

Dyer–Lashof operations on Tate cohomology of finite groups

MARTIN LANGER

Let $k = \mathbb{F}_p$ be the field with $p > 0$ elements, and let G be a finite group. By exhibiting an E_∞ -operad action on $\text{Hom}(P, k)$ for a complete projective resolution P of the trivial kG -module k , we obtain power operations of Dyer–Lashof type on Tate cohomology $\hat{H}^*(G; k)$. Our operations agree with the usual Steenrod operations on ordinary cohomology $H^*(G)$. We show that they are compatible (in a suitable sense) with products of groups, and (in certain cases) with the Evens norm map. These theorems provide tools for explicit computations of the operations for small groups G . We also show that the operations in negative degree are nontrivial.

As an application, we prove that at the prime 2 these operations can be used to determine whether a Tate cohomology class is productive (in the sense of Carlson) or not.

20J06, 55S12

1 Introduction

Let $k = \mathbb{F}_p$ be the field with p elements. For every finite group G , let $\hat{H}^*(G) = \hat{H}^*(G, k)$ denote the graded Tate cohomology algebra of G over k . Then \hat{H}^* is functorial with respect to injective group homomorphisms. The starting point of our discussion will be the following Theorem.

Theorem 1.1 *There is a family of k -linear operations Q_s (and βQ_s for $p \geq 3$) for all integers s on Tate cohomology \hat{H}^* , satisfying the following properties.*

- (1) *The operations Q_s are natural with respect to injective group homomorphisms.*
- (2) *The operation Q_s lowers the degree by $2s(p-1)$ (by s if $p=2$), and βQ_s lowers the degree by $2s(p-1)-1$ for $p > 2$.*
- (3) *$Q_s(x) = 0$ if $2s < -|x|$ (if $s < -|x|$ for $p=2$).*
- (4) *If $p > 2$, then $\beta Q_s(x) = 0$ if $2s \leq -|x|$.*
- (5) *$Q_s(x) = x^p$ if $2s = -|x|$ (if $s = -|x|$ for $p=2$).*
- (6) *$Q_s(1) = 0$ unless $s \neq 0$, where $1 \in \hat{H}^0(G)$ is the unit element.*

(7) The internal Cartan formula holds:

$$Q_s(xy) = \sum_{i+j=s} Q_i(x)Q_j(y),$$

$$\beta Q_s(xy) = \sum_{i+j=s} \beta Q_i(x)Q_j(y) + (-1)^{|x|} Q_i(x)\beta Q_j(y) \quad \text{for all } x, y \in \widehat{H}^*(G).$$

(8) The Adem relations hold: For $r > ps$,

$$Q_r Q_s = \sum_i (-1)^{r+i} (pi - r, r - (p - 1)s - i - 1) Q_{r+s-i} Q_i$$

and for $r \geq ps$ and $p > 2$

$$Q_r \beta Q_s = \sum_i (-1)^{r+i} (pi - r, r - (p - 1)s - i) \beta Q_{r+s-i} Q_i - \sum_i (-1)^{r+i} (pi - r - 1, r - (p - 1)s - i) Q_{r+s-i} \beta Q_i.$$

Here the convention is that $(a, b) = 0$ if a or b is negative, and $(a, b) = \binom{a+b}{b}$ otherwise.

(9) On classes of nonnegative degree, the operations agree with the Steenrod operations on $H^*(BG; k) = H^*(G)$. More precisely, for $|x| \geq 0$ we have

$$Q_{-n}(x) = \text{Sq}^n(x) \quad \text{for } p = 2 \text{ and } n \geq 0,$$

$$Q_{-n}(x) = P^n(x), \beta Q_{-n}(x) = \beta P^n(x) \quad \text{for } p > 2 \text{ and } n \geq 0,$$

$$Q_{-n}(x) = 0, \beta Q_{-n}(x) = 0 \quad \text{for } n < 0.$$

We define the total operation $Q = \sum_i Q_i$. To be more precise, we define for every \mathbb{Z} -graded ring R the ring R^\vee to be

$$R^\vee = \left\{ (x_n)_{n \in \mathbb{Z}} \in \prod_{n \in \mathbb{Z}} R^n \mid \text{there is some } N \in \mathbb{Z} \text{ with } x_n = 0 \text{ for all } n \geq N \right\},$$

where R^n denotes the homogeneous elements of degree n in R . Similarly, for later reference, we define R^\wedge to be the ring of all those sequences $(x_n)_{n \in \mathbb{Z}}$ which vanish in sufficiently negative degrees. Now for $x \in \widehat{H}^*(G)$ we have $Q(x) \in \widehat{H}^*(G)^\vee$ by part (3) of Theorem 1.1, and the Cartan formula reads $Q(xy) = Q(x)Q(y)$ for all x, y . If $p = 2$ then we will sometimes use the notation

$$\text{Sq}_i(x) = Q_{i-|x|}(x)$$

so that $\text{Sq}_i(x) = 0$ for all $i < 0$ and $\text{Sq}_0(x) = x^2$.

Example 1.2 Let $p = 2$, and let $G = \mathbb{Z}/2\mathbb{Z}$ be the cyclic group of order 2. We can easily compute all the operations on $\hat{H}^*(G)$ using the statements of the theorem only. It is known that $\hat{H}^*(G) \cong k[s^{\pm 1}]$ for the unique nonzero class s of degree 1; see Cartan and Eilenberg [6, XII.Section 7]. We know $Q(s) = s + s^2$, so $1 = Q(1) = Q(s^{-1}s) = Q(s^{-1})(s + s^2)$. Using that $Q(s^{-1}) = s^{-2} + (\text{terms of degree less than } -2)$ we obtain

$$Q(s^{-1}) = s^{-2} + s^{-3} + s^{-4} + \dots$$

More generally we get for all integers i

$$Q(s^i) = (s + s^2)^i = s^{2i}(s^{-1} + 1)^i = \sum_{j \geq 0} \binom{i}{j} s^{2i-j},$$

so that $Q_{j-i}(s^i) = \binom{i}{j} s^{2i-j}$ for all $j \geq 0$. Here we use the generalized binomial coefficient

$$\binom{i}{j} = \frac{i(i-1)\cdots(i-j+1)}{j!} \quad \text{for integers } i, j \text{ with } j \geq 0.$$

Example 1.3 Slightly more complicated, but still an immediate consequence of the theorem is the case $G = \mathbb{Z}/p\mathbb{Z}$ for odd primes p . Here $\hat{H}^*(G) \cong k[s^{\pm 1}] \otimes \Lambda(u)$, where s is of degree 2 and u is exterior of degree 1. Let us define $\beta Q = \sum_i \beta Q_i$; then from the topological fact $\beta(u) = s$ we get for integers i

$$\begin{aligned} Q(s^i) &= \sum_{j \geq 0} \binom{i}{j} s^{pi-j}, & Q(s^i u) &= Q(s^i)u, \\ \beta Q(s^i) &= 0, & \beta Q(s^i u) &= Q(s^i)s. \end{aligned}$$

Example 1.4 Let us do an example of a noncommutative group. Let $G = Q_8$ be the quaternion group with 8 elements, and let $p = 2$. Then it is known that $\hat{H}^*(G) \cong k[s^{\pm 1}, x, y]/(x^2 + xy + y^2, x^3)$ with degrees $|x| = |y| = 1$ and $|s| = 4$. We immediately get $Q(x) = x + x^2$ and $Q(y) = y + y^2$. Every automorphism of $H^1(G)$ is realized by a group automorphism; this implies that $\text{Sq}^1(s) = 0$ and $\text{Sq}^2(s) = 0$. From the Adem relation $\text{Sq}^3(s) = \text{Sq}^1 \text{Sq}^2(s)$ it then follows that $Q(s) = s + s^2$. By the same methods as above, one easily deduces the operations on all of $\hat{H}^*(G)$.

Remark 1.5 We will prove Theorem 1.1 by establishing an E_∞ -operad action on $\text{Hom}_{kG}(P, k)$, the cochains of a complete projective resolution P of the trivial kG -module k . There is another way of constructing Dyer–Lashof operations on Tate cohomology, using equivariant stable homotopy theory as follows. In the homotopy category of G -spectra (see Lewis, May, Steinberger and McClure [13]) let $k_G = H\mathbb{F}_p$ denote the Eilenberg–Mac Lane spectrum, regarded as a G -spectrum with “trivial

G -action". Associated with k_G there is a Tate spectrum $t = F(EG_+, k_G) \wedge \tilde{E}G$ (see Greenlees and May [10]) with the property that $\hat{H}^*(G; k) \cong [S, t]_G^*$ (see Greenlees [9]). Then McClure [16] shows that t is an E'_∞ -ring spectrum, that is, we have a nonequivariant operad acting equivariantly, which can be used to define power operations on $\hat{H}^*(G)$. The author does not know whether the operations defined in that topological manner agree with the rather algebraically defined operations of this paper.

Remark 1.6 Benson and Greenlees [3] define an action of the Steenrod algebra on Tate cohomology. As the operations of this paper, their operations agree with the usual Steenrod operations on elements of positive degree. However, in the negative range their operations *raise* the degree, whereas ours lower the degree. For instance, in Example 1.2, the total squaring operation $Sq_{B,G}$ in the sense of Benson and Greenlees satisfies the Cartan formula $Sq_{B,G}(s^{-1})(s^2 + s) = 1$, but now $Sq_{B,G}(s^{-1})$ lies in the ring $\hat{H}^*(G)^\wedge$, so for degree reasons we get

$$Sq_{B,G}(s^{-1}) = s^{-1} + 1 + s + s^2 + \dots$$

which is the inverse of $s + s^2$ in the ring $\hat{H}^*(G)^\wedge$; this should be compared with Example 1.2, where $Q(s^{-1})$ was the inverse of $s + s^2$ in the ring $\hat{H}^*(G)^\vee$.

From now on, assume that the order of G is divisible by p . Let us define a graded submodule $M^*(G)$ of a shift of $\hat{H}^*(G)$ as follows:

$$M^n(G) = \begin{cases} \hat{H}^{n-1}(G) & \text{if } n \leq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then $M^*(G)$ inherits the Dyer–Lashof operations from $\hat{H}^*(G)$, because classes of negative degrees are mapped to classes of negative degrees (or to 0) by the Q_i . Via the identification $M^*(G) \cong (\hat{H}^*(G)/H^*(G))[1]$ we can also view $M^*(G)$ as a left $H^*(G)$ -module. For finite groups G_1 and G_2 , we have the Künneth isomorphism $H^*(G_1 \times G_2) \cong H^*(G_1) \otimes H^*(G_2)$ which is known to be an isomorphism of modules over the Steenrod algebra. We also have the isomorphism $M^*(G_1 \times G_2) \cong M^*(G_1) \otimes M^*(G_2)$ which is an isomorphism of modules over $H^*(G_1 \times G_2)$. Even more is true:

Theorem 1.7 *For finite groups G_1 and G_2 , the Künneth isomorphism $M^*(G_1 \times G_2) \cong M^*(G_1) \otimes M^*(G_2)$ is an isomorphism of modules over the Dyer–Lashof algebra. In other words, $Q(\alpha \otimes \beta) = Q(\alpha) \otimes Q(\beta)$ for all $\alpha \in M^*(G_1)$ and $\beta \in M^*(G_2)$.*

Example 1.8 Let us consider the case $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ at the prime $p = 2$. Let φ_i be a generator of $M^{-i}(\mathbb{Z}/2\mathbb{Z})$; it corresponds to $s^{-i-1} \in \hat{H}^*(\mathbb{Z}/2\mathbb{Z})$, but this

notation suggests the existence of an internal product which we do not have on M^* . Let us write $\varphi_{ij} \in \widehat{H}^{-i-j-1}(G)$ for the element $\varphi_i \otimes \varphi_j \in M^*(G)$; then $\widehat{H}^*(G)$ is the commutative graded algebra generated by polynomial classes x, y of degree 1 (coming from the two factors of G) and the classes φ_{ij} subject to the relations

$$\begin{aligned} \varphi_{ij}x &= \begin{cases} \varphi_{i-1,j} & \text{if } i \geq 1, \\ 0 & \text{otherwise,} \end{cases} \\ \varphi_{ij}y &= \begin{cases} \varphi_{i,j-1} & \text{if } j \geq 1, \\ 0 & \text{otherwise,} \end{cases} \\ \varphi_{ij}\varphi_{i'j'} &= 0. \end{aligned}$$

The total square on φ_0 is given by $Q(\varphi_0) = \varphi_1 + \varphi_2 + \dots$; see Example 1.2. By the theorem, the total square on $\varphi_0 \otimes \varphi_0 = \varphi_{00} \in \widehat{H}^*(G)$ is given by $Q(\varphi_{00}) = \sum_{i,j \geq 1} \varphi_{ij}$. More generally we get the formula

$$Q(\varphi_{ij}) = \sum_{k,l \geq 0} \binom{k+i}{k} \binom{l+j}{j} \varphi_{2i+k+1, 2j+l+1}.$$

In particular $Sq_0 = 0$ and $Sq_1(\varphi_{ij}) = \varphi_{2i+1, 2j+1}$.

In the same spirit we can prove:

Corollary 1.9 *Suppose that the group G is a direct product of r groups of order divisible by p , and let $a \in \widehat{H}^*(G)$ with $|a| < 0$. Then $Q_j(a) = 0$ if p is odd and $|a| + 2j < r - 1$. If $p = 2$, then $Q_j(a) = 0$ if $|a| + j < r - 1$, so that Sq_j vanishes on elements of negative degree if $j < r - 1$.*

Proof Let $G = G_1 \times \dots \times G_r$, and take an element of the form $a = a_1 \otimes \dots \otimes a_r$ with $a_i \in \widehat{H}^{|a_i|}(G_i)$ and $|a_i| < 0$. Then $|a| = |a_1| + \dots + |a_r| + r - 1$. Now $Q(a_i)$ is a sum of elements of degrees at most $p|a_i|$ by part (3) of Theorem 1.1. Therefore, $Q(a) = Q(a_1) \otimes Q(a_2) \dots \otimes Q(a_r)$ can be written as sum of elements $b = b_1 \otimes \dots \otimes b_r$ with $|b_i| \leq p|a_i|$, so that

$$|b| = |b_1| + \dots + |b_r| + (r - 1) \leq p|a_1| + \dots + p|a_r| + (r - 1) = p|a| - (p - 1)(r - 1).$$

We have therefore shown that $Q_j(a) = 0$ whenever $|Q_j(a)| > p|a| - (p - 1)(r - 1)$. If p is odd, then this is equivalent to

$$|a| - 2j(p - 1) > p|a| - (p - 1)(r - 1),$$

which in turn simplifies to $r - 1 > 2j + |a|$. If $p = 2$ then $Q_j(a) = 0$ if

$$|a| - j > 2|a| - (r - 1)$$

which means $r - 1 > |a| + j$. This implies the result. □

Remark 1.10 Notice that, unlike the ordinary Steenrod operations (see Evens [8]), the operations Q_i are not compatible with transfers. For instance, if we embed $K = \mathbb{Z}/2\mathbb{Z} \times \{0\} \subset \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = V$, then the diagram

$$\begin{array}{ccc} \widehat{H}^{-1}(K) & \xrightarrow[\cong]{Q_1} & \widehat{H}^{-2}(K) \\ \text{tr}_{K,V} \downarrow & & \text{tr}_{K,V} \downarrow \neq 0 \\ \widehat{H}^{-1}(V) & \xrightarrow[=0]{Q_1} & \widehat{H}^{-2}(V) \end{array}$$

cannot commute.

