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Inequivalent handlebody-knots with homeomorphic complements

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We distinguish the handlebody-knots $5_1, 6_4$ and $5_2, 6_{13}$ in the table, due to Ishii et al, of irreducible handlebody-knots up to six crossings. Furthermore, we construct two infinite families of handlebody-knots, each containing one of the pairs $5_1, 6_4$ and $5_2, 6_{13}$, and show that any two handlebody-knots in each family have homeomorphic complements but they are not equivalent.

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1 Introduction

Given a knot in S^3, its regular neighborhood is a knotted solid torus. Conversely, an embedded solid torus in S^3 uniquely determines a knot. Thus we may regard an embedded solid torus as a knot in S^3. Instead of an embedded solid torus in S^3, consider an embedded handlebody. We may regard it as a kind of a knot. Following Ishii, Kishimoto, Moriuchi and Suzuki [3], we say that a handlebody embedded in S^3 is a *handlebody-knot*.

Throughout this paper, by a handlebody-knot we will mean a genus two handlebody embedded in S^3. A handcuff graph or a θ–curve Γ in a handlebody-knot H is called a *spine* if H is a regular neighborhood of Γ. The spine of H is not uniquely determined, but any two spines are related by a finite sequence of isotopies and IH-moves (see Ishii [2]), where an IH-move is a local move on a spatial trivalent graph depicted in Figure 1.

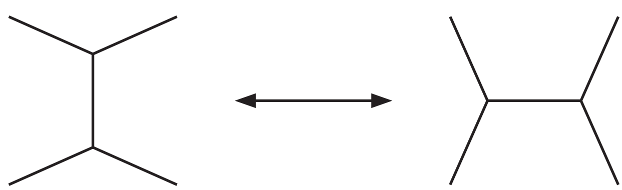

Figure 1

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Two handlebody-knots H_1 and H_2 are said to be *equivalent* if there exists an isotopy of S^3 that takes H_1 to H_2, or equivalently if there exists an orientation-preserving automorphism h of S^3 such that $h(H_1) = H_2$. A handlebody-knot H is *reducible* if there exists a 2–sphere S in S^3 such that $S \cap H$ is a disk separating H into two solid tori. Otherwise, it is *irreducible*. Note that H is irreducible if $S^3 - \mathrm{int}(H)$ is ∂–irreducible.

As done for knots, we can use regular diagrams of spines of a handlebody-knot to define the crossing number of the handlebody-knot. Ishii, Kishimoto, Moriuchi and Suzuki recently give a table of handlebody-knots such that any irreducible handlebody-knot with six or fewer crossings or its mirror image is equivalent to one of the handlebody-knots in the table. See [3, Table 1]. By using some invariants, they distinguish all handlebody-knots in their table except only for the two pairs $(5_1, 6_4)$ and $(5_2, 6_{13})$. See Figure 2.

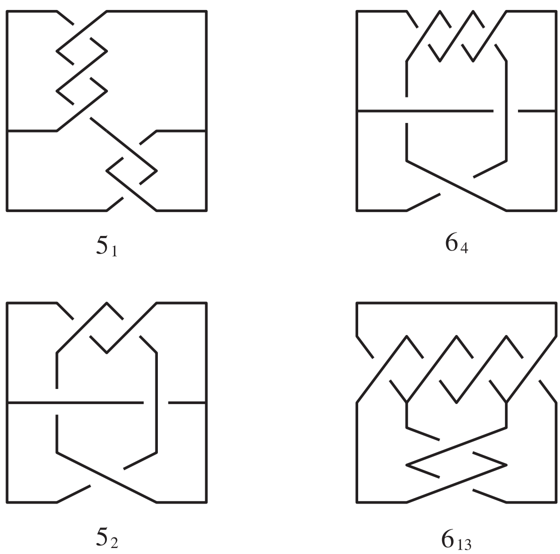


Figure 2

Consider the handcuff graphs Φ_n, Ψ_n in S^3, shown in Figure 3, where a rectangle labeled by an integer n denotes a vertical right-handed twist of two strings with $2n$ crossings. Let V_n and W_n denote regular neighborhoods of Φ_n and Ψ_n, respectively. Put $X_n = S^3 - \mathrm{int}(V_n)$ and $Y_n = S^3 - \mathrm{int}(W_n)$.

Let $\Theta_n = \Phi_n$ or Ψ_n, and let $Z_n = X_n$ or Y_n correspondingly. The handcuff graph Θ_n consists of two vertices and three edges, two forming loops and one connecting the two loops. One of the two loops bounds a disk intersecting the vertical twist in two points.

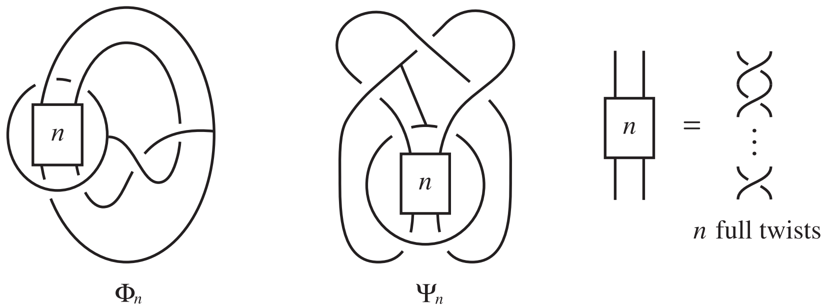


Figure 3

By twisting along the disk, one can transform Θ_n into Θ_m for any other integer m. This shows that Z_n is homeomorphic to Z_m.

For any submanifold M of S^3, denote by M^* the mirror image of M. We say that M is *amphicheiral* if an isotopy of S^3 takes M to M^*. The main result of the present paper is the following.

Theorem 1.1 *Let n and m be distinct integers.*

(1) *No two of V_n, V_n^*, V_m, V_m^* are equivalent.*

(2) *No two of W_n, W_n^*, W_m, W_m^* are equivalent.*

In particular, V_n and W_n are not amphicheiral for each integer n.

By calculating fundamental groups, one can show that X_0 and Y_0 are not homeomorphic. This implies that V_n and W_m are not equivalent for any integers n and m.

It is a celebrated result of Gordon and Luecke that if two knots in S^3 have homeomorphic complements then the homeomorphism between the two complements extends to an automorphism of S^3 [1]. In contrast, Motto [5] showed that handlebody-knots are not determined by their complements. We remark that our infinite families of inequivalent handlebody-knots are also of this type.

We can now distinguish the handlebody-knots $5_1, 6_4$, and $5_2, 6_{13}$ in the table due to Ishii et al.

Corollary 1.2 (1) *No two of $5_1, 5_1^*, 6_4, 6_4^*$ are equivalent.*

(2) *No two of $5_2, 5_2^*, 6_{13}, 6_{13}^*$ are equivalent.*

In particular, $5_1, 5_2, 6_4, 6_{13}$ are not amphicheiral.

