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The twisted Alexander polynomial for finite abelian covers over three manifolds with boundary

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We provide the twisted Alexander polynomials of finite abelian covers over three-dimensional manifolds whose boundary is a finite union of tori. This is a generalization of a well-known formula for the usual Alexander polynomial of knots in finite cyclic branched covers over the three-dimensional sphere.

57M25; 57M27

1 Introduction

The classical Alexander polynomial is defined for null–homologous knots in rational homology spheres, where null–homologous means that the homology class of a knot is trivial in the first homology group with $\mathbb{Z}$–coefficients of the ambient space.

If the pair $(\widehat{M}, \widehat{K})$ of a rational homology sphere $\widehat{M}$ and a null-homologous knot $\widehat{K}$ in $\widehat{M}$ is given by a finite cyclic branched cover over S^3 branched along a knot K, where $\widehat{K}$ is the lift of K, then we can compute the Alexander polynomial of $\widehat{K}$ by using the well–known formula

$$\Delta_{\widehat{K}}(t) = \prod_{\xi \in \{x \in \mathbb{C} \mid x^k = 1\}} \Delta_K(\xi t) \quad \text{up to a factor } \pm t^a \ (a \in \mathbb{Z})$$

where k is the order of the covering transformation group, ξ runs all over the kth roots of unity and $\Delta_K(t)$ is the Alexander polynomial of K. Such formulas have been investigated from the viewpoint of Reidemeister torsion for a long time. In particular, V Turaev gave a formula for the Alexander polynomial of $\widehat{K}$ in a finite cyclic branched cover over S^3, and a generalization in the case of links in general three–dimensional manifolds (we refer to Turaev [8, Theorems 1.9.2 and 1.9.3]).

The purpose of this paper is to provide the generalization of the above formula giving the Alexander polynomial of a knot in a finite cyclic branched cover over S^3 to a formula for the twisted Alexander polynomial of finite abelian covers, which is a special kind of Reidemeister torsion. Especially, we also consider the twisted Alexander polynomial

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for a link in a three–dimensional manifold from the viewpoint of Reidemeister torsion in the same way as V Turaev. But to deal with finite abelian covers beyond finite cyclic covers, we adopt the approach of J Porti in his work [5]. Porti gave a new proof of Mayberry–Murasugi's formula, which gives the order of the first homology group of finite abelian branched covers over S^3 branched along links, by using Reidemeister torsion theory. We call the twisted Alexander polynomial *the polynomial torsion* regarded as a kind of Reidemeister torsion.

In this paper, we are interested in the Reidemeister torsion for a finite sheeted abelian covering. We are mainly intested in link exteriors in homology three–spheres and their abelian covers. Our main theorem (see Theorem 4.1) is stated for an abelian cover $\widehat{M} \to M$ between two three–dimensional manifolds whose boundary is a finite union of tori as follows

$$\Delta_{\widehat{M}}^{\widehat{\varphi}\otimes\widehat{\rho}}(\mathbf{t}) = \epsilon \cdot \prod_{\xi \in \widehat{G}} \Delta_M^{(\varphi\otimes\rho)\otimes\xi}(\mathbf{t})$$

where $\Delta_{\widehat{M}}^{\widehat{\varphi}\otimes\widehat{\rho}}(\mathbf{t})$ and $\Delta_M^{(\varphi\otimes\rho)\otimes\xi}(\mathbf{t})$ are the signed twisted Alexander polynomials, $\widehat{G}$ is the set of homomorphisms from the covering transformation group G to the non–zero complex numbers and ϵ is a sign determined by the homology orientations of $\widehat{M}$ and M.

To be more precise, we need two homomorphisms of the fundamental group to define the twisted Alexander polynomial of a manifold. The symbol φ denotes a surjective homomorphism from $\pi_1(M)$ to a multiplicative group $\mathbb{Z}^n$ and ρ denotes a representation of $\pi_1(M)$, *ie*, a homomorphism from $\pi_1(M)$ to a linear automorphism group $\mathrm{Aut}(V)$ of some vector space V (see Section 3 for the definition of the polynomial torsion). In the definition of the twisted Alexander polynomial of $\widehat{M}$, we use the pull–backs $\widehat{\varphi}$ and $\widehat{\rho}$ of φ and ρ to $\pi_1(\widehat{M})$. The homomorphisms φ and ξ determine variables in the twisted Alexander polynomial of $\widehat{M}$. In our main theorem, we assume that the composition of ξ with the quotient homomorphism $\pi_1(M) \to \pi_1(M)/\pi_1(\widehat{M}) \simeq G$ factors through homomorphism φ (see Section 4).

When we choose $\widehat{M} \to M$ as a finite cyclic cover of a knot exterior E_K of K in S^3, φ as the abelianization homomorphism $\pi_1(E_K) \to \pi_1(E_K)/[\pi_1(E_K), \pi_1(E_K)] \simeq \mathbb{Z}$ and ρ as the one–dimensional trivial representation, Theorem 4.1 reduces to the classical formula for the Alexander polynomial of $\widehat{K}$, where $\widehat{K}$ is the lift of the knot in the finite cyclic branched cover over S^3

$$\Delta_{\widehat{K}}(t) = \prod_{\xi \in \{x \in \mathbb{C} \mid x^k = 1\}} \Delta_K(\xi t),$$

up to a factor $\pm t^a$ $(a \in \mathbb{Z})$, where k is the order of $\pi_1(M)/\pi_1(\widehat{M})$. Our formula also provides the Alexander polynomial of a link in finite abelian branched covers over S^3 branched along the link.

