

Normalizers of parabolic subgroups of Coxeter groups

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We improve a bound of Borchers on the virtual cohomological dimension of the nonreflection part of the normalizer of a parabolic subgroup of a Coxeter group. Our bound is in terms of the types of the components of the corresponding Coxeter subdiagram rather than the number of nodes. A consequence is an extension of Brink’s result that the nonreflection part of a reflection centralizer is free. Namely, the nonreflection part of the normalizer of parabolic subgroup of type D_5 or $A_{m \text{ odd}}$ is either free or has a free subgroup of index 2.

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Suppose Π is a Coxeter diagram, J is a subdiagram and $W_J \subseteq W_\Pi$ is the corresponding inclusion of Coxeter groups. The normalizer $N_{W_\Pi}(W_J)$ has been described in detail by Borchers [2] and Brink and Howlett [4]. Such normalizers have significant applications to working out the automorphism groups of Lorentzian lattices and K3 surfaces; see [2] and its references. $N_{W_\Pi}(W_J)$ falls into 3 pieces: W_J itself, another Coxeter group W_Ω and a group Γ_Ω of diagram automorphisms of W_Ω . The last two groups are called the “reflection” and “nonreflection” parts of the normalizer. Borchers bounded the virtual cohomological dimension of Γ_Ω by $|J|$. Our Theorems 1, 3 and 4 give stronger bounds, in terms of the types of the components of J rather than the number of nodes. There are choices involved in the definition of W_Ω and Γ_Ω , and our bound in Theorem 3 applies regardless of how these choices are made (Theorem 1 is a special case). Theorem 4 improves this bound when W_Ω is “maximal”. In this case, when $J = D_5$ or $A_{m \text{ odd}}$, Γ_Ω turns out to either be free or have an index 2 subgroup that is free. This extends Brink’s result [3] that Γ_Ω is free when $J = A_1$.

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We follow the notation of Borchers [2] and refer to Humphreys [5] for general information about Coxeter groups. Suppose (W_Π, Π) is a Coxeter system, which is to say that W_Π is a Coxeter group and Π is a standard set of generators. The Coxeter diagram is the graph whose nodes are Π , with an edge between $s_i, s_j \in \Pi$ labeled by the order m_{ij} of $s_i s_j$, when $m_{ij} > 2$. W_Π acts isometrically on a real inner product space V_Π with basis (the simple roots) Π and inner products defined in terms of the m_{ij} . The (open) Tits cone K is an open convex subset of V_Π^* on which

W_Π acts properly discontinuously with fundamental chamber C_Π . (Our C_Π and K are “missing” the faces corresponding to infinite parabolic subgroups of W_Π .) The standard generators act on V_Π^* by reflections across the hyperplanes containing the facets of C_Π , and they also act on V_Π by reflections. For a root α (ie, a W_Π -image of a simple root) we write α^\perp for α 's mirror, meaning the fixed-point set in K of the reflection associated to α .

Now let $J \subseteq \Pi$ be a spherical subdiagram, ie, one corresponding to a finite subgroup of W_Π , and let W_{\min} be the group generated by the reflections in W_Π that act trivially on $V_J \subseteq V_\Pi$. This is the “reflection” part of $N_{W_\Pi}(W_J)$, or rather the strictest possible interpretation of this idea. It corresponds to Borchers’ W_Ω in the case that the groups he calls Γ_Π and Γ_J are trivial; see the discussion after Lemma 2. Let $J^\perp := \bigcap_{\alpha \in J} \alpha^\perp$, pick a component C_{\min}° of the complement of W_{\min} 's mirrors in J^\perp , and define C_{\min} as its closure (in J^\perp). By definition, W_{\min} is a Coxeter group, and the general theory of these groups shows that C_{\min} is a chamber for it. The “nonreflection” part of $N_{W_\Pi}(W_J)$ means the subgroup Γ_{\min} of W_Π preserving J (regarded as a set of roots) and sending C_{\min} to itself. The reason for the first condition is to discard the trivial part of $N_{W_\Pi}(W_J)$, namely W_J itself. That is, $W_{\min}:\Gamma_{\min}$ is a complement to W_J in $N_{W_\Pi}(W_J)$. We write Γ_{\min}^\vee for the subgroup of Γ_{\min} acting trivially on J (equivalently, on V_J). The reason for passing to this (finite-index) subgroup is that Γ_{\min} often contains torsion and therefore has infinite cohomological dimension for boring reasons.