Proof The sequences of pictures in Figure 4(a),(b) show that V_0 and V_{-1} are respectively equivalent to 5_1 and 6_4, and the sequences of pictures in Figure 4(c),(d) show that W_0 and W_1 are respectively equivalent to 5_2 and 6^*_{13}. Hence the result immediately follows from Theorem 1.1. □

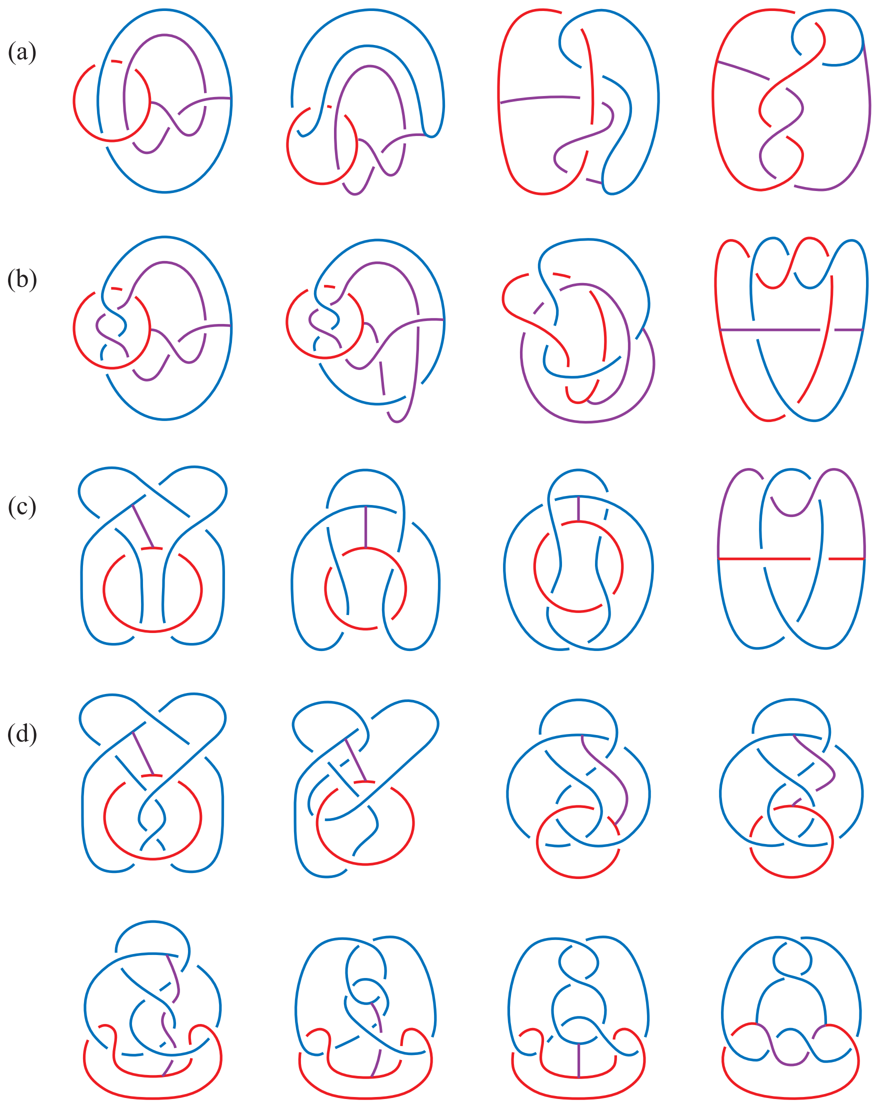


Figure 4

Some figures in this paper are best viewed in color; readers confused by figures in a black-and-white version are recommended to view the electronic version.

2 Curves in the boundary of a genus two handlebody

A properly embedded disk in a 3–manifold M is *essential* if it is not isotopic to a disk in ∂M. A properly embedded compact surface in M, which is neither a disk nor a sphere, is *essential* if it is incompressible and is not ∂–parallel. Given a set $\{c_1, \dots, c_n\}$ of disjoint simple loops in ∂M, $M[c_1 \cup \dots \cup c_n]$ will denote the 3–manifold obtained by attaching 2–handles to M along disjoint neighborhoods of $c_1, \dots, c_n$.

Throughout this section, H will denote a genus two handlebody. A simple loop in ∂H is called a *primitive curve* if there exists a disk in H, called a *dual disk*, that intersects the loop in a single point.

Lemma 2.1 *Let c_1, c_2 be two disjoint nonisotopic primitive curves in ∂H. If there are two disjoint nonisotopic essential disks D_1, D_2 of H each of which is a common dual disk of c_1 and c_2, then the fundamental group of $H[c_1 \cup c_2]$ is either the infinite cyclic group or the cyclic group of order 2.*

Proof The two disks D_1, D_2 cut H into a 3–ball B and $c_1 \cup c_2$ into four arcs. Let D_i^+, D_i^- be the copies of D_i on ∂B for $i = 1, 2$. There are two cases; the four arcs together with the four disks $D_1^\pm, D_2^\pm$ form two cycles of length 2 or a single cycle of length 4. See Figure 5. One easily sees that the fundamental group of $H[c_1 \cup c_2]$ is the infinite cyclic group in the first case and it is the cyclic group of order 2 in the latter case. □

An element x of the free group F of rank 2 is called a *primitive element* if there exists an element $y \in F$ such that x, y generate F.

Lemma 2.2 *Let A be an essential separating annulus in H. Let c_1, c_2 be two essential simple loops in ∂H which are disjoint from ∂A. Suppose that A separates c_1 and c_2. Then one of c_1 and c_2 represents a proper power of a primitive element of the free group $\pi_1(H)$.*

Proof By Kobayashi [4, Lemma 3.2(i)], A cuts H into a solid torus H_1 and a genus two handlebody H_2. Since A separates c_1 and c_2, we may assume $c_1 \subset H_1$ and $c_2 \subset H_2$. Let A_i be the copy of A in ∂H_i for $i = 1, 2$. Then the core of A_1 is parallel to c_1 in ∂H_1, and the core of A_2 represents a primitive element of the free group $\pi_1(H_2)$.

If c_1 were a meridian curve of H_1 then A would be compressible in H. If c_1 were homotopic to the core of H_1 then A would be ∂–parallel in H. Hence c_1 is homotopic in H_1 to $n\ (\geq 2)$ times around the core of H_1.

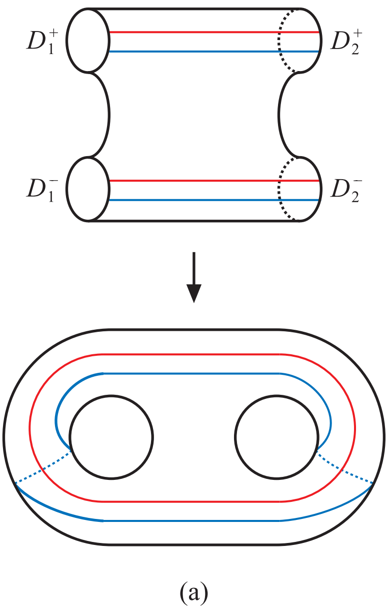


Figure 5

Let x be a generator of the infinite cyclic group $\pi_1(H_1)$, and let y, z be two elements generating the free group $\pi_1(H_2)$. Here, we may assume that x^n is represented by the core of A_1 (or c_1) and y is represented by the core of A_2. By the Van Kampen's theorem, $\pi_1(H)$ has three generators x, y, z and one relation $x^n = y$. Thus $\pi_1(H)$ is the free group on x and z, and c_1 represents x^n in the group $\pi_1(H)$. □

Lemma 2.3 *Let c_1, c_2 be two simple loops in ∂H which are not contractible in H. Suppose that there exists a properly embedded disk D in $H - c_1 \cup c_2$ which splits H into two solid tori, each containing one of c_1 and c_2. Then any such disk is isotopic to D in $H - c_1 \cup c_2$.*

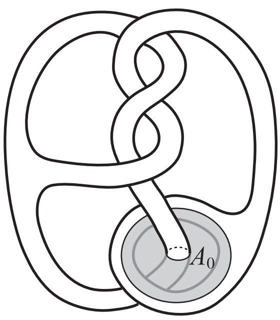
Proof Let E be a properly embedded disk in $H - c_1 \cup c_2$ which splits H into two solid tori H_1 and H_2 with $c_i \subset H_i$ for each $i = 1, 2$. Suppose that E is not isotopic to D in $H - c_1 \cup c_2$.

