a filtration of KGL_τ taking the form

$$\begin{aligned}\cdots \to \underline{\mathrm{Hom}}_{\Sigma^\infty\mathbf{P}^\infty_+}&(\Sigma^\infty\mathcal{X}^\tau_+, \Sigma^{2i+2,i+1}\mathsf{kgl}) \\ &\to \underline{\mathrm{Hom}}_{\Sigma^\infty\mathbf{P}^\infty_+}(\Sigma^\infty\mathcal{X}^\tau_+, \Sigma^{2i,i}\mathsf{kgl}) \\ &\to \cdots \to \underline{\mathrm{Hom}}_{\Sigma^\infty\mathbf{P}^\infty_+}(\Sigma^\infty\mathcal{X}^\tau_+, \mathsf{KGL}) = \mathsf{KGL}_\tau.\end{aligned} \tag{15}$$

Our next objective is to identify the filtration quotients $\mathsf{Q}_i(\mathcal{X}^\tau)$ of the tower (14) and $\mathsf{Q}^i(\mathcal{X}^\tau)$ of the tower (15). Note that the tower (14) gives rise to an exact couple by applying homotopy groups for a fixed weight:

$$\begin{tikzcd}
\pi_*\Sigma^\infty\mathcal{X}^\tau_+ \wedge_{\Sigma^\infty\mathbf{P}^\infty_+} \Sigma^{2i+2,i+1}\mathsf{kgl} \arrow[rr] & & \pi_*\Sigma^\infty\mathcal{X}^\tau_+ \wedge_{\Sigma^\infty\mathbf{P}^\infty_+} \Sigma^{2i,i}\mathsf{kgl} \arrow[dl] \\
& \pi_*\mathsf{Q}_i(\mathcal{X}^\tau) \arrow[ul] &
\end{tikzcd}$$

Similarly, the tower (15) gives rise to an exact couple featuring the quotients $\mathsf{Q}^i(\mathcal{X}^\tau)$. Following a standard process we obtain spectral sequences with target graded groups KGL^τ_* and KGL^*_τ. In the following we analyze these spectral sequences in detail when the base scheme is a perfect field.

From now on we assume that the base scheme S is a perfect field. Using the slice computations of KGL in [28; 50; 51], there is an exact triangle of $\Sigma^\infty\mathbf{P}^\infty_+$–modules

$$\Sigma^{2,1}\mathsf{kgl} \to \mathsf{kgl} \to \mathsf{M}\mathbf{Z} \to \Sigma^{3,1}\mathsf{kgl}.$$

Thus the filtration quotient $\mathsf{Q}_i(\mathcal{X}^\tau)$ is isomorphic to

$$\Sigma^\infty\mathcal{X}^\tau_+ \wedge_{\Sigma^\infty\mathbf{P}^\infty_+} \Sigma^{2i,i}\mathsf{M}\mathbf{Z},$$

whereas the filtration quotient $\mathsf{Q}^i(\mathcal{X}^\tau)$ is isomorphic to

$$\underline{\mathrm{Hom}}_{\Sigma^\infty\mathbf{P}^\infty_+}(\Sigma^\infty\mathcal{X}^\tau_+, \mathsf{M}\mathbf{Z}).$$

Lemma 5.2 *The unit map $\mathbf{1} \to \Sigma^\infty\mathbf{P}^\infty_+$ induces an isomorphism on zero slices.*

Proof Induction on the cofiber sequence

$$\Sigma^\infty\mathbf{P}^{n-1}_+ \to \Sigma^\infty\mathbf{P}^n_+ \to \Sigma^{2n,n}\mathbf{1}$$

gives the isomorphism

$$s_0\mathbf{1} \cong s_0\Sigma^\infty\mathbf{P}^n_+.$$

To conclude we use that s_0 commutes with homotopy colimits; cf [43, Lemma 4.4]. □

Lemma 5.3 *The diagram of E_∞–ring spectra*

$$\begin{tikzcd}
\Sigma^\infty\mathbf{P}^\infty_+ \arrow[r] \arrow[dr] & \mathsf{kgl} \arrow[r] & \mathsf{M}\mathbf{Z} \\
& \mathbf{1} \arrow[ur] &
\end{tikzcd}$$

commutes.

Proof By Lemma 5.2 and the isomorphism $s_0\mathbf{1} \cong \mathsf{M}\mathbf{Z}$, applying the zero slice functor to the maps $\Sigma^\infty \mathbf{P}^\infty_+ \to \mathsf{kgl}$ and $\Sigma^\infty \mathbf{P}^\infty_+ \to \mathbf{1}$ produces diagrams of E_∞–ring spectra:

$$\begin{tikzcd} \Sigma^\infty \mathbf{P}^\infty_+ \arrow[r] \arrow[d] & \mathsf{kgl} \arrow[d] \\ \mathsf{M}\mathbf{Z} \arrow[r, "="] & \mathsf{M}\mathbf{Z} \end{tikzcd} \qquad \begin{tikzcd} \Sigma^\infty \mathbf{P}^\infty_+ \arrow[r] \arrow[d] & \mathbf{1} \arrow[d] \\ \mathsf{M}\mathbf{Z} \arrow[r, "="] & \mathsf{M}\mathbf{Z} \end{tikzcd}$$

In these diagrams, the vertical maps are induced by the natural transformation $f_0 \to s_0$ [20, Theorem 5.17]. Now, since the construction of E_∞–structures on zero slices from [20, Theorem 5.1] is not transparently functorial, a trick involving colored operads seems to be required in order to verify commutativity of the two diagrams. This follows by applying the (co)localization machinery of [20] to the two-colored operad whose algebras comprise maps between E_∞–algebras; cf [20, Theorem 5.20]. □