Organization The outline of the paper is as follows. Section 2 deals with some reviews on the sign–determined Reidemeister torsion for a manifold. In Section 3, we give the definition of the polynomial torsion (the twisted Alexander polynomial) for a manifold whose boundary is a finite union of tori. In Section 4, we consider the polynomial torsion of finite abelian covering spaces (see Theorem 4.1).

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2 Preliminaries

2.1 The Reidemeister torsion

We review the basic notions and results about the sign–determined Reidemeister torsion introduced by V Turaev which are needed in this paper. Details can be found in Milnor's survey [3] and in Turaev's monograph [10].

Torsion of a chain complex Let $C_* = (0 \to C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} C_0 \to 0)$ be a chain complex of finite dimensional vector spaces over a field $\mathbb{F}$. Choose a basis $\mathbf{c}^i$ of C_i and a basis $\mathbf{h}^i$ of the ith homology group $H_i(C_*)$. The torsion of C_* with respect to these choices of bases is defined as follows.

For each i, let $\mathbf{b}^i$ be a set of vectors in C_i such that $d_i(\mathbf{b}^i)$ is a basis of $B_{i-1} = \operatorname{im}(d_i\colon C_i \to C_{i-1})$ and let $\widetilde{\mathbf{h}}^i$ denote a lift of $\mathbf{h}^i$ in $Z_i = \ker(d_i\colon C_i \to C_{i-1})$. The set of vectors $d_{i+1}(\mathbf{b}^{i+1})\widetilde{\mathbf{h}}^i\mathbf{b}^i$ is a basis of C_i. Let $[d_{i+1}(\mathbf{b}^{i+1})\widetilde{\mathbf{h}}^i\mathbf{b}^i/\mathbf{c}^i] \in \mathbb{F}^*$ denote the determinant of the transition matrix between those bases (the entries of this matrix are coordinates of vectors in $d_{i+1}(\mathbf{b}^{i+1})\widetilde{\mathbf{h}}^i\mathbf{b}^i$ with respect to $\mathbf{c}^i$). The *sign-determined Reidemeister torsion* of C_* (with respect to the bases $\mathbf{c}^*$ and $\mathbf{h}^*$) is the following alternating product (see Turaev's book [9, Definition 3.1]):

$$\mathrm{Tor}(C_*, \mathbf{c}^*, \mathbf{h}^*) = (-1)^{|C_*|} \cdot \prod_{i=0}^{n} [d_{i+1}(\mathbf{b}^{i+1})\widetilde{\mathbf{h}}^i\mathbf{b}^i/\mathbf{c}^i]^{(-1)^{i+1}} \in \mathbb{F}^*. \tag{2-1}$$

Here

$$|C_*| = \sum_{k \geqslant 0} \alpha_k(C_*)\beta_k(C_*),$$

where $\alpha_i(C_*) = \sum_{k=0}^{i} \dim C_k$ and $\beta_i(C_*) = \sum_{k=0}^{i} \dim H_k(C_*)$.

The torsion $\mathrm{Tor}(C_*, \mathbf{c}^*, \mathbf{h}^*)$ does not depend on the choices of $\mathbf{b}^i$ nor on the lifts $\widetilde{\mathbf{h}}^i$. Note that if C_* is acyclic (*ie* if $H_i = 0$ for all i), then $|C_*| = 0$.

Torsion of a CW-complex Let W be a finite CW-complex and (V, ρ) be a pair of a vector space with an inner product over $\mathbb{F}$ and a homomorphism of $\pi_1(W)$ into $\mathrm{Aut}(V)$. The vector space V turns into a right $\mathbb{Z}[\pi_1(W)]$–module denoted V_ρ by using the right action of $\pi_1(W)$ on V given by $v \cdot \gamma = \rho(\gamma)^{-1}(v)$, for $v \in V$ and $\gamma \in \pi_1(W)$. The complex of the universal cover with integer coefficients $C_*(\widetilde{W}; \mathbb{Z})$ also inherits a left $\mathbb{Z}[\pi_1(W)]$–module structure via the action of $\pi_1(W)$ on $\widetilde{W}$ as the covering group. We define the V_ρ–twisted chain complex of W to be

$$C_*(W; V_\rho) = V_\rho \otimes_{\mathbb{Z}[\pi_1(W)]} C_*(\widetilde{W}; \mathbb{Z}).$$

The complex $C_*(W; V_\rho)$ computes the *V_ρ–twisted homology* of W which is denoted by $H_*(W; V_\rho)$.