If E is disjoint from D then D and E are parallel in H, that is, they cut off a 1–handle $D \times I$ from H. Since neither c_1 nor c_2 is contractible in H, $\partial D \times I$ does not meet any of c_1 and c_2. This means that $D \times I$ is, in fact, the parallelism between D and E in $H - c_1 \cup c_2$. This contradicts our assumption on E.

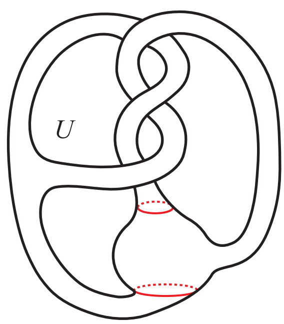
We may assume that the intersection $D \cap E$ is transverse and minimal up to isotopy of E. Then a standard disk swapping argument shows that $D \cap E$ has no circle components. An arc component of $D \cap E$, outermost in D, cuts off a subdisk of D. Surgery on E along the subdisk yields two disks, both of which are disjoint from $c_1 \cup c_2$. Let E' be any of these disks. Then E' lies in a solid torus H_i for some $i = 1, 2$. By the minimality of $|D \cap E|$, E' is parallel in $H - c_1 \cup c_2$ to neither E nor a disk in ∂H. Hence E' is a meridian disk of the solid torus H_i, cutting it into a 3–ball in which c_i lies. This implies that c_i is contractible in H, a contradiction. □

3 V_n and V_m ($n \neq m$) are not equivalent

Consider Φ_0. The drawings in Figure 4(a) depict an isotopy from V_0 to 5_1, showing that there exists a properly embedded nonseparating annulus A_0 in X_0 as shown in Figure 6(a). Cutting X_0 along A_0 gives a new compact 3–manifold U as shown in Figure 6(b), where the two loops in ∂U are the cores of the two copies A_0^+ and A_0^- of A_0 in ∂U. Let $c^\pm$ be the loops. After an isotopy, U becomes the complement of a standardly embedded genus two handlebody in S^3 (see Figure 7), so U itself is a genus two handlebody.



(a)



(b)

Figure 6

Let $C = c^+ \cup c^-$. Take three essential nonseparating disks X, Y, Z in U as shown in Figure 8(a). These three disks divide U into two 3–balls $B^\pm$ and C into arcs. See Figure 8(b). Let $X^\pm, Y^\pm, Z^\pm$ be copies of X, Y, Z in $\partial B^\pm$. Then $C^\pm = C \cap B^\pm$ consists of five arcs, two connecting $X^\pm$ and $Y^\pm$, two connecting $X^\pm$ and $Z^\pm$, and one connecting $Y^\pm$ and $Z^\pm$. Set $\Delta = X \cup Y \cup Z$ and $\Delta^\pm = X^\pm \cup Y^\pm \cup Z^\pm$. Then $\partial B^\pm - (\Delta^\pm \cup C^\pm)$ is a union of (open) disks.

Lemma 3.1 *U does not contain an essential disk or annulus or a properly embedded Möbius band which is disjoint from C.*

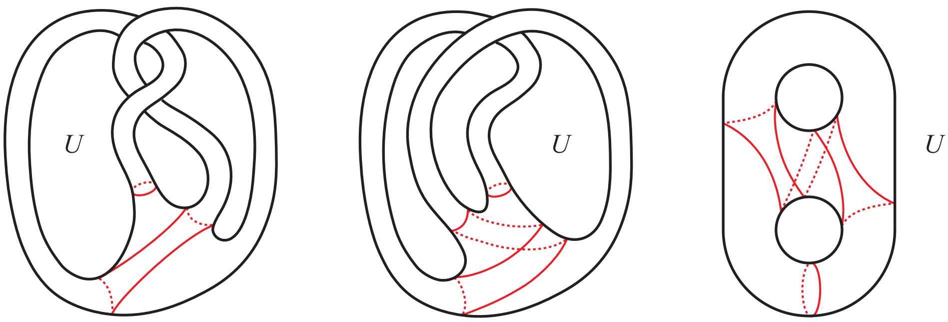
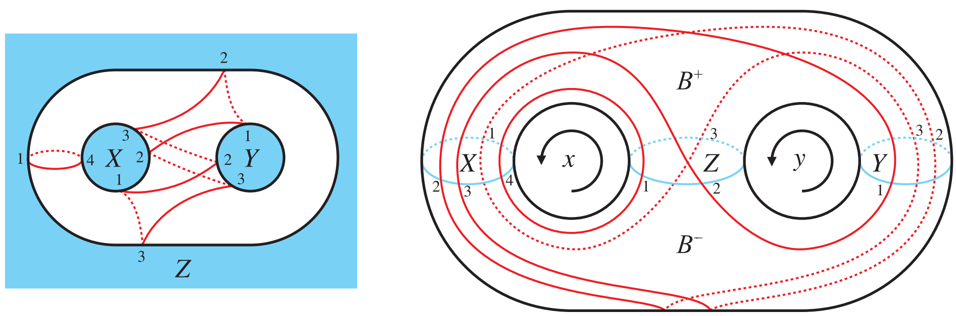


Figure 7



(a) (b)

Figure 8

Proof Assume for contradiction that U contains such a surface F.

First, suppose that F is a disk. The intersection $F \cap \Delta$ may be assumed to be transverse and minimal among all essential disks of U that are disjoint from C. Note that $F \cap \Delta \neq \varnothing$, since otherwise F would be properly embedded in either B^+ or B^- with $\partial F \cap (\Delta^{\pm} \cup C^{\pm}) = \varnothing$ and hence F would be parallel to a disk in ∂U. By the minimality of $|F \cap \Delta|$, F has no circle components of intersection with Δ. An arc component of intersection, outermost in F, cuts off a disk F' from F. Any two disks in $\Delta^{\pm}$ are joined by an arc in $C^{\pm}$, so the arc $F' \cap \partial U$ together with an arc in $\partial\Delta$ bounds a disk in ∂U that is disjoint from C. This disk could be used to reduce $|F \cap \Delta|$, contradicting the minimality assumption. Hence F is not a disk.

The fundamental group $\pi_1(U)$ is a free group generated by two elements x and y, where x and y are respectively represented by the cores of the 1–handles $N(X)$ and $N(Y)$, attached to the 3–ball $N(Z)$. See Figure 8(b). The two loops c^+ and c^- represent two group elements x and $xyxy^{-1}x^{-1}y^{-1}$. Hence the 3–manifold

$Q = U[c^+ \cup c^-]$ has a trivial fundamental group, so it is a 3–ball. Since F is disjoint from C, F is properly embedded in Q. No Möbius bands can be properly embedded in a 3–ball, so F must be an annulus. Since every properly embedded annulus in a 3–ball is separating, F must be separating in U. Splitting U along F, we get a solid torus U_1 and a genus two handlebody U_2, where the core of the copy of F in ∂U_1 winds the solid torus U_1 at least two times in the longitudinal direction. See [4, Lemma 3.2(i)].