Theorem 5.4 *There exists an isomorphism in the motivic stable homotopy category between the filtration quotient $\mathsf{Q}_i(\mathscr{X}^\tau)$ of (14) and the $(2i,i)$–suspension of the motive $\mathsf{M}\mathbf{Z} \wedge \Sigma^\infty \mathscr{X}_+$ of $\mathscr{X}$. Likewise, there exists an isomorphism between the filtration quotient $\mathsf{Q}^i(\mathscr{X}^\tau)$ of (15) and the $(2i,i)$–suspension of the internal hom $\underline{\mathrm{Hom}}(\Sigma^\infty \mathscr{X}_+, \mathsf{M}\mathbf{Z})$.*

Proof It suffices to consider the case $i=0$. The 0–th filtration quotient $\mathsf{Q}_0(\mathscr{X}^\tau)$ identifies with $\Sigma^\infty \mathscr{X}^\tau_+ \wedge_{\Sigma^\infty \mathbf{P}^\infty_+} \mathsf{M}\mathbf{Z}$. By Lemma 5.3 there is an isomorphism

$$\Sigma^\infty \mathscr{X}^\tau_+ \wedge_{\Sigma^\infty \mathbf{P}^\infty_+} \mathsf{M}\mathbf{Z} \cong (\Sigma^\infty \mathscr{X}^\tau_+ \wedge_{\Sigma^\infty \mathbf{P}^\infty_+} \mathbf{1}) \wedge_{\mathbf{1}} \mathsf{M}\mathbf{Z}.$$

Lemma 3.6 implies there is an isomorphism

$$(\Sigma^\infty \mathscr{X}^\tau_+ \wedge_{\Sigma^\infty \mathbf{P}^\infty_+} \mathbf{1}) \wedge_{\mathbf{1}} \mathsf{M}\mathbf{Z} \cong \Sigma^\infty \mathscr{X}_+ \wedge \mathsf{M}\mathbf{Z}.$$

The proof of the statement for $\mathsf{Q}^i(\mathscr{X}^\tau)$ proceeds similarly by comparing the module categories over $\Sigma^\infty \mathbf{P}^\infty_+$ and $\mathsf{M}\mathbf{Z}$ via the isomorphisms

$$\begin{aligned} \mathsf{Q}^0(\mathscr{X}^\tau) &\cong \underline{\mathrm{Hom}}_{\Sigma^\infty \mathbf{P}^\infty_+}(\Sigma^\infty \mathscr{X}^\tau_+, \mathsf{M}\mathbf{Z}) \cong \underline{\mathrm{Hom}}_{\mathsf{M}\mathbf{Z}}(\Sigma^\infty \mathscr{X}^\tau_+ \wedge_{\Sigma^\infty \mathbf{P}^\infty_+} \mathsf{M}\mathbf{Z}, \mathsf{M}\mathbf{Z}) \\ &\cong \underline{\mathrm{Hom}}_{\mathsf{M}\mathbf{Z}}(\Sigma^\infty \mathscr{X}_+ \wedge \mathsf{M}\mathbf{Z}, \mathsf{M}\mathbf{Z}) \cong \underline{\mathrm{Hom}}(\Sigma^\infty \mathscr{X}_+, \mathsf{M}\mathbf{Z}). \end{aligned}$$ □

The isomorphisms in Theorem 5.4 are clearly functorial in $\mathscr{X}$ and τ. It is important to note that the filtration quotients $\mathsf{Q}_i(\mathscr{X}^\tau)$ and $\mathsf{Q}^i(\mathscr{X}^\tau)$ are independent of the twist.

Theorem 5.4 implies there exist spectral sequences

$$\mathsf{M}\mathbf{Z}_*(\Sigma^\infty \mathscr{X}_+) \Longrightarrow \mathsf{KGL}^\tau_*(\mathscr{X}), \tag{16}$$

$$\mathsf{M}\mathbf{Z}^*(\Sigma^\infty \mathscr{X}_+) \Longrightarrow \mathsf{KGL}_\tau^*(\mathscr{X}) \tag{17}$$

relating motivic homology and cohomology to motivic twisted K–theory. In what follows we shall discuss the convergence properties of (16) and (17). Our approach makes use of the notion of "very effectiveness" which is of independent interest in motivic homotopy theory over any base scheme S. In order to make this precise we introduce the following subcategory of $\mathbf{SH}(S)$.

Definition 5.5 The very effective motivic stable homotopy category $\mathbf{SH}(S)^{\mathbf{Veff}}$ is the smallest full subcategory of $\mathbf{SH}(S)$ that contains all suspension spectra of smooth schemes of finite type over S and is closed under extensions and homotopy colimits.

We note that $\mathbf{SH}(S)^{\mathbf{Veff}}$ is not a triangulated category since it is not closed under simplicial desuspension. However, it is a subcategory of the effective motivic stable homotopy category, which we denote by $\mathbf{SH}(S)^{\mathbf{eff}}$. We remark that $\mathbf{SH}(S)^{\mathbf{Veff}}$ forms the homologically positive part of t–structures on $\mathbf{SH}(S)$ and $\mathbf{SH}(S)^{\mathbf{eff}}$.

