

Products of Greek letter elements dug up from the third Morava stabilizer algebra

RYO KATO

KATSUMI SHIMOMURA

In [3], Oka and the second author considered the cohomology of the second Morava stabilizer algebra to study nontriviality of the products of beta elements of the stable homotopy groups of spheres. In this paper, we use the cohomology of the third Morava stabilizer algebra to find nontrivial products of Greek letters of the stable homotopy groups of spheres: $\alpha_1\gamma_t$, $\beta_2\gamma_t$, $\langle\alpha_1, \alpha_1, \beta_{p/p}^p\rangle\gamma_t\beta_1$ and $\langle\beta_1, p, \gamma_t\rangle$ for t with $p \nmid t(t^2 - 1)$ for a prime number $p > 5$.

55Q45; 55Q51

1 Introduction

Greek letter elements are well known generators of the stable homotopy groups of spheres localized at a prime p . Studying products among these elements is an interesting subject, and studied by several authors. For example, at an odd prime p , all products of alpha elements are trivial. In [3], we used $H^*S(2)$ to study nontriviality of the product of beta elements. In this paper, we use $H^*S(3)$ to find relations of Greek letters. The multiplicative structure of $H^*S(3)$ was given by Yamaguchi [7], but unfortunately, it has some typos. So here, our computation is based on Ravenel's [4].

Let $\beta_{p/p}$ be the generator of the E_2 -term $E_2^{2,p^2q}(S)$ of the Adams–Novikov spectral sequence converging to the homotopy groups $\pi_*(S)$ of the sphere spectrum S . Hereafter, $q = 2p - 2$ as usual. A relation given by Toda (see [4]) implies that $\beta_{p/p}$ dies in the Adams–Novikov spectral sequence at a prime $p > 2$. At the prime two, $\beta_{2/2}^2 = 0$ by Miller, Ravenel and Wilson [2, Proposition 8.22], while at the primes three and five, Ravenel showed that $\beta_{p/p}^p$ survives to a homotopy element of $\pi_*(S)$ and $\alpha_1\beta_{p/p}^p = 0$ for the generator α_1 of $\pi_{q-1}(S)$. Here, we show the following:

1.1 Theorem *At a prime $p > 3$, $\beta_{p/p}^p$ survives to $\pi_{(p^3-1)q-2}(S)$ and $\alpha_1\beta_{p/p}^p = 0$.*

1.2 Corollary *At a prime $p > 3$, the Toda bracket $\langle\alpha_1, \alpha_1, \beta_{p/p}^p\rangle (= \alpha_1\beta_{p^2/p^2})$ is defined.*

1.3 Remark It is already known that $\alpha_1\beta_{p^2/p^2}$ survives in the Adams–Novikov spectral sequence by the work of R Cohen [1]. Corollary 1.2 states that Cohen’s element is a Toda bracket $\langle \alpha_1, \alpha_1, \beta_{p/p}^p \rangle$.

At the prime 3, Ravenel showed these results in [4].

Let β_1, β_2 and γ_t ($t > 0$) be the generators of Coker J of dimensions $pq - 2, (2p + 1)q - 2$ and $(tp^2 + (t - 1)p + t - 2)q - 3$, respectively.

1.4 Theorem Let $p > 5$, and t be a positive integer with $p \nmid t(t^2 - 1)$. Then, the elements $\alpha_1\gamma_t, \beta_2\gamma_t, \langle \alpha_1, \alpha_1, \beta_{p/p}^p \rangle \beta_1\gamma_t$ and $\langle \beta_1, p, \gamma_t \rangle$ generate subgroups of the stable homotopy groups of spheres isomorphic to \mathbb{Z}/p . Besides, even in the case $p \mid (t + 1)$, the elements $\beta_1\gamma_t$ and $\langle \beta_1, p, \gamma_t \rangle$ are generators of order p .

Note that $\langle \beta_1, p, \gamma_t \rangle = \langle \gamma_t, p, \beta_1 \rangle$. Also, if $t = 1$, then $\langle \gamma_1, p, \beta_1 \rangle = 0$, while $\beta_2\gamma_1$ is nontrivial (see Section 5).

From here on, we assume that the prime number p is greater than three.