Theorem 1 Γ_{\min}^\vee acts freely on a contractible cell complex of dimension at most

$$(1) \quad \#A_1 + \#D_{m>4} + \#E_6 + \#I_2(5) + 2(\#A_{m>1} + \#D_4),$$

where $\#X_m$ means the number of components of J isomorphic to a given Coxeter diagram X_m . In particular, the cohomological dimension of Γ_{\min}^\vee is at most (1).

Borchers’ result [2, Theorem 4.1] has $|J|$ in place of (1), but treats a more general group Γ_Ω , of which Γ_{\min} is a special case. The more general case follows from this one, in Theorem 3 below.

Proof First we prove for $x \in C_{\min}^\circ$ that its stabilizer $\Gamma_{\min,x}^\vee$ is trivial. The W_Π -stabilizer of x is some W_Π -conjugate W_x of a spherical parabolic subgroup of W_Π . So W_x acts on V_Π as a finite Coxeter group. It is well-known that any vector stabilizer in such an action is generated by reflections, so the subgroup $W_{x,J}$ fixing J pointwise is generated by reflections. Observe that any reflection in $W_{x,J}$ lies in W_{\min} . Since x lies in the interior C_{\min}° of C_{\min} , it is fixed by no reflection in W_{\min} , so there can be no

reflection in $W_{x,J}$, so $W_{x,J} = 1$. It is easy to see that $W_{x,J}$ contains $\Gamma_{\min,x}^\vee$, so we have proven that Γ_{\min}^\vee acts freely on C_{\min}° .

The component C_{\min}° is contractible because it is convex, and it obviously admits an equivariant deformation-retraction to its dual complex. So it suffices to show that the dual complex has dimension at most (1). Suppose $\phi \subseteq J^\perp$ is a face of a chamber of W_Π , with codimension in J^\perp larger than (1); we must show $\phi \cap C_{\min}^\circ = \emptyset$. For some $w \in W_\Pi$, $w\phi$ is a face of C_Π whose corresponding set of simple roots $I' \subseteq \Pi$ contains $J' := w(J) \cong J$. By the codimension hypothesis on ϕ , $|I'| - |J'|$ is more than (1). Applying the lemma below to J' and I' , we see that $W_{I'}$ contains a reflection r fixing J' pointwise. Since $r \in W_{I'}$, its mirror contains $w\phi$. So $w^{-1}rw$ is a reflection fixing J pointwise (so it lies in W_{\min}), whose mirror contains ϕ . Since C_{\min}° is a component of the complement of the mirrors of W_{\min} , it is disjoint from ϕ , as desired. \square

Lemma 2 *If J lies in a spherical Coxeter diagram $I \subseteq \Pi$ whose cardinality exceeds that of J by more than (1), then W_I contains a reflection fixing J pointwise.*

Remark Equality in (1) holds when I extends the A_m, D_m, E_6 and $I_2(5)$ components of J by $A_1 \rightarrow A_2, A_{m>1} \rightarrow D_{m+2}, D_4 \rightarrow E_6, D_{m>4} \rightarrow D_{m+1}, E_6 \rightarrow E_7$ and $I_2(5) \rightarrow H_3$. One can check in these cases that the conclusion of the lemma fails.