Lemma 5.6 *The subcategory* $\mathbf{SH}(S)^{\mathbf{Veff}}$ *of* $\mathbf{SH}(S)$ *is closed under the smash product.*

Proof To begin with, suppose $\mathsf{E} \in \mathbf{SH}(S)^{\mathbf{Veff}}$ and $X \in \mathbf{Sm}$. Then $\Sigma^\infty X_+ \wedge \mathsf{E}$ lies in $\mathbf{SH}(S)^{\mathbf{Veff}}$ by the following "induction" argument on the form of E. It clearly holds when $\mathsf{E} = \Sigma^\infty Y_+$ for some $Y \in \mathbf{Sm}$. Suppose $\mathsf{E} = \operatorname{hocolim} \mathsf{E}_i$ and $\Sigma^\infty X_+ \wedge \mathsf{E}_i \in \mathbf{SH}(S)^{\mathbf{Veff}}$. Then $\Sigma^\infty X_+ \wedge \mathsf{E} \in \mathbf{SH}(S)^{\mathbf{Veff}}$ because $\mathbf{SH}(S)^{\mathbf{Veff}}$ is closed under homotopy colimits. Furthermore, if in a triangle

$$\mathsf{A} \to \mathsf{E} \to \mathsf{B} \to \mathsf{A}[1],$$

$\Sigma^\infty X_+ \wedge \mathsf{A} \in \mathbf{SH}(S)^{\mathbf{Veff}}$ and likewise for B, then $\Sigma^\infty X_+ \wedge \mathsf{E} \in \mathbf{SH}(S)^{\mathbf{Veff}}$ because $\mathbf{SH}(S)^{\mathbf{Veff}}$ is closed under extensions by definition. A similar "induction" argument in the first variable shows now that for all objects $\mathsf{F}, \mathsf{E} \in \mathbf{SH}(S)^{\mathbf{Veff}}$ the smash product $\mathsf{F} \wedge \mathsf{E} \in \mathbf{SH}(S)^{\mathbf{Veff}}$. □

For the definition of the algebraic cobordism spectrum MGL, refer to [48]. One of the reasons why the category $\mathbf{SH}(S)^{\mathbf{Veff}}$ is of interest is that it contains MGL for general base schemes.

Theorem 5.7 *The algebraic cobordism spectrum* MGL *is very effective.*

In fact our proof of Theorem 5.7 shows the following stronger statement: The cofiber of the unit map $\mathbf{1} \to \mathsf{MGL}$ is contained in $\Sigma_T \mathbf{SH}(S)^{\Delta}_{\geq 0}$, where $\mathbf{SH}(S)^{\Delta}_{\geq 0}$ is the smallest full saturated subcategory of $\mathbf{SH}(S)$ that contains the suspension spectra $\Sigma^{2i,i}\mathbf{1}$ for every $i \geq 0$ and is closed under homotopy colimits and extensions. The notation Σ_T refers to suspension with respect to the Tate object, ie $\Sigma_T = \Sigma^{2,1}$ in the usual bigrading.

Lemma 5.8 *Let r be an integer and suppose*

$$\Sigma^{2r,r}\mathbf{1} \to \mathsf{A} \to \mathsf{B} \to \Sigma^{2r+1,r}\mathbf{1},$$
$$\mathsf{A} \to \mathsf{E} \to \mathsf{F} \to \mathsf{A}[1]$$

are triangles in $\mathbf{SH}(S)$. If $\mathsf{B}, \mathsf{F} \in \Sigma_T^{r+1}\mathbf{SH}(S)^{\Delta}_{\geq 0}$ then the cofiber of $\Sigma^{2r,r}\mathbf{1} \to \mathsf{E}$ lies in $\Sigma_T^{r+1}\mathbf{SH}(S)^{\Delta}_{\geq 0}$.

Proof This follows since the cofiber of $\Sigma^{2r,r}\mathbf{1} \to \mathsf{E}$ is an extension of F by B and the category $\Sigma_T^{r+1}\mathbf{SH}(S)^{\Delta}_{\geq 0}$ is closed under extensions. □

Let $\mathsf{G}(n,d)$ denote the Grassmannian parametrizing locally free quotients of rank d of the trivial bundle of rank n. Recall there is a universal subsheaf $\mathcal{K}_{n,d}$ of $\mathcal{O}^n$ and a natural map $\iota\colon \mathsf{G}(n,d) \to \mathsf{G}(n+1,d)$ that classifies the subbundle $\mathcal{K}_{n,d} \oplus \mathcal{O}$ of $\mathcal{O}^{n+1}$. Denote by $\overline{\iota}$ the canonical point of $\mathsf{G}(n,d)$ obtained by the composite map

$$* \cong \mathsf{G}(d,d) \xrightarrow{\iota} \mathsf{G}(d+1,d) \xrightarrow{\iota} \cdots \xrightarrow{\iota} \mathsf{G}(n,d).$$

We are interested in vector bundles of a particular type over Grassmannians.