Let us consider the following application of our operations. Let G be a finite group, and let $\zeta: \Omega^n k \rightarrow k$ be a surjective map representing a Tate cohomology class $[\zeta] \in \widehat{H}^n(G)$. Define L_ζ to be the kernel of ζ ; we therefore get an exact triangle

$$(1.11) \quad \Omega k \xrightarrow{\eta} L_\zeta \xrightarrow{\iota} \Omega^n k \xrightarrow{\zeta} k.$$

Following Carlson [5, Section 9] we call the class $[\zeta]$ *productive* if ζ annihilates the cohomology of L_ζ , that is, the map $\zeta \otimes \text{id}_{L_\zeta}: \Omega^n k \otimes L_\zeta \rightarrow L_\zeta$ is stably zero. It is known that, for all primes $p \geq 3$, a nonzero class $[\zeta]$ is productive if and only if its degree n is even (see Carlson [4, Theorem 4.1]). The case $p = 2$ is more complicated, and we will show in Section 6 that the operations Q can be used to determine whether a class is productive or not:

Theorem 1.12 *Let $p = 2$, and let G be a finite group. A cohomology class $[\zeta] \in \widehat{H}^n(G)$ is productive if and only if $\text{Sq}_1(\zeta)$ is divisible by ζ in $\widehat{H}^*(G)$.*

Remark 1.13 The “only if” part of this theorem has been conjectured and independently proven in the case of ordinary cohomology classes by Yalçın [17], using connections to the existence of diagonal approximations of certain chain complexes.

1.1 Notation and conventions

Throughout the paper, p is a prime number and $k = \mathbb{F}_p$ is the prime field of characteristic p . Some of the results also hold for arbitrary fields of characteristic p , but then certain k -vector spaces have to be twisted by the Frobenius map. Groups labelled G, K, L are assumed to be finite. All modules are right modules, unless mentioned otherwise. We will mainly work in $\text{mod-}kG$, the category of right kG -modules, with its tensor product \otimes and internal Hom-object Hom_k obtained from the Hopf algebra structure on kG . We will use several known results about projective modules without further notice (eg, projective is the same as injective, the tensor product of a projective

and an arbitrary module is projective, and arbitrary products and sums of projectives are projective). The ground field k is considered as an object in $\mathfrak{mod}\text{-}kG$ by the trivial G -action. We denote by $\underline{\mathfrak{mod}}\text{-}kG$ the stable module category, obtained from $\mathfrak{mod}\text{-}kG$ by dividing out those morphisms which factor through a projective module. Homomorphisms in $\underline{\mathfrak{mod}}\text{-}kG$ between modules X, Y are denoted by $\underline{\text{Hom}}(X, Y)$. The category $\underline{\mathfrak{mod}}\text{-}kG$ is a triangulated category with shift functor $\Sigma = \Omega^{-1}$, and Tate cohomology can be defined as $\hat{H}^n(G) = \underline{\text{Hom}}(\Omega^n k, k)$, with the composition product as multiplication. A morphism $X \rightarrow Y$ in $\mathfrak{mod}\text{-}kG$ is called a *stable equivalence* if it induces an isomorphism in the stable category. See Carlson [5] for an introduction to the stable module category.

In this paper, we use the notation \otimes for the internal tensor product \otimes_k of $\mathfrak{mod}\text{-}kG$, but Hom is used for the k -vector space of kG -linear maps, that is, $\text{Hom} = \text{Hom}_{kG}$. Furthermore, the symbol ∂ is used for the differential of chain complexes over kG , whereas d often denotes the differential of cochain complexes over k .

1.2 Plan of the paper

In Section 2, we will construct the E_∞ -operad acting on $\text{Hom}_{kG}(P, k)$ for a projective resolution P of the trivial kG -module k . We also compare the Dyer–Lashof operations obtained from that action with the usual Steenrod operations that we have on $H^*(G) \cong H^*(BG)$, thereby completing the proof of Theorem 1.1. In Section 3 we prove Theorem 1.7 about products of groups. In Section 4 we give a description of negative Tate Ext-groups in terms of complexes of projective modules, a tool we need for the proofs in the later sections. The duals of certain operations are shown to commute with the Evens norm map in Section 5, where we also show that our operations are nontrivial in negative degrees. Finally, in the last section we prove Theorem 1.12.

Acknowledgements Most of the paper evolved from parts of my PhD thesis written at the University of Bonn under supervision of Stefan Schwede. I would like to thank him for suggesting that project, and for his interest and helpful comments on this paper. Furthermore, I would like to thank Wolfgang Lück for the financial support during my stay in Münster. Finally, I thank the referee for various helpful comments and suggestions.

2 The operad

2.1 Resolutions

Let k be a field of characteristic p , and let G be a finite group. Let M be a kG -module. A *complete projective resolution* of M is a long exact sequence of projective

kG -modules

$$\cdots \longleftarrow P_{-2} \longleftarrow P_{-1} \longleftarrow P_0 \xleftarrow{\partial_1} P_1 \longleftarrow P_2 \longleftarrow \cdots$$

$$\begin{array}{ccc} & \swarrow & \searrow \\ & M & \end{array}$$

ε

such that ε is the cokernel map of ∂_1 . The map ε is called *augmentation* and can be viewed as a chain map $\varepsilon: P \rightarrow M$, where M is regarded as a complex concentrated in degree 0. If N is another module, then a map $\varepsilon': P_0 \rightarrow N$ (or, equivalently, a chain map $\varepsilon': P \rightarrow N$) will be called *quasi-augmentation* if there is a stable equivalence $f: M \rightarrow N$ such that $f \circ \varepsilon = \varepsilon'$.

There is a dual notion using injectives. A *complete injective resolution* of M is a long exact sequence of injective kG -modules

$$\cdots \longleftarrow I_{-2} \longleftarrow I_{-1} \xleftarrow{\partial_0} I_0 \longleftarrow I_1 \longleftarrow I_2 \longleftarrow \cdots$$

$$\begin{array}{ccc} & \swarrow & \searrow \\ & M & \end{array}$$

η

in which η is the inclusion of the kernel of ∂_0 . The map η is called *coaugmentation* and can be viewed as a chain map $\eta: M \rightarrow I$. If N is another module, then a map $\eta': N \rightarrow I$ will be called *quasi-coaugmentation* if there is a stable equivalence $f: N \rightarrow M$ with $\eta \circ f = \eta'$.

Since projectives are the same as injectives, the notions of complete resolutions only differ in the position of the resolved module M . If P is a complete projective resolution of the trivial module k , and N is another kG -module, then the cohomology groups of the complex $\text{Hom}_{kG}(P, N)$ define the Tate cohomology of G with coefficients in N , that is, $\widehat{H}^n(G; N) \cong H^n \text{Hom}_{kG}(P, N)$.

Proposition 2.1 *Let P and Q be complete projective resolutions, and let $\varepsilon: P \rightarrow M$ be a quasi-augmentation. If for some chain transformation $f: Q \rightarrow P$ the composite*

$$Q_0 \xrightarrow{f_0} P_0 \xrightarrow{\varepsilon} M$$

is zero, then f is null-homotopic. The corresponding statement holds for injective resolutions.

We omit the straightforward proof.

Whenever C is a cochain complex of kG -modules, we define the dual complex C^\vee as $(C^\vee)_n = \text{Hom}(C_{-n}, k)$ with the induced differentials. If P is a complete projective

resolution of M with (quasi-augmentation ε , then P^\vee is a complete injective resolution of M^\vee with (quasi-coaugmentation ε^\vee , and the same is true with the roles of projective and injective interchanged).

Let k be the trivial kG -module, and choose complete injective resolutions I and I' of k with coaugmentations η, η' . The tensor product $I \otimes I'$ is defined to be the complex with modules $(I \otimes I')_n = \bigoplus_{i+j=n} I_i \otimes I'_j$ and differential $\partial_{I \otimes I'} = \partial_I \otimes \text{id} + \text{id} \otimes \partial_{I'}$ (note here that evaluation of the differential involves the usual sign, ie, $(\text{id} \otimes \partial)(x \otimes y) = (-1)^{|x|} x \otimes \partial y$). It is known that the tensor product $I \otimes I'$ is a complete injective resolution of k with quasi-coaugmentation $\eta \otimes \eta'$ (see Krause [11, Section 8]).

Now let P and P' be complete projective resolutions of k , and assume that all modules P_i, P'_i are finitely generated. Let us define a new tensor product $P \boxtimes P' = (P^\vee \otimes P'^\vee)^\vee$; more explicitly, $(P \boxtimes P')_n = \prod_{i+j=n} P_i \otimes P'_j$. By the considerations above, this is a complete projective resolution of k with quasi-augmentation $\varepsilon \boxtimes \varepsilon'$: $P \boxtimes P' \rightarrow k$. These definitions and observations can be generalized to \boxtimes -products of finitely many complete projective resolutions.

Remark 2.2 The \boxtimes -product can be used to define the multiplication on Tate cohomology. By usual homological algebra, the identity map on k can be lifted to a commutative diagram as follows:

$$\begin{array}{ccc}
 P & \xrightarrow{\Delta} & P \boxtimes P \\
 \varepsilon \downarrow & & \downarrow \varepsilon \boxtimes \varepsilon \\
 k & \xlongequal{\quad} & k
 \end{array}$$

Such a lift is unique up to homotopy. A more explicit construction of Δ is given in the proof of Theorem 4.1 in Cartan and Eilenberg [6, XII], where it is also shown that Δ induces the Tate cohomology product in the following way: given cycles $f, g \in \text{Hom}_{kG}^*(P, k)$ we get a cycle $(f \boxtimes g) \circ \Delta \in \text{Hom}_{kG}^*(P, k)$ representing $[f] \cdot [g]$.

2.2 Motivation for the definition of the operad

Let P be a complete projective resolution of k by finitely generated kG -modules. Before we start with the actual construction of an E_∞ -structure on $\text{Hom}_{kG}^*(P, k)$, let us point out a major issue in the construction of power operations which does not turn up in the case of ordinary cohomology $H^*(G)$. For simplicity, let us assume that $p = 2$ throughout this motivational part. Let us naively transfer to Tate cohomology the construction of Sq_1 as it is done in ordinary cohomology. We know that the identity map of k can be lifted to a map $\Delta: P \rightarrow P^{\boxtimes 2}$ as in Remark 2.2, and any two such liftings are homotopic. Therefore, if T denotes the twist map of $P^{\boxtimes 2}$, then we know that

$(1 - T) \circ \Delta$ is the boundary of some map $\Delta_1: P \rightarrow P^{\boxtimes 2}$ of degree -1 . If $\zeta: P \rightarrow k$ is a chain map of degree n representing some cohomology class $[\zeta] \in \widehat{H}^n(G)$, then we know that $\zeta^{\boxtimes 2} \circ \Delta_1$ is a chain map of degree $2n$, and we could define $\text{Sq}_1(\zeta)$ to be the class represented by that map. The problem is here that there is an ambiguity in the choice of the map Δ_1 , and any two such choices differ by a chain map $P \rightarrow P^{\boxtimes 2}$ of degree -1 . Therefore, $\text{Sq}_1(\zeta)$ is only well-defined up to some element in $\zeta^2 \cdot \widehat{H}^{-1}(G)$. This problem does not occur in ordinary cohomology simply because $H^{-1}(G)$ is zero. We therefore have to rigidify our choice of Δ_1 in order to get actual operations. To do so, observe that a chain map $P \rightarrow P^{\boxtimes 2}$ of degree -1 certainly represents the zero class if the composite $P_{-1} \rightarrow (P^{\boxtimes 2})_0 \twoheadrightarrow P_0 \otimes P_0 \rightarrow k$ vanishes, so that one possibility is to require the map $P_{-1} \rightarrow P_0 \otimes P_0$ to be zero. The next step is to elaborate this idea, and because we want an E_∞ -structure, we need to do so in an ‘‘operadic’’ way.

2.3 Definition of the operad

As before, let P be a complete projective resolution of k by finitely generated kG -modules. We are now going to define an acyclic operad which acts on $\text{Hom}_{kG}^*(P, k)$. To do so, we will work in the category of (increasing degree) differential graded modules over k (or, equivalently, the category of cochain complexes of k -vector spaces) with its symmetric monoidal tensor product \otimes . Recall that if X and Y are chain complexes of kG -modules with differential ∂ , then we get such a differential graded module $\text{Hom}^*(X, Y)$ by defining

$$\text{Hom}^n(X, Y) = \prod_{j \in \mathbb{Z}} \text{Hom}_{kG}(X_{n+j}, Y_j)$$

with differential $d(f) = \partial f - (-1)^n f \partial$.

Let us recall some basics about operads; see, eg, Kříž and May [12] for an introduction. A symmetric operad \mathcal{C} is given by a differential graded module $\mathcal{C}(j)$ for every integer $j \geq 0$ together with a Σ_j -action, equivariant structure maps

$$\mathcal{C}(j) \otimes \mathcal{C}(i_1) \otimes \dots \otimes \mathcal{C}(i_j) \rightarrow \mathcal{C}(i_1 + \dots + i_j)$$

for all j, i_1, \dots, i_j , and a unit map $k \rightarrow \mathcal{C}(j)$ for each j ; all these maps have to satisfy certain coherence diagrams. A typical example of such an operad is the so-called *coendomorphism-operad* $\mathcal{F}(j) = \text{Hom}^*(P, P^{\boxtimes j})$ for $j \geq 0$, whose structure maps are given by

$$\begin{aligned} \text{Hom}(P, P^{\boxtimes j}) \otimes \text{Hom}(P, P^{\boxtimes i_1}) \otimes \dots \otimes \text{Hom}(P, P^{\boxtimes i_j}) &\rightarrow \text{Hom}(P, P^{\boxtimes (i_1 + \dots + i_j)}) \\ g \otimes f_1 \otimes \dots \otimes f_j &\mapsto (f_1 \boxtimes \dots \boxtimes f_j) \bullet g. \end{aligned}$$

We have written \bullet here because we want to stress that the Koszul sign rule also applies to this situation; whenever a and b are composable maps of certain degrees, we write $a \bullet b$ for $(-1)^{|a||b|} \cdot a \circ b$, so that expressions like $b \otimes a \mapsto a \bullet b$ indeed yield maps of chain complexes. The symmetric group Σ_j acts on $P^{\boxtimes j}$ by permutation of the factors (note that this also involves the usual signs), and we therefore get an action of Σ_j on $\text{Hom}(P, P^{\boxtimes j})$. The unit map $k \rightarrow \text{Hom}(P, P)$ is given by the identity of P . The operad we are up to will be a suboperad of the coendomorphism-operad \mathcal{F} .