2 $H^*S(3)$ revisited

We begin with recalling some notation from Ravenel’s green book [4]. Let BP denote the Brown–Peterson spectrum. Then, the pair

$$(BP_*, BP_*(BP)) = (\mathbb{Z}_{(p)}[v_1, v_2, \dots], BP_*[t_1, t_2, \dots])$$

is a Hopf algebroid. Here, the degrees of v_i and t_i are $2p^i - 2$. The structure maps act as follows:

$$\begin{aligned} \eta_R(v_1) &= v_1 + pt_1 \\ \eta_R(v_2) &\equiv v_2 + v_1t_1^p + pt_2 \pmod{(p^2, v_1^p)} \\ \eta_R(v_3) &\equiv v_3 + v_2t_1^{p^2} + v_1t_2^p + pt_3 \\ &\quad - pv_1v_2^{p-1}(t_2 + t_1^{p+1}) \pmod{(p^2, v_1^2, v_2^p)} \\ \Delta(t_1) &= t_1 \otimes 1 + 1 \otimes t_1 \\ (2.1) \quad \Delta(t_2) &= t_2 \otimes 1 + t_1 \otimes t_1^p + 1 \otimes t_2 - v_1b_{10} \\ \Delta(t_3) &\equiv t_3 \otimes 1 + t_2 \otimes t_1^{p^2} + t_1 \otimes t_2^p + 1 \otimes t_3 \pmod{(v_1, v_2)} \\ \Delta(t_4) &\equiv t_4 \otimes 1 + t_3 \otimes t_1^{p^3} + t_2 \otimes t_2^{p^2} + t_1 \otimes t_3^p + 1 \otimes t_4 \\ &\quad - v_3b_{12} \pmod{(v_1, v_2)} \\ \Delta(t_5) &= t_5 \otimes 1 + t_4 \otimes t_1^{p^4} + t_3 \otimes t_2^{p^3} + t_2 \otimes t_3^{p^2} + t_1 \otimes t_4^p + 1 \otimes t_5 \\ &\quad - v_3b_{22} - v_4b_{13} \pmod{(p, v_1, v_2)} \end{aligned}$$

for

$$\begin{aligned}
 (2.2) \quad b_{1k} &= \frac{1}{p} (\Delta(t_1)^{p^{k+1}} - t_1^{p^{k+1}} \otimes 1 - 1 \otimes t_1^{p^{k+1}}) \\
 &= \frac{1}{p} \sum_{i=1}^{p^{k+1}-1} \binom{p^{k+1}}{i} t_1^i \otimes t_1^{p^{k+1}-i}, \\
 b_{2k} &= \frac{1}{p} (\Delta(t_2)^{p^{k+1}} - t_2^{p^{k+1}} \otimes 1 - t_1^{p^{k+1}} \otimes t_1^{p^{k+2}} - 1 \otimes t_2^{p^{k+1}} - v_1^{p^{k+1}} b_{1k+1}).
 \end{aligned}$$

Let $K(3)_* = F_p[v_3, v_3^{-1}]$ have the BP_* -module structure given by $v_i v_3^s = v_3^s v_i = v_3^{s+1}$ if $i = 3$, and $= 0$ otherwise, and

$$\begin{aligned}
 \Sigma(3) &= K(3)_* \otimes_{BP_*} BP_*(BP) \otimes_{BP_*} K(3)_* \\
 &= K(3)_*[t_1, t_2, \dots] / (v_3 t_i^{p^3} - v_3^{p^i} t_i : i > 0) \quad (\text{by [4, 6.1.16]})
 \end{aligned}$$

is the Hopf algebra with structure inherited from $BP_*(BP)$. Define the Hopf algebra $S(3)$ by $S(3) = \Sigma(3) \otimes_{K(3)_*} F_p$, where $K(3)_*$ acts on F_p by $v_3 \cdot 1 = 1$. Then,

$$S(3) = F_p[t_1, t_2, \dots] / (t_i^{p^3} - t_i : i > 0).$$

Now we abbreviate $\text{Ext}_{S(3)}(F_p, F_p)$ to $H^*S(3)$.

Consider integers $d_i (= d_{3,i}$ in [4, 6.3.1])

$$d_i = \begin{cases} 0 & i \leq 0, \\ \max(i, p d_{i-3}) & i > 0. \end{cases}$$

Then, there is a unique increasing filtration of the Hopf algebroid $S(3)$ with $\text{deg } t_i^{p^j} = d_i$ for $0 \leq j < 3$.

2.3 Theorem (Ravenel [4, 6.3.2]) *The associated Hopf algebra $E^0 S(3)$ is isomorphic to the truncated polynomial algebra of height p on the elements $t_i^{p^j}$ for $i > 0$ and $j \in \mathbb{Z}/3$, with coproduct defined by*

$$\Delta(t_i^{p^j}) = \begin{cases} \sum_{k=0}^i t_k^{p^j} \otimes t_{i-k}^{p^{k+j}} & i \leq 3, \\ t_i^{p^j} \otimes 1 + 1 \otimes t_i^{p^j} + b_{i-3, j+2} & i > 3. \end{cases}$$

Let $L(3)$ be the Lie algebra without restriction with basis $x_{i,j}$ for $i > 0$ and $j \in \mathbb{Z}/3$ and bracket given by

$$[x_{i,j}, x_{k,l}] = \begin{cases} \delta_{i+j}^l x_{i+k, j} - \delta_{k+l}^j x_{i+k, l} & \text{for } i+k \leq 3, \\ 0 & \text{otherwise,} \end{cases}$$

where $\delta_j^i = 1$ if $i \equiv j \pmod 3$ and 0 otherwise, and $L(3, k)$ the quotient of $L(3)$ obtained by setting $x_{i,j} = 0$ for $i > k$. Then, Ravenel noticed in [4, 6.3.8]:

2.4 Theorem $H^*(L(3, k))$ for $k \leq 3$ is the cohomology of the exterior complex $E(h_{i,j})$ on one-dimensional generators $h_{i,j}$ with $i \leq k$ and $j \in \mathbb{Z}/3$, with coboundary

$$d(h_{i,j}) = \sum_{s=1}^{i-1} h_{s,j} h_{i-s,s+j}.$$

From now on, we abbreviate $h_{i,j}$ to h_{ij} , and h_{1j} to h_j .

Under the above filtration, Ravenel constructed the May spectral sequences:

2.5 Theorem (Ravenel [4, 6.3.4, 6.3.5]) *There are spectral sequences*

- (a) $E_2 = H^*(L(3, 3)) \implies H^*(E_0S(3))$,
- (b) $E_2 = H^*(E_0S(3)) \implies H^*(S(3))$.

Since these spectral sequences collapse, $H^*S(3)$ is additively isomorphic to $H^*L(3, 3)$. Therefore, we have a projection

$$(2.6) \quad \pi: H^*S(3) \rightarrow E^0 H^*S(3) = H^*(E_0S(3)) = H^*L(3, 3).$$

Note that the Massey product $\langle h_i, h_{i+1}, h_{i+2}, h_i \rangle$ is homologous to $v_3^{(2-p)p^i} b_{i+2}$ represented by $v_3^{(2-p)p^i} b_{1,i+2}$ of (2.2), and π assigns the Massey product to $b_{i+2} \in H^*L(3, 3)$. Ravenel determined in [4, 6.3.34] the additive structure of $H^*L(3, 3)$. In particular, we have the following:

2.7 Theorem $H^*L(3, 3)$ contains submodules generated by the linear independent elements

$$h_1 k_1 \zeta_3, \quad b_0 k_1 \zeta_3, \quad h_0 l, \quad k_0 l, \quad h_0 b_0 b_2 l \quad \text{and} \quad h_1 l.$$

Here, $l = h_2 h_{21} h_{30}$, $k_i = h_{2i} h_{i+1}$ ($i = 0, 1$), $b_0 = h_1 h_{32} + h_{21} h_{20} + h_{31} h_1$, $b_2 = h_0 h_{31} + h_{20} h_{22} + h_{30} h_0$ and $\zeta_3 = h_{30} + h_{31} + h_{32}$.

Proof In the table of the proof of [4, 6.3.34], we find the elements

$$h_0, \quad h_1, \quad k_0, \quad b_0, \quad b_2, \quad l \quad l' = h_0 h_{22} h_{31}, \quad \text{and} \quad \zeta_3,$$

along with the first element $h_1 k_1 \zeta_3$ of the theorem. We also have the element $-h_1 k_1 h_{30} = h_1 h_2 h_{21} h_{30}$ in the table, which is the last element $h_1 l$ of the theorem. Also in the table are $h_1 k_1 h_{31}$ and $h_1 k_1 h_{32}$. We see that $b_0 k_1 = -h_1 k_1 h_{31} + h_1 k_1 h_{32}$ and so the second element is given by $b_0 k_1 \zeta_3 = -h_1 k_1 h_{31} \zeta_3 + h_1 k_1 h_{32} \zeta_3$.

The element $h_0b_0b_2l\zeta_3$ is computed as

$$h_0h_2h_{21}h_{30}(h_1h_{32} + h_{21}h_{20} + h_{31}h_1)(h_0h_{31} + h_{20}h_{22} + h_{30}h_0)(h_{30} + h_{31} + h_{32}) - 2h_0h_1h_2h_{20}h_{21}h_{22}h_{30}h_{31}h_{32}.$$

Therefore, $h_0b_0b_2l$ is the dual of the generator $-\frac{1}{2}\zeta_3$, and the elements $h_0b_0b_2l$ and h_0l are generators. Similarly, a computation

$$\begin{aligned} k_0ll'\zeta_3 &= h_{20}h_1h_2h_{21}h_{30}h_0h_{22}h_{31}(h_{30} + h_{31} + h_{32}) \\ &= h_0h_1h_2h_{20}h_{21}h_{22}h_{30}h_{31}h_{32} \end{aligned}$$

shows that k_0l is the dual of the generator $l'\zeta_3$. □

2.8 Lemma In $H^*L(3, 3)$, $h_0k_1 = 0$ and $k_0k_1 = 0$.

Proof From the proof of [4, 6.3.34], we read off the relations $h_0k_1 = e_{30}h_2$ and $k_0k_1 = e_{30}g_1$ in $H^*L(3, 2)$. Since e_{30} cobounds h_{30} in $H^*L(3, 3)$, the lemma follows. □