Let $\{e^i_1, \dots, e^i_{n_i}\}$ be the set of i–dimensional cells of W. We lift them to the universal cover and we choose an arbitrary order and an arbitrary orientation for the cells $\{\widetilde{e}^i_1, \dots, \widetilde{e}^i_{n_i}\}$. If we choose an orthonormal basis $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ of V, then we consider the corresponding basis

$$\mathbf{c}^i = \{\mathbf{v}_1 \otimes \widetilde{e}^i_1, \dots, \mathbf{v}_m \otimes \widetilde{e}^i_1, \cdots, \mathbf{v}_1 \otimes \widetilde{e}^i_{n_i}, \dots, \mathbf{v}_m \otimes \widetilde{e}^i_{n_i}\}$$

of $C_i(W; V_\rho) = V_\rho \otimes_{\mathbb{Z}[\pi_1(W)]} C_*(\widetilde{W}; \mathbb{Z})$. We call the basis $\mathbf{c}^* = \oplus_i \mathbf{c}^i$ *a geometric basis* of $C_*(W; V_\rho)$. Now choosing for each i a basis $\mathbf{h}^i$ of the V_ρ–twisted homology $H_i(W; V_\rho)$, we can compute the torsion

$$\mathrm{Tor}(C_*(W; V_\rho), \mathbf{c}^*, \mathbf{h}^*) \in \mathbb{F}^*.$$

We mainly consider the torsion of acyclic chain complexes $C_*(W;V_\rho)$, *ie*, the homology group $H_*(W;V_\rho)=\mathbf{0}$. For acyclic chain complex $C_*(W;V_\rho)$, this definition only depends on the combinatorial class of W, the conjugacy class of ρ, the choices of $\mathbf{c}^*$. The basis $\mathbf{c}^*$ for $C_*(W;V_\rho)$ depends on the following choices:

(1) an order of cells $\{e^i_j\}$ and an orientation of each $\{e^i_j\}$;

(2) a lift $\widetilde{e}^i_j$ of e^i_j and;

(3) an orthonormal basis of the vector space V.

We summarize the effect of changing these choices to $\mathrm{Tor}(C_*(W;V_\rho),\mathbf{c}^*,\varnothing)$ in the following three remarks.

Remark 2.1 We have the same $\mathrm{Tor}(C_*(W;V_\rho),\mathbf{c}^*,\varnothing)$ for all orthonormal bases of V since the effect of change of orthonormal bases in V is given by multiplying the determinant of the bases change matrix with power of $\chi(W)$. If the Euler characteristic $\chi(W)$ is zero, then we have the same torsion for any basis of V.

Remark 2.2 The torsion $\mathrm{Tor}(C_*(W;V_\rho),\mathbf{c}^*,\varnothing)$ depends on the choice of the lifts $\widetilde{e}^i_j$ under the action of $\pi_1(W)$ by ρ. The effect of different lift of a cell is expressed as the determinant of $\rho(\gamma)$ for some γ in $\pi_1(W)$. To avoid this problem, we often use representations into $\mathrm{SL}(V)$.

Remark 2.3 To define the Reidemeister torsion, we order the cells $\{e^i_j\}$ and choose an orientation of each e^i_j, if we choose a different order and different orientations of cells, we could change the torsion sign. To remove this sign ambiguity, that only occurs when m is odd, we use the fact that the sign of the torsions $\mathrm{Tor}(C_*(W;\mathbb{R}),\mathbf{c}^*_{\mathbb{R}},\mathbf{h}^*_{\mathbb{R}})$ and $\mathrm{Tor}(C_*(W;\mathbb{R}),\mathbf{c}^*,\mathbf{h}^*)$ change in the same way.

Therefore we usually consider the torsion $\mathrm{Tor}(C_*(W;V_\rho),\mathbf{c}^*,\varnothing)$ up to the above indeterminacy, namely up to a factor $\pm\det\rho(\gamma)$ for some γ in $\pi_1(W)$.

We can construct the additional sign term referred to in Remark 2.3 as follows. The cells $\{\widetilde{e}^i_j \mid 0\leqslant i\leqslant \dim W, 1\leqslant j\leqslant n_i\}$ are in one–to–one correspondence with the cells of W, their order and orientation are induced an order and an orientation for the cells $\{e^i_j \mid 0\leqslant i\leqslant \dim W, 1\leqslant j\leqslant n_i\}$. Again, corresponding to these choices, we get a basis $\mathbf{c}^i_{\mathbb{R}}$ over $\mathbb{R}$ of $C_i(W;\mathbb{R})$.

Choose a *homology orientation* of W, which is an orientation of the real vector space $H_*(W;\mathbb{R})=\bigoplus_{i\geqslant 0}H_i(W;\mathbb{R})$. Let $\mathfrak{o}$ denote this chosen orientation. Provide each vector space $H_i(W;\mathbb{R})$ with a reference basis $\mathbf{h}^i_{\mathbb{R}}$ such that the basis $\{\mathbf{h}^0_{\mathbb{R}},\dots,\mathbf{h}^{\dim W}_{\mathbb{R}}\}$

of $H_*(W;\mathbb{R})$ is positively oriented with respect to $\mathfrak{o}$. Compute the sign–determined Reidemeister torsion $\mathrm{Tor}(C_*(W;\mathbb{R}), \mathbf{c}^*_{\mathbb{R}}, \mathbf{h}^*_{\mathbb{R}}) \in \mathbb{R}^*$ of the resulting based and homology based chain complex and consider its sign

$$\tau_0 = \mathrm{sgn}\big(\mathrm{Tor}(C_*(W;\mathbb{R}), \mathbf{c}^*_{\mathbb{R}}, \mathbf{h}^*_{\mathbb{R}})\big) \in \{\pm 1\}.$$