An operad \mathcal{C} is called *unital* if $\mathcal{C}(0) = k$. In that case, the $\mathcal{C}(j)$ have augmentations coming from the operad structure maps

$$\mathcal{C}(j) \cong \mathcal{C}(j) \otimes \mathcal{C}(0)^j \rightarrow \mathcal{C}(0) = k.$$

The operad is called *acyclic* if the augmentations are quasi-isomorphisms of chain complexes. An operad \mathcal{C} is called an E_∞ -operad if it is acyclic and for every j , $\mathcal{C}(j)$ is free as a $k\Sigma_j$ -module. A differential graded module A is called a \mathcal{C} -algebra if there are structure maps

$$\mathcal{C}(j) \otimes A^j \rightarrow A$$

for every $j \geq 0$ which are associative, unital and equivariant (see [12, Section 2] for details). Our goal is to define an acyclic operad \mathcal{C} (and later an E_∞ -operad) and a \mathcal{C} -algebra structure on $A = \text{Hom}^*(P, k)$. This structure can then be used to define the operations Q_i on $H^*A \cong \widehat{H}^*(G)$, and also for proving most of Theorem 1.1.

Let us begin with the definition of \mathcal{C} . For every nonnegative integer j , we define a differential graded submodule $\mathcal{C}(j)$ of $\text{Hom}^*(P, P^{\boxtimes j})$ as follows:

$$\begin{aligned} \mathcal{C}(j)^m &= 0 && \text{for } m > 0, \\ \mathcal{C}(j)^0 &= \{f \in \text{Hom}^0(P, P^{\boxtimes j}) \mid df = 0\}, \\ \mathcal{C}(j)^m &= \left\{ f \in \text{Hom}^m(P, P^{\boxtimes j}) \left| \begin{array}{l} P_i \xrightarrow{\text{proj} \circ f} P_{s_1} \otimes P_{s_2} \otimes \cdots \otimes P_{s_j} \\ \text{vanishes for all } i < 0 \text{ and} \\ \text{all } s_1, \dots, s_j \geq 0 \end{array} \right. \right\} && \text{for } m < 0. \end{aligned}$$

In order to check that $\mathcal{C}(j)$ is indeed a differential graded submodule, we have to prove $d\mathcal{C}(j)^m \subseteq \mathcal{C}(j)^{m+1}$. This is clear for $m \geq -1$, and in case $m < -1$, the map

$$P_i \xrightarrow{\text{proj} \circ df} P_{s_1} \otimes \cdots \otimes P_{s_j}$$

is the sum of

$$P_i \xrightarrow{\partial} P_{i-1} \xrightarrow{\text{proj} \circ f} P_{s_1} \otimes \cdots \otimes P_{s_j}$$

and maps

$$P_i \xrightarrow{\text{proj} \circ f} P_{s_1} \otimes \cdots \otimes P_{s_{t+1}} \otimes \cdots \otimes P_{s_j} \xrightarrow{\text{id} \otimes \partial \otimes \text{id}} P_{s_1} \otimes \cdots \otimes P_{s_j},$$

all of which are zero by assumption.

Next we show that \mathcal{C} is a suboperad of the co-endomorphism operad \mathcal{F} . In order to do so, we only need to show that it is closed under the structure maps, the Σ -action, and the unit. The latter two are immediate consequences of the definition, so let us take $g \in \mathcal{C}(j)$, $f_i \in \mathcal{C}(j_i)$ for $i = 1, \dots, j$ and prove that $(f_1 \boxtimes \dots \boxtimes f_j) \bullet g \in \mathcal{C}(j_1 + \dots + j_j)$. If one of the chosen elements is of positive degree, then the composition is zero. If all the chosen elements are of degree zero, then they are chain transformations and so is the composition. Now we can assume that the composition is of negative degree, and we have to show that the composite

$$P_i \xrightarrow{g} P_{s_1} \otimes \dots \otimes P_{s_j} \xrightarrow{f_1 \boxtimes \dots \boxtimes f_j} P_{t_{1,1}} \otimes \dots \otimes P_{t_{1,j_1}} \otimes \dots \otimes P_{t_{j,1}} \otimes \dots \otimes P_{t_{j,j_j}}$$

is zero for all $i < 0$ and $t_{l,n} \geq 0$. If s_l is negative, then $f_l: P_{s_l} \rightarrow P_{t_{l,1}} \otimes \dots \otimes P_{t_{l,j_l}}$ vanishes and so does the composition. But if all the s_l 's are nonnegative, then g is zero, so we are done.

The operad \mathcal{C} is unital, that is, $\mathcal{C}(0)$ is isomorphic to k concentrated in degree 0. Here we use the convention $P^{\boxtimes 0} = k$; then $\mathcal{C}(0)^m = 0$ unless $m = 0$, in which case

$$\mathcal{C}(0)^0 = \{ f \in \text{Hom}^0(P, k) \mid df = 0 \} \cong k \langle \varepsilon \rangle .$$

So we get augmentations $\mathcal{C}(j) \cong \mathcal{C}(j) \otimes \mathcal{C}(0)^j \rightarrow \mathcal{C}(0) \cong k$ given by postcomposition with $\varepsilon^{\boxtimes j}$.

2.4 Acyclicity of the operad

We are now going to show that the augmentations $\mathcal{C}(j) \rightarrow k$ are quasi-isomorphisms.

To do so, let us consider another complete projective resolution Q of k , constructed as follows. Let us define P^+ to be the nonnegative part of P , that is $P_n^+ = P_n$ for $n \geq 0$ with the induced differentials. Then $k \leftarrow P^+ : \varepsilon$ is an acyclic augmented complex, and by the Künneth theorem $k \leftarrow (P^+)^{\otimes j} : \varepsilon^{\otimes j}$ is also acyclic. Next, we define a complex $\dots \leftarrow R_{-2} \leftarrow R_{-1} \leftarrow R_0$ by setting $R_n = P_n$ for $n < 0$ and $R_0 = k$, the differential $R_{-1} \leftarrow R_0$ being the coaugmentation of P . Then R is acyclic, and by the Künneth theorem $R^{\otimes j}$ is also acyclic. Note that $(R^{\otimes j})_0 = k$, so we can splice the complexes $R^{\otimes j}$ and $k \leftarrow (P^+)^{\otimes j}$ to get a complex Q , which then is a complete projective resolution of k . There is a chain map $\Phi: P^{\boxtimes j} \rightarrow Q$ which in nonnegative degrees is given by projections, and in negative degrees the maps

$$P_{s_1} \otimes P_{s_2} \otimes \dots \otimes P_{s_j} \rightarrow R_{s_1} \otimes R_{s_2} \otimes \dots \otimes R_{s_j}$$

are zero unless all the s_i 's are nonpositive, in which case the map is the tensor product of identity maps and the augmentation $\varepsilon: P_0 \rightarrow R_0 = k$.

Since the composition

$$P^{\boxtimes j} \xrightarrow{\Phi} Q \xrightarrow{\varepsilon} k$$

equals the quasi-augmentation $\varepsilon^{\boxtimes j}$, we get that Φ is a chain homotopy equivalence. Therefore, the induced map $\lambda: \text{Hom}^*(P, P^{\boxtimes j}) \rightarrow \text{Hom}^*(P, Q)$ is a quasi-isomorphism. Moreover, λ is surjective because Φ is levelwise onto.

Proposition 2.3 *Suppose that $\lambda: A \rightarrow B$ is a surjective quasi-isomorphism of differential graded modules, and let $C \subseteq B$ be a differential graded submodule of B . Then the restriction $\lambda: \lambda^{-1}(C) \rightarrow C$ is a quasi-isomorphism as well.*

Proof Let us denote by K the kernel of λ . Since λ is a quasi-isomorphism, the long exact sequence in homology implies that $H^*(K) = 0$. Since K is also the kernel of $\lambda|_{\lambda^{-1}(C)}$, using the long exact sequence in homology again we get that the restriction of λ is a quasi-isomorphism. \square

Now the idea is to choose a dg submodule C of $\text{Hom}^*(P, Q)$ quasi-isomorphic to k , and such that $\lambda^{-1}(C)$ is (close to) our $\mathcal{C}(j)$. Define

$$\begin{aligned} C^m &= 0 && \text{for } m > 0, \\ C^0 &= \{f \in \text{Hom}^0(P, Q) \mid df = 0\} \\ C^m &= \{f \in \text{Hom}^m(P, Q) \mid P_i \xrightarrow{f} Q_j \text{ is zero for all } i < 0 \leq j\} && \text{for } m < 0. \end{aligned}$$

Then C is indeed a dg submodule of $\text{Hom}^*(P, Q)$.

Proposition 2.4 *We have $H^*(C) \cong k$.*

Proof Clearly, $H^m(C) = 0$ for $m > 0$. Let $m < 0$, and let $f \in C^m$ be a cocycle. Define $g: P_{i+m-1} \rightarrow Q_i$ to be zero for all $i = 0, 1, \dots, -m$. By common homological algebra we can extend g to a chain null-homotopy for f (the condition needed for the inductive construction of g is that $\partial g \partial = f$ at the two boundary points of the domain on which g has been defined, and this condition is clearly satisfied). Then $dg = f$ with $g \in C$, and hence $H^m(C) = 0$ for $m > 0$.

Finally, we claim that the image of $d: C^{-1} \rightarrow C^0$ is the same as the image of $d: \text{Hom}^{-1}(P, Q) \rightarrow C^0$ (then it follows that $H^0(C) \cong H^0(G) \cong k$). Let $f \in \text{Hom}^{-1}(P, Q)$; then the bottom row in the diagram

$$\begin{array}{ccccccc} P_0 & \xrightarrow{\partial} & P_1 & \xrightarrow{f} & Q_0 & \xrightarrow{\partial} & Q_{-1} \\ & \searrow \varepsilon & & \nearrow \eta & & \searrow \varepsilon^{\otimes j} & & \nearrow & \\ & & k & \xrightarrow{\hspace{2cm}} & k & & & & \end{array}$$

is stably trivial and therefore the zero map (we assume here that $|G|$ is divisible by p , which is the only interesting case). Therefore the upper row vanishes, and by usual homological algebra there is a cocycle $g \in \text{Hom}^{-1}(P, Q)$ with $f_0 = g_0: P_{-1} \rightarrow Q_0$. Then $f - g \in C^{-1}$ and $d(f - g) = df$, so we are done. \square

We finally use a method of chopping off the positive part of a dg module. Given a dg module A , define $F(A)$ to be the dg submodule given by

$$F(A)^m = \begin{cases} 0 & \text{if } m > 0, \\ \text{cycles of } A^0 & \text{if } m = 0, \\ A^m & \text{if } m < 0. \end{cases}$$

(This can be viewed as the (co)connected cover of A .) Then the inclusion $F(A) \subseteq A$ induces an isomorphism $H^*(F(A)) \cong H^*(A)$ in nonpositive degrees.

Proposition 2.5 *The augmentation $\mathcal{C}(j) \rightarrow k$ is a quasi-isomorphism. Thus, the operad \mathcal{C} is acyclic.*

Proof Note that $\mathcal{C}(j) = F(\lambda^{-1}(C))$, so $H^*(\mathcal{C}(j)) \cong H^*(C) \cong k$ by Proposition 2.4. Since there is a cocycle $f \in \mathcal{C}(j)^0$ such that

$$P \xrightarrow{f} P \boxtimes j \xrightarrow{\varepsilon \boxtimes j} k$$

equals the augmentation ε , the map $\mathcal{C}(j) \rightarrow k$ is onto in H^0 and therefore a quasi-isomorphism. \square

For every operad \mathcal{A} , the module $\mathcal{A}(0)$ is an algebra over \mathcal{A} via the action map $\mathcal{A}(j) \otimes \mathcal{A}(0)^{\otimes j} \rightarrow \mathcal{A}(0)$. In particular, $\text{Hom}^*(P, k)$ is an algebra over the co-endomorphism operad $\text{Hom}^*(P, P^{\otimes j})$, and we can restrict the operad action to the suboperad \mathcal{C} . Hence, $\text{Hom}^*(P, k)$ is a \mathcal{C} -algebra.

Proposition 2.6 *The operad \mathcal{C} induces an E_∞ -structure on $\text{Hom}^*(P, k)$ in such a way that the product on $H^* \text{Hom}^*(P, k)$ agrees with the composition product of the Tate cohomology ring $\hat{H}^*(G)$.*

Proof The operad \mathcal{C} might itself not be Σ -free, so we have to choose an approximation of \mathcal{C} by an E_∞ -operad. One possible way of doing so (see [14, Section 1]) is to choose an arbitrary E_∞ -operad \mathcal{E}' and tensor its augmentation $\mathcal{E}' \rightarrow k$ with \mathcal{C} . Then $\mathcal{E} = \mathcal{C} \otimes \mathcal{E}'$ is an E_∞ -operad acting on $\text{Hom}^*(P, k)$ via the action of \mathcal{C} pulled back along the morphism of operads $\mathcal{E} \rightarrow \mathcal{C}$. The statement about the product follows from the fact that the element $\Delta \in \mathcal{C}(2)^0 \subseteq \text{Hom}^0(P, P \boxtimes P)$ given in Remark 2.2 generates the cohomology $H^0(\mathcal{C}(2))$ and induces the right product on $H^* \text{Hom}^*(P, k)$. \square

2.5 Comparison with Steenrod reduced powers

For the proof of part (9) of Theorem 1.1 we need to recall the construction of Steenrod operations in the cohomology of cocommutative Hopf algebras. Let \tilde{P} be an ordinary projective resolution of k , viewed as a complex $\cdots \leftarrow 0 \leftarrow \tilde{P}_0 \leftarrow \tilde{P}_1 \leftarrow \cdots$. Then $\tilde{P}^{\otimes j}$ is a projective resolution of k for all j . Consider the suboperad $\mathcal{A}(j) = F(\text{Hom}(\tilde{P}, \tilde{P}^{\otimes j}))$ of the coendomorphism-operad $\text{Hom}(\tilde{P}, \tilde{P}^{\otimes j})$. Then \mathcal{A} is acyclic, and $\text{Hom}(\tilde{P}, k)$ is an \mathcal{A} -algebra in the obvious way. Using an E_∞ -approximation of \mathcal{A} , this operad action defines the Steenrod operations on $H^* \text{Hom}^*(\tilde{P}, k) \cong H^*(G)$.