Proposition 5.9 *Suppose $\mathcal{E}$ is a vector bundle of rank r over the Grassmannian $\mathsf{G}(n,d)$ which is a finite sum of copies of $\mathcal{K}_{n,d}$ and its dual $\mathcal{K}'_{n,d}$ and $\mathcal{O}$. Then $\overline{\iota}^*\mathcal{E}$ is canonically trivialized. Furthermore the cofiber of the map between the suspension spectra of Thom spaces $\Sigma^{2r,r}\mathbf{1} \to \Sigma^\infty\mathsf{Th}(\mathcal{E})$ lies in $\Sigma_T^{r+1}\mathbf{SH}(S)^{\Delta}_{\geq 0}$.*

Proof We outline an argument which is reminiscent of the one for [43, Proposition 3.6]. The first step of the proof consists of showing there is an exact triangle

$$\Sigma^\infty\mathsf{Th}(\iota^*\mathcal{E}) \to \Sigma^\infty\mathsf{Th}(\mathcal{E}) \to \Sigma^\infty\mathsf{Th}(\mathcal{E}_{\mathsf{G}(n,d)} \oplus \mathcal{K}'_{n,d}) \to \Sigma^\infty\mathsf{Th}(\iota^*\mathcal{E})[1]$$

for the canonical map $\iota\colon \mathsf{G}(n,d+1) \to \mathsf{G}(n+1,d+1)$ (that classifies the subbundle $\mathcal{K}_{n,d+1} \oplus \mathcal{O} \subseteq \mathcal{O}^{n+1}$). By induction we deduce that the cofiber of the canonical map $\Sigma^{2r,r}\mathbf{1} \to \Sigma^\infty\mathsf{Th}(\iota^*\mathcal{E})$ lies in $\Sigma_T^{r+1}\mathbf{SH}(S)^{\Delta}_{\geq 0}$. Again by applying induction, it follows that $\Sigma^\infty\mathsf{Th}(\mathcal{E}_{\mathsf{G}(n,d)} \oplus \mathcal{K}'_{n,d}) \in \Sigma_T^{r+j}\mathbf{SH}(S)^{\Delta}_{\geq 0}$, where $j = n-d > 0$. The proposition follows now from Lemma 5.8. □

Next we give a proof of Theorem 5.7.

Proof We denote by $\xi_n = \operatorname{colim}_d \mathcal{K}_{n+d,d}$ the universal vector bundle over the infinite Grassmannian $\mathsf{B}\mathbf{GL}_n = \operatorname{colim}_d \mathsf{G}(n+1,d)$, and write

$$\mathsf{MGL} = \operatorname{hocolim}_n \Sigma^{-2n,-n}\Sigma^\infty\mathsf{Th}(\xi_n) = \operatorname{hocolim}_{n,d} \Sigma^{-2n,-n}\Sigma^\infty\mathsf{Th}(\mathcal{K}_{n+d,d}).$$

The unit map $\mathbf{1} \to \mathsf{MGL}$ is in turn induced by the maps

$$\Sigma^{-2n,-n}\Sigma^{\infty}\mathsf{Th}(\overline{\iota}^{*}\mathcal{K}_{n+d,d}) \to \Sigma^{-2n,-n}\Sigma^{\infty}\mathsf{Th}(\mathcal{K}_{n+d,d}).$$

By Proposition 5.9 the cofibers of these maps are contained in $\Sigma_T\mathbf{SH}(S)^{\Delta}_{\geq 0}$. Since cofiber sequences are compatible with homotopy colimits, this finishes the proof. □

Lemma 5.10 *Let $\mathsf{E} \in \mathbf{SH}(S)^{\mathbf{Veff}}$ and suppose S is the spectrum of a perfect field. Then the homotopy group $\pi_{p,q}(\mathsf{E}) = 0$ for $p < q$.*

Proof For suspension spectra of smooth projective schemes of finite type the claimed vanishing is stated in [31, Section 5.3]. Suppose $\mathsf{E} = \operatorname{hocolim} \mathsf{E}_i$ where every E_i satisfies the conclusion of the lemma. Minor variations of [29, Corollary 4.4.2.4, Proposition 4.4.2.6] allows us to assuming the homotopy colimit is either a coproduct or a homotopy pushout. For coproducts the result is clear, while for homotopy colimits the corresponding long exact sequence of homotopy sheaves implies the vanishing. For a general extension $\mathsf{A} \to \mathsf{E} \to \mathsf{B} \to \mathsf{A}[1]$, where the vanishing holds for A and B, the corresponding long exact sequence of homotopy groups implies the result. □

We denote by $\mathbf{SH}(S)^{\mathbf{proj}}$ the full thick subcategory of $\mathbf{SH}(S)$ generated by the objects $\Sigma^i_T\Sigma^{\infty}X_+$ for $X \in \mathbf{Sm}$ a projective scheme and $i \in \mathbf{Z}$.

Proposition 5.11 *Suppose the base scheme S is a perfect field. Let*

$$\cdots \to \mathsf{E}_{i+1} \to \mathsf{E}_i \to \mathsf{E}_{i-1} \to \cdots \to \mathsf{E}$$

be a tower of motivic spectra such that $\operatorname{hocolim} \mathsf{E}_i = \mathsf{E}$ *and denote the corresponding filtration quotients by Q_i. Suppose that $\mathsf{E}_i \in \Sigma^i_T\mathbf{SH}(S)^{\mathbf{Veff}}$ and $X \in \mathbf{SH}(S)^{\mathbf{proj}}$. If for each fixed n the groups* $\operatorname{Hom}(X, \mathsf{E}_i[n])$ *stabilize as i tends to minus infinity, then the spectral sequence of the tower with E_2–term* $\operatorname{Hom}(X, \mathsf{Q}_*[*])$ *and target graded group* $\operatorname{Hom}(X, \mathsf{E}[*])$ *converges strongly.*

Proof Smashing the tower with the Spanier-Whitehead dual $D(X)$ of X produces a tower with terms $D(X) \wedge \mathsf{E}_i \in \Sigma^{i+n}_T\mathbf{SH}(S)^{\mathbf{Veff}}$ for a fixed integer n. Hence we may assume that X is the sphere spectrum because smooth projective schemes of finite type over S are dualizable [24]. The spectral sequence obtained from the exact couple associated to the tower is strongly convergent due to Lemma 5.10. □

For the complex cobordism spectrum MU, fix an isomorphism $\mathsf{MU}_* \cong \mathbf{Z}[\mathsf{x}_1, \mathsf{x}_2, \dots]$ where $|\mathsf{x}_i| = i$ (this is half of the usual topological grading) and consider the canonical map $\mathsf{MU}_* \to \mathsf{MGL}_*$.