3 Greek letter elements

Let $E_r^{s,t}(X)$ denote the E_r -term of the Adams–Novikov spectral sequence converging to the homotopy group $\pi_{t-s}(X)$ of a spectrum X . Then the E_2 -term is $\text{Ext}_{BP_*(BP)}(BP_*, BP_*(X))$. We here consider the Ext-group $\text{Ext}_{BP_*(BP)}(BP_*, M)$ for a $BP_*(BP)$ -comodule M as the cohomology of the cobar complex $\Omega_{BP_*(BP)}^*M$ (cf [2]). Consider a sequence $A = (a_0, a_1, \dots, a_n)$ of nonnegative integers so that the sequence $p^{a_0}, v_1^{a_1}, \dots, v_n^{a_n}$ is invariant and regular. For such a sequence A , Miller, Ravenel and Wilson introduced in [2] n -th Greek letter elements $\eta_s^{(n)}(A)$ in the Adams–Novikov E_2 -term $E_2^{n,t(A)}(S)$ by

$$(3.1) \quad \eta_s^{(n)}(A) = \delta_{A,1} \cdots \delta_{A,n}(v_n^{a_n}) \in E_2^{n,t(A)}(S)$$

for

$$v_n^{a_n} \in \text{Ext}_{BP_*(BP)}^{0,2a_n(p^n-1)}(BP_*, BP_*/I(A, n)).$$

Here, $s(A) = a_n/a_{n-1}, a_{n-2}, \dots, a_0$ and $t(A) = 2a_n(p^n - 1) - 2 \sum_{i=0}^{n-1} a_i(p^i - 1)$, $I(A, k)$ denotes the ideal of BP_* generated by $p^{a_0}, v_1^{a_1}, \dots, v_{k-1}^{a_{k-1}}$, and $\delta_{A,k+1}$ is the connecting homomorphism associated to the short exact sequence

$$0 \rightarrow BP_*/I(A, k) \xrightarrow{v_k^{a_k}} BP_*/I(A, k) \rightarrow BP_*/I(A, k+1) \rightarrow 0.$$

In particular, we write $\alpha = \eta^{(1)}$, $\beta = \eta^{(2)}$ and $\gamma = \eta^{(3)}$. So far, only when $n \leq 3$, many conditions for that Greek letter elements survives to homotopy elements are known. We abbreviate $\eta_{s(A)}^{(n)}$ to $\eta_{a_n}^{(n)}$ if $A = (1, \dots, 1, a_n)$ as usual. For example, we consider β -elements defined by

$$(3.2) \quad \begin{aligned} \beta_s &= \delta_{(1,1),1}(\beta'_s) \in E_2^{2,t(1,1,s)}(S) \\ &\text{for } \beta'_s = \delta_{(1,1),2}(v_2^s) \in E_2^{1,t(1,1,s)}(V(0)), \\ \beta_{p^i/p^i} &= \beta_{p^i/p^i,1} = \delta_{(1,p^i),1} \delta_{(1,p^i),2}(v_2^{p^i}) \in E_2^{2,t(1,p^i,p^i)}(S). \end{aligned}$$

Hereafter we assume that the prime p is greater than three. We have the Smith–Toda spectrum $V(k)$ for $k = 0, 1, 2$ defined by the cofiber sequences

$$(3.3) \quad \begin{aligned} S &\xrightarrow{p} S \xrightarrow{i} V(0) \xrightarrow{j} \Sigma S, \\ \Sigma^q V(0) &\xrightarrow{\alpha} V(0) \xrightarrow{i_1} V(1) \xrightarrow{j_1} \Sigma^{q+1} V(0), \\ \Sigma^{(p+1)q} V(1) &\xrightarrow{\beta} V(1) \xrightarrow{i_2} V(2) \xrightarrow{j_2} \Sigma^{(p+1)q+1} V(1). \end{aligned}$$

Here, $\alpha \in [V(0), V(0)]_q$ is the Adams map and $\beta \in [V(1), V(1)]_{(p+1)q}$ is the v_2 -periodic element due to L Smith. Note that the BP_* -homology of these spectra are $BP_*(V(k)) = BP_*/I_{k+1}$ for the ideal I_k of BP_* generated by v_i for $0 \leq i < k$ with $v_0 = p$. We consider the Bousfield–Ravenel localization functor L_3 with respect to $v_3^{-1}BP$. The E_2 -term $E_2^*(L_3 V(2))$ of $L_3 V(2)$ is isomorphic to $K(3)_* \otimes H^* S(3)$, whose structure is given in [4] (see also [7]), and we consider the composite

$$r: E_2^*(S) \xrightarrow{t_*} E_2^*(V(2)) \xrightarrow{\eta} E_2^*(L_3 V(2)) \xrightarrow{\rho} H^*(S(3)) \xrightarrow{\pi} H^* L(3, 3).$$

Here the first map is induced from the inclusion $\iota: S \rightarrow V(2)$ to the bottom cell, the second is from the localization map, the third is obtained by setting $v_3 = 1$ and the last is the projection (2.6).