Neither x nor $xyxy^{-1}x^{-1}y^{-1}$ is a proper power of a primitive element of the group $\pi_1(U)$. Thus it follows from Lemma 2.2 that the two loops c^+ and c^- are not separated by F. Since c^+ and c^- are not parallel in ∂U, they are contained in U_2. Hence F splits Q into U_1 and $U_2[c^+ \cup c^-]$. In particular, F cuts off the solid torus U_1 from the 3–ball Q so that the core of the copy of F in ∂U_1 is homotopic to at least two times around the core of U_1. This is impossible. □

Lemma 3.2 *A_0 is incompressible and ∂–incompressible in X_0.*

Proof Since each of c^+ and c^- represents a nontrivial element of the free group $\pi_1(U)$, A_0 is incompressible. Suppose that A_0 is ∂–compressible. Then there exists a properly embedded disk D in U intersecting C in a single point. We may assume that D intersects c^+. Then the frontier of a neighborhood of $D \cup c^+$ in U is an essential separating disk in U that is disjoint from C, contradicting Lemma 3.1. Hence A_0 is ∂–incompressible. □

Lemma 3.3 *X_0 is irreducible and ∂–irreducible. Hence X_n is irreducible and ∂–irreducible for any integer n.*

Proof It is clear that X_0 is irreducible. If X_0 is ∂–reducible then any compressing disk for ∂X_0 can be isotoped to be disjoint from A_0. Then it lies in U as an essential disk disjoint from $c^+ \cup c^-$. This contradicts Lemma 3.1. □

Since X_n is ∂–irreducible, V_n is an irreducible handlebody-knot.

Lemma 3.4 *A_0 is a unique properly embedded nonseparating annulus in X_0 up to isotopy.*

Proof Let A be a properly embedded nonseparating annulus in X_0 that is not isotopic to A_0. The ∂–irreducibility of X_0 implies that A is incompressible and ∂–incompressible.

We may assume that A had been chosen to intersect A_0 transversely and minimally among all properly embedded nonseparating annuli in X_0. Note that A must intersect A_0, otherwise A would survive in U and be incompressible, so by Lemma 3.1 A would be parallel to either A_0^+ or A_0^- in U and hence be parallel to A_0 in X_0, contradicting the choice of A.

Suppose that there are circle components of $A \cap A_0$ that are inessential on both A and A_0. Let α be a circle component of $A \cap A_0$ that is innermost on A_0 among all such circle components. Then α bounds a disk D in A and a disk D_0 in A_0. Note that the interior of D_0 is disjoint from A, since otherwise an innermost component of $A \cap D_0$ on D_0 would bound a compressing disk for A. We now obtain a new nonseparating annulus $(A - D) \cup D_0$, which is properly embedded in X_0 and can be isotoped so as to intersect A_0 transversely with fewer components of intersection. This contradicts the choice of A. Hence each circle component of $A \cap A_0$, if it exists, is essential on at least one of A and A_0. Suppose that there are circle components of $A \cap A_0$ that are essential on one of the annuli A and A_0, and inessential on the other annulus. Let β be a circle component of $A \cap A_0$ that is innermost on (say) A among all such circle components (the argument for the case $\beta \subset A_0$ is similar). Then β bounds a disk E in A. Since no circle components of $A \cap A_0$ are inessential on both A and A_0, the interior of E misses A_0 by the choice of β. This implies that E is a compressing disk for A_0, a contradiction. We conclude that all circle components of $A \cap A_0$, if they exist, are essential on both A and A_0.

A similar argument, using an outermost arc component of intersection instead of an innermost circle component and using the ∂–incompressibility of $A \cup A_0$ instead of the incompressibility, shows that all arc components of $A \cap A_0$, if they exist, are essential on both A and A_0. Thus all the components of $A \cap A_0$ are either circles or arcs.

First, suppose that they are all circles. Take an annulus cut off from A by an outermost component of $A \cap A_0$ in A, and surger A_0 along this annulus. The resulting surface is a union of two annuli disjoint from A_0. Let A_0' be any one of these two annuli. Since one boundary circle of A_0' is isotopic to that of A_0 (or A), A_0' must be incompressible in X_0 and hence in U. By Lemma 3.1, A_0' must be ∂–parallel in U, which implies that A_0' is either ∂–parallel in X_0 or parallel to A_0. In any case, we can reduce $|A \cap A_0|$, giving a contradiction.

Now suppose that all components of $A \cap A_0$ are arcs that are essential on both A and A_0. The arcs divide A into rectangles $R_1, \dots, R_n$, where $n = |A \cap A_0|$. Consider $R = R_1$. We may regard R as a properly embedded disk in U whose boundary intersects $C = c^+ \cup c^-$ in two points. There are two cases; ∂R intersects each of c^+ and c^- in a single point, or ∂R intersects only one of c^+ and c^-, say, c^+. In the former case, each of c^+ and c^- is a primitive curve in U, that is, it is a generator of the free group $\pi_1(U)$ of rank two, but it is easy to see from Figure 8(b) that one of c^+ and c^- is not a generator.

In the latter case, the two points in $\partial R \cap c^+$ split c^+ into two arcs a_1 and a_2. Let S_i $(i = 1, 2)$ be a properly embedded surface in U obtained from R by attaching a band along a_i and then pushing the interior of the resulting surface into the interior of U. Note that S_i is disjoint from C for each $i = 1, 2$. The two ends of a_i must lie on the same side of R (then S_i is an annulus), otherwise S_i would be a Möbius band, contradicting Lemma 3.1.

If R were ∂–parallel in U then we could reduce $|A \cap A_0|$. Thus R is an essential disk in U. First, suppose that R is a nonseparating disk in U. Consider any S_i and recall that S_i is obtained from the nonseparating disk R by attaching a band. Any such annulus has boundary circles which are not mutually parallel in ∂U and at least one of which is essential in ∂U. Since the two boundary circles of S_i are not mutually parallel in ∂U, S_i is not ∂–parallel in U. Since at least one boundary circle of S_i is essential in ∂U, S_i is incompressible in U, otherwise a compression of S_i would yield an essential disk in U disjoint from C, contradicting Lemma 3.1. Hence S_i is an essential annulus. This contradicts Lemma 3.1 again.

Suppose that R is an essential separating disk in U. Then R splits U into two solid tori U_1 and U_2, where S_i can be pushed into U_i. If the core of some S_i winds U_i at least two times in the longitudinal direction, then S_i is an essential annulus in U, contradicting Lemma 3.1. Thus the core of each S_i is homotopic to the core of U_i. This implies that $c^+ = a_1 \cup a_2$ is a primitive curve in U. Since c^- does not intersect $R \cup c^+$, c^- is also a primitive curve in U. See Figure 9. This contradicts our observation that one of c^+ and c^- is not a primitive curve in U. □

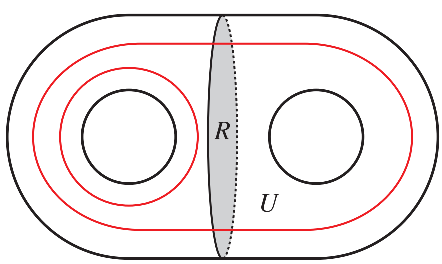


Figure 9

Lemma 3.5 V_0 *is not amphicheiral.*

Proof Assume that there exists an orientation-preserving automorphism h of S^3 that takes V_0 to V_0^* (and then X_0 to X_0^*). Take a regular neighborhood $N(A_0)$ of the nonseparating annulus A_0 in X_0. Put $A_h = h(A_0)$ and $N(A_h) = h(N(A_0))$. Then $\widetilde{V}_h = V_0^* \cup N(A_h)$ is the image of $\widetilde{V}_0 = V_0 \cup N(A_0)$ under the automorphism h. The frontier of $N(A_0)$ in X_0 consists of two annuli whose cores c^+ and c^- run along $\partial \widetilde{V}_0$ as shown in Figure 6(b), where U in the figure may be considered as the closed complement of $\widetilde{V}_0$. Each core $c^\pm$ bounds a disk $D^\pm$ in $\widetilde{V}_0$. Let $c_h^\pm = h(c^\pm)$ and $D_h^\pm = h(D^\pm)$. Then $c_h^\pm$ are the cores of the frontier annuli of $N(A_h)$ in X_0^* and they bound disks $D_h^\pm$.