Proposition 5.12 *Over fields of characteristic zero there is a natural isomorphism from the quotient of* MGL *by the sequence* $(\mathsf{x}_i)_{i\geq 2}$ *to* $\mathsf{kgl} = f_0\mathsf{KGL}$.

Proof The orientation map $\mathsf{MGL} \to \mathsf{KGL}$ sends $\mathsf{x}_i \in \mathsf{MGL}_{2i,i}$ to 0 in $\mathsf{KGL}_{2i,i}$ for $i \geq 2$; see eg [44, Example 2.2]. Hence there is a naturally induced map from the quotient of MGL by the sequence $(\mathsf{x}_i)_{i\geq 2}$ to KGL. Since the quotient $\mathsf{MGL}/(\mathsf{x}_i)_{i\geq 2}$ is an effective spectrum we obtain the desired map to kgl. As shown in [43, Proposition 5.4] this map induces an isomorphism on all slices. (The proof in loc. cit. employs the work of Hopkins and Morel on quotients of MGL.) For any X of $\mathbf{SH}(S)^{\mathbf{proj}}$ we may consider the spectral sequences obtained by taking homs into the respective slice filtrations of kgl and the quotient. Theorem 5.7 and Proposition 5.11 ensure that the spectral sequence for the quotient is strongly convergent. For kgl, strong convergence holds by [50, Proposition 5.5]. (Note that [50, Conjecture 4] is proven in [28]; cf the introduction in loc. cit. for a discussion.) Our claim follows now by comparing the target graded groups of these spectral sequences. □

Corollary 5.13 *Over fields of characteristic zero the connective K–theory spectrum* kgl *is very effective.*

Proof Combine Theorem 5.7 and Proposition 5.12 with the fact that very effectiveness is preserved under homotopy colimits. □

Lemma 5.14 *The motivic spectrum* $\Sigma^\infty\mathscr{X}^\tau_+ \wedge_{\Sigma^\infty\mathbf{P}^\infty_+} \mathsf{kgl}$ *is very effective.*

Proof For $n \geq 0$, Corollary 5.13 shows the smash product $\Sigma^\infty\mathscr{X}^\tau_+ \wedge (\Sigma^\infty\mathbf{P}^\infty_+)^{\wedge n} \wedge \mathsf{kgl}$ is very effective since $\mathbf{SH}(S)^{\mathbf{Veff}}$ is closed under smash products in $\mathbf{SH}(S)$ according to Lemma 5.6. When n varies,

$$n \mapsto \Sigma^\infty\mathscr{X}^\tau_+ \wedge (\Sigma^\infty\mathbf{P}^\infty_+)^{\wedge n} \wedge \mathsf{kgl}$$

defines a simplicial object in motivic symmetric spectra. Its homotopy colimit is very effective and identifies with the smash product $\Sigma^\infty\mathscr{X}^\tau_+ \wedge_{\Sigma^\infty\mathbf{P}^\infty_+} \mathsf{kgl}$. □

We denote by $\mathbf{Ho}(\Sigma^\infty\mathbf{P}^\infty_+ - \mathbf{Mod})^{\mathbf{proj}}$ the full thick subcategory of the homotopy category $\mathbf{Ho}(\Sigma^\infty\mathbf{P}^\infty_+ - \mathbf{Mod})$ generated by the objects $\Sigma^i_T\Sigma^\infty\mathbf{P}^\infty_+ \wedge \Sigma^\infty X_+$ for $X \in \mathbf{Sm}$ projective and $i \in \mathbf{Z}$.

Lemma 5.15 *Suppose* $\Sigma^\infty\mathscr{X}^\tau_+$ *is an object of* $\mathbf{Ho}(\Sigma^\infty\mathbf{P}^\infty_+ - \mathbf{Mod})^{\mathbf{proj}}$. *Then there exists an integer* $n \in \mathbf{Z}$ *such that* $\underline{\mathrm{Hom}}_{\Sigma^\infty\mathbf{P}^\infty_+}(\Sigma^\infty\mathscr{X}^\tau_+, \mathsf{kgl})$ *lies in* $\Sigma^n_T\mathbf{SH}(S)^{\mathbf{Veff}}$.

Proof First, for every $X \in \mathbf{Sm}$, $i, j \in \mathbf{Z}$, there is an $n_1 \in \mathbf{Z}$ such that $\Sigma^{i,j}\Sigma^\infty X_+ \in \Sigma_T^{n_1}\mathbf{SH}(S)^{\mathbf{Veff}}$. For X projective we get that $D(\Sigma^\infty X_+) \in \Sigma_T^{n_2}\mathbf{SH}(S)^{\mathbf{Veff}}$ for some $n_2 \in \mathbf{Z}$ by [24, Appendix] and we conclude that

$$\underline{\mathrm{Hom}}_{\Sigma^\infty \mathbf{P}^\infty_+}(\Sigma^{i,j}\Sigma^\infty \mathbf{P}^\infty_+ \wedge \Sigma^\infty X_+, \mathsf{kgl}) \cong \Sigma^{-i,-j} D(\Sigma^\infty X_+) \wedge \mathsf{kgl}$$

lies in $\Sigma_T^{n_3}\mathbf{SH}(S)^{\mathbf{Veff}}$ for some $n_3 \in \mathbf{Z}$ by Corollary 5.13. This shows the result for all of the generators of $\mathbf{Ho}(\Sigma^\infty\mathbf{P}^\infty_+ - \mathbf{Mod})^{\mathbf{proj}}$. The general case follows routinely by taking cones and direct summands. □

Remark 5.16 In the following we use implicitly the equivalence between the statements $\Sigma^\infty\mathscr{X}_+$ is compact.