We define the *sign–refined twisted Reidemeister torsion* of W (with respect to $\mathfrak{o}$) to be

$$\tau_0^m \cdot \mathrm{Tor}(C_*(W;V_\rho), \mathbf{c}^*, \varnothing) \in \mathbb{F}^* \tag{2-2}$$

where $m = \dim_{\mathbb{F}} V$. This sign refinement also works for the twisted chain complex $C_*(W;V_\rho)$ with non–trivial homology group. When the dimension of V is even, we do not need the sign refinement, *ie*, the torsion $\mathrm{Tor}(C_*(W;V_\rho), \mathbf{c}^*, \varnothing)$ is determined up to $\det\rho(\gamma)$ for some γ in $\pi_1(W)$.

One can prove that the sign–refined Reidemeister torsion is invariant under cellular subdivision, homeomorphism and simple homotopy equivalences. In fact, it is precisely the sign $(-1)^{|C_*|}$ in Equation (2-1) which ensures all these important invariance properties to hold (see Turaev's monograph [10]).

3 Definition of the polynomial torsion

In this section we define the polynomial torsion. This gives a point of view from the Reidemeister torsion to polynomial invariants of topological spaces.

Hereafter M denotes a compact and connected three–dimensional manifold such that its boundary ∂M is empty or a disjoint union of b two–dimensional tori:

$$\partial M = T_1^2 \cup \ldots \cup T_b^2.$$

In the sequel, we denote by V a vector space over $\mathbb{C}$ and by ρ a representation of $\pi_1(M)$ into $\mathrm{Aut}(V)$, and such that $\det\rho(\gamma) = 1$ for all $\gamma \in \pi_1(M)$.

Next we introduce a twisted chain complex with some variables. It will be done by using a $\mathbb{Z}[\pi_1(M)]$–module with variables to define a new twisted chain complex. We regard $\mathbb{Z}^n$ as the multiplicative group generated by n variables $t_1, \ldots, t_n$, *ie*,

$$\mathbb{Z}^n = \big\langle t_1, \ldots, t_n \mid t_i t_j = t_j t_i \ (\forall i, j)\big\rangle$$

and consider a surjective homomorphism $\varphi\colon \pi_1(M) \to \mathbb{Z}^n$. We often abbreviate the n variables $(t_1, \ldots, t_n)$ to $\mathbf{t}$ and the rational function field $\mathbb{C}(t_1, \ldots, t_n)$ to $\mathbb{C}(\mathbf{t})$.

When we consider the right action of $\pi_1(M)$ on $V(\mathbf{t}) = \mathbb{C}(\mathbf{t}) \otimes V$ by the tensor representation

$$\varphi \otimes \rho^{-1}\colon \pi_1(M) \to \mathrm{Aut}(V(\mathbf{t})), \quad \gamma \mapsto \varphi(\gamma) \otimes \rho^{-1}(\gamma),$$

we have the associated twisted chain $C_*(M; V_\rho(\mathbf{t}))$ given by

$$C_*(M; V_\rho(\mathbf{t})) = V_\rho(\mathbf{t}) \otimes_{\mathbb{Z}[\pi_1(M)]} C_*(\widetilde{M}; \mathbb{Z})$$

where $f \otimes v \otimes \gamma \cdot \sigma$ is identified with $f\varphi(\gamma) \otimes \rho(\gamma)^{-1}(v) \otimes \sigma$ for any $\gamma \in \pi_1(M)$, $\sigma \in C_*(\widetilde{M}; \mathbb{Z})$, $v \in V$ and $f \in \mathbb{C}(\mathbf{t})$. We call this complex *the $V_\rho(\mathbf{t})$–twisted chain complex* of M.

Definition 3.1 Fix a homology orientation on M. If $C_*(M; V_\rho(\mathbf{t}))$ is acyclic, then the sign–refined Reidemeister torsion of $C_*(M; V_\rho(\mathbf{t}))$

$$\Delta_M^{\varphi\otimes\rho}(t_1, \ldots, t_n) = \tau_0^m \cdot \mathrm{Tor}(C_*(M; V_\rho(\mathbf{t})), \mathbf{c}^*, \varnothing) \in \mathbb{C}(t_1, \ldots, t_n) \setminus \{0\}$$

is called the *polynomial torsion* of M.

Observe that the sign–refined Reidemeister torsion $\Delta_M^{\varphi\otimes\rho}(t_1, \ldots, t_n)$ is determined up to a factor $t_1^{m_1} \ldots t_n^{m_n}$ like the classical Alexander polynomial is.

Example 3.2 (J Milnor [4], P Kirk & C Livingston [2]) Suppose that M is the knot exterior $E_K = S^3 \setminus N(K)$ of a knot K in S^3 where $N(K)$ is an open tubular neighbourhood of K.