Note that A_h is a properly embedded nonseparating annulus in X_0^*. By Lemma 3.4 A_0^* is a unique properly embedded nonseparating annulus in X_0^* up to isotopy. Hence A_h and A_0^* are isotopic in X_0^*.

Note that $\mathrm{cl}(\widetilde{V}_0 - N(D^\pm))$ is an embedded solid torus in S^3. The core of the solid torus is either the unknot or the right-handed trefoil according to the choice of the disks D^+ and D^-. We may assume that the core is the unknot for D^- and the right-handed trefoil for D^+. See Figure 10. Similarly, $\mathrm{cl}(\widetilde{V}_h - N(D_h^\pm))$ is a solid torus embedded in S^3 whose core is either the unknot or the left-handed trefoil. The orientation-preserving automorphism h takes $\mathrm{cl}(\widetilde{V}_0 - N(D^+))$ to $\mathrm{cl}(\widetilde{V}_h - N(D_h^+))$ or $\mathrm{cl}(\widetilde{V}_h - N(D_h^-))$. This implies that the right-handed trefoil is equivalent to the unknot or the left-handed trefoil, both of which are impossible. □

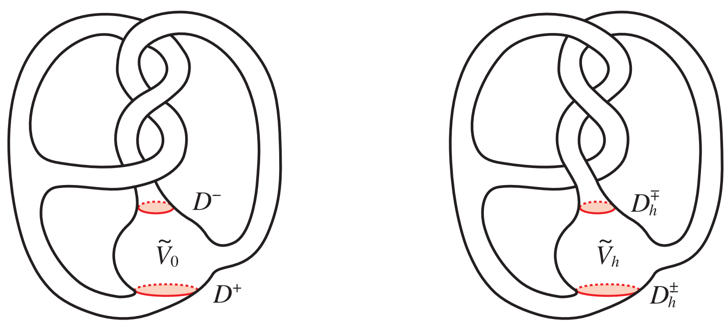


Figure 10

Recall that twisting V_0 n times along the shaded disk in Figure 11(a) defines a homeomorphism $\sigma_k\colon X_0 \to X_k$. By Lemma 3.4, $A_k = \sigma_k(A_0)$ is up to isotopy a unique nonseparating annulus in X_k. Note that $A_k \subset S^3$ is an unknotted annulus with k full twists and its boundary is the $(2,2k)$–torus link (if $k = \pm 1$, the boundary is the Hopf link). See Figure 11(b).

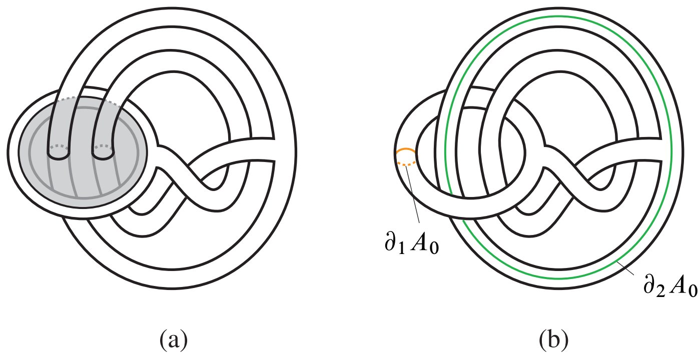


Figure 11

Let c_k, d_k be the two loop edges of Φ_k and e_k the nonloop edge. Then V_k is a union of two solid tori $V_{k,1} = N(c_k), V_{k,2} = N(d_k)$, and a 1–handle $H_k = \mathrm{cl}(N(e_k) - V_{k,1} \cup V_{k,2})$. It may be assumed that $V_{k,1}$ contains the boundary of the shaded disk in Figure 11(a). Each boundary component of A_k is not contractible in V_k if $k \neq 0$, and a cocore disk D_k for the 1–handle H_k splits V_k into two solid tori, isotopic to $V_{k,1}$ and $V_{k,2}$, each of which contains one boundary component of A_k. Let $\partial_i A_k (i = 1, 2)$ denote the boundary component of A_k lying in $V_{k,i}$. See Figure 11(b).

Lemma 3.6 *There exists an orientation-preserving automorphism of the pair* (S^3, V_{-1}) *which interchanges* $V_{-1,1}$ *and* $V_{-1,2}$.

Proof Figure 4(b) allows us to regard V_{-1} as 6_4. It is easy to see that an involution on $(S^3, 6_4)$ is defined by rotating 6_4 through π about a vertical axis. The involution is the desired automorphism. □

Proof of Theorem 1.1(1) First, assume that V_n is amphicheiral for some nonzero integer n (V_0 is not amphicheiral by Lemma 3.5), that is, there is an orientation-preserving homeomorphism of pairs $(S^3, V_n) \to (S^3, V_n^*)$. Note that A_n and A_n^* are up to isotopy unique nonseparating annuli in X_n and X_n^*, respectively. Hence composing with an orientation-preserving automorphism of the pair (S^3, V_n^*), if necessary, we may assume that the homeomorphism takes A_n to A_n^*. In other words, A_n and A_n^* are isotopic in S^3. However, one of the annuli A_n and A_n^* has right-handed $|n|$ full twists and the other left-handed $|n|$ full twists, so they cannot be isotopic. This gives a contradiction. Therefore V_n is not equivalent to its mirror image for any integer n.

Let n, m be distinct integers, and assume that there is a homeomorphism of pairs $h\colon (S^3, V_n) \to (S^3, V_m)$, where h may or may not preserve the orientation of S^3.

Similarly as above, we may assume that $h(A_n) = A_m$. Then $h(\partial A_n) = \partial A_m$, which means that h takes a $(2, 2n)$–torus link to a $(2, 2m)$–torus link. Hence $m = n$ or $m = -n$. The former contradicts the assumption that n and m are distinct. If $n = 0$ then h must preserve the orientation of S^3 by Lemma 3.5, so h is isotopic to the identity of S^3 and we have nothing to prove. Hence we may assume that $m = -n$ and $n \neq 0$. Since the twists of A_n and A_{-n} are reversed, h must be orientation-reversing.

By Lemma 2.3 $D_{\pm n}$, a cocore disk of the 1–handle $H_{\pm n}$ in $V_{\pm n}$, is up to isotopy a unique essential separating disk in $V_{\pm n}$ which separates the two boundary components of $A_{\pm n}$, so it may be assumed up to isotopy of V_{-n} that $h(D_n) = D_{-n}$ and moreover $h(H_n) = H_{-n}$. This implies that h takes each solid torus $V_{n,i} (i = 1, 2)$ to one of the two solid tori $V_{-n,1}$ and $V_{-n,2}$. Note that $\partial_1 A_{\pm n}$ is homotopic to $\pm n$ times the core of $V_{\pm n,1}$, while $\partial_2 A_{\pm n}$ is homotopic to the core of $V_{\pm n,2}$. Hence when $|n| \geq 2$, $h(\partial_i A_n) = \partial_i A_{-n}$ for each $i = 1, 2$, which implies $h(V_{n,i}) = V_{-n,i}$. When $|n| = 1$, by composing h with an orientation-preserving automorphism of the pair (S^3, V_{-1}) given in Lemma 3.6 we may assume that $h(V_{n,i}) = V_{-n,i}$ for each $i = 1, 2$. In particular, we may always assume that c_n, the core of $V_{n,1}$, is mapped by h onto c_{-n}, the core of $V_{-n,1}$. Consider the composition

$$(S^3, V_n) \xrightarrow{h} (S^3, V_{-n}) \xrightarrow{r} (S^3, V^*_{-n}),$$

where r is a reflection. See Figure 12. Let f be the restriction of the composition $r \circ h$ onto the pair $(S^3 - V_{n,1}, V_n - V_{n,1})$. Then $f\colon (S^3 - V_{n,1}, V_n - V_{n,1}) \to (S^3 - V^*_{-n,1}, V^*_{-n} - V^*_{-n,1})$ is an orientation-preserving homeomorphism of pairs.