Theorem 5.17 *Let S be a field of characteristic zero. Suppose $\Sigma^\infty\mathscr{X}_+$ is compact. Then the motivic twisted K–theory spectral sequence*

$$\mathsf{M}\mathbf{Z}_*(\Sigma^\infty\mathscr{X}_+) \Longrightarrow \mathsf{KGL}^\tau_*(\mathscr{X})$$

in (16) *is strongly convergent.*

Proof The proof follows by reference to Proposition 5.11 in the case when $X = S^{0,q}$. Two assumptions need to be checked, ie $E_i = \Sigma^{2i,i}\Sigma^\infty\mathscr{X}_+ \wedge_{\Sigma^\infty\mathbf{P}^\infty_+} \mathsf{kgl} \in \Sigma^i_{\mathbf{P}^1}\mathbf{SH}(S)^{\mathbf{Veff}}$ and stability. Very effectiveness and Lemma 5.14 verify that the first assumption holds. Second, the stabilization condition is equivalent to the fact that for a fixed n, the group $\pi_{n,q}\mathsf{Q}_i(\mathscr{X}^\tau) \neq 0$ for only finitely many i. The latter follows from the corresponding statement in the untwisted case because $\Sigma^\infty\mathscr{X}_+$ is strongly dualizable. Namely, letting $i \ll 0$ vary, the groups $\mathrm{Hom}(\Sigma^{n,q}D(\Sigma^\infty\mathscr{X}_+), f_i\mathsf{KGL})$ become isomorphic. □

Remark 5.18 In the proof of Theorem 5.17 we used implicitly the equivalence between $\Sigma^\infty\mathscr{X}_+$ being compact, strongly dualizable, and an object of $\mathbf{SH}(S)^{\mathbf{proj}}$. See eg [39] for more details.

Theorem 5.19 *Let S be a field of characteristic zero. Suppose $\Sigma^\infty\mathscr{X}^\tau_+$ is compact in $\mathbf{Ho}(\Sigma^\infty\mathbf{P}^\infty_+ - \mathbf{Mod})$, equivalently strongly dualizable. Then the motivic twisted K–theory spectral sequence*

$$\mathsf{M}\mathbf{Z}^*(\Sigma^\infty\mathscr{X}_+) \Longrightarrow \mathsf{KGL}^*_\tau(\mathscr{X})$$

in (17) *is strongly convergent.*

Proof We first note that the assumptions imply that $\Sigma^\infty\mathscr{X}_+$ is strongly dualizable by Lemma 3.6. The proof proceeds now along the lines of the proof of Theorem 5.17 with a reference to Lemma 5.15 for very effectiveness. □

Remark 5.20 The motivic twisted sphere $(S^{3,1})^{\tau_n}$ satisfies the condition in Theorems 5.17 and 5.19; cf. Section 4.

We end this section by discussing the closely related approach of the slice spectral sequence for KGL^{τ}. Recall that the slices of any motivic spectrum fit into the slice tower constructed by Voevodsky [49].

An identification of the zero-slice of KGL^{τ} would in turn determine all the slices $s_n\mathsf{KGL}^{\tau}$ by $(2,1)$–periodicity, ie there is an isomorphism in the motivic stable homotopy category

$$s_n\mathsf{KGL}^{\tau} \cong \Sigma^{2n,n} s_0\mathsf{KGL}^{\tau}.$$

This follows from the evident KGL–module structure on KGL^{τ} and the Bott periodicity isomorphism $\beta\colon \Sigma^{2,1}\mathsf{KGL} \to \mathsf{KGL}$ furnishing the composite isomorphism

$$\Sigma^{2,1}\mathsf{KGL}^{\tau} \to \Sigma^{2,1}\mathsf{KGL}^{\tau} \wedge \mathsf{KGL} \to \mathsf{KGL}^{\tau} \wedge \Sigma^{2,1}\mathsf{KGL} \to \mathsf{KGL}^{\tau} \wedge \mathsf{KGL} \to \mathsf{KGL}^{\tau}.$$

The same comments apply to KGL_{τ}. If the base scheme is a perfect field, then all of the slices $s_n\mathsf{KGL}^{\tau}$ and $s_n\mathsf{KGL}_{\tau}$ are in fact motives, ie modules over the integral motivic Eilenberg–Mac Lane spectrum $\mathsf{M}\mathbf{Z}$; cf [28; 34; 38; 39; 51; 52]. However, except for example when $\mathcal{X}$ is the point and τ the trivial twist, the slice spectral sequences cannot coincide with the spectral sequence constructed earlier in this section. Indeed the corresponding filtration quotients are different because by weight considerations the smash product $\mathcal{X} \wedge \mathsf{M}\mathbf{Z}$ is not a zero slice in general.

6 Further problems and questions

We end the main body of the paper by discussing some problems and questions related to motivic twisted K–theory.

A pressing question left open in the previous section is to identify the d_1–differentials in the spectral sequences.

Problem 6.1 Express the d_1–differentials in the slice spectral sequence for KGL_{τ} in terms of motivic Steenrod squares and the twist τ.