If the representation $\rho \in \mathrm{Hom}(\pi_1(E_K); \mathbb{Q})$ is the trivial homomorphism and φ is the abelianization of $\pi_1(E_K)$, *ie*, $\varphi\colon \pi_1(E_K) \to H_1(E_K; \mathbb{Z}) \simeq \langle t \rangle$, then the twisted chain complex $C_*(E_K; \mathbb{Q}(t)_\rho)$ is acyclic and the Reidemeister torsion $\Delta_{E_K}^{\varphi\otimes\rho}(t)$ is expressed as a rational function which is the Alexander polynomial $\Delta_K(t)$ divided by $(t-1)$ (see Turaev's book [9] and monograph [10]).

Example 3.3 Suppose now that M is the link exterior $E_L = S^3 \setminus N(L)$ of a link L in S^3. We suppose that L has n components, where $n \geqslant 2$. We denote by μ_i the meridian of the ith component. Consider the abelianization $\varphi\colon \pi_1(E_L) \to \mathbb{Z}^n$ defined by $\varphi(\mu_i) = t_i$. Let $\rho\colon \pi_1(E_L) \to \mathrm{GL}(1; \mathbb{C}) = \mathbb{C} \setminus \{0\}$ be the one–dimensional representation such that $\rho(\mu_i) = \xi_i$. Then the twisted chain complex $C_*(E_L; \mathbb{C}(\mathbf{t})_\rho)$ is acyclic and the Reidemeister torsion $\Delta_{E_L}^{\varphi\otimes\rho}(t_1, \ldots, t_n)$ is given by (up to $\pm(\xi_1^{-1}t_1)^{k_1} \ldots (\xi_n^{-1}t_n)^{k_n}$, $k_i \in \mathbb{Z}$)

$$\Delta_{E_L}^{\varphi\otimes\rho}(t_1, \ldots, t_n) = \Delta_L(\xi_1^{-1}t_1, \ldots, \xi_n^{-1}t_n)$$

where $\Delta_L(t_1, \ldots, t_n)$ is the Alexander polynomial of L.

4 Torsion for finite sheeted abelian coverings

4.1 Statement of the result

Let $\widehat{M}$ be a finite sheeted abelian covering of M, where M denotes a compact and connected three–dimensional manifold such that its boundary ∂M is empty or a disjoint union of b two–dimensional tori:

$$\partial M = T_1^2 \cup \ldots \cup T_b^2.$$

We denote by p the induced homomorphism from $\pi_1(\widehat{M})$ to $\pi_1(M)$ by the covering map $\widehat{M} \to M$. The associated deck transformation group is a finite abelian group G of order $|G|$. We endow the manifolds M and $\widehat{M}$ with some arbitrary homology orientations.

We have the following exact sequence:

$$1 \longrightarrow \pi_1(\widehat{M}) \xrightarrow{p} \pi_1(M) \xrightarrow{\pi} G \longrightarrow 1. \tag{4-1}$$

When we consider the polynomial torsion for $\widehat{M}$, we use the pull–back of homomorphisms of $\pi_1(M)$ as homomorphisms of $\pi_1(\widehat{M})$. We denote by φ a surjective homomorphism from $\pi_1(M)$ to $\mathbb{Z}^n$ and by $\widehat{\varphi}$ the pull-back by p. We also suppose that π factors through φ. Our situation is summarized as follows:

$$\begin{tikzcd} \pi_1(\widehat{M}) \arrow[r, "p_*"] \arrow[dr, "\widehat{\varphi}"'] & \pi_1(M) \arrow[r, "\pi"] \arrow[d, "\varphi"] & G \\ & \mathbb{Z}^n \arrow[ur, dashed] & \end{tikzcd}$$

Similarly we use the symbol $\widehat{\rho}$ for the pull–back of $\rho\colon \pi_1(M) \to \mathrm{Aut}(V)$ by p, where V is a vector space. For homomorphisms of the quotient group $G \simeq \pi_1(M)/\pi_1(\widehat{M})$, we use the Pontrjagin dual of G which is the set of all representations $\xi\colon G \to \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ from G to non–zero complex numbers. Let $\widehat{G}$ denote this space.

We give the statement of the polynomial torsion for abelian coverings via that of the based manifold.

Theorem 4.1 *With the above notation, we suppose that the twisted chain complex $C_*(M; V_\rho(\mathbf{t}))$ is acyclic. Then the twisted chain complex $C_*(\widehat{M}; V_{\widehat{\rho}}(\mathbf{t}))$ is also acyclic and the polynomial torsion is expressed as*

$$\Delta_{\widehat{M}}^{\widehat{\varphi}\otimes\widehat{\rho}}(\mathbf{t}) = \epsilon \cdot \prod_{\xi \in \widehat{G}} \Delta_M^{(\varphi\otimes\rho)\otimes\xi}(\mathbf{t}) \tag{4-2}$$

where ϵ *is a sign equal to* $\tau_0(\widehat{M})^m \cdot \tau_0(M)^{m|G|}$ *and* $m = \dim V$.

Remark 4.2 As we already observed, the sign term in Equation (4-2) is not relevant when m is even.