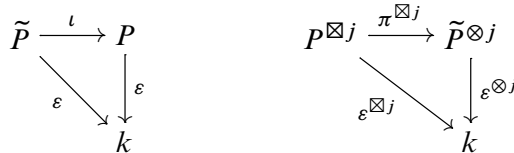
Extend \tilde{P} to a complete projective resolution P of k . We are now going to write down a quasi-isomorphism of unital operads $\mathcal{C} \rightarrow \mathcal{A}$. Let us begin with a function Ψ which maps an element $f \in \text{Hom}^*(P, P^{\boxtimes j})$ to the element in $\text{Hom}^*(\tilde{P}, \tilde{P}^{\otimes j})$ given by the composition

$$\tilde{P} \xrightarrow{\iota} P \xrightarrow{f} P^{\boxtimes j} \xrightarrow{\pi^{\boxtimes j}} \tilde{P}^{\otimes j}.$$

Notice here that the inclusion map ι is not quite a chain map; its differential $d\iota$ in $\text{Hom}(\tilde{P}, P)$ is zero everywhere except for $\tilde{P}_0 \rightarrow P_{-1}$. On the other hand, the projection map π is a chain map, and therefore

$$d(\pi f \iota) = \pi d(f) \iota \pm \pi f d(\iota)$$

in $\text{Hom}^*(\tilde{P}, \tilde{P}^{\otimes j})$. Now assume that $f \in \mathcal{C}(j)$; then either f is of nonnegative degree, in which case $\pi f d(\iota)$ is zero (because π vanishes in negative degrees), or f is of negative degree, but then f is zero as maps $P_{-1} \rightarrow P_{s_1} \otimes P_{s_2} \otimes \cdots \otimes P_{s_j}$ for all $s_i \geq 0$, and π is zero on all other factors of $P^{\boxtimes j}$ of interest. Hence $d(\Psi(f)) = \Psi(df)$, so that Ψ restricted to $\mathcal{C}(j)$ is indeed a map of dg modules. We get a map $\Psi: \mathcal{C} \rightarrow \mathcal{A}$ of unital operads, and we need to show that Ψ commutes with the augmentations of $\mathcal{C}(j)$ and $\mathcal{A}(j)$. This follows from the following commutative diagrams:



Proof of Theorem 1.1 Everything except part (9) is a consequence of Proposition 2.6 and the fact that E_∞ -structures can be used to construct power operations with the desired properties; see, eg, Kříž and May [12, I, Section 7], Cohen, Lada and May [7, I, Section 1] and May [15]. For part (9), note that by construction of the operations Q_i via \mathcal{C} and the Steenrod operations via \mathcal{A} we get the desired statement for $n \geq 0$. To prove $Q_{-n}(x) = 0$ and $\beta Q_{-n}(x) = 0$ for $n < 0$ it is enough to notice that for elements f in $\mathcal{C}(p)$ we have that $f: P_{\text{neg}} \rightarrow P_{|x|}^{\otimes p}$ vanishes. \square

3 Products of groups

This section is devoted to the proof of Theorem 1.7. Let G be any finite group whose order is divisible by p . As a first step, we shall define a new operad action defining some power operations on $M^*(G)$. In the second step we prove that these operations agree with the Dyer–Lashof operations coming from $\widehat{H}^*(G)$.

Let P be a complete projective resolution of k as a kG -module. We denote by \bar{P} the complex $\cdots \leftarrow P_{-2} \leftarrow P_{-1} \leftarrow 0 \leftarrow 0 \leftarrow \cdots$, with the P_{-1} sitting in degree 0 and with differential $\partial_{\bar{P}} = -\partial_P$, and let $\eta: k \rightarrow \bar{P}$ be the coaugmentation. For $j \geq 1$ define the differential graded module $\mathcal{B}(j) = F(\text{Hom}^*(\bar{P}, \bar{P}^{\otimes j}))$. Also put $\mathcal{B}(0) = k$; we want to turn \mathcal{B} into a unital operad, so we need to define the structure maps

$$\gamma: \mathcal{B}(j) \otimes \mathcal{B}(i_1) \otimes \cdots \otimes \mathcal{B}(i_j) \rightarrow \mathcal{B}(i_1 + \cdots + i_j).$$

As long as all i_s 's are positive, we simply take the usual structure maps of the coendomorphism-operad $\text{Hom}^*(\bar{P}, \bar{P}^{\otimes j})$. If one of the i_s 's is zero, then we put $\gamma = 0$ unless $i_1 = i_2 = \cdots = i_j = 0$, in which case

$$(3.1) \quad \gamma: \mathcal{B}(j) \otimes \mathcal{B}(0)^{\otimes j} = \mathcal{B}(j) \rightarrow \text{Hom}_{kG}(k, k) = k = \mathcal{B}(0)$$

sends a chain map $\bar{P} \rightarrow \bar{P}^{\otimes j}$ in $\mathcal{B}(j)$ to the induced map $k \rightarrow k^{\otimes j} = k$ on zero-cycles. It is now straightforward to check that \mathcal{B} is indeed a unital symmetric operad. Also, \mathcal{B} is acyclic because by usual homological algebra the augmentations $\mathcal{B}(j) \rightarrow k$ are quasi-isomorphisms.

Now $\text{Hom}^*(\bar{P}, k)$ is a \mathcal{B} -algebra, therefore we obtain Dyer–Lashof operations on $H^* \text{Hom}^*(\bar{P}, k) \cong M^*(G)$ which we are now going to compare with those obtained from \mathcal{C} . Let $\iota \in \text{Hom}^1(\bar{P}, P)$ be the inclusion, and let $\pi \in \text{Hom}^{-1}(P, \bar{P})$ be the projection map. Then $d\iota = 0$, but $d\pi \neq 0$. Let K be the cochain complex of k -vector spaces generated by an element x of degree -1 which is mapped by the differential to a nontrivial element y in degree 0:

$$\begin{array}{ccccccc} \cdots & \longleftarrow & 0 & \longleftarrow & k \langle y \rangle & \longleftarrow & k \langle x \rangle & \longleftarrow & 0 & \longleftarrow & \cdots \\ & & & & y & \longleftarrow & x & & & & \end{array}$$

Let $Y = K^{\otimes p}$, and then define the augmented cochain complex X by the formula $X_i = Y_{i-1}$ for all $i \leq 0$ with augmentation $X_0 \rightarrow Y_0 = k \langle y^p \rangle$. Then X is an acyclic augmented complex of $k\Sigma_p$ -modules. The map of cochain complexes $K \rightarrow \text{Hom}^*(P, \bar{P})$ given by $x \mapsto \pi$ induces a map of cochain complexes $\varphi: Y \rightarrow \text{Hom}^*(P^{\boxtimes p}, \bar{P}^{\boxtimes p})$. Let us define $\sigma: X \otimes \mathcal{C}(p) \rightarrow \mathcal{B}(p)$ by the formula $\sigma(v \otimes f) = (-1)^{|f|} \varphi(v) \circ f \circ \iota$ (the sign coming from shifting Y to X).

Proposition 3.2 The map $\sigma: X \otimes \mathcal{C}(p) \rightarrow \mathcal{B}(p)$ enjoys the following properties:

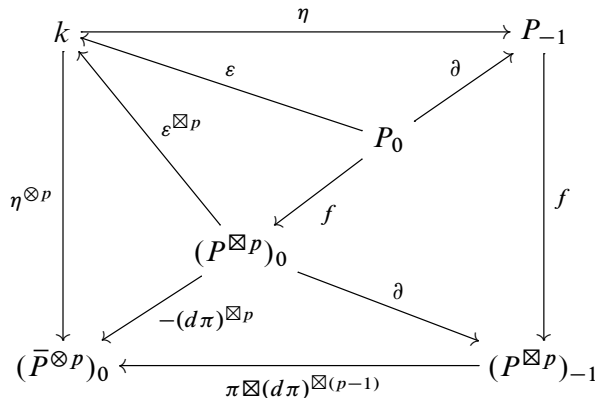
- (a) It is a Σ_p -equivariant cochain map lifting the identity of k .
- (b) For every $f \in \mathcal{C}(p)$ we have $\sigma(x^p \otimes f) = (-1)^{|f|} \pi^{\boxtimes p} \circ f \circ \iota$.
- (c) For every element $w \in X \otimes \mathcal{C}(p)$ of bidegree (m, n) with $m > -p + 1$, and for every cocycle $a \in \text{Hom}^*(\bar{P}, k)$, we have $\gamma(\sigma(w) \otimes a^{\otimes p}) = 0$, where γ is the operad action of \mathcal{B} on $\text{Hom}^*(\bar{P}, k)$.

Proof Part (b) follows from the definition. To show (c), let $w = w_1 \otimes \dots \otimes w_p \otimes f \in X \otimes \mathcal{C}(p)$ with $w_i \in \{x, y\}$ for all i . Up to a sign, $\gamma(\sigma(w) \otimes a^{\otimes p})$ is given by the composition

$$\bar{P} \xrightarrow{\iota} P \xrightarrow{f} P^{\boxtimes p} \xrightarrow{u_1 \boxtimes \dots \boxtimes u_p} \bar{P}^{\otimes p} \xrightarrow{a^{\otimes p}} k,$$

where $u_i = \pi$ if $w_i = x$ and $u_i = d\pi$ if $w_i = y$. From the condition on the bidegree of w we know that at least one of the u_i 's equals $d\pi$, so that $a \circ u_i = 0$, which implies (c).

For (a), let $f \in \mathcal{C}(p)$ be a cocycle in degree 0 mapping to 1 under the augmentation $\mathcal{C}(p) \rightarrow k$; then consider the following diagram:



All smaller parts commute, and since $\epsilon: P_0 \rightarrow k$ is surjective we can deduce that the exterior square commutes. Therefore, σ indeed lifts the identity of k . Also, σ is Σ_p -equivariant and is a cochain map because $d\iota = 0$, so (a) is proved. \square

Proposition 3.3 The operad action of \mathcal{B} on $\text{Hom}^*(\bar{P}, k)$ and the action of \mathcal{C} on $\text{Hom}^*(P, k)$ define the same operations on $M^*(G)$.

Proof Let $\epsilon: X \rightarrow k[1-p]$ be the $k\Sigma_p$ -linear chain map given by $x^p \mapsto 1$, and define $\tau = \epsilon \otimes \text{id}_{\mathcal{C}}: X \otimes \mathcal{C}(p) \rightarrow \mathcal{C}(p)[1-p]$. By suitably shifting the action of $\mathcal{C}(p)$ on the negative part of $\text{Hom}^*(P, k)$ we get a map defined by

$$\begin{aligned} \gamma_{\mathcal{C}}: \mathcal{C}(p)[1-p] \otimes \text{Hom}^*(\bar{P}, k)^{\otimes p} &\rightarrow \text{Hom}^*(\bar{P}, k) \\ f \otimes w &\mapsto (-1)^{|f| \cdot (|w|+1)} w \circ \pi^{\boxtimes p} \circ f \circ \iota \end{aligned}$$

for all $f \in \mathcal{C}(p)[1-p]$ and $w \in \text{Hom}^*(\bar{P}, k)^{\otimes p}$. The sign is due to the Koszul sign rule, and the check that this is indeed a map of chain complexes uses the fact that $a \circ (d\pi) = 0$ for all $a \in \text{Hom}(P, k)$. Now $\gamma_{\mathcal{C}}$ can be used to construct the power operations on $M^*(G)$ as follows. Let W be the standard free resolution of the trivial kC_p -module k (where C_p denotes the cyclic group of order p), so that W_i is generated by a single element e_i . Since $X \otimes \mathcal{C}(p)$ is an acyclic augmented complex of kC_p -modules, we can lift the identity of k to a C_p -equivariant chain map $\vartheta: W \rightarrow X \otimes \mathcal{C}(p)$. We then have a diagram like this:

$$\begin{array}{ccc} W & \xrightarrow{\vartheta} & X \otimes \mathcal{C}(p) \\ & & \swarrow \sigma \quad \searrow \tau \\ & & \mathcal{B}(p) \quad \mathcal{C}(p)[1-p] \end{array}$$

For cocycles $a \in \text{Hom}^*(\bar{P}, k)$, define $D_i^{\mathcal{C}}(a)$ to be the cohomology class of the cocycle $\gamma_{\mathcal{C}}(\tau\vartheta(e_i) \otimes a^p) \in \text{Hom}^*(\bar{P}, k)$, and define $D_i^{\mathcal{B}}(a)$ to be the class of $\gamma_{\mathcal{B}}(\sigma\vartheta(e_i) \otimes a^p) \in \text{Hom}^*(\bar{P}, k)$. We need to show that $D_i^{\mathcal{C}} = D_i^{\mathcal{B}}$, and for this it suffices to prove the identity

$$\gamma_{\mathcal{B}}(\sigma(w) \otimes a^p) = \gamma_{\mathcal{C}}(\tau(w) \otimes a^p)$$

for all $w \in X \otimes \mathcal{C}(p)$ and $a \in \text{Hom}^*(\bar{P}, k)$. We can write $w = x^p \otimes f + \sum_i u_i \otimes f_i$ with $u_i \in X$ of degree $|u_i| > 1-p$, and $f, f_i \in \mathcal{C}(p)$. By Proposition 3.2 we have that

$$\gamma_{\mathcal{B}}(\sigma(w) \otimes a^p) = \gamma_{\mathcal{B}}(\sigma(x^p \otimes f) \otimes a^p) = (-1)^{|f|} \gamma_{\mathcal{B}}((\pi^{\boxtimes p} \circ f \circ \iota) \otimes a^p).$$

On the other hand, $\gamma_{\mathcal{C}}(\tau(w) \otimes a^p) = \gamma_{\mathcal{C}}(f \otimes a^p)$, and all these expressions equal $(-1)^{|f|(|a^p|+1)}$ times the composition

$$\bar{P} \xrightarrow{\iota} P \xrightarrow{f} P^{\boxtimes p} \xrightarrow{\pi^{\boxtimes p}} \bar{P}^{\otimes p} \xrightarrow{a^{\otimes p}} k. \quad \square$$

Proof of Theorem 1.7 Let us write $G = G_1 \times G_2$. Choose complete projective resolutions P and Q for k as trivial kG_1 - and kG_2 -module, respectively. Then $k \rightarrow \bar{P} \otimes \bar{Q}$ is the negative part of a projective resolution of k as kG -module. We denote by \mathcal{B}^{G_1} , \mathcal{B}^{G_2} and \mathcal{B}^G the operads constructed above using these resolutions; then we get a quasi-isomorphism of unital operads $\mathcal{B}^{G_1} \otimes \mathcal{B}^{G_2} \rightarrow \mathcal{B}^G$ by tensoring

morphisms. Let us denote by $A_1 = \text{Hom}_{kG_1}^*(\bar{P}, k)$, $A_2 = \text{Hom}_{kG_2}^*(\bar{Q}, k)$ and $A = \text{Hom}_{kG}^*(\bar{P} \otimes \bar{Q}, k)$ the corresponding \mathcal{B} -algebras, then the commutative diagram

$$\begin{CD} \mathcal{B}^{G_1}(p) \otimes A_1^p \otimes \mathcal{B}^{G_2}(p) \otimes A_2^p @>>> A_1 \otimes A_2 \\ @VVV @VVV \\ \mathcal{B}^G(p) \otimes A^p @>>> A \end{CD}$$

implies the desired result. □

4 An alternative description of negative Ext-groups

Let $n > 0$. It is well-known that $\text{Ext}_{kG}^n(A, B) = \underline{\text{Hom}}_{kG}(\Omega^n A, B)$ admits a description via extensions of B by A . We will now give a similar description of $\widehat{\text{Ext}}_{kG}^{-n}(A, B) \cong \underline{\text{Hom}}_{kG}(A, \Omega^n B)$, which will be used throughout the next two sections. Let us define a category $\mathcal{K}_n(A, B)$, whose objects are all the chain complexes

$$C: \quad A \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \dots \longrightarrow P_1 \longrightarrow B$$

with projective modules P_1, P_2, \dots, P_n , and a morphism of two such complexes is a commutative diagram as follows:

$$\begin{CD} C @>>> A @>>> P_n @>>> P_{n-1} @>>> \dots @>>> P_1 @>>> B \\ @VVV @| @VVV @VVV @VVV @VVV @| \\ C' @>>> A @>>> P'_n @>>> P'_{n-1} @>>> \dots @>>> P'_1 @>>> B \end{CD}$$

For objects C and C' , let us write $C \approx C'$ if there is a morphism $C \rightarrow C'$ in $\mathcal{K}_n(A, B)$. Define the relation \sim on $\mathcal{K}_n(A, B)$ to be the equivalence relation generated by \approx , and put $K_n(A, B) = \mathcal{K}_n(A, B) / \sim$, the connected components of $\mathcal{K}_n(A, B)$.