3.4 Lemma *The map r assigns the Greek letter elements as follows:*

$$\begin{aligned} r(\alpha_1) &= h_0, & r(\beta_1) &= -b_0, & r(\beta_2) &= 2k_0, \\ r(\gamma_t) &= -t(t^2 - 1)l - t(t - 1)k_1 \zeta_3 & \text{and } r(\beta_{p/p}) &= -b_1. \end{aligned}$$

We also have $\beta'_1 = h_1 - v_1^{p-1} h_0 \in E_2^{1,pq}(V(0))$ for the generators h_i of $E_2^{1,p^i q}(V(0))$ represented by $t_1^{p^i}$.

Proof First we consider the images of the Greek letter elements under the map $\iota_*: E_2^*(S) \rightarrow E_2^*(V(2))$. In the cobar complex $\Omega_{BP_*(BP)}^*BP_*$, by (2.1),

$$\begin{aligned} d(v_1) &= pt_1, \quad d(v_2^{p^i}) \equiv v_1^{p^i}t_1^{p^{i+1}} - v_1^{p^{i+1}}t_1^{p^i} \pmod{(p)} \text{ for } i \geq 0, \\ d(v_2^2) &\equiv 2v_1v_2t_1^p + v_1^2t_1^{2p} \pmod{(p, v_1^p)}, \\ d(v_3^t) &\equiv tv_2v_3^{t-1}t_1^{p^2} + \binom{t}{2}v_2^2v_3^{t-2}t_1^{2p^2} + \binom{t}{3}v_2^3v_3^{t-3}t_1^{3p^2} \pmod{(p, v_1, v_2^4)}, \end{aligned}$$

which imply

$$\begin{aligned} \delta_{(1,1),1}(v_1) &= [t_1], & \delta_{(1,1),2}(v_2) &= [t_1^p - v_1^{p-1}t_1], \\ \delta_{(1,1),2}(v_2^2) &= [2v_2t_1^p + v_1t_1^{2p} + v_1^{p-1}y], & \delta_{(1,p),2}(v_2^p) &= [t_1^{p^2} - v_1^{p^2-p}t_1^p], \\ \delta_{(1,1,1),3}(v_3^t) &= [tv_3^{t-1}t_1^{p^2} + \binom{t}{2}v_2v_3^{t-2}t_1^{2p^2} + \binom{t}{3}v_2^2v_3^{t-3}t_1^{3p^2} + v_2^3z] = \bar{\gamma}_t, \end{aligned}$$

for cochains $y \in \Omega_{BP_*(BP)}^1BP_*/(p)$ and $z \in \Omega_{BP_*(BP)}^1BP_*/(p, v_1)$. Here, $[x]$ denotes a cohomology class represented by a cocycle x . The first one shows $\alpha_1 = h_0$, and the second gives the last statement of the lemma. We further see that $d(t_1^{p^k}) = -pb_{1k-1}$ for $k \geq 1$ and $d(v_k) \equiv pt_k \pmod{I((2, 1, 1), k)}$ for $k = 2, 3$ by (2.1) in $\Omega_{BP_*(BP)}^1BP_*$. Moreover, $[b_{1k}]$'s are assigned to b_k in $H^*L(3, 3)$ under the projection π , and we obtain

$$\begin{aligned} r\delta_{(1,p^{k-1}),1}(h_k - v_1^{p^k-p^{k-1}}h_{k-1}) &= -b_{k-1} \quad \text{for } k = 1, 2, \\ r\delta_{(1,1),1}([2v_2t_1^p + v_1t_1^{2p}]) &= 2k_0, \\ \delta_{(1,1,1),2}(\bar{\gamma}_t) &= [t(t-1)v_3^{t-2}t_2^p \otimes t_1^{p^2} + \binom{t}{2}v_3^{t-2}t_1^p \otimes t_1^{2p^2} + w] = \gamma'_t, \\ r\delta_{(1,1,1),1}(\gamma'_t) &= t(t-1)(t-2)h_{30}k_1 + t(t-1)r\delta_{(1,1,1),1}(\left[t_2^p \otimes t_1^{p^2} + \frac{1}{2}t_1^p \otimes t_1^{2p^2} \right]). \end{aligned}$$

Here, w is a linear combination of terms in the ideal $(v_1, v_2)^2$. Thus the relations other than $r(\gamma_t)$ follows.