Remark 4.3 (Explanation of Formula (4-2) with variables) If we denote by $\overline{\xi}$ the composition $\xi \circ \overline{\pi}$ as in the following commutative diagram

$$\begin{tikzcd} \pi_1(M) \arrow[r, "\pi"] \arrow[d, "\varphi"'] & G \arrow[r, "\xi"] & \mathbb{C} \\ \mathbb{Z}^n \arrow[ur, "\overline{\pi}"'] & & \end{tikzcd}$$

then Formula (4-2) can be written concretely as follows:

$$\Delta_{\widehat{M}}^{\widehat{\varphi}\otimes\widehat{\rho}}(t_1, \dots, t_n) = \epsilon \cdot \prod_{\xi \in \widehat{G}} \Delta_M^{\varphi\otimes\rho}(t_1\overline{\xi}(t_1), \dots, t_n\overline{\xi}(t_n))$$

In the special case where $n = 1$, $G = \mathbb{Z}/q\mathbb{Z}$ and $\widehat{M}$ is the q–fold cyclic covering M_q of M, then we have that $\overline{\xi}(t) = e^{2\pi k\sqrt{-1}/q}$, for $k = 0, \dots, q-1$. Hence we have the following covering formula for the polynomial torsion.

Corollary 4.4 *Suppose that* $\varphi(\pi_1(M)) = \langle t \rangle$ *and* $\widehat{\varphi}(\pi_1(M_q)) = \langle s \rangle \subset \langle t \rangle$, *where we suppose that* $s = t^q$. *We have*

$$\Delta_{M_q}^{\widehat{\varphi}\otimes\widehat{\rho}}(s) = \Delta_{M_q}^{\widehat{\varphi}\otimes\rho}(t^q) = \epsilon \cdot \prod_{k=0}^{q-1} \Delta_M^{\varphi\otimes\rho}(e^{2\pi k\sqrt{-1}/q}t).$$

The torsion $\Delta_{M_q}^{\widehat{\varphi}\otimes\rho}(t^q)$ in Corollary 4.4 can be regarded as a kind of the total twisted Alexander polynomial introduced in Silver and Williams [7]. Hirasawa and Murasugi [1] worked on the total twisted Alexander polynomial for abelian representations as in Example 3.2 and they observed the similar formula as in Corollary 4.4 in terms of the total Alexander polynomial and the Alexander polynomial of a knot in the cyclic branched coverings over S^3.

4.2 Proof of Theorem 4.1

We use the same notation as in Remark 4.3.

First observe the following key facts:

- the universal cover $\widetilde{M}$ of M is also the one of $\widehat{M}$,

- the torsion $\Delta_{\widehat{M}}^{\widehat{\varphi}\otimes\rho}$ is computed using the twisted complex

$$V_\rho(t_1,\dots,t_n)\otimes_{\mathbb{Z}[\pi_1(\widehat{M})]} C_*(\widetilde{M};\mathbb{Z}),$$

- whereas the torsion $\Delta_M^{\varphi\otimes\rho}$ is computed using

$$V_\rho(t_1,\dots,t_n)\otimes_{\mathbb{Z}[\pi_1(M)]} C_*(\widetilde{M};\mathbb{Z}).$$

Lemma 4.5 *Let $x\in V_\rho(t_1,\dots,t_n)$, $c\in C_*(\widetilde{M};\mathbb{Z})$. For $\gamma\in\pi_1(M)$, $x\gamma^{-1}\otimes_{\pi_1(\widehat{M})}\gamma c$ only depends on $\pi(\gamma)\in G$. For $g\in G$, choose $\gamma\in\pi_1(M)$ such that $\pi(\gamma)=g$ and set*

$$g\star\big(x\otimes_{\pi_1(\widehat{M})}c\big)=x\gamma^{-1}\otimes_{\pi_1(\widehat{M})}\gamma c. \tag{4-3}$$

This defines a natural action of G on $V_\rho(t_1,\dots,t_n)\otimes_{\pi_1(\widehat{M})}C_*(\widetilde{M};\mathbb{Z})$.

Further observe that, since for any lift γ of g, γ is not contained in $p(\pi_1(\widehat{M}))$, we can not reduce the right hand side in Equation (4-3).

Proof Take another lift γ' in $\pi_1(M)$ of $g\in G$. Since $\gamma'=\widehat{\gamma}\gamma$ for some $\widehat{\gamma}\in p(\pi_1(\widehat{M}))$, we can see that $x\gamma'^{-1}\otimes_{\pi_1(\widehat{M})}\gamma' c=x\gamma^{-1}\otimes_{\pi_1(\widehat{M})}\gamma c$. □

The proof of Theorem 4.1 is based on the following technical lemma.