Let us fix a projective resolution of B :

$$(4.1) \quad P: \quad 0 \longrightarrow \Omega^n B \xrightarrow{i} P_n \longrightarrow P_{n-1} \longrightarrow \dots \longrightarrow P_1 \longrightarrow B \longrightarrow 0$$

Theorem 4.2 *The map $\Phi: \text{Hom}_{kG}(A, \Omega^n B) \rightarrow \mathcal{K}_n(A, B)$ which associates to each map $f: A \rightarrow \Omega^n B$ the complex*

$$A \xrightarrow{i \circ f} P_n \rightarrow P_{n-1} \rightarrow \dots \rightarrow P_1 \rightarrow B$$

induces a bijection

$$\underline{\text{Hom}}_{kG}(A, \Omega^n B) \xleftrightarrow{1:1} K_n(A, B)$$

which is natural in G .

To prove this, we need the following proposition.

Proposition 4.3 Consider two finite chain complexes $A = (0 \rightarrow A_{n+1} \rightarrow \dots \rightarrow A_0 \rightarrow 0)$ and $B = (0 \rightarrow B_{n+1} \rightarrow \dots \rightarrow B_0 \rightarrow 0)$, where A_i is projective for $i = 1, 2, \dots, n$, and B is exact. Let $f, g: A \rightarrow B$ be chain maps satisfying $f_0 = g_0: A_0 \rightarrow B_0$. Then the classes of f_{n+1} and g_{n+1} in $\underline{\text{Hom}}_{kG}(A_{n+1}, B_{n+1})$ are the same.

The proof is standard homological algebra, and we omit it.

Proof of Theorem 4.2 As a first step, we show Φ induces a map $\underline{\text{Hom}}_{kG}(A, \Omega^n B) \rightarrow K_n(A, B)$. Suppose we are given $f' \in \text{Hom}_{kG}(A, \Omega^n B)$ such that $f' - f$ factors through some projective module R :

$$f' - f: A \xrightarrow{u} R \xrightarrow{w} \Omega^n B$$

Then the complexes $\Phi(f)$ and $\Phi(f')$ differ in their first map only; let us denote these by $\alpha, \alpha': A \rightarrow P_n$, respectively. From the commutative diagram

$$\begin{array}{ccccccccccc}
 A & \xrightarrow{\alpha} & P_n & \longrightarrow & P_{n-1} & \longrightarrow & P_{n-2} & \longrightarrow & \cdots & \longrightarrow & B \\
 \parallel & & \uparrow \begin{pmatrix} \text{id} \\ 0 \end{pmatrix} & & \parallel & & \parallel & & & & \parallel \\
 A & \xrightarrow{(\alpha \ u)} & P_n \oplus R & \xrightarrow{\begin{pmatrix} \partial \\ 0 \end{pmatrix}} & P_{n-1} & \longrightarrow & P_{n-2} & \longrightarrow & \cdots & \longrightarrow & B \\
 \parallel & & \downarrow \begin{pmatrix} \text{id} \\ \partial \circ w \end{pmatrix} & & \parallel & & \parallel & & & & \parallel \\
 A & \xrightarrow{\alpha'} & P_n & \longrightarrow & P_{n-1} & \longrightarrow & P_{n-2} & \longrightarrow & \cdots & \longrightarrow & B
 \end{array}$$

we get that $\Phi(f) \sim \Phi(f')$. Therefore, we obtain a map $\underline{\text{Hom}}_{kG}(A, \Omega^n B) \rightarrow K_n(A, B)$ which we also denote by Φ .

To construct an inverse for Φ , start with some object $C = (A \rightarrow Q_* \rightarrow B) \in \mathcal{K}_n(A, B)$. Since the Q_i 's are projective and (4.1) is exact, we can lift the identity on B to a map of chain complexes $f: C \rightarrow P$:

$$\begin{array}{ccccccccccc}
 A & \longrightarrow & Q_n & \longrightarrow & Q_{n-1} & \longrightarrow & \cdots & \longrightarrow & Q_1 & \longrightarrow & B \\
 \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \parallel \\
 \Omega^n B & \longrightarrow & P_n & \longrightarrow & P_{n-1} & \longrightarrow & \cdots & \longrightarrow & P_1 & \longrightarrow & B
 \end{array}$$

By Proposition 4.3, the stable class of the resulting map $f_{n+1}: A \rightarrow \Omega^n B$ is independent of the choice of the lift; let us write $\Psi(C) = f_{n+1} \in \underline{\text{Hom}}_{kG}(A, B)$. Suppose we are given a morphism $g: C' \rightarrow C$ in $\mathcal{K}_n(A, B)$. Then $f \circ g$ is a lift of the identity on B to a map of chain complexes $C' \rightarrow P$. Since $g_{n+1} = \text{id}_A$, we

have $\Psi(C') = (f \circ g)_{n+1} = f_{n+1} = \Psi(C)$. Therefore, we have constructed a map $\Psi: K_n(A, B) \rightarrow \underline{\text{Hom}}_{kG}(A, \Omega^n B)$. The proofs of $\Psi \circ \Phi = \text{id}$ and $\Phi \circ \Psi = \text{id}$ are immediate. \square

Example 4.4 Suppose that p divides the order of the group G . Then it is known that $\widehat{H}^{-1}(G) \cong \underline{\text{Hom}}_{kG}(k, \Omega k)$ is isomorphic to k . Under the bijection of Theorem 4.2, a canonical generator of that vector space is given by the complex

$$k \xrightarrow{\sum_{g \in G} g} kG \xrightarrow{\epsilon} k,$$

where ϵ is the augmentation of kG .

Proposition 4.5 Suppose we have a commutative diagram

$$\begin{array}{ccccccccc} A & \longrightarrow & P_n & \longrightarrow & \cdots & \longrightarrow & P_1 & \longrightarrow & B \\ g \downarrow & & \downarrow & & & & \downarrow & & \downarrow f \\ 0 & \longrightarrow & D & \longrightarrow & E_n & \longrightarrow & \cdots & \longrightarrow & E_1 & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

in $\text{mod-}kG$. Assume further that the P_i 's are projective, so that the upper row represents some element $\alpha \in \underline{\text{Hom}}_{kG}(A, \Omega^n B)$, and assume that the lower row is exact, therefore representing some element $\beta \in \underline{\text{Hom}}_{kG}(\Omega^n C, D)$. Then the diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & \Omega^n B \\ g \downarrow & & \downarrow \Omega^n f \\ D & \xleftarrow{\beta} & \Omega^n C \end{array}$$

commutes stably.

Proof Choose projective resolutions $\Omega^n B \rightarrow Q_* \rightarrow B$ and $\Omega^n C \rightarrow R_* \rightarrow C$. By the usual “projective to acyclic” argument, we get a diagram

$$\begin{array}{ccccccc} A & \longrightarrow & P_* & \longrightarrow & B & & \\ \bar{\alpha} \downarrow & & \downarrow & & \parallel & & \\ \Omega^n B & \longrightarrow & Q_* & \longrightarrow & B & & \\ \Omega^n f \downarrow & & \downarrow & & \downarrow f & & \\ \Omega^n C & \longrightarrow & R_* & \longrightarrow & C & & \\ \bar{\beta} \downarrow & & \downarrow & & \parallel & & \\ D & \longrightarrow & E_* & \longrightarrow & C & & \end{array}$$

where $\bar{\alpha}$ and $\bar{\beta}$ are unstable representatives of α and β , respectively. The result follows from Proposition 4.3. \square

Remark 4.6 Suppose we have an exact sequence $A \hookrightarrow P_n \rightarrow \dots \rightarrow P_1 \twoheadrightarrow B$ with projective modules P_1, \dots, P_n . Then we can view this as an extension representing some stable isomorphism $\Omega^n B \rightarrow A$; but we can also consider this as an element of $\mathcal{K}_n(A, B)$, representing some stable isomorphism $A \rightarrow \Omega^n B$; by the previous proposition, the two maps are stable inverses of each other.

We have a composition product $\mathcal{K}_n(B, C) \times \mathcal{K}_m(A, B) \rightarrow \mathcal{K}_{n+m}(A, C)$ similar to the Yoneda splice: given $E: A \rightarrow P_* \rightarrow B$ and $E': B \rightarrow Q_* \rightarrow C$ we define $E' \circ E$ to be the complex

$$E' \circ E: \quad A \rightarrow P_* \begin{array}{c} \longrightarrow \\ \searrow \\ \end{array} \begin{array}{c} Q_* \\ \nearrow \\ \end{array} \rightarrow C.$$

B

This product is compatible with the equivalence relation \sim and therefore induces a product

$$K_n(B, C) \times K_m(A, B) \rightarrow K_{n+m}(A, C).$$

Proposition 4.7 *The composition products on K_* and $\widehat{\text{Ext}}_{kG}^{-*}$ coincide under the bijection of Theorem 4.2.*

Proof Let us start with complexes $A \rightarrow P_* \rightarrow B$ and $B \rightarrow Q_* \rightarrow C$ representing stable maps $\alpha: A \rightarrow \Omega^m B$ and $\beta: B \rightarrow \Omega^n C$, respectively. Choose projective resolutions $\Omega^n C \rightarrow R_* \rightarrow C$ and $\Omega^{n+m} C \rightarrow T_* \rightarrow \Omega^n C$. Then we can lift the identity map on C to commutative diagrams as follows:

$$\begin{array}{ccccc} A & \longrightarrow & P_* & \longrightarrow & B \\ \bar{\gamma} \downarrow & & \downarrow & & \downarrow \bar{\beta} \\ \Omega^{n+m} C & \longrightarrow & T_* & \longrightarrow & \Omega^n C \end{array} \qquad \begin{array}{ccccc} B & \longrightarrow & Q_* & \longrightarrow & C \\ \bar{\beta} \downarrow & & \downarrow & & \parallel \\ \Omega^n C & \longrightarrow & R_* & \longrightarrow & C \end{array}$$

Here, $\bar{\beta}$ and $\bar{\gamma}$ are unstable representatives of β and some γ . Note that the extension $\Omega^{n+m} C \rightarrow T_* \rightarrow \Omega^n C$ represents the identity map $\text{id} \in \underline{\text{Hom}}_{kG}(\Omega^m \Omega^n C, \Omega^{n+m} C)$. By Proposition 4.5, the left diagram shows that $\gamma = \beta\alpha$. After splicing the two diagrams the result follows from Proposition 4.3. □

There is also a way of composing an element $x \in \widehat{\text{Ext}}_{kG}^{-n}(A, B)$ given as a complex $A \rightarrow P_* \rightarrow B$ with an element of $y \in \widehat{\text{Ext}}_{kG}^m(B, C)$ (with $m > 0$) given as an extension $C \hookrightarrow M_* \twoheadrightarrow B$:

Proposition 4.8 Suppose $m < n$. The identity map of B can be lifted to a diagram

$$\begin{array}{ccccccccccc}
 A & \longrightarrow & P_n & \longrightarrow & \cdots & \longrightarrow & P_{m+1} & \longrightarrow & P_m & \longrightarrow & \cdots & \longrightarrow & P_1 & \longrightarrow & B \\
 & & & & & & \downarrow & & \downarrow & & & & \downarrow & & \parallel \\
 & & & & & & C & \longrightarrow & M_m & \longrightarrow & \cdots & \longrightarrow & M_1 & \longrightarrow & B
 \end{array}$$

and for any such lifting, the complex $A \rightarrow P_n \rightarrow \cdots \rightarrow P_{m+1} \rightarrow C$ represents the composition $y \cdot x \in \widehat{\text{Ext}}_{kG}^{m-n}(A, C)$.

Proof Existence of the lifting is common homological algebra. For the second statement choose a projective resolution $\Omega^{n-m}C \rightarrow R_* \rightarrow C$; then we have the following commutative diagram:

$$\begin{array}{ccccccccccc}
 A & \longrightarrow & P_n & \longrightarrow & \cdots & \longrightarrow & P_{m+1} & \longrightarrow & P_m & \longrightarrow & \cdots & \longrightarrow & P_1 & \longrightarrow & B \\
 \downarrow \bar{y} & & \downarrow & & & & \downarrow & \searrow & \downarrow & & & & \downarrow & & \parallel \\
 \Omega^{n-m}C & \longrightarrow & R_{n-m} & \longrightarrow & \cdots & \longrightarrow & R_1 & \longrightarrow & M_m & \longrightarrow & \cdots & \longrightarrow & M_1 & \longrightarrow & B \\
 & & & & & & \nearrow & C & \searrow & & & & & &
 \end{array}$$

The complex in question represents the stable class of the map \bar{y} . The bottom row represents $y \in \underline{\text{Hom}}_{kG}(\Omega^n B, \Omega^{n-m}C)$, the upper row represents $x \in \underline{\text{Hom}}_{kG}(A, \Omega^n B)$. The result follows from Proposition 4.5. \square

Proposition 4.9 Suppose that we have a commutative diagram

$$\begin{array}{ccccccc}
 A & \longrightarrow & P_n & \longrightarrow & \cdots & \longrightarrow & P_1 & \longrightarrow & B \\
 f \downarrow & & \downarrow & & & & \downarrow & & \downarrow g \\
 A' & \longrightarrow & Q_n & \longrightarrow & \cdots & \longrightarrow & Q_1 & \longrightarrow & B'
 \end{array}$$

with projective modules P_i, Q_i for $i = 1, 2, \dots, n$. Then the rows represent maps $x: A \rightarrow \Omega^n B$ and $y: A' \rightarrow \Omega^n B'$, respectively, and $y \circ f = \Omega^n(g) \circ x$ in $\text{mod-}kG$.

Proof Choose a projective resolution $\Omega^n B' \rightarrow R_* \rightarrow B'$. By usual homological algebra, we get a diagram

$$\begin{array}{ccccc}
 A & \longrightarrow & P_* & \longrightarrow & B \\
 f' \downarrow & & \downarrow & & \downarrow g \\
 A' & \longrightarrow & Q_* & \longrightarrow & B' \\
 y \downarrow & & \downarrow & & \parallel \\
 \Omega^n B' & \longrightarrow & R_* & \longrightarrow & B'
 \end{array}$$

and then the result follows from Proposition 4.5. \square

Remark 4.10 There is a similar statement for extensions. Suppose that we have a diagram as in Proposition 4.9, but this time with exact rows and the P_i 's and Q_i 's are not necessarily projective. Then the rows represent maps $x: \Omega^n B \rightarrow A$ and $y: \Omega^n B' \rightarrow A'$, and $f \circ x = y \circ \Omega^n(g)$ in $\underline{\text{mod}}\text{-}kG$.

Remark 4.11 If $k \rightarrow I_0 \rightarrow I_1 \rightarrow \dots \rightarrow I_j$ is an injective resolution and $\lambda: I_j \rightarrow k$ represents some cohomology class in $\widehat{H}^{-1-j}(G) \cong H^j \text{Hom}(I, k)$, then the complex

$$k \rightarrow I_0 \rightarrow \dots \rightarrow I_j \xrightarrow{\lambda} k$$

represents the same class.