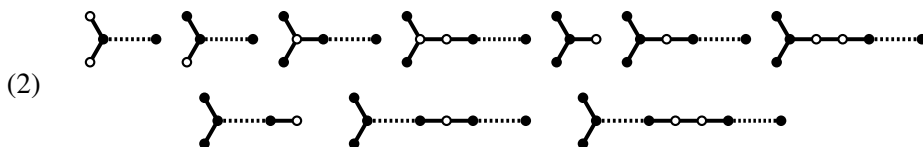
Proof We may suppose $I = \Pi$, by discarding the rest of Π . Working one component at a time, it suffices to prove the lemma under the additional hypothesis that Π is connected. We now consider the various possibilities for Π , and suppose W_Π contains no reflections fixing V_J pointwise. That is, we assume $W_{\min} = 1$. In each case we will show that $|\Pi| - |J|$ is at most (1).

The $\Pi = A_n$ case is a model for the rest. Suppose the component of J nearest one end of Π has type A_m and does not contain that end. Then it must be adjacent to that end (since $W_{\min} = 1$), so together with the end it forms an A_{m+1} . We conjugate by the long word in $W(A_{m+1})$, which exchanges the two A_m diagrams in A_{m+1} and fixes the roots in the other components of J . The result is that we may suppose without loss that J contains that end of Π . Repeating the argument to move the other components of J toward that end, we may suppose that there is exactly one node of Π between any two consecutive components of J . And the other end of Π is either in J or adjacent to it. It is now clear that $|\Pi| - |J|$ is the number of components of J , or one less than this. Since every component of J has type A , $|\Pi| - |J|$ is at most (1). This finishes the proof in the $\Pi = A_n$ case.

If $\Pi = B_n = C_n$ then we begin by shifting any type A components of J as far as possible from the double bond. If J has no B_m then J contains one end of the double

bond, and we get $|\Pi| - |J|$ equal to the number of components of J , all of which have type A . If J has a B_m then the node after it (if there is one) must be adjacent to some type A component of J . This is because $W(B_{m+1})$ contains a reflection acting trivially on V_{B_m} . This is easy to see in the model of $W(B_{m+1})$ as the isometry group of \mathbb{Z}^{m+1} . It follows that $|\Pi| - |J|$ is the number of components of J of type A .

In the $\Pi = D_{n>3}$ case, one can use the shifting trick to reduce to one of the cases



where the filled nodes are those in J and the dashes indicate a chain of nodes with no two consecutive unfilled nodes. (Except for the dashes on the left in the last 3 diagrams, which indicate chains of filled nodes.) In every case we get

$$|\Pi| - |J| \leq \#A_1 + \#D_{m \geq 4} + 2\#A_{m > 1}.$$

The most interesting case is $A_{n-2} \rightarrow D_n$, at the top left.

We will treat the case $\Pi = E_8$ and leave the similar E_6 and E_7 cases to the reader. If J has a D_4 , D_5 or E_6 component, then it must also have a type A component, and then $|\Pi| - |J| \leq 2\#D_4 + \#D_5 + \#A_{m \geq 1}$, as desired. J cannot be D_6 or E_7 , because then W_{\min} would contain the reflection in the lowest root of E_8 , which extends E_8 to the affine diagram \tilde{E}_8 . So we may suppose J 's components have type A . In order for $|\Pi| - |J|$ to exceed (1), we must have $J = A_{m \leq 5}, A_3A_1, A_2A_1$ or $A_1^{m \leq 3}$. But none of these cases can occur, because in each of them we may shift J 's components around so that some node of Π is not joined to J .

The remaining cases are $\Pi = F_4, H_3, H_4$ and I_2 , the last case including $G_2 = I_2(6)$. The facts required to treat these cases are that if $J = B_2$ or B_3 in $\Pi = F_4$ then W_{\min} contains a reflection, and similarly in the $J = H_3 \subseteq H_4 = \Pi$ case. The first fact is visible inside a B_3 or B_4 root system inside F_4 . To see the second, observe that the root stabilizer in H_4 contains Coxeter groups of types A_2 and $I_2(5)$, visible in the centralizers of the two end reflections of H_4 (which are conjugate). So the root stabilizer can only be $W(H_3)$, which is to say that the H_3 root system is orthogonal to a root. □