We note that b_{20} in (2.2) corresponds to $h_{21}h_{30} + h_{31}h_{21}$ by $\Delta(t_5)^p$ in (2.1). Since $d(t_2^p) = -t_1^p \otimes t_1^{p^2} + v_1^p b_{11} - pb_{20}$ by (2.1), we obtain

$$r\delta_{(1,1,1),1}([t_2^p \otimes t_1^{p^2} + \frac{1}{2}t_1^p \otimes t_1^{2p^2}]) = -(h_{21}h_{30} + h_{31}h_{21})h_2 + h_{21}b_1 = -3l - k_1\zeta_3,$$

which shows the relation on $r(\gamma_t)$. □

Recall the cofiber sequences (3.3) and the v_3 -periodic element $\gamma \in [V(2), V(2)]_{q_3}$ ($q_3 = (p^2 + p + 1)q$) due to H Toda. Then, the Greek letter elements in homotopy are

defined by

$$(3.5) \quad \alpha_t = j\alpha^t i, \quad \beta_t = j\beta'_t \quad \text{for } \beta'_t = j_1\beta^t i_1 i \quad \text{and} \quad \gamma_t = jj_1 j_2 \gamma^t i_2 i_1 i$$

for $t > 0$, and the Greek elements in the E_2 -term survives to the same named one in homotopy by the Geometric Boundary Theorem (cf [4]).

Proof of Theorem 1.4 We begin with noticing that the element b_i in $H^*L(3, 3)$ is the image of the Massey product $\langle h_i, h_{i+1}, h_{i+2}, h_i \rangle$ under π , which is homologous to b_i represented by b_{1i} in (2.2). We further note that the Toda brackets $\langle \alpha_1, \alpha_1, \beta_{p/p}^p \rangle$ and $\langle \beta_1, p, \gamma_t \rangle$ are detected by $\alpha_1 b_2$ and $h_1 \gamma_t$ of $E_2^*(S)$, respectively. Indeed, in the first bracket, $d_{2p-1}(b_2) = \alpha_1 \beta_{p/p}^p$ by Corollary 4.4 below, and in the second bracket, $\langle \beta_1, p, \gamma_t \rangle = j \langle \beta'_1, p, \gamma_t \rangle$. Under the condition on t , Lemmas 3.4, 2.7 and 2.8 imply that each element of $\alpha_1 \gamma_t, \beta_2 \gamma_t, \alpha_1 b_2 \gamma_t \beta_1$ and $h_1 \gamma_t$, as well as $\beta_1 \gamma_t$, generates a submodule isomorphic to \mathbb{Z}/p of the E_2 -term $E_2^*(S)$. These are, of course, permanent cycles, and nothing kills them in the Adams–Novikov spectral sequence since each element has a filtration degree less than $2p - 1$. □

4 $\beta_{p/p}^p$ in the homotopy of spheres

Let X and \bar{X} be the $(p-1)q-$ and $(p-2)q-$ skeletons of the Brown–Peterson spectrum BP . Then, we have the cofiber sequences

$$(4.1) \quad S \xrightarrow{\iota} X \xrightarrow{\kappa} \Sigma^q \bar{X} \xrightarrow{\lambda} S^1 \quad \text{and} \quad \bar{X} \xrightarrow{\iota'} X \xrightarrow{\kappa'} S^{(p-1)q} \xrightarrow{\lambda'} \Sigma \bar{X}.$$

Then,

$$BP_*(X) = BP_*[x]/(x^p) \quad \text{and} \quad BP_*(\bar{X}) = BP_*[x]/(x^{p-1})$$

as subcomodules of $BP_*(BP)$, where x corresponds to t_1 . From [4, Chapter 7], we read off the following:

$$(4.2) \quad b_1^p = 0 \in E_2^{2p, p^3q}(X), \text{ which implies}$$

$$E_2^{2s+e, tq}(X) = 0 \quad \text{if } s \geq p \text{ and } t < (s-1)p^2 + (s+1+e)p.$$

4.3 Lemma $b_0: E_2^{2s+e, tq}(S) \rightarrow E_2^{2s+2+e, (t+p)q}(S)$ is monomorphic if $s \geq p$ and $t \leq (s-1)p^2 + (s+e)p$.

Proof Note that $b_0 = \lambda \lambda'$, and the lemma follows from (4.2) and the exact sequences

$$E_2^{2s+e, (t+p-1)q}(X) \xrightarrow{\kappa'} E_2^{2s+e, tq}(S) \xrightarrow{\lambda'} E_2^{2s+1+e, (t+p-1)q}(\bar{X}),$$

$$E_2^{2s+e+1, (t+p)q}(X) \rightarrow E_2^{2s+e+1, (t+p-1)q}(\bar{X}) \xrightarrow{\lambda} E_2^{2s+2+e, (t+p)q}(S)$$

induced from the cofiber sequences in (4.1). □

Ravenel showed that $d_{2p-1}(\beta_{p^2/p^2}) \equiv \alpha_1 \beta_{p/p}^p \pmod{\text{Ker } \beta_1^p}$ in the Adams–Novikov spectral sequence for $\pi_*(S)$ [4, 6.4.1]. Here, the mapping β_1^p on $E_2^{2p+1, (p^3+1)q}(S)$ is a monomorphism by Lemma 4.3:

4.4 Corollary *In the Adams–Novikov spectral sequence for $\pi_*(S)$,*

$$d_{2p-1}(\beta_{p^2/p^2}) = \alpha_1 \beta_{p/p}^p \in E_{2p-1}^{2p+1, (p^3+1)q}(S) = E_2^{2p+1, (p^3+1)q}(S).$$

Proof of Theorem 1.1 Consider the first cofiber sequence in (4.1). The Adams–Novikov E_2 -term $E_2^{sq+3, (p^3+s)q}(X)$ vanishes for $s > 0$ by (4.2), so the element $\iota_*(\beta_{p^2/p^2}) \in E_2^{2, p^3q}(X)$ survives to a homotopy element ${}^X\beta_{p^2/p^2} \in \pi_*(X)$. In general, we see that:

(4.5) Let $\bar{\iota}: S \rightarrow \bar{X}$ denote the inclusion to the bottom cell. Then, $\lambda_*\bar{\iota}(x) = \alpha_1 x$ for $x \in E_2^*(S)$.

Put $\bar{\beta}_{p/p} = \bar{\iota}_*(\beta_{p/p}) \in E_2^{2, p^2q}(\bar{X})$, and we see that $\lambda_*(\bar{\beta}_{p/p}) = \alpha_1 \beta_{p/p}^p$, and so we see that $\bar{\beta}_{p/p}^p$ detects an essential homotopy element $\kappa_*({}^X\beta_{p^2/p^2}) \in \pi_*(\bar{X})$ by Corollary 4.4 and Shimomura [5], which we also denote by $\bar{\beta}_{p/p}^p$.

Now turn to the second cofiber sequence in (4.1). The relation $b_1^p = 0$ of (4.2) yields a cochain $y = \sum_{i=0}^{p-1} x^i y_i \in \Omega^{2p-1}BP_*(X)$ such that $d(y) = b_1^p$, where $y_i \in \Omega^{2p-1}BP_*$. It follows that $d(\bar{y}) = b_1^p - d(x^{p-1})y_{p-1} \in \Omega^{2p}BP_*(\bar{X})$ for $\bar{y} = \sum_{i=0}^{p-2} x^i y_i \in \Omega^{2p-1}BP_*(\bar{X})$. In particular $d(y_{p-1}) = 0 \in \Omega^{2p-1}BP_*$ and $d(y_{p-2}) = (1-p)t_1 \otimes y_{p-1}$. By definition, these imply $\lambda'_*(y_{p-1}) = b_1^p$. Consider the exact sequence obtained by applying the homotopy groups to the second cofiber sequence. Then, $\iota'_*(\bar{\beta}_{p/p}^p) = 0$ by (4.2), and so $\bar{\beta}_{p/p}^p$ must be pulled back to an element $\xi \in \pi_*(S)$ detected by y_{p-1} . Since $b_0 = \lambda\lambda'$, $b_0 y_{p-1} = h_0 b_1^p$, and $\langle h_0, \dots, h_0 \rangle y_{p-1} = h_0 \langle h_0, \dots, h_0, y_{p-1} \rangle$, we see that

$$b_1^p \equiv \langle h_0, \dots, h_0, y_{p-1} \rangle \not\equiv 0 \in E_2^{2p, p^3q}(S) \pmod{\text{ker } h_0}.$$

Put $b_1^p = \langle h_0, \dots, h_0, y_{p-1} \rangle + c$ for $c \in \text{ker } h_0 \subset E_2^{2p, p^3q}(S)$. Then, $b_1^p - c$ survives to $\beta_{p/p}^p \in \pi_*(S)$.

The element $\alpha_1 \beta_{p/p}^p$ is detected by $h_0(b_1^p - c) = h_0 b_1^p$ in the Adams–Novikov E_2 -term, which is killed by b_2 by Corollary 4.4. □

5 Remarks

5.1 A relation on Toda bracket

The relation $\langle \beta_s, p, \gamma_t \rangle = \langle \gamma_t, p, \beta_s \rangle$ follows immediately from results of Toda: By definition, $\langle \beta_s, p, \gamma_t \rangle = j\beta_{(s)}\gamma_{(t)}i$ and $\langle \gamma_t, p, \beta_s \rangle = j\gamma_{(t)}\beta_{(s)}i$ for $\beta_{(s)} = j_1\beta^s i_1$

and $\gamma_{(t)} = j_1 j_2 \gamma^t i_2 i_1$. Since $V(2)$ and $V(3)$ are $V(0)$ -module spectra, $\theta(\beta) = 0$ and $\theta(\gamma) = 0$ by Toda [6, Lemma 2.3]. Similarly, $\theta(i_k) = 0$ and $\theta(j_k) = 0$ for $k = 1, 2$. Therefore, [6, Lemma 2.2] implies $\theta(\beta_{(s)}) = 0$ and $\theta(\gamma_{(t)}) = 0$. Therefore, $\beta_{(s)}\gamma_{(t)} = \gamma_{(t)}\beta_{(s)}$ by [6, Corollary 2.7] as desired.