5 The Evens norm map and the dual operations

In this part we are going to show that some of the dual operations Q_i^* on ordinary group cohomology are compatible with the Evens norm map in certain cases. For simplicity we restrict to the case $p = 2$. Recall (see, eg, Benson [1, Section 4.1]) that the Evens norm map is a function

$$\text{norm}_{K,G}: H^i(K) \rightarrow H^{ni}(G)$$

for all $i \geq 0$, where G is a finite group and $K \leq G$ is a subgroup of index n . It can be defined as follows: let $x \in H^i(K) = \text{Ext}_{kK}^i(k, k)$ be represented by an exact sequence $k = E_i \rightarrow E_{i-1} \rightarrow \dots \rightarrow E_0 \rightarrow k$, which we think of as an augmented complex $E \rightarrow k$. Then define $x^{\otimes n}$ to be the augmented complex $E^{\otimes n} \rightarrow k$, which is an exact sequence of $k(\Sigma_n \wr K)$ -modules. It therefore represents some class in $H^{ni}(\Sigma_n \wr K)$. We then choose a suitable inclusion $\iota: G \hookrightarrow \Sigma_n \wr K$ and define $\text{norm}_{K,G}(x) = \iota^*(x^{\otimes n})$.

In the following, we often implicitly identify $H^*(G)$ with the dual of $\widehat{H}^{-1-*}(G)$ by the use of Tate duality. In particular, we have dual operations $Q_i^*: H^{i+j}(G) \rightarrow H^j(G)$.

Theorem 5.1 *Let $k = \mathbb{F}_2$, and let K be a subgroup of index n of a finite group G .*

(1) *For all $i \geq 0$ the diagram*

$$\begin{CD} H^i(K) @>Q_i^*>> H^0(K) \\ @V(-)^{\otimes n}VV @VV(-)^{\otimes n}V \\ H^{ni}(\Sigma_n \wr K) @>Q_{ni}^*>> H^0(\Sigma_n \wr K) \end{CD}$$

commutes.

- (2) If K is a central factor of G (eg, a central subgroup or a direct factor), then for $x \in H^i(K)$ we have that $\text{norm}_{K,G} Q_i^*(x) = Q_{ni}^*(\text{norm}_{K,G} x)$.
- (3) For $x \in H^i(K)$ we have that $Q_i^*(x) = Q_{ni}^*(x^n) \in H^0(K)$.

Remark 5.2 Recall that K is a central factor of G if and only if the product of K with its centralizer is the whole group G . The condition we really need for the proof of part (2) is that we can choose coset representatives for K in G which commute with all elements of order 2 in K . This is true if K is a central factor, but it is also true in other cases like $\mathbb{Z}/4\mathbb{Z} \subseteq Q_8$. The condition is not satisfied for $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \subset D_8$ (the dihedral group with 8 elements), and we will see in Remark 5.18 that the conclusion fails in that case.

Corollary 5.3 If the order of the finite group G equals an odd multiple of 2^i with $i \geq 1$, then the operation $Q_n: \hat{H}^{-1}(G) \rightarrow \hat{H}^{-1-n}(G)$ is nontrivial whenever n is divisible by 2^i . □

Proof Let $P \leq G$ be a 2–Sylow subgroup, which is of order 2^i . The commutative diagram

$$\begin{array}{ccc} \hat{H}^{-1}(G) & \xrightarrow{Q_n} & \hat{H}^{-1-n}(G) \\ \cong \downarrow \text{res}_{P,G} & & \downarrow \text{res}_{P,G} \\ \hat{H}^{-1}(P) & \xrightarrow{Q_n} & \hat{H}^{-1-n}(P) \end{array}$$

shows that it is enough to consider the case of a 2–group P . Let $K \leq P$ be a central cyclic subgroup of order 2; then the following commutative diagram proves the claim:

$$\begin{array}{ccc} H^{2^i j}(P) & \xrightarrow{Q_{2^i j}^*} & H^0(P) \\ \text{norm} \uparrow & & \text{norm} \uparrow \cong \\ H^j(K) & \xrightarrow{Q_j^*} & H^0(K) \end{array} \quad \square$$

For the proof of Theorem 5.1 we use a reinterpretation of $Q_i: \hat{H}^{-1}(G) \rightarrow \hat{H}^{-1-i}(G)$:

Proposition 5.4 Let G be a finite group of order divisible by $p = 2$, and let $\varphi \in \hat{H}^{-1}(G)$ be the canonical generator. For every $i \geq 0$, the complex

$$k \xrightarrow{N^{\otimes 2}} kG^{\otimes 2} \xrightarrow{1+T} kG^{\otimes 2} \xrightarrow{1+T} \dots \xrightarrow{1+T} kG^{\otimes 2} \xrightarrow{\varepsilon^{\otimes 2}} k$$

with $i + 1$ projective modules $kG^{\otimes 2}$ represents the element $Q_i(\varphi) \in \hat{H}^{-1-i}(G)$. Here, T denotes the twist map interchanging the two factors of $kG^{\otimes 2}$, and ε is the augmentation map.

Proof We use the definition of Q_i using the operad \mathcal{B} from Section 3. Let P be a complete projective resolution of k as trivial kG -module, where we assume that $P_{-1} = kG$ and $k \hookrightarrow P_{-1}$ is the norm map N . Let W be the standard free resolution of k as trivial $k\Sigma_2$ -module, with one generator e_j in degree j for every $j \geq 0$. Choose a Σ_2 -equivariant chain map $\Psi: W \rightarrow \mathcal{B}(2)$ lifting the identity of k , and define $\alpha_j: P_{-1-j} \rightarrow P_{-1}^{\otimes 2}$ to be the degree 0-part of the map $\Psi(e_j) \in \mathcal{B}(2) \subseteq \text{Hom}^*(\bar{P}, \bar{P}^{\otimes 2})$. We get a commutative diagram

$$\begin{array}{ccccccccccc}
 k & \xrightarrow{N} & P_{-1} & \longrightarrow & P_{-2} & \longrightarrow & P_{-3} & \longrightarrow & \cdots & \longrightarrow & P_{-i} & \xrightarrow{\lambda} & k \\
 \parallel & & \downarrow \alpha_0 & & \downarrow \alpha_1 & & \downarrow \alpha_2 & & & & \downarrow \alpha_{i-1} & & \parallel \\
 k & \xrightarrow{N^{\otimes 2}} & P_{-1}^{\otimes 2} & \xrightarrow{1+T} & P_{-1}^{\otimes 2} & \xrightarrow{1+T} & P_{-1}^{\otimes 2} & \xrightarrow{1+T} & \cdots & \xrightarrow{1+T} & P_{-1}^{\otimes 2} & \xrightarrow{\varepsilon^{\otimes 2}} & k
 \end{array}$$

By definition, the class of λ in $H^i \text{Hom}(\bar{P}, k) \cong \hat{H}^{-1-i}(G)$ represents $Q_i(\varphi)$. The commutative diagram then shows the claim by Remark 4.11. \square

Proposition 5.5 *Let G be a finite group of order divisible by $p = 2$, and let $v: k \rightarrow kG$ be the norm map. If $\alpha: kG^{\otimes 2} \rightarrow k$ is a map for which*

$$k \xrightarrow{v^{\otimes 2}} kG^{\otimes 2} \xrightarrow{\alpha} k$$

is a complex, then that complex represents $\sum_{g \in G} \alpha(1 \otimes g) \in k = \hat{H}^{-1}(G)$.

Proof Let $b = \sum_g \alpha(1 \otimes g)$; then the result follows from the commutative diagram

$$\begin{array}{ccccc}
 k & \xrightarrow{v^{\otimes 2}} & kG^{\otimes 2} & \xrightarrow{\alpha} & k \\
 \parallel & & \uparrow 1 \otimes v & & \parallel \\
 k & \xrightarrow{v} & kG & \xrightarrow{b\varepsilon} & k \\
 \downarrow b & & \downarrow b & & \parallel \\
 k & \xrightarrow{v} & kG & \xrightarrow{\varepsilon} & k
 \end{array}$$

and Example 4.4 and Proposition 4.9. \square

Proposition 5.6 *Suppose that $\sigma \in \Sigma_n$ satisfies $\sigma^2 = 1$ and $\sigma \neq 1$, and let K be a finite group. Define the map $f: \mathbb{Z}/2\mathbb{Z} \times K \rightarrow \Sigma_n \wr K$ to be $(u, g) \mapsto (\sigma^u; g, g, \dots, g)$. Then there is some $m \geq 1$ such that for every $x \in H^i(K)$ we have $f^*(x^{\otimes n}) = x^{n-2m} (\sum_{r=0}^{|x|} \text{Sq}^r(x) z^{|x|-r})^m$, where $H^*(\mathbb{Z}/2\mathbb{Z}) = k[z]$.*

Proof We can assume that σ is of the form $(1\ 2)(3\ 4)\cdots(2m-1\ 2m)$ for some m . For every i and every group L denote by $\Psi_{i,L}$ the map $L \rightarrow \Sigma_i \wr L$ given by $l \mapsto (\text{id}; l, l, \dots, l)$. Let h be the composition

$$\mathbb{Z}/2\mathbb{Z} \times K \rightarrow \Sigma_2 \wr K \xrightarrow{\Psi_{m, \Sigma_2 \wr K}} \Sigma_m \wr (\Sigma_2 \wr K) \hookrightarrow \Sigma_{2m} \wr K,$$

the first map being given by $(u; g) \mapsto (\tau^u; g; g)$, where τ is the generator of Σ_2 . Also let j be the composition

$$\mathbb{Z}/2\mathbb{Z} \times K \xrightarrow{\text{proj}} K \xrightarrow{\Psi_{n-2m, K}} \Sigma_{n-2m} \wr K;$$

then we get a composition

$$\mathbb{Z}/2\mathbb{Z} \times K \xrightarrow{h \times j} \Sigma_{2m} \wr K \times \Sigma_{n-2m} \wr K \hookrightarrow \Sigma_n \wr K$$

which equals f . Now let $x \in H^i(K)$; then $x^{\otimes n} \in H^{ni}(\Sigma_n \wr K)$ restricts to

$$(x^{\otimes 2m}) \otimes (x^{\otimes (n-2m)}) \in H^{ni}(\Sigma_{2m} \wr K \times \Sigma_{n-2m} \wr K).$$

Now

$$h^*(x^{\otimes 2m}) = \text{res } \Psi_{m, \Sigma_2 \wr K}^*((x^{\otimes 2})^{\otimes m}) = (\text{res } x^{\otimes 2})^m = (\text{norm}_{K, \mathbb{Z}/2\mathbb{Z} \times K}(x))^m.$$

On the other hand, $j^*(x^{\otimes (n-2m)}) = x^{n-2m}$, so it remains to show that

$$\text{norm}_{K, \mathbb{Z}/2\mathbb{Z} \times K}(x) = \sum_{r=0}^{|x|} \text{Sq}^r(x) z^{|x|-r},$$

which is done, eg, in [1, Section 4.4]; note that this can actually be used to define the Steenrod operations on ordinary group cohomology. □

Consider the augmentation $k\Sigma_n \rightarrow k$ as an augmented chain complex; then the augmented chain complex $k\Sigma_n \otimes k\Sigma_n \rightarrow k$ is a chain complex of right $\Sigma_2 \times \Sigma_n$ -modules, where Σ_n acts diagonally and Σ_2 acts by permuting the factors. Let W be the standard free resolution of k as trivial $k\Sigma_2$ -module. By endowing W with a trivial right Σ_n -action, we can consider $W \rightarrow k$ and hence also $W \otimes k\Sigma_n \otimes k\Sigma_n \rightarrow k$ as augmented chain complexes of right $\Sigma_2 \times \Sigma_n$ -modules. As such the latter consists entirely of free modules, and we can therefore lift the identity map of k to a map of chain complexes $\vartheta: W \otimes k\Sigma_n \otimes k\Sigma_n \rightarrow W^{\otimes n}$, where Σ_n acts on $W^{\otimes n}$ by permuting the factors and Σ_2 acts diagonally. Finally note that $k\Sigma_n$ is a right $\Sigma_n \wr K$ -module via the projection map $\Sigma_n \wr K \rightarrow \Sigma_n$, and $kK^{\otimes n}$ is also a right $\Sigma_n \wr K$ -module. Therefore the tensor product $k\Sigma_n \otimes kK^{\otimes n}$ is a right $k(\Sigma_n \wr K)$ -module, which is free

of rank one. We can now form the following map ξ of augmented chain complexes over $k(\Sigma_n \wr K)$:

$$\begin{array}{ccc}
 W \otimes_{\Sigma_2} (k(\Sigma_n \wr K))^{\otimes 2} & \xrightarrow{\cong} & W \otimes_{\Sigma_2} (k \Sigma_n \otimes k K^{\otimes n})^{\otimes 2} \xrightarrow{\cong} (W \otimes k \Sigma_n \otimes k \Sigma_n) \otimes_{\Sigma_2} (k K^{\otimes n})^{\otimes 2} \\
 & \searrow \xi & \downarrow \vartheta \otimes \text{twist} \\
 & & (W \otimes_{\Sigma_2} k K^{\otimes 2})^{\otimes n} \xleftarrow{\text{twist}} W^{\otimes n} \otimes_{\Sigma_2} (k K^{\otimes 2})^{\otimes n}
 \end{array}$$

In the following, we consider triples $(E, \bar{\beta}, \bar{\gamma})$ where $E \rightarrow k$ is an exact sequence $k = E_i \hookrightarrow E_{i-1} \rightarrow \dots \rightarrow E_0 \twoheadrightarrow k$ of kK -modules, $\bar{\beta}: W \otimes_{\Sigma_2} k K^{\otimes 2} \rightarrow E$ is a map of augmented chain complexes, and $\bar{\gamma}$ is defined to be $\bar{\gamma} = \bar{\beta}^{\otimes n} \circ \xi$. Then $\bar{\gamma}: W \otimes_{\Sigma_2} (k(\Sigma_n \wr K))^{\otimes 2} \rightarrow E^{\otimes n}$ is a map of augmented chain complexes over $k(\Sigma_n \wr K)$. Define β to be the composite

$$k K^{\otimes 2} \cong W_i \otimes_{\Sigma_2} k K^{\otimes 2} \xrightarrow{\bar{\beta}_i} E_i = k,$$

and similarly define $\gamma: k(\Sigma_n \wr K)^{\otimes 2} \rightarrow k$. Furthermore, for every group L let us define the subset $L' = \{l \in L \mid l^2 = 1\} \subseteq L$.