The greater generality obtained by Borchers is the following. Let Γ_Π be a group of diagram automorphisms of Π , acting on V_Π and K in the obvious way. The goal is to understand $N_{W_\Pi: \Gamma_\Pi}(W_J)$. Again we discard the boring part of this normalizer by passing to the subgroup W'_J preserving the set of roots $J \subseteq \Pi$. Let W_Ω be any

subgroup of W'_J which contains W_{\min} and is generated by elements which act on J^\perp by reflections. We define C°_Ω , C_Ω and Γ_Ω as for C°_{\min} , C_{\min} and Γ_{\min} , and define Γ^\vee_Ω as the subgroup of $\Gamma_\Omega \cap W_\Pi$ acting trivially on J . (Borcherds left Γ^\vee_Ω unnamed and defined W_Ω in terms of auxiliary groups $R \trianglelefteq \Gamma_J \subseteq \text{Aut } J$; his W_Ω has the properties assumed here.) The inclusion $W_{\min} \subseteq W_\Omega$ is the source of the subscript “min”, but note that C_{\min} and Γ_{\min} are larger than C_Ω and Γ_Ω . We can now recover Borcherds’ result [2, Theorem 4.1] with our (1) in place of $|J|$.

Theorem 3 *Theorem 1 holds with Γ_{\min}^\vee replaced by Γ^\vee_Ω .*

Proof The freeness of the action follows from the same argument. (This is why Γ^\vee_Ω is defined as a subgroup of $\Gamma_\Omega \cap W_\Pi$ rather than just Γ_Ω .) The essential point for the rest of the proof is that W_Ω contains W_{\min} , so the decomposition of J^\perp into chambers of W_Ω refines that of W_{\min} . This shows $C^\circ_\Omega \subseteq C^\circ_{\min}$. So the dual complex of C°_Ω has dimension at most that of C°_{\min} , and we can apply Theorem 1. □

The point of considering W_Ω rather than W_{\min} is that it is larger and so Γ_Ω will be smaller than Γ_{\min} . This is good since the nonreflection part is more mysterious than the reflection part. So it is natural to define W_{\max} by setting $\Gamma_\Pi = 1$ and taking W_Ω as large as possible, ie, W_{\max} is the subgroup of W'_J generated by the transformations which act on J^\perp by reflections.

This is the largest possible “universal” W_Ω , although a larger W_Ω is possible if Π admits suitable diagram automorphisms. For example, Γ_Π might contain elements acting on C_Π by reflections. I don’t know other examples, although probably there are some.

We define C°_{\max} , C_{\max} , Γ_{\max} and Γ^\vee_{\max} as above. The next theorem follows from Lemma 5 in exactly the same way that Theorem 1 follows from Lemma 2.

Theorem 4 *The dimension of the dual complex of C°_{\max} , hence the cohomological dimension of Γ^\vee_{\max} , is bounded above by*

$$(3) \qquad \#D_5 + \#A_{m \text{ odd}} + 2\#A_{m \text{ even}}. \qquad \square$$

Remarks (i) If J has no A_m or D_5 component then $\Gamma^\vee_{\max} = 1$ and Γ_{\max} is finite. This is Borcherds’ [2, Example 5.6].

(ii) If $J = D_5$ or $A_{m \text{ odd}}$ then $\Gamma^\vee_{\max} \subseteq N_{W_\Pi}(W_J)$ is free. Also, since $|\text{Aut } J| \leq 2$, Γ^\vee_{\max} has index 1 or 2 in Γ_{\max} . Therefore the nonreflection part Γ_{\max} of $N_{W_\Pi}(W_J)$ has a free subgroup of index 1 or 2.

(iii) If $J = A_1$ then $\Gamma_{\min} = \Gamma_{\min}^{\vee} = \Gamma_{\max} = \Gamma_{\max}^{\vee}$ has cohomological dimension ≤ 1 . This recovers Brink's result [3] that Γ_{\min} is free.

(iv) If $J = A_{m \text{ even}}$ then the conclusion $\dim(\text{dual of } C_{\min}^{\circ}) \leq 2$ suggests that Γ_{\max} is often comprehensible, like the $J = A_6$ example of [2, Example 5.4].