Lemma 4.6 *The map*

$$\Phi\colon V_\rho(\mathbf{t})\otimes_{\mathbb{Z}[\pi_1(\widehat{M})]}C_*(\widetilde{M};\mathbb{Z})\to(V_\rho(\mathbf{t})\otimes_{\mathbb{C}}\mathbb{C}[G])\otimes_{\mathbb{Z}[\pi_1(M)]}C_*(\widetilde{M};\mathbb{Z}) \tag{4-4}$$

given by

$$\Phi\big(x\otimes_{\pi_1(\widehat{M})}c\big)=(x\otimes 1)\otimes c$$

is an isomorphism of complexes of $\mathbb{C}[G]$–modules where the action of G on the twisted complex $(V_\rho(\mathbf{t})\otimes_{\mathbb{C}}\mathbb{C}[G])\otimes_{\mathbb{Z}[\pi_1(M)]}C_*(\widetilde{M};\mathbb{Z})$ is given by

$$g\cdot(x\otimes g'\otimes c)=x\otimes gg'\otimes c.$$

and the right action of $\gamma\in\pi_1(M)$ on $V_\rho(\mathbf{t})\otimes_{\mathbb{C}}\mathbb{C}[G]$ is defined by

$$((f\otimes v)\otimes g)\cdot\gamma=f\varphi(\gamma)\otimes\rho^{-1}(\gamma)(v)\otimes\pi(\gamma)g,$$

where $f\in\mathbb{C}(t)$, $v\in V$ and $g\in G$.

Proof of Lemma 4.6 We first observe that Φ is a well–defined chain map of $\mathbb{C}$–vector spaces since $\pi_1(\widehat{M})$ is a normal subgroup of $\pi_1(M)$. By the definition, we can see that $\Phi(x \otimes_{\pi_1(\widehat{M})} \gamma c) = \Phi(x\gamma \otimes_{\pi_1(\widehat{M})} c)$ for any γ in $\pi_1(\widehat{M})$. Hence Φ is well–defined. From $\Phi(\partial(x \otimes_{\pi_1(\widehat{M})} c)) = \Phi(x \otimes_{\pi_1(\widehat{M})} \partial c) = (x \otimes 1) \otimes \partial c = \partial((x \otimes 1) \otimes c) = \partial(\Phi(x \otimes c))$ it follows that $\Phi \circ \partial = \partial \circ \Phi$. The G–equivariance of Φ follows from

$$\begin{aligned}\Phi(g \star (x \otimes_{\pi_1(\widehat{M})} c)) &= \Phi(x\gamma^{-1} \otimes_{\pi_1(\widehat{M})} \gamma c)\\ &= (x\gamma^{-1} \otimes 1) \otimes \gamma c\\ &= (x \otimes g) \otimes c\\ &= g \cdot \Phi(x \otimes_{\pi_1(\widehat{M})} c).\end{aligned}$$

We can prove that Φ is an isomorphism by taking its inverse Ψ as $\Psi((x \otimes g) \otimes c) = g \star (x \otimes c)$. □

We mention bases of the chain complex $V_\rho(\mathbf{t}) \otimes_{\mathbb{Z}[\pi_1(\widehat{M})]} C_*(\widetilde{M};\mathbb{Z})$ before the next step. The following basis

$$\widehat{\mathbf{c}}^* = \bigcup_{i \geqslant 0} \left\{ x_k \otimes_{\pi_1(\widehat{M})} \gamma_g \widetilde{e}^i_j \;\middle|\; 1 \leqslant j \leqslant n_i, g \in G, 1 \leqslant k \leqslant m \right\} \tag{4-5}$$

is the geometric basis used to compute the polynomial torsion $\Delta^{\widehat{\varphi}\otimes\widehat{\rho}}_{\widehat{M}}$. When we consider the bases change from the basis in Equation (4-5) to the basis in the next equation

$$\left\{ g \star \left(x_k \otimes_{\pi_1(\widehat{M})} \widetilde{e}^i_j \right) = x_k \gamma_g^{-1} \otimes_{\pi_1(\widehat{M})} \gamma_g \widetilde{e}^i_j \;\middle|\; 1 \leqslant j \leqslant n_i, g \in G, 1 \leqslant k \leqslant m \right\}, \tag{4-6}$$

we can see that the action of γ_g^{-1} arises the change in $\Delta^{\widehat{\varphi}\otimes\widehat{\rho}}_{\widehat{M}}$ by multiplying its determinant powered the Euler characteristic of M. Since the Euler characteristic of M is zero, the polynomial torsion $\Delta^{\widehat{\varphi}\otimes\widehat{\rho}}_{\widehat{M}}$ can also be computed using the basis in Equation (4-6). Finally observe that Φ maps the basis in Equation (4-6) to the geometric basis

$$\mathbf{c}^*_G = \bigcup_{i \geqslant 0} \left\{ (x_k \otimes g) \otimes \widetilde{e}^i_j \;\middle|\; 1 \leqslant j \leqslant n_i, g \in G, 1 \leqslant k \leqslant m \right\}, \tag{4-7}$$

thus

$$\Delta^{\widehat{\varphi}\otimes\widehat{\rho}}_{\widehat{M}} = \tau_0(\widehat{M})^m \cdot \mathrm{Tor}((V_\rho(\mathbf{t}) \otimes_{\mathbb{C}} \mathbb{C}[G]) \otimes_{\mathbb{Z}[\pi_1(M)]} C_*(\widetilde{M};\mathbb{Z}), \mathbf{c}^*_G, \varnothing).$$

Now, we want to compute the torsion of $(V_\rho(\mathbf{t}) \otimes_{\mathbb{C}} \mathbb{C}[G]) \otimes_{\mathbb{Z}[\pi_1(M)]} C_*(\widetilde{M};\mathbb{Z})$ in terms of polynomial torsions of M. To this end we use the decomposition along orthogonal idempotents of the group ring $\mathbb{C}[G]$, see Serre [6] for details. Associated to