Remark 6.2 The d_3–differentials in the Atiyah–Hirzebruch spectral sequence for twisted K–theory were identified by Atiyah and Segal [5] as the difference between Sq^3 and the twisting. In the same paper the higher differentials are determined in terms of Massey products. One may ask if also the higher differentials in the slice spectral sequence can be described in terms of motivic Massey products.

Problem 6.3 For twists τ and τ' construct products

$$\mathsf{KGL}_{\tau} \wedge \mathsf{KGL}_{\tau'} \to \mathsf{KGL}_{\tau+\tau'}$$

and investigate its properties.

In Remark 4.5 we noted that all the motivic twisted K–groups of the identity map of $\mathsf{BB}\mathbf{G}_{\mathrm{m}}$ are trivial. More generally, if τ is any twisting of $\mathsf{BB}\mathbf{G}_{\mathrm{m}}$ one may ask if the motivic τ–twisted K–groups are trivial. This is the content of the next problem asking when $\mathsf{BB}\mathbf{G}_{\mathrm{m}}$ is a point for motivic twisted K–theory.

Problem 6.4 For which twists of $\mathsf{BB}\mathbf{G}_{\mathrm{m}}$ is the associated motivic twisted K–theory trivial?

Remark 6.5 The corresponding problem in topology has an affirmative solution for all twists by work of Anderson and Hodgkin [2]. By analogy, work on Problem 6.4 is likely to involve a computation of the KGL–homology of the motivic Eilenberg–Mac Lane spaces $K(\mathbf{Z}/n, 2)$ for $n \geq 1$ any integer.

Twisted equivariant K–theory for compact Lie groups is closely related to loop groups [15]. It is natural to ask for a generalization of our construction of motivic twisted K–theory to an equivariant setting involving group schemes.

Problem 6.6 Develop a theory of motivic twisted equivariant K–theory.

The last problem we suggest is a very basic one. The construction of motivic twisted K–theory should generalize to other examples. We wish to single out hermitian K–theory as a closely related example of much interest. In this case we expect the twistings arise from homotopy classes of maps from $\mathcal{X}$ to the classifying space $\mathsf{BB}\mu_2$, ie elements of the second mod–2 motivic cohomology group $\mathsf{M}\mathbf{Z}^{2,1}(\mathcal{X}; \mathbf{Z}/2)$ of weight one.

Problem 6.7 Develop a theory of motivic twisted hermitian K–theory.

7 Graded Adams Hopf algebroids

Recall that a Hopf algebroid is a cogroupoid object in the category of commutative rings [36, Appendix A1]. Let (A, Γ) be a Hopf algebroid. If the left unit $\eta_L\colon A \to \Gamma$ classifying the domain is flat, or equivalently the right unit $\eta_R\colon A \to \Gamma$ classifying the codomain is flat, then (A, Γ) is called a flat Hopf algebroid. If $A \to B$ is a ring map, we write $B \otimes_A \Gamma$ for the tensor product when Γ is given an A–module structure via η_L and $\Gamma \otimes_A B$ when Γ is given an A–module structure via η_R. An (A,Γ)–comodule comprises an A–module M together with a coassociative and unital map of left A–modules $M \to \Gamma \otimes_A M$ (see eg [36, Appendix A1]). The category of (A,Γ)–comodules with the evident notion of a morphism is an abelian category provided Γ is a flat right A–module via η_R.

Likewise, a graded Hopf algebroid is a cogroupoid object in the category of graded commutative rings [36, Appendix A1]. The notions of flat graded Hopf algebroids and comodules over a graded Hopf algebroid are defined exactly as in the ungraded setting.

The examples of Hopf algebroids of main interest in stable homotopy theory are so-called "Adams Hopf algebroids." In the graded setting we make the following definition: A graded Hopf algebroid (A,Γ) is called a graded Adams Hopf algebroid if Γ is the colimit of a filtered system of graded comodules which are finitely generated and projective as graded A–modules.

Proposition 7.1 *The pair* $(\mathsf{KGL}_*, \mathsf{KGL}_*\mathsf{KGL})$ *is a flat graded Adams Hopf algebroid.*

Proof We give two proofs of this result.

Since homology commutes with sequential colimits we get that $\mathsf{KGL}_*(\mathbf{P}^\infty)$ is a filtered colimit of comodules which are finitely generated free KGL_*–modules. Hence the same holds for $\mathsf{KGL}_*\mathsf{KGL}$ by using the Bott tower (1) for $\Sigma^\infty \mathbf{P}^\infty_+$ as a model for KGL.

For the second proof, recall the base change isomorphism in (5),

$$\mathsf{KGL}_*\mathsf{KGL} \cong \mathsf{KGL}_* \otimes_{\mathsf{KU}_*} \mathsf{KU}_*\mathsf{KU}.$$

By the topological analogue of the first proof we see that $(\mathsf{KU}_*, \mathsf{KU}_*\mathsf{KU})$ is a flat Adams Hopf algebroid. We conclude by pulling back the filtered colimit to the tensor product. $\square$

Proposition 7.2 *Let (A,Γ) be a graded Hopf algebroid and $A \to B$ a graded ring map. Suppose B is a graded (A,Γ)–comodule algebra.*

(i) *The pair $(B, B \otimes_A \Gamma)$ is a graded Hopf algebroid.*

(ii) *If (A,Γ) is flat, then so is $(B, B \otimes_A \Gamma)$.*

(iii) *If (A,Γ) is a graded Adams Hopf algebroid, then so is $(B, B \otimes_A \Gamma)$.*

Proof For C be a graded (commutative) algebra, let $X = \mathrm{Hom}(A,C)$, $M = \mathrm{Hom}(\Gamma,C)$ and $Y = \mathrm{Hom}(B,C)$. Then (X,M) is a groupoid and Y is a set over X equipped with an M–action. It is easily seen that the pair $(Y, Y \times_X M)$ acquires the structure of a groupoid. This settles the first part.