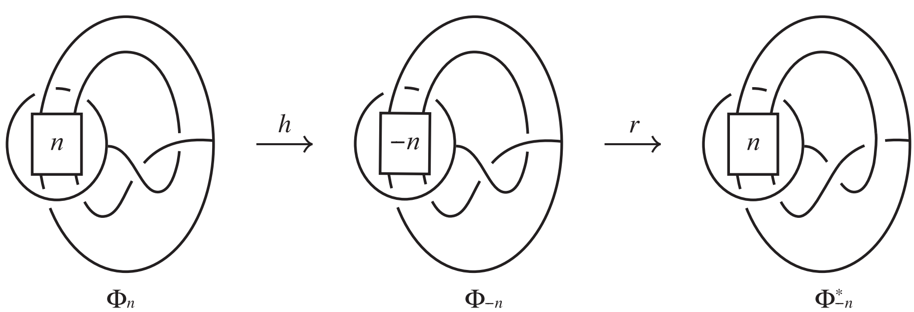


Figure 12

Note that (S^3, V_n) is obtained from (S^3, V_0) by $1/n$–surgery on c_0. Also, (S^3, V^*_{-n}) is obtained from (S^3, V^*_0) by $1/n$–surgery on c^*_0. These two surgeries define two orientation-preserving homeomorphisms of pairs as follows:

$$(S^3 - V_{0,1}, V_0 - V_{0,1}) \xrightarrow{g} (S^3 - V_{n,1}, V_n - V_{n,1}),$$
$$(S^3 - V^*_{0,1}, V^*_0 - V^*_{0,1}) \xrightarrow{g^*} (S^3 - V^*_{-n,1}, V^*_{-n} - V^*_{-n,1}).$$

For example, twisting n times along the shaded disk in Figure 11(a) defines g. The composition $(g^*)^{-1} \circ f \circ g$ is an orientation-preserving homeomorphism from $(S^3 - V_{0,1}, V_0 - V_{0,1})$ to $(S^3 - V^*_{0,1}, V^*_0 - V^*_{0,1})$. Note that the composition takes a meridian of c_0 to a meridian of c^*_0. Hence $(g^*)^{-1} \circ f \circ g$ extends to an orientation-preserving homeomorphism of pairs from (S^3, V_0) to (S^3, V^*_0). This contradicts Lemma 3.5. □

4 W_n and W_m $(n \neq m)$ are not equivalent

Consider Ψ_0. An isotopy of S^3 gives the pictures in Figure 13, showing that there exists a nonseparating annulus A_0 in Y_0. Cutting Y_0 along A_0 gives a genus two handlebody U. Let $A^{\pm}_0$ be the two copies of A_0 in ∂U and $c^{\pm}$ the cores of $A^{\pm}_0$. See Figure 14(a) for $c^{\pm}$, where U is the outside of the standardly embedded genus two surface and Y_0 can be recovered by gluing the annulus neighborhoods $A^{\pm}_0$ of $c^{\pm}$ in the manner indicated in the figure. An external view of $(U, c^{\pm})$ is illustrated in Figure 14(b), that is, U is the inside of the standardly embedded genus two surface in the figure.

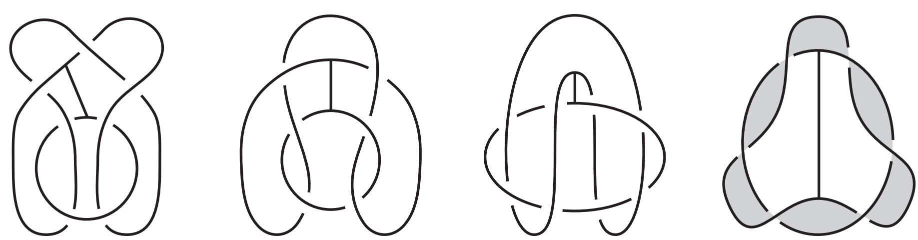


Figure 13

Lemma 4.1 *U does not contain an essential disk or a properly embedded nonseparating annulus disjoint from $c^+ \cup c^-$.*

Proof First, note that both $c^{\pm}$ are primitive curves in U, so $U[c^{\pm}]$ are solid tori. Also, it is easy to see that the fundamental group of $U[c^+ \cup c^-]$ is cyclic with order 3.

Assume that there exists an essential disk D in U disjoint from $c^+ \cup c^-$. If D is a nonseparating disk in U then it is also nonseparating in $U[c^+ \cup c^-]$ and hence

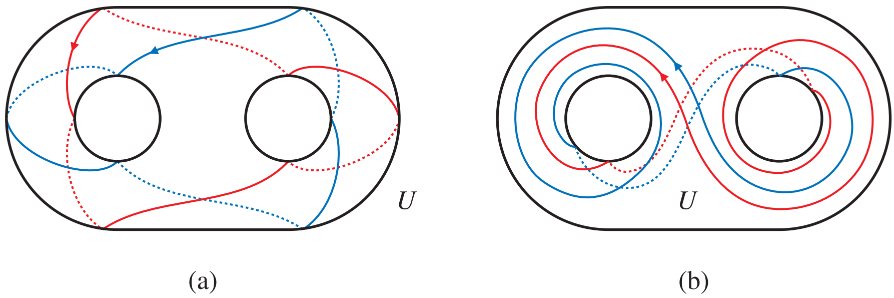


Figure 14

the fundamental group of $U[c^+ \cup c^-]$ contains an element of infinite order, contradicting the observation above. Hence D separates U into two solid tori U^+ and U^-. Since U does not contain a nonseparating disk disjoint from $c^+ \cup c^-$, both U^+ and U^- intersect $c^+ \cup c^-$ and hence we may assume that $c^\pm \subset U^\pm$. Then $\mathbb{Z}_3 \cong \pi_1(U[c^+ \cup c^-]) \cong \pi_1(U^+[c^+]) * \pi_1(U^-[c^-])$, so either $\pi_1(U^+[c^+]) \cong \mathbb{Z}_3$, $\pi_1(U^-[c^-]) = 1$ or $\pi_1(U^+[c^+]) = 1$, $\pi_1(U^-[c^-]) \cong \mathbb{Z}_3$. In the first case, since $U[c^+]$ is the union of $U^+[c^+]$ and U^- along the disk D, its fundamental group is $\pi_1(U[c^+]) \cong \pi_1(U^+[c^+]) * \pi_1(U^-) \cong \mathbb{Z}_3 * \mathbb{Z}$. This contradicts our observation that $U[c^+]$ is a solid torus. In the latter case, we get a contradiction in a similar way. Therefore we conclude that U does not contain an essential disk disjoint from $c^+ \cup c^-$.