$\xi \in \widehat{G}$, we define:

$$f_\xi = \frac{1}{|G|} \sum_{g \in G} \xi(g^{-1}) g \in \mathbb{C}[G].$$

The properties of f_ξ are the following

$$f_\xi^2 = f_\xi, \quad f_\xi f_{\xi'} = 0 \,(\text{ if } \xi \neq \xi'), \quad \sum_{\xi \in \widehat{G}} f_\xi = 1$$

and

$$g \cdot f_\xi = \xi(g) f_\xi, \text{ for all } g \in G.$$

We have the following $\mathbb{C}[G]$–modules decomposition of the group ring as a direct sum according to its representations:

$$\mathbb{C}[G] = \bigoplus_{\xi \in \widehat{G}} \mathbb{C}[f_\xi]. \tag{4-8}$$

Here each factor is the 1–dimensional $\mathbb{C}$–vector space which is isomorphic to the $\mathbb{C}[G]$–module associated to $\xi\colon G \to \mathbb{C}^*$.

Following Porti [5, Section 3], corresponding to the decomposition in Equation (4-8) we have a decomposition of complexes of $\mathbb{C}[G]$–modules:

$$(V_\rho(\mathbf{t}) \otimes_{\mathbb{C}} \mathbb{C}[G]) \otimes_{\mathbb{Z}[\pi_1(M)]} C_*(\widetilde{M};\mathbb{Z}) = \bigoplus_{\xi \in \widehat{G}} (V_\rho(\mathbf{t}) \otimes_{\mathbb{C}} \mathbb{C}[f_\xi]) \otimes_{\mathbb{Z}[\pi_1(M)]} C_*(\widetilde{M};\mathbb{Z}).$$

Remark 4.7 This decomposition implies that $(V_\rho(\mathbf{t}) \otimes_{\mathbb{C}} \mathbb{C}[G]) \otimes_{\mathbb{Z}[\pi_1(M)]} C_*(\widetilde{M};\mathbb{Z})$ is acyclic, since one can see that each chain complex $(V_\rho(\mathbf{t}) \otimes_{\mathbb{C}} \mathbb{C}[f_\xi]) \otimes_{\mathbb{Z}[\pi_1(M)]} C_*(\widetilde{M};\mathbb{Z})$ is acyclic from our assumptions and a change of variables.

The geometric basis in Equation (4-7) induces a basis compatible with the decomposition in Equation (4-8) by replacing $\{g \mid g \in G\}$ by $\{f_\xi \mid \xi \in \widehat{G}\}$. The change of bases cancels when we compute the torsion because Euler characteristic is zero, see [5, Lemma 5.2]. And thus decomposition in Equation (4-8) implies that (in the natural geometric bases):

$$\begin{aligned} \Delta_{\widehat{M}}^{\widehat{\varphi}\otimes\widehat{\rho}} &= \tau_0(\widehat{M})^m \cdot \mathrm{Tor}((V_\rho(\mathbf{t}) \otimes_{\mathbb{C}} \mathbb{C}[G]) \otimes_{\mathbb{Z}[\pi_1(M)]} C_*(\widetilde{M};\mathbb{Z}), \mathbf{c}_G^*, \varnothing) \\ &= \tau_0(\widehat{M})^m \cdot \prod_{\xi \in \widehat{G}} \mathrm{Tor}((V_\rho(\mathbf{t}) \otimes_{\mathbb{C}} \mathbb{C}[f_\xi]) \otimes_{\mathbb{Z}[\pi_1(M)]} C_*(\widetilde{M};\mathbb{Z}), \mathbf{c}^*, \varnothing). \end{aligned} \tag{4-9}$$

Each factor in the right hand side is related to the polynomial torsion of M and its relation is given by the following claim.

Lemma 4.8 *We have:*

$$\Delta_M^{(\varphi\otimes\rho)\otimes\xi} = \tau_0(M)^m \cdot \mathrm{Tor}((V_\rho(\mathbf{t}) \otimes_{\mathbb{C}} \mathbb{C}[f_\xi]) \otimes_{\mathbb{Z}[\pi_1(M)]} C_*(\widetilde{M};\mathbb{Z}), \mathbf{c}^*, \varnothing).$$

Proof of Lemma 4.8 One can observe that, as a $\mathbb{Z}[\pi_1(M)]$–module, $V_\rho(\mathbf{t}) \otimes_{\mathbb{C}} \mathbb{C}[f_\xi]$ is isomorphic to $V_\rho(\mathbf{t})$ simply by replacing the action $\varphi\otimes\rho$ by $(\varphi\otimes\rho)\otimes\xi$. This proves the equality of torsions. □

Proof of Theorem 4.1 Combining Equation (4-9) and Lemma 4.8, we obtain

$$\Delta_{\widehat{M}}^{\widehat{\varphi}\otimes\widehat{\rho}} = \tau_0(\widehat{M})^m \cdot \tau_0(M)^{m|G|} \cdot \prod_{\xi\in\widehat{G}} \Delta_M^{(\varphi\otimes\rho)\otimes\xi}$$

which achieves the proof of Formula (4-2). □

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