The second part follows by a standard base change argument, while the third point follows by pulling back the graded subcomodules which are finitely generated projective as graded A–modules. □

The graded Hopf algebroid of primary interest in this paper is the following example.

Proposition 7.3 *The pair*

$$(\mathsf{KGL}_*(\mathbf{P}^\infty), \mathsf{KGL}_*\mathsf{KGL} \otimes_{\mathsf{KGL}_*} \mathsf{KGL}_*(\mathbf{P}^\infty))$$

is a flat graded Adams Hopf algebroid.

Proof Note first that $\mathsf{KGL}_*(\mathbf{P}^\infty)$ has the structure of graded comodule algebra over $(\mathsf{KU}_*, \mathsf{KU}_*\mathsf{KU})$. The result follows from Propositions 7.1 and 7.2. □

Remark 7.4 The left unit map denoted by $\eta_{\mathsf{KGL}_*(\mathbf{P}^\infty)}$ is determined by the composite map

$$\mathsf{KGL} \wedge \Sigma^\infty \mathbf{P}^\infty_+ \cong \mathsf{KGL} \wedge \mathbf{1} \wedge \Sigma^\infty \mathbf{P}^\infty_+ \to \mathsf{KGL} \wedge \mathsf{KGL} \wedge \Sigma^\infty \mathbf{P}^\infty_+$$

and the right unit map by

$$\mathsf{KGL} \wedge \Sigma^\infty \mathbf{P}^\infty_+ \cong \mathbf{1} \wedge \mathsf{KGL} \wedge \Sigma^\infty \mathbf{P}^\infty_+ \to \mathsf{KGL} \wedge \mathsf{KGL} \wedge \Sigma^\infty \mathbf{P}^\infty_+.$$

(Here we make use of the unit map from the motivic sphere spectrum $\mathbf{1}$ to KGL.)

We use the isomorphism

$$\mathsf{KGL}_*(\mathsf{KGL} \wedge \Sigma^\infty \mathbf{P}^\infty_+) \cong \mathsf{KGL}_*\mathsf{KGL} \otimes_{\mathsf{KGL}_*} \mathsf{KGL}_*(\mathbf{P}^\infty).$$

Passing to KGL–homology under the left unit map yields a map

$$\mathsf{KGL}_*(\mathbf{P}^\infty) \to \mathsf{KGL}_*\mathsf{KGL} \otimes_{\mathsf{KGL}_*} \mathsf{KGL}_*(\mathbf{P}^\infty)$$

displaying $\mathsf{KGL}_*(\mathbf{P}^\infty)$ as a comodule over $\mathsf{KGL}_*\mathsf{KGL}$.

The augmentation is determined by the multiplication on KGL via the map

$$\mathsf{KGL} \wedge \mathsf{KGL} \wedge \Sigma^\infty \mathbf{P}^\infty_+ \to \mathsf{KGL} \wedge \Sigma^\infty \mathbf{P}^\infty_+.$$

The following is the graded version of the notion of Landweber exactness introduced by Hovey and Strickland [23, Definition 2.1].

Definition 7.5 Suppose (A, Γ) is a flat graded Hopf algebroid. Then a graded ring map $A \to B$ is Landweber exact over (A, Γ) if the functor $- \otimes_A B$ from graded (A, Γ)–comodules to graded B–modules is exact.

By abuse of notation we let η_L denote the composite map

$$A \xrightarrow{\eta_L} \Gamma \cong \Gamma \otimes_A A \to \Gamma \otimes_A B.$$

The next lemma is well known. For the convenience of the reader we shall sketch a proof since the result is employed in the proof of Theorem 4.3.

Lemma 7.6 *Suppose (A, Γ) is a flat graded Hopf algebroid. Then a graded ring map $A \to B$ is Landweber exact over (A, Γ) if and only if the map $\eta_L \colon A \to \Gamma \otimes_A B$ is flat.*

Proof The only if implication holds because $\Gamma \otimes_A -$ preserves monomorphisms between graded A–modules. Conversely, for every graded A–comodule M, the graded coaction map $M \to \Gamma \otimes_A M$ is a retraction. Thus for a monomorphism of graded comodules $M \to N$ the map $B \otimes_A M \to B \otimes_A N$ is a retract of $B \otimes_A \Gamma \otimes_A M \to B \otimes_A \Gamma \otimes_A N$. □

Remark 7.7 In the proof of Theorem 2.2 we could have worked with the Hopf algebroid $(\mathsf{KU}_*(\mathbf{CP}^\infty), \mathsf{KU}_*\mathsf{KU} \otimes_{\mathsf{KU}_*} \mathsf{KU}_*(\mathbf{CP}^\infty))$ by restricting the comodule structure and using (ungraded) Landweber exactness. In this way one can bootstrap a proof of Theorem 2.2 more directly from [27] by using base change isomorphisms with no mention of graded Hopf algebroids. In the same spirit, we note there is an isomorphism

$$\mathsf{KGL}^\tau_*(\mathscr{X}) \cong \mathsf{KGL}_*(\mathscr{X}^\tau) \otimes_{\mathsf{KU}_*(\mathbf{CP}^\infty)} \mathsf{KU}_*.$$

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