Assume that there exists a properly embedded nonseparating annulus A in U which is disjoint from $c^+ \cup c^-$. Since A is disjoint from $c^+ \cup c^-$, A survives in $U[c^+ \cup c^-]$ as a properly embedded nonseparating annulus. Capping off the boundary sphere of $U[c^+ \cup c^-]$ with a 3–ball, we get a 3–manifold without boundary, in which A extends to a nonseparating sphere. But the fundamental group of the 3–manifold is the cyclic group of order 3 and hence the 3–manifold cannot contain a nonseparating sphere, a contradiction. □

Lemma 4.2 *Let $D_0 \subset U$ be the disk illustrated in Figure 15. Then up to isotopy D_0 is a unique properly embedded disk in U which is commonly dual to c^+ and c^-.*

Proof Let D be a common dual disk of c^+ and c^- that is not isotopic to D_0. We may assume that D intersects D_0 transversely and the intersection $D \cap D_0$ is minimal among all such disks. If D were disjoint from D_0, then by Lemma 2.1 $\pi_1(U[c^+ \cup c^-]) \cong \mathbb{Z}$ or $\mathbb{Z}_2$, contradicting the fact that $\pi_1(U[c^+ \cup c^-]) \cong \mathbb{Z}_3$.

By the minimality of $|D \cap D_0|$, the intersection $D \cap D_0$ has no circle components. An outermost arc of intersection in D_0 cuts off a subdisk from D_0 which intersects $c^+ \cup c^-$ in at most one point. Surgery on D along the subdisk produces two disks D_1, D_2.

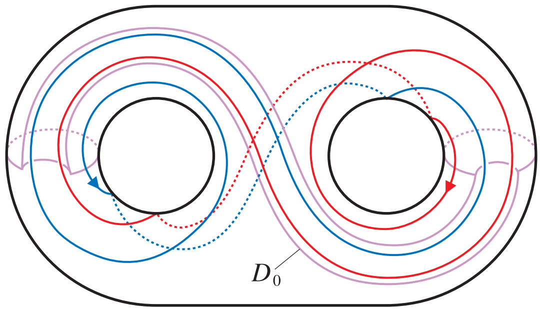


Figure 15

One of these disks, say, D_1 intersects $c^+ \cup c^-$ in at most two points. Note that D_1 is essential in U, otherwise $|D \cap D_0|$ could be reduced. By Lemma 4.1 D_1 cannot be disjoint from $c^+ \cup c^-$. If D_1 had exactly one point of intersection with $c^+ \cup c^-$ then there would exist an essential (separating) disk in U disjoint from $c^+ \cup c^-$, contradicting Lemma 4.1. Hence D_1 intersects $c^+ \cup c^-$ in two points, and so does the other disk D_2. One of the two disks D_1 and D_2 is a common dual disk of c^+ and c^-, and the other intersects one of c^+ and c^- in two points. The former disk contradicts the minimality of $|D \cap D_0|$. □

Lemma 4.3 *A_0 is incompressible and ∂–incompressible in Y_0.*

Proof One sees from Figure 14(b) that both $c^\pm$ are primitive curves in U, so A_0 is incompressible. Suppose that A_0 is ∂–compressible. Let D be a ∂–compressing disk for A_0. Then D is an essential disk in U which intersects $c^+ \cup c^-$ in a single point. We may assume that D intersects c^+ but not c^-. Then c^+ becomes a longitudinal curve of the solid torus $U[c^-]$, since D, a meridian disk of $U[c^-]$, intersects c^+ in a single point. This implies that $U[c^+ \cup c^-]$ is a 3–ball. But in the proof of Lemma 4.1 we already observed that the fundamental group of $U[c^+ \cup c^-]$ is the cyclic group of order 3. □

Lemma 4.4 *Y_0 is irreducible and ∂–irreducible. Hence Y_n is irreducible and ∂–irreducible for any integer n.*

Proof The same argument as in the proof of Lemma 3.3 applies here by using Lemma 4.1 instead of Lemma 3.1. □

Since Y_n is ∂–irreducible, W_n is an irreducible handlebody-knot.

Lemma 4.5 *A_0 is a unique properly embedded nonseparating annulus in Y_0 up to isotopy.*

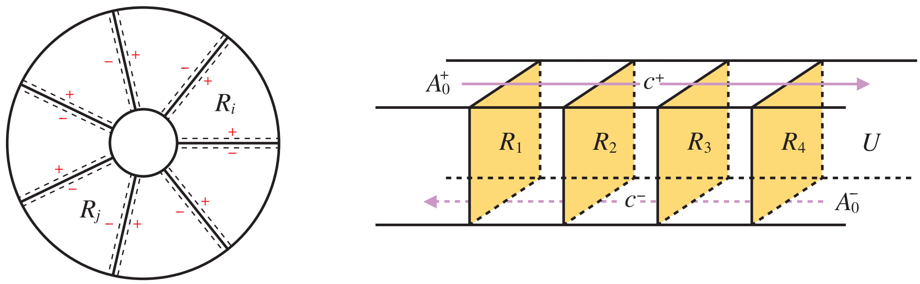
Proof Let A be a properly embedded nonseparating annulus in Y_0 which is not isotopic to A_0. The ∂–irreducibility of Y_0 implies that A is incompressible and ∂–incompressible.

The intersection $A \cap A_0$ may be assumed to be transverse and minimal up to isotopy. Suppose that the intersection is empty. Then A lies in U and is disjoint from $c^+ \cup c^-$. Also, A is incompressible and not ∂–parallel in U, since otherwise A would be compressible in Y_0 or parallel to A_0 or an annulus in ∂Y_0. By Lemma 4.1 A is separating in U. Since A is nonseparating in Y_0, A must separate c^+ and c^-. It follows from Lemma 2.2 that one of c^+ and c^- represents a proper power of a primitive element of $\pi_1(U)$, contradicting the fact that both $c^\pm$ are primitive curves in U. Hence $A \cap A_0$ is not empty.

The same argument as in the third and fourth paragraphs in the proof of Lemma 3.4 applies to show that all the components of $A \cap A_0$ are essential on both A and A_0 and that they are all either circles or arcs. First, suppose that they are all circles. Then surgery on A_0 along an annulus cut off from A by an outermost component of $A \cap A_0$ in A yields two properly embedded annuli A_1, A_2 in Y_0 which are disjoint from A_0. Each annulus $A_i (i = 1, 2)$ is not isotopic to A_0 by the minimality assumption on $|A \cap A_0|$. Since we already observed that any nonseparating annulus in Y_0 which is not isotopic to A_0 cannot be disjoint from A_0, each A_i is separating in Y_0. This implies that A_0 is separating in Y_0, a contradiction.

Now suppose all the components of $A \cap A_0$ are arcs that are essential on both A and A_0. Then the arcs cut A into rectangles $R_1, \dots, R_n$. Each rectangle R_i can be considered as a properly embedded disk in U, which is essential by the minimality of $A \cap A_0$. Also, each ∂R_i intersects $c^+ \cup c^-$ in two points. There are two possibilities for the intersection of each ∂R_i with $c^+ \cup c^-$; for each i, either ∂R_i intersects each of c^+ and c^- in a single point or ∂R_i intersects one of c^+ and c^- in two points and misses the other.

Suppose that some R_i intersects one of the cores c^+ and c^- in two points. Note that each arc of $A \cap A_0$ has two copies in ∂U, one in A_0^+ and the other in A_0^-. This implies that some $R_j (j \neq i)$ intersects the other core in two points. See Figure 16(a). We may assume that R_i has two points of intersection with c^+ (and then R_j has two points of intersection with c^-). Then R_i is disjoint from c^-, implying that R_i is a properly embedded disk in the solid torus $U[c^-]$. Also, c^+ is a simple loop in $\partial U[c^-]$ intersecting R_i in two points. Since a 2–handle addition on $U[c^-]$ along c^+ results in the 3–manifold $U[c^+ \cup c^-]$ with $\pi_1(U[c^+ \cup c^-]) \cong \mathbb{Z}_3$, R_i must be ∂–parallel in $U[c^-]$. This implies that R_i is separating in U. Similarly, R_j is separating in U. Since any two disjoint separating essential disks in a genus two handlebody are parallel, R_i and R_j are parallel in U. Since R_j is disjoint from c^+, R_i can be isotoped to be disjoint from c^+ (and still from c^-). This contradicts Lemma 4.1.