Lemma 5 *If J lies in a spherical Coxeter diagram $I \subseteq \Pi$, whose cardinality exceeds that of J by more than (3), then W_I contains an element preserving the set J of roots and acting on J^{\perp} by a reflection.*

Proof This is essentially the same as for Lemma 2, using the following additional ingredients. For example, when $I = D_n$ one can use them to show that the fifth, seventh, eighth and tenth cases of (2) are impossible, while the first can only occur when n is even.

First, if $J = E_6 \subseteq E_7 = I$ then W_I contains the negation of V_I , which we follow by the long word in W_J to send $-J$ back to J . The composition is the claimed element of W_I . The same argument applies if $J = I_2(5) \subseteq H_3 = I$.

Second, if $J = A_{m \text{ odd}} \subseteq D_{m+2} = I$ as in the first diagram of (2), then consider the long word in W_I . It negates J and exchanges and negates the two simple roots in $I - J$. Following this by the long word in W_J yields the claimed element of W_I . (When m is even, the long word in W_I negates the simple roots in $I - J$ without exchanging them, so it doesn't act on J^{\perp} by a reflection.)

Third, if $J = D_{m \geq 3} \subseteq D_{m+1} = I$ then consider the model of W_I as the group generated by permutations and evenly many negations of $m + 1$ coordinates, with W_J the corresponding subgroup for the first m coordinates. Letting σ be the negation of the last two coordinates, and following it by the element of W_J sending $\sigma(J)$ back to J , gives the claimed element of W_I . \square

There is a nice geometrical interpretation of the freeness of Γ_{\min} in the case $J = A_1$, developed further in [1]. Namely, the natural map $C_{\min}^{\circ} \rightarrow C_{\min}^{\circ} / \Gamma_{\min} \subseteq K / W_{\Pi} = C_{\Pi}$ is the universal cover of its image. The image is got by discarding all the codimension 2 faces of C_{Π} corresponding to even bonds in Π , discarding all codimension 3 faces, and taking the component corresponding to J . This identifies Γ_{\min} with the fundamental group of J 's component of the "odd" subgraph of Π in a natural manner.

One can extend this picture to the case $J \neq A_1$, but complications arise. First, one must take W_{Ω} to be normal in $W_{\Pi} : \Gamma_{\Pi}$. Second, while $C_{\Omega}^{\circ} \rightarrow C_{\Omega}^{\circ} / \Gamma_{\Omega}^{\vee}$ is a covering space, the image $C_{\Omega}^{\circ} / \Gamma_{\Omega}$ of C_{Ω}° in C_{Π} is the quotient of $C_{\Omega}^{\circ} / \Gamma_{\Omega}^{\vee}$ by the finite group $\Gamma_{\Omega} / \Gamma_{\Omega}^{\vee}$. Usually, $C_{\Omega}^{\circ} \rightarrow C_{\Omega}^{\circ} / \Gamma_{\Omega}$ is only an orbifold cover since Γ_{Ω} often has torsion. The

top-dimensional strata of $C_{\Omega}^{\circ}/\Gamma_{\Omega}^{\vee}$ correspond to the “associates” of the inclusion $J \rightarrow \Pi$ in the sense of [2; 4]. Suppose $J' \subseteq \Pi$ is (the image of) an associate and I' is a spherical diagram containing it. Then the face of C_{Π} corresponding to I' , minus lower-dimensional faces, lies in $C_{\Omega}^{\circ}/\Gamma_{\Omega}$ just if $W_{I'}$ contains no element preserving J' , acting on it in a manner constrained by the choice of W_{Ω} , and acting on J'^{\perp} by a reflection. From this perspective, Lemmas 2 and 5 amount to working out two cases of Borchers’ notion of “ R -reflectivity”. The orbifold structure on $C_{\Omega}^{\circ}/\Gamma_{\Omega}$ is essentially the same information as Borchers’ classifying category for Γ_{Ω} .

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