Proposition 5.7 *If $(E, \bar{\beta}, \bar{\gamma})$ is a triple as above, then*

$$\sum_{g \in K} \beta(1 \otimes g) = \sum_{g \in K'} \beta(1 \otimes g) \quad \text{and} \quad \sum_{l \in L} \gamma(1 \otimes l) = \sum_{l \in L'} \gamma(1 \otimes l).$$

Proof The formula

$$(5.8) \quad \bar{\beta}_i(w \otimes_{\Sigma_2} (g \otimes h)) = \varepsilon(w) \beta(g \otimes h),$$

holds because it is true for $w = 1 \in \Sigma_2$ and for $w = 1 - \tau \in k \Sigma_2$ (where τ is the generator of Σ_2) since $\bar{\beta}$ is a chain map. The formula implies that $\beta(1 \otimes g) = \beta(g \otimes 1) = \beta(1 \otimes g^{-1})$, and therefore $\sum_{g \in K} \beta(1 \otimes g) = \sum_{g \in K'} \beta(1 \otimes g)$. The same proof applies to γ . □

Proposition 5.9 *There exist constants $c_{n,i,\sigma}$ (for all $\sigma \in \Sigma_n$), not depending on K , with the following property: for all triples $(E, \bar{\beta}, \bar{\gamma})$ as above and all elements $g = (\sigma, k_1, k_2, \dots, k_n) \in \Sigma_n \wr K$ we have that*

$$\gamma(1 \otimes g) = c_{n,i,\sigma} \cdot \beta(1 \otimes k_1) \cdots \beta(1 \otimes k_n).$$

Some of the constants will be determined later in Proposition 5.13.

Proof We can write $\vartheta_{ni}(1 \otimes 1 \otimes \sigma) = \sum_s w_{s,1} \otimes \cdots \otimes w_{s,n}$ where $w_{s,j} \in W$. Then $\gamma(1 \otimes g)$ equals

$$\gamma(1 \otimes g) = \sum_s \bar{\beta}^{\otimes n} \left((w_{s,1} \otimes_{\Sigma_2} (1 \otimes k_1)) \otimes \cdots \otimes (w_{s,n} \otimes_{\Sigma_2} (1 \otimes k_n)) \right).$$

If the degree of one of the $w_{s,j}$ is bigger than i , then the corresponding s -th summand vanishes because $\bar{\beta}$ is the zero map. Therefore, we are only interested in the case where all $w_{s,j}$ are of degree i , in which case we can simplify by (5.8)

$$\begin{aligned} \gamma(1 \otimes g) &= \left(\sum_s \varepsilon(w_{s,1}) \cdots \varepsilon(w_{s,n}) \right) \cdot \beta(1 \otimes k_1) \beta(1 \otimes k_2) \cdots \beta(1 \otimes k_n) \\ &= c_{n,i,\sigma} \cdot \beta(1 \otimes k_1) \beta(1 \otimes k_2) \cdots \beta(1 \otimes k_n), \end{aligned}$$

where $c_{n,i,\sigma}$ is some constant in k not depending on the group K . □

Proposition 5.10 *Let L be any finite group, and suppose that $f: L \rightarrow \Sigma_n \wr K$ is an injective group homomorphism and $c \in k$ is some constant. Suppose that for all triples $(E, \bar{\beta}, \bar{\gamma})$ as above we have that $c \cdot \sum_{g \in K} \beta(1 \otimes g) = \sum_{l \in L} \gamma(1 \otimes f(l))$. Then for all $x \in H^i(K)$ the formula $Q_{ni}^* f^*(x^{\otimes n}) = c \cdot f^*(Q_i^*(x)^{\otimes n}) \in H^0(L)$ holds.*

Proof We may assume that the order of both K and L is divisible by $p = 2$. Then we identify $\hat{H}^0(K)$ and $\hat{H}^0(L)$ with k , so that we have to prove

$$Q_{ni}^* f^*(a^{\otimes n}) = c \cdot Q_i^*(a)$$

for all $a \in H^i(K)$ (recall that $k = \mathbb{F}_2$). Let E be an exact sequence representing $a \in H^i(K)$. Since the modules of the augmented complex $W \otimes kK^{\otimes 2} \rightarrow k$ are free, we can lift the identity of k to a chain map $\bar{\beta}$:

$$\begin{array}{ccccccc} k & \xrightarrow{\nu^2} & kK^{\otimes 2} & \xrightarrow{1+T} & kK^{\otimes 2} & \xrightarrow{1+T} & \cdots & \xrightarrow{1+T} & kK^{\otimes 2} & \xrightarrow{\varepsilon^2} & k \\ & & \beta \downarrow & & \downarrow & & & & \downarrow & & \parallel \\ & & k & \longrightarrow & E_{i-1} & \longrightarrow & \cdots & \longrightarrow & E_0 & \longrightarrow & k \end{array}$$

Here $\nu: k \rightarrow kK$ is the norm map, and the upper row represents $Q_i(\kappa) \in \hat{H}^{-1-i}(K)$ for the generator $\kappa \in \hat{H}^{-1}(K)$ (by Proposition 5.4). Due to Proposition 4.8 the product $Q_i(\kappa)a \in \hat{H}^{-1}(K)$ is represented by the complex

$$k \xrightarrow{\nu^2} kK^{\otimes 2} \xrightarrow{\beta} k.$$

Therefore, by Proposition 5.5, $Q_i(\kappa)a = \sum_{g \in K} \beta(1 \otimes g)\kappa$ and hence

$$(5.11) \quad Q_i^*(a) = \sum_{g \in K} \beta(1 \otimes g) \in k.$$

As before, we get a triple $(E, \bar{\beta}, \bar{\gamma})$ in such a way that the diagram of kL -modules

$$\begin{array}{ccccccc}
 k & \xrightarrow{\mu^2} & kL \otimes^2 & \xrightarrow{1+T} & kL \otimes^2 & \xrightarrow{1+T} & \cdots & \xrightarrow{1+T} & kL \otimes^2 & \xrightarrow{\varepsilon^2} & k \\
 & & \downarrow (kf)^{\otimes 2} & & \downarrow & & & & \downarrow & & \parallel \\
 & & k(\Sigma_n \wr K)^{\otimes 2} & \xrightarrow{1+T} & k(\Sigma_n \wr K)^{\otimes 2} & \xrightarrow{1+T} & \cdots & \xrightarrow{1+T} & k(\Sigma_n \wr K)^{\otimes 2} & \xrightarrow{\varepsilon^2} & k \\
 & & \downarrow \gamma & & \downarrow & & & & \downarrow & & \parallel \\
 & & k & \longrightarrow & (E^{\otimes n})_{ni-1} & \longrightarrow & \cdots & \longrightarrow & (E^{\otimes n})_0 & \longrightarrow & k
 \end{array}$$

commutes, where $\mu: k \rightarrow KL$ is the norm map, such that the upper row represents $Q_{ni}(\lambda) \in \widehat{H}^{-1-ni}(L)$, where $\lambda \in \widehat{H}^{-1}(L)$ is the generator. As above, Propositions 4.8 and 5.5 show that $Q_{ni}(\lambda)f^*(a^{\otimes n}) = \sum_{l \in L} \gamma(1 \otimes f(l))\lambda$, so that

$$(5.12) \quad Q_{ni}^*(f^*(a^{\otimes n})) = \sum_{l \in L} \gamma(1 \otimes f(l)) \in k.$$

Combining formulas (5.11) and (5.12) we get the desired result. □

We will now exploit this fact for several maps f .

Proposition 5.13 *The constants $c_{n,i,\sigma}$ satisfy*

$$(5.14) \quad c_{n,i,\text{id}} = 1,$$

$$(5.15) \quad c_{n,i,\sigma} = 0 \quad \text{if } \sigma^2 = 1 \text{ and } \sigma \neq \text{id},$$

$$(5.16) \quad c_{n,i,\sigma} = c_{n,i,\sigma^{-1}} \quad \text{for all } \sigma.$$

Proof of Proposition 5.13 and Theorem 5.1 As a first step, take $L = K$ and let $f: K \rightarrow \Sigma_n \wr K$ be given by $g \mapsto (\text{id}; g, \dots, g)$ for all $g \in K$. Then $x^n = f^*(x^{\otimes n})$, and the computation

$$\begin{aligned}
 \sum_{g \in K} \gamma(1 \otimes f(g)) &= \sum_{g \in K} \gamma(1 \otimes (\text{id}; g, \dots, g)) = c_{n,i,\text{id}} \sum_{g \in K} \beta(1 \otimes g)^n \\
 &= c_{n,i,\text{id}} \sum_{g \in K} \beta(1 \otimes g)
 \end{aligned}$$

shows that $Q_i^*(x) = c_{n,i,\text{id}} Q_{ni}^*(x^n) \in H^0(K)$. If we put $K = \mathbb{Z}/2\mathbb{Z}$, then the computations in Example 1.2 show that the constant $c_{n,i,\text{id}}$ equals 1, so we have proved (5.14) and Theorem 5.1(3).

As a second step, let us take $L = \mathbb{Z}/2 \times K$ and let $f: \mathbb{Z}/2\mathbb{Z} \times K \rightarrow \Sigma_n \wr K$ to be given by $(u, g) \mapsto (\sigma^u; g, g, \dots, g)$, where $\sigma \in \Sigma_n$ is some fixed element of order 2. Then

$$\begin{aligned} \sum_{g \in \mathbb{Z}/2\mathbb{Z} \times K} \gamma(1 \otimes f(g)) &= \sum_{g \in K} \gamma(1 \otimes (\text{id}; g, \dots, g)) + \gamma(1 \otimes (\sigma; g, \dots, g)) \\ &= (1 + c_{n,i,\sigma}) \sum_{g \in K} \beta(1 \otimes g). \end{aligned}$$

We take $K = \mathbb{Z}/2\mathbb{Z}$, but we keep the notation K in order to distinguish from the other factor $\mathbb{Z}/2\mathbb{Z}$. We have $H^*(K) \cong k[x]$ and $H^*(\mathbb{Z}/2\mathbb{Z}) \cong k[z]$ for one-dimensional classes x and z . By Proposition 5.6, we know that $f^*((x^i)^{\otimes n}) = f^*(x^{\otimes n})^i = x^{(n-2m)i}(x^2 + xz)^{mi}$. By the computations in Example 1.8, applying Q_{ni}^* to such a polynomial in x, z equals the sum of the evaluations of that polynomial at $(x, z) = (1, 1), (0, 1)$ and $(1, 0)$; therefore, $Q_{ni}^*(f^*((x^i)^{\otimes n})) = 1 \in H^0(\mathbb{Z}/2\mathbb{Z} \times K)$. This implies (5.15).

In order to prove (5.16), take a situation in which $l \in K$ is of order 2, and β is such that $\beta(1 \otimes l) \neq 0$. Then put $g = (\sigma; l, l, \dots, l)$ and the result follows from $\gamma(1 \otimes g) = \gamma(1 \otimes g^{-1})$. Up to this point, we have proved Proposition 5.13 completely.

Now we prove Theorem 5.1(1). Take $f: L \rightarrow \Sigma_n \wr K$ to be the identity map of $\Sigma_n \wr K$ and compute

$$\begin{aligned} \sum_{g=(\sigma;k_1,\dots,k_n) \in \Sigma_n \wr K} \gamma(1 \otimes g) &= \sum_{\sigma} c_{n,i,\sigma} \cdot \sum_{k_1,\dots,k_n \in K} \beta(1 \otimes k_1)\beta(1 \otimes k_2) \cdots \beta(1 \otimes k_n) \\ &= \sum_{\sigma} c_{n,i,\sigma} \left(\sum_{g \in K} \beta(1 \otimes g) \right)^n. \end{aligned}$$

By Proposition 5.13, $\sum_{\sigma \in \Sigma_n} c_{n,i,\sigma} = 1$, which proves Theorem 5.1(1).

Let $\pi: \Sigma_n \wr K \rightarrow \Sigma_n$ be the projection map. For the proof of Theorem 5.1(2), choose a set $\{g_j\}$ of coset representatives of K in G with the property that all the g_j 's commute with all elements of order 2 in K . Then for each $g \in G$, there are unique elements $k_1, \dots, k_n \in K$ and $\sigma \in \Sigma_n$ such that $g g_j = g_{\sigma(j)} k_j$ for all j , and we get an injection $f: G \hookrightarrow \Sigma_n \wr K$ by $g \mapsto (\sigma; k_1, \dots, k_n)$. Then $\text{norm}_{K,G}(x) = f^*(x^{\otimes n})$, and we

need to investigate

$$\begin{aligned} \sum_{g \in G} \gamma(1 \otimes g) &= \sum_{g \in G'} \gamma(1 \otimes g) && \text{by Proposition 5.7} \\ &= \sum_{\substack{g \in G' \\ \pi(f(g)) = \text{id}}} \gamma(1 \otimes g) && \text{by (5.14) and (5.15).} \end{aligned}$$

But if $\pi(f(g)) = \text{id}$, then $gg_j = g_jk_j$ for all j , which means $gg_j = k_jg_j$ for all j by our condition on K . Therefore, we get $g = k_j$ for all j , so that $g \in K$ and $f(g) = (\text{id}; g, g, \dots, g)$. Conversely, if $g \in K$, then $f(g) = (\text{id}; g, g, \dots, g)$ by our condition on K . Therefore,

$$\sum_{g \in G} \gamma(1 \otimes g) = \sum_{g \in K'} \gamma(1 \otimes (\text{id}; g, g, \dots, g)) = \sum_{g \in K'} \beta(1 \otimes g)^n = \sum_{g \in K'} \beta(1 \otimes g).$$

This proves (2) of Theorem 5.1. □

Example 5.17 Let us work out in detail the operations Q on the generator of $\hat{H}^{-1}(G)$ in the case $G = D_8$, the dihedral group with 8 elements. The structure of the cohomology ring $H^*(G)$ is known to be $H^*(G) \cong k[a, b, c]/(ab)$ where $|a| = |b| = 1$ and $|c| = 2$ (see, eg, Carlson [5, Theorem 7.8]). From Benson and Carlson [2, Theorem 3.1 and Lemma 2.1] we get that $\hat{H}^-(G) \cdot \hat{H}^-(G) = 0$ and that $\hat{H}^n(G) \cdot \hat{H}^m(G) = 0$ for $n < 0 \leq n + m$. Consider the k -basis $\{a^i c^j, b^i c^j\}_{i,j \geq 0}$ of $H^*(G)$ and let us define $\{\varphi_{a^i c^j}, \varphi_{b^i c^j}\}$ to be the dual basis; in particular, φ_1 is the canonical generator of $\hat{H}^{-1}(G)$. Using Tate duality, one derives the relations

$$\begin{aligned} c\varphi_{a^i c^j} &= \begin{cases} \varphi_{a^i c^{j-1}} & \text{if } j > 0, \\ 0 & \text{otherwise,} \end{cases} \\ a\varphi_{a^i c^j} &= \begin{cases} \varphi_{a^{i-1} c^j} & \text{if } i > 0, \\ 0 & \text{otherwise,} \end{cases} \\ a\varphi_{b^i c^j} &= 0, \end{aligned}$$

and similarly for a and b interchanged. All these facts together completely determine the multiplicative structure of $\hat{H}^*(G)$.