5.2 On the action of γ_1

Note that $\gamma_1 = \alpha_1 \beta_{p-1}$. Then, $\alpha_1 \gamma_1 = \alpha_1^2 \beta_{p-1} = 0$ and $\langle \alpha_1, \alpha_1, \beta_{p/p}^p \rangle \beta_1 \gamma_1 = -\alpha_1 \langle \alpha_1, \alpha_1, \beta_{p/p}^p \rangle \beta_1 \beta_{p-1} = -\langle \alpha_1, \alpha_1, \alpha_1 \rangle \beta_{p/p}^p \beta_1 \beta_{p-1} = 0$ since $\langle \alpha_1, \alpha_1, \alpha_1 \rangle = 0$, and $\langle \gamma_1, p, \beta_1 \rangle = \beta_{p-1} \langle \alpha_1, p, \beta_1 \rangle = \beta_{p-1} j \alpha_1 \beta_1 i_1 i = 0$.

For $t \geq 2$,

$$\begin{aligned} \beta_t &= \delta_{(1,1),1} \delta_{(1,1),2}(v_2^t) = \delta_{(1,1),1} \left(\left[t v_2^{t-1} t_1^p + \binom{t}{2} v_1 v_2^{t-2} t_1^{2p} + v_1^2 x \right] \right) \\ &\equiv \left[t(t-1) v_2^{t-2} t_2 \otimes t_1^p - t v_2^{t-1} b_0 + \binom{t}{2} v_2^{t-2} t_1 \otimes t_1^{2p} \right] \pmod{(p, v_1)} \\ &\equiv t(t-1) v_2^{t-2} k_0 - t v_2^{t-1} b_0 \pmod{(p, v_1)} \end{aligned}$$

and $\alpha_1 \beta_2 \beta_{p-1} \in E_2^5(S^0)$ is projected to $h_0(2k_0 - 2v_2 b_0)(2v_2^{p-3} k_0 + v_2^{p-2} b_0) = -2v_2^{p-2} h_0 k_0 b_0 - 2h_0 v_2^{p-1} b_0^2$ in $E_2^5(V(2))$ under the induced map i_* from the inclusion $i: S^0 \rightarrow V(2)$ to the bottom cell. Here, $k_0 = [t_2 \otimes t_1^p + \frac{1}{2} t_1 \otimes t_1^{2p}]$. Then, this element is detected by $-2v_2^{p-2} k_0 \in E_1^3 = E_2^{2,(p^2+p-1)q}(X \wedge V(2))$ in the small descent spectral sequence. The killer of this element, if any, stays in the E_1 -terms

$$\begin{aligned} E_1^2 &= E_2^{2,(p^2+p)q}(X \wedge V(2)), & E_1^1 &= E_2^{3,(p^2+2p-1)q}(X \wedge V(2)), \\ E_1^0 &= E_2^{4,(p^2+2p)q}(X \wedge V(2)). \end{aligned}$$

These are zero, and we see that the product is not zero.

References

- [1] **RL Cohen**, *Odd primary infinite families in stable homotopy theory*, Mem. Amer. Math. Soc. 30, no. 242, Amer. Math. Soc. (1981) MR603393
- [2] **HR Miller, DC Ravenel, WS Wilson**, *Periodic phenomena in the Adams–Novikov spectral sequence*, Ann. of Math. 106 (1977) 469–516 MR0458423
- [3] **S Oka, K Shimomura**, *On products of the β -elements in the stable homotopy of spheres*, Hiroshima Math. J. 12 (1982) 611–626 MR676562
- [4] **DC Ravenel**, *Complex cobordism and stable homotopy groups of spheres*, Pure and Applied Math. 121, Academic Press, Orlando, FL (2000) MR860042

- [5] **K Shimomura**, *On differential of a generalized Adams spectral sequence*, J. Fac. Educ. Tottori Univ. (Nat. Sci.) 41 (1992) 119–131
- [6] **H Toda**, *Algebra of stable homotopy of Z_p -spaces and applications*, J. Math. Kyoto Univ. 11 (1971) 197–251 MR0293631
- [7] **A Yamaguchi**, *The structure of the cohomology of Morava stabilizer algebra $S(3)$* , Osaka J. Math. 29 (1992) 347–359 MR1173994

*Graduate school of Mathematics, Nagoya University
Furo-cho, Chikusa-ku, Nagoya 464-8602, Japan*

*Department of Mathematics, Faculty of Science, Kochi University
2-5-1, Akebono, Kochi 780-8520, Japan*

ryo_kato_1128@yahoo.co.jp, katsumi@kochi-u.ac.jp

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