(a) (b)

Figure 16

Hence each ∂R_i intersects each c^+ and c^- in a single point, that is, each R_i is commonly dual to c^+ and c^-. By Lemma 4.2 all the rectangles $R_1, \dots, R_n$ are isotopic to the disk D_0 in Figure 15 and hence they are mutually parallel in U. Let $a_i^\pm = R_i \cap A_0^\pm$ for $i = 1, \dots, n$. We may assume that $R_1, \dots, R_n$ had been labeled so that $a_1^+, \dots, a_n^+$ appear in A_0^+ successively along the orientation of c^+. Then $a_1^-, \dots, a_n^-$ appear in A_0^- successively along the reversed orientation of c^-, since the algebraic intersection number of ∂D_0 with the two oriented loops $c^+ \cup c^-$ is zero. See Figure 16(b). In Y_0, the arcs $a_1^+, \dots, a_n^+$ and the arcs $a_1^-, \dots, a_n^-$ are identified in pair to form A. The identification defines a permutation σ of $\{1, \dots, n\}$ such that a_i^+ is identified with $a_{\sigma(i)}^-$. In fact, $\sigma(i) \equiv -i + k \mod n$ for some integer k.

Suppose that n is odd. By replacing k with $k+n$, if necessary, we may assume that k is even. Then $\sigma(k/2) \equiv -k/2 + k \equiv k/2 \bmod n$. This implies $n = 1$, otherwise we would obtain a disconnected surface from the rectangles $R_1, \dots, R_n$ by identifying a_i^+ and $a_{\sigma(i)}^-$ $(i = 1, \dots, n)$. Even if $n = 1$, the identification produces a Möbius band because the two oriented loops c^+ and c^- intersect oppositely with ∂R_1. This gives a contradiction.

Suppose that n is even. The complementary regions of $R_1 \cup \cdots \cup R_n$ in U can be alternately colored black and white. If $\sigma(i) \equiv -i + k \mod n$ for some odd integer k then black regions match with black regions and white regions match with white regions, implying that A is separating in Y_0. Hence k is even. Then $\sigma(k/2) \equiv k/2 \bmod n$, and two opposite sides a_k^+ and a_k^- of R_k are identified to form a Möbius band. This is also impossible. □

Proof of Theorem 1.1(2) Let $\partial_1 A_0$ and $\partial_2 A_0$ denote the two boundary components of A_0 as shown in Figure 17. After an isotopy, the two loops appear in ∂Y_0 as shown in the last drawing in the figure.

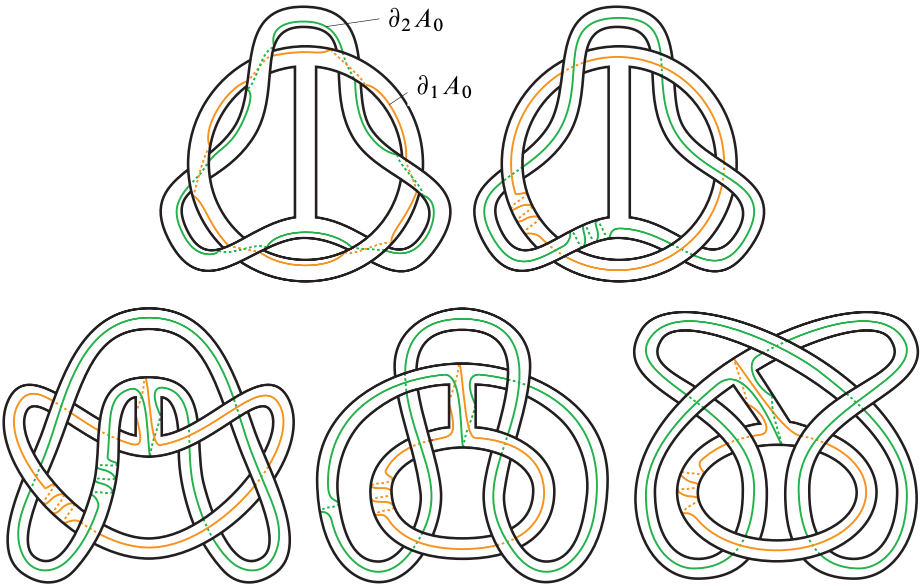


Figure 17

Recall that twisting W_0 n times along the shaded disk in Figure 18 defines a homeomorphism $\sigma_n\colon Y_0 \to Y_n$. By Lemma 4.5, $A_n = \sigma_n(A_0)$ is a unique properly embedded nonseparating annulus in Y_n up to isotopy. Let $\partial_i A_n = \sigma_n(\partial_i A_0)$ for $i = 1, 2$. The core of A_n is an embedded circle in S^3, isotopic to any boundary component of A_n in S^3 along a half of A_n. One easily sees that $\partial_1 A_n$ is a $(3, 3n-1)$–torus knot, and so is the core.

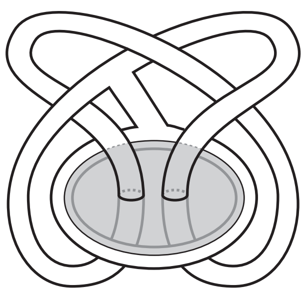


Figure 18

Assume that W_n is amphicheiral. Then there is an orientation-preserving homeomorphism of pairs $(S^3, W_n) \to (S^3, W_n^*)$. Since A_n and A_n^* are respectively up to isotopy unique nonseparating annuli in Y_n and Y_n^* by Lemma 4.5, composing with an orientation-preserving automorphism of the pair (S^3, W_n^*), if necessary, we may assume that the homeomorphism takes A_n to A_n^*. This implies that A_n and A_n^* are isotopic in S^3. In particular, their cores are isotopic. The core of A_n is a $(3, 3n{-}1)$–torus knot, while that of A_n^* is the mirror image of a $(3, 3n{-}1)$–torus knot. It is well known that every nontrivial torus knot is not amphicheiral. If $n \neq 0$ then a $(3, 3n{-}1)$–torus knot is not the trivial knot, so it is not amphicheiral. Hence $n = 0$. However, ∂A_0 is a $(2, -6)$–torus link (see the first drawing in Figure 17), while ∂A_0^* is the mirror image of a $(2, -6)$–torus link. The two torus links are not isotopic, a contradiction. Hence W_n is not amphicheiral for any integer n.

Let n and m be distinct integers. Then neither of the $(3, 3n{-}1)$–torus knot and its mirror image is isotopic to the $(3, 3m{-}1)$–torus knot. Hence a similar argument as above shows that neither of W_n and W_n^* is equivalent to W_m. □

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References

[1] **C M Gordon**, **J Luecke**, *Knots are determined by their complements*, J. Amer. Math. Soc. 2 (1989) 371–415 MR965210

[2] **A Ishii**, *Moves and invariants for knotted handlebodies*, Algebr. Geom. Topol. 8 (2008) 1403–1418 MR2443248

[3] **A Ishii**, **K Kishimoto**, **H Moriuchi**, **M Suzuki**, *A table of genus two handlebody-knots up to six crossings*, J. Knot Theory Ramifications 21 (2012) Art. ID 1250035, 9pp

[4] **T Kobayashi**, *Structures of the Haken manifolds with Heegaard splittings of genus two*, Osaka J. Math. 21 (1984) 437–455 MR752472

[5] **M Motto**, *Inequivalent genus 2 handlebodies in S^3 with homeomorphic complement*, Topology Appl. 36 (1990) 283–290 MR1070707

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