From $Q_1(\varphi_1) = \varphi_1^2 = 0$ we get that $Q_1^*: H^1(G) \rightarrow H^0(G)$ is zero. Therefore, by Theorem 5.1(3), we get $Q_2^*(a^2) = Q_2^*(b^2) = 0$. Now notice that $G \cong \Sigma_2 \wr \mathbb{Z}/2\mathbb{Z}$, so that Theorem 5.1 implies that $Q_2^*: H^2(G) \rightarrow H^0(G)$ is onto and hence $Q_2^*(c) = 1$. We have therefore determined $Q_2(\varphi_1) = \varphi_c$. More generally, note that $Q(\varphi_1) = Q(a\varphi_a) = Q(a)Q(\varphi_a) = (a + a^2)Q(\varphi_a)$ is divisible by a and, by symmetry, also

by b . This fact already implies that $Q_i(\varphi_1)$ is a multiple of φ_{c^j} for some j . Together with $Q_{2i}^*(c^i) = Q_2^*(c)^i = 1$ we get

$$\begin{aligned} Q_{2i}(\varphi_1) &= \varphi_{c^i} & \text{for } i \geq 1, \\ Q_{2i+1}(\varphi_1) &= 0 & \text{for } i \geq 0. \end{aligned}$$

Remark 5.18 Let us prove that Theorem 5.1(2) is not true for arbitrary subgroups K of G . Take $K = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $G = D_8$, and let us write norm for $\text{norm}_{K,G}$. We know that $H^*(K) \cong k[x, y]$ for some one-dimensional classes x, y . Suppose that Theorem 5.1(2) would hold in that case. Then $Q_2^*(\text{norm}(x)) = \text{norm}(Q_1^*(x)) = 0$, so that $\text{norm}(x) = \alpha a^2 + \beta b^2$ for some $\alpha, \beta \in k$. Similarly, $\text{norm}(y) = \alpha' a^2 + \beta' b^2$ for some $\alpha', \beta' \in k$. But then $\text{norm}(xy) = \text{norm}(x) \text{norm}(y) = \alpha\alpha' a^4 + \beta\beta' b^4$ and hence

$$0 = Q_4^*(\text{norm}(xy)) \neq \text{norm}(Q_2^*(xy)) = \text{norm}(1),$$

a contradiction.

6 Productive elements at the prime 2

We are now going to give the proof of Theorem 1.12, which relies on the following commutative diagram.

Proposition 6.1 *Under the conditions of Theorem 1.12, the following diagram commutes stably:*

$$\begin{array}{ccc} \Omega^n k \otimes L_\zeta & \xrightarrow{\zeta \otimes \text{id}} & L_\zeta \\ \text{id} \otimes \iota \downarrow & & \uparrow \eta \\ \Omega^n k \otimes \Omega^n k & \xrightarrow{\text{Sq}_1(\zeta)} & \Omega k \end{array}$$

Proof of Theorem 1.12 We assume that $[\zeta] \neq 0$. If $\text{Sq}_1(\zeta)$ is divisible by ζ , then there is a map $\alpha: \Omega^n k \otimes \Omega^n k \rightarrow \Omega^{n+1} k$ such that $\text{Sq}_1(\zeta) = \zeta\alpha$. But then $\eta \text{Sq}_1(\zeta) = \eta\zeta\alpha = 0$ because $\eta\zeta = 0$. By Proposition 6.1 we get that ζ is productive.

Conversely, suppose that ζ is productive, so that $\eta \text{Sq}_1(\zeta) \cdot (\text{id} \otimes \iota) = 0$ by Proposition 6.1. Since the triangle

$$\Omega^n k \otimes L_\zeta \xrightarrow{\text{id} \otimes \iota} \Omega^n k \otimes \Omega^n k \xrightarrow{\text{id} \otimes \zeta} \Omega^n k$$

is exact, we get that $\eta \text{Sq}_1(\zeta) = \lambda \cdot (\text{id} \otimes \zeta)$ for some map $\lambda: \Omega^n k \rightarrow L_\zeta$. When we apply the homological functor $\underline{\text{Hom}}_{kG}(\Omega^n k, -)$ to the triangle (1.11), we get a long exact sequence

$$\underline{\text{Hom}}_{kG}(\Omega^n k, \Omega k) \xrightarrow{\eta_*} \underline{\text{Hom}}_{kG}(\Omega^n k, L_\zeta) \xrightarrow{\iota_*} \underline{\text{Hom}}_{kG}(\Omega^n k, \Omega^n k) \xrightarrow{\zeta_*} \underline{\text{Hom}}_{kG}(\Omega^n k, k).$$

Here ζ_* can be viewed as $\zeta: \widehat{H}^0(G) \rightarrow \widehat{H}^n(G)$ which is injective because the class $[\zeta]$ is nonzero. By exactness, $\iota_* = 0$ and η_* is surjective. In particular, $\lambda = \eta\rho$ for some map $\rho: \Omega^n k \rightarrow \Omega k$. Altogether we have that

$$\eta(\text{Sq}_1(\zeta) - \rho(\text{id} \otimes \zeta)) = \lambda(\text{id} \otimes \zeta) - \eta\rho(\text{id} \otimes \zeta) = 0,$$

and therefore $\text{Sq}_1(\zeta) - \rho(\text{id} \otimes \zeta) = \zeta\sigma$ for some map $\sigma: \Omega^n k \otimes \Omega^n k \rightarrow \Omega^{n+1} k$. But then $\text{Sq}_1(\zeta) = [\rho][\zeta] + [\zeta][\sigma]$, so that $\text{Sq}_1(\zeta)$ is divisible by $[\zeta]$. \square

Remark 6.2 Before we start proving that the diagram commutes, let us draw some analogies to the topological world. Let us define k/ζ to be some choice of cone of $\zeta: \Omega^n k \rightarrow k$. Then the commutative square of Proposition 6.1 is a shift of the diagram on the left-hand side:

$$\begin{array}{ccc} \Omega^n k \otimes k/\zeta & \xrightarrow{\zeta} & k/\zeta \\ \downarrow & & \uparrow \\ \Omega^n k \otimes \Omega^{n-1} k & \xrightarrow{\text{Sq}_1(\zeta)} & k \end{array} \qquad \begin{array}{ccc} S/2 & \xrightarrow{2} & S/2 \\ \text{pinch} \downarrow & & \uparrow \text{incl} \\ \Sigma S & \xrightarrow{\eta} & S \end{array}$$

Note the similarity to the topological situation on the right-hand side, which takes place in the stable homotopy category. Here, S denotes the sphere spectrum, $S/2$ is the mod-2-Moore spectrum, a cone of multiplication by 2 on S , and $\eta = \text{Sq}_1(2)$ is the Hopf map.

The rest of this section is devoted to the proof of Proposition 6.1. Let $p = 2$, and let G be a finite group. Let P be a complete projective resolution of the trivial kG -module k , and let this resolution define the modules $\Omega^n k$.

Proposition 6.3 *There is an unstably commutative diagram*

$$\begin{array}{ccccc} \Omega^n k \otimes \Omega^n k & \longrightarrow & Q & \longrightarrow & k \\ \parallel & & \downarrow & & \parallel \\ \Omega^n k \otimes \Omega^n k & \xrightarrow{1+T} & \Omega^n k \otimes \Omega^n k & \xrightarrow{\zeta \otimes 2} & k \end{array}$$

in which Q is projective, and the upper row is a complex representing $\text{Sq}_1(\zeta) \in \widehat{H}^{2n-1}(G)$.

Proof We use the operad \mathcal{C} given in Section 2.3 for the definition of $\text{Sq}_1(\zeta)$. Choose a morphism of augmented Σ_2 –chain complexes $\Delta: W \rightarrow \mathcal{C}(2)$, where W is the standard free resolution of k as trivial Σ_2 –module. When we consider $\Omega^n k$ as a chain complex concentrated in degree 0, then the differential of P induces a chain map $\pi_n: P \rightarrow \Omega^n k$ of degree n . Let $\Delta_i = \Delta(e_i)$, where e_i is the generator of W_i . We get a commutative diagram

$$\begin{array}{ccccc} P & \xrightarrow{\Delta_0} & P \boxtimes 2 & \xrightarrow{1+T} & P \boxtimes 2 \\ \downarrow \pi_{2n} & & \downarrow \pi_n^{\boxtimes 2} & & \downarrow \pi_n^{\boxtimes 2} \\ \Omega^{2n} k & \xrightarrow{\psi} & (\Omega^n k)^{\otimes 2} & \xrightarrow{1+T} & (\Omega^n k)^{\otimes 2} \end{array}$$

where ψ is a stable equivalence and the upper row equals $d\Delta_1$. Since π_n is a chain map, this diagram restricts to the following commutative diagram in dimension $2n$:

$$\begin{array}{ccccc} P_{2n} & \xrightarrow{\partial} & P_{2n-1} & \xrightarrow{\Delta_1} & (P \boxtimes 2)_{2n} \\ \downarrow \pi_{2n} & & & & \downarrow \pi_n^{\boxtimes 2} \\ \Omega^{2n} k & \xrightarrow{\psi} & (\Omega^n k)^{\otimes 2} & \xrightarrow{1+T} & (\Omega^n k)^{\otimes 2} \end{array}$$

We define $\lambda = \pi_n^{\boxtimes 2} \circ \Delta_1$; then $\zeta^{\otimes 2} \circ \lambda: P_{2n-1} \rightarrow k$ represents $\text{Sq}_1(\zeta)$ by definition. Since ψ is a stable equivalence, we can choose a map $\omega: (\Omega^n k)^{\otimes 2} \rightarrow \Omega^{2n} k$ such that $\psi \omega - \text{id}$ factors as

$$\psi \omega - \text{id}: (\Omega^n k)^{\otimes 2} \xrightarrow{\alpha} R \xrightarrow{\beta} (\Omega^n k)^{\otimes 2}$$

for some projective module R . Then we have a commutative diagram

$$\begin{array}{ccccc} \Omega^{2n} k & \xrightarrow{\text{incl}} & P_{2n-1} & \xrightarrow{\zeta^{\otimes 2} \circ \lambda} & k \\ \omega \uparrow & & \uparrow (\text{id}, 0) & & \parallel \\ \Omega^n k \otimes \Omega^n k & \xrightarrow{(\text{incl} \circ \omega)} & P_{2n-1} \oplus R & \xrightarrow{(\zeta^{\otimes 2} \circ \lambda, 0)} & k \\ \parallel & & \downarrow (\lambda, (1+T) \circ \beta) & & \parallel \\ \Omega^n k \otimes \Omega^n k & \xrightarrow{1+T} & \Omega^n k \otimes \Omega^n k & \xrightarrow{\zeta^{\otimes 2}} & k \end{array}$$

proving the claim. □

Proof of Proposition 6.1 Define the map $\kappa: \Omega^n k \otimes \Omega^n k \rightarrow L_\zeta$ by $a \otimes b \mapsto \zeta(a)b + \zeta(b)a$. Then the upper left triangle in the diagram

$$\begin{array}{ccc}
 \Omega^n k \otimes L_\zeta & \xrightarrow{\zeta \otimes \text{id}} & L_\zeta \\
 \text{id} \otimes \iota \downarrow & \nearrow \kappa & \uparrow \eta \\
 \Omega^n k \otimes \Omega^n k & \xrightarrow{\text{Sq}_1(\zeta)} & \Omega^n k
 \end{array}$$

commutes, and we want to show that the bottom right triangle also commutes. To do so, we extend the diagram of Proposition 6.3 as follows:

$$\begin{array}{ccccc}
 \Omega^n k \otimes \Omega^n k & \longrightarrow & Q & \longrightarrow & k \\
 \parallel & & \downarrow & & \parallel \\
 \Omega^n k \otimes \Omega^n k & \xrightarrow{1+T} & \Omega^n k \otimes \Omega^n k & \xrightarrow{\zeta \otimes 2} & k \\
 \kappa \downarrow & & \downarrow \zeta \otimes \text{id} & & \parallel \\
 L_\zeta & \xrightarrow{\iota} & \Omega^n k & \xrightarrow{\zeta} & k
 \end{array}$$

The bottom row is an extension representing $[\eta] \in \text{Ext}_{kG}^1(k, L_\zeta)$, so we are done by Proposition 4.5. □

References

- [1] **D J Benson**, *Representations and cohomology, II: Cohomology of groups and modules*, second edition, Cambridge Studies in Adv. Math. 31, Cambridge Univ. Press (1998) MR1634407
- [2] **D J Benson, J F Carlson**, *Products in negative cohomology*, J. Pure Appl. Algebra 82 (1992) 107–129 MR1182934
- [3] **D J Benson, J P C Greenlees**, *The action of the Steenrod algebra on Tate cohomology*, J. Pure Appl. Algebra 85 (1993) 21–26 MR1207065
- [4] **J F Carlson**, *Products and projective resolutions*, from: “The Arcata Conference on Representations of Finite Groups (Arcata, CA, 1986)”, (P Fong, editor), Proc. Sympos. Pure Math. 47, Amer. Math. Soc. (1987) 399–408 MR933376
- [5] **J F Carlson**, *Modules and group algebras*, Lectures in Math. ETH Zürich, Birkhäuser, Basel (1996) MR1393196 Notes by R Suter
- [6] **H Cartan, S Eilenberg**, *Homological algebra*, Princeton Landmarks in Math., Princeton Univ. Press (1999) MR1731415 With an appendix by D A Buchsbaum

- [7] **F R Cohen, T J Lada, J P May**, *The homology of iterated loop spaces*, Lecture Notes in Math. 533, Springer, Berlin (1976) MR0436146
- [8] **L Evens**, *Steenrod operations and transfer*, Proc. Amer. Math. Soc. 19 (1968) 1387–1388 MR0233347
- [9] **J P C Greenlees**, *Representing Tate cohomology of G -spaces*, Proc. Edinburgh Math. Soc. 30 (1987) 435–443 MR908451
- [10] **J P C Greenlees, J P May**, *Generalized Tate cohomology*, Mem. Amer. Math. Soc. 113, no. 543, Amer. Math. Soc. (1995) MR1230773
- [11] **H Krause**, *The stable derived category of a Noetherian scheme*, Compos. Math. 141 (2005) 1128–1162 MR2157133
- [12] **I Kříž, J P May**, *Operads, algebras, modules and motives*, Astérisque 233, Soc. Math. France (1995) MR1361938
- [13] **L G Lewis, Jr, J P May, M Steinberger, J E McClure**, *Equivariant stable homotopy theory*, Lecture Notes in Math. 1213, Springer, Berlin (1986) MR866482
- [14] **MA Mandell**, *E_∞ algebras and p -adic homotopy theory*, Topology 40 (2001) 43–94 MR1791268
- [15] **J P May**, *A general algebraic approach to Steenrod operations*, from: “The Steenrod Algebra and its Applications (Proc. Conf. to Celebrate NE Steenrod’s Sixtieth Birthday, Battelle Memorial Inst., Columbus, OH, 1970)”, (F P Peterson, editor), Lecture Notes in Math. 168, Springer, Berlin (1970) 153–231 MR0281196
- [16] **J E McClure**, *E_∞ -ring structures for Tate spectra*, Proc. Amer. Math. Soc. 124 (1996) 1917–1922 MR1307549
- [17] **E Yalçın**, *Productive elements in group cohomology*, Homology Homotopy Appl. 13 (2011) 381–401 MR2845636

Mathematisches Institut, Rheinische Friedrich-Wilhelms-Universität Bonn
Endenicher Allee 60, D-53115 Bonn, Germany

mlanger@math.uni-bonn.de

<http://www.math.uni-bonn.de/people/mlanger/>

Received: 18 June 2011 Revised: 22 November 2011

