A Jørgensen–Thurston theorem for homomorphisms

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We provide a description of the structure of the set of homomorphisms from a finitely generated group to any torsion-free (3–dimensional) Kleinian group with uniformly bounded finite covolume. This is analogous to the Jørgensen–Thurston Theorem in hyperbolic geometry.

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1 Introduction

In classical 3–dimensional hyperbolic geometry, the Jørgensen–Thurston Theorem [16, Theorem 5.12.1] asserts that for every $V > 0$, every complete orientable hyperbolic 3–manifold of volume at most V is isometric to a hyperbolic Dehn filling of one of a finite collection of complete orientable hyperbolic 3–manifolds of volume at most V. This can be restated using Kleinian groups (cf Theorem 2.1). We always speak of Kleinian groups in the classical sense, meaning discrete subgroups of $\mathrm{PSL}_2(\mathbb{C})$.

In this note, we prove the following theorem:

Theorem 1.1 *Suppose G is a finitely generated group, and suppose $V > 0$. Then there exist finitely many groups $R_1, \dots, R_k$, each of which is a Dehn extension of a torsion-free Kleinian group of covolume at most V, such that for any torsion-free Kleinian group H of covolume at most V, there is some $1 \le i \le k$, so that every homomorphism $\phi\colon G \to H$ factors through R_i. Namely, there is some $\psi\colon G \to R_i$ such that $\phi = \iota_{i,H} \circ \psi$, where $\iota_{i,H}\colon R_i \twoheadrightarrow H$ is an extended-filling epimorphism associated to R_i and H, independent of ϕ.*

As R_i's are all torsion-free hyperbolic groups relative to abelian subgroups (see the reference and the explanation in the proof of Lemma 4.1), and the structure of $\mathrm{Hom}(G, R_i)$ can be described using the Makanin–Razborov diagram by the work of Groves [7] (cf Sela [14] for an original version for free groups). In this sense, Theorem 1.1 implies a uniform description of the structure of homomorphisms from a finitely generated group to torsion-free Kleinian groups of uniformly bounded volume.

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When G is finitely presented, Theorem 1.1 is a quick consequence of the factorization result obtained by Agol and Liu [1, Theorem 3.2] (Section 3). To obtain the finitely generated case, we show that generic Dehn extensions of finite-covolume torsion-free Kleinian groups are stable limits of congruent extended filling (Lemma 4.1). This is also a special case of some more general results for Dehn fillings of relatively hyperbolic groups (cf Lemma 4.1). It follows from a trick of representation varieties that such Dehn extensions are all subgroups of $\mathrm{SL}_2(\mathbb{C})$ and that the finitely generated case can be reduced to the finitely presented case (Lemma 4.3).

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2 Preliminaries

In this section, we recall some notions and results related to the topic of this paper.

2.1 The Jørgensen–Thurston Theorem

For convenience of our discussion, we adopt the following notation and state the results in terms of Kleinian groups. See Thurston [16] for facts mentioned in this subsection.

Let Γ be a torsion-free Kleinian group of finite covolume, namely, which has a fundamental domain in $\mathbb{H}^3$ of finite volume. Then Γ has at most finitely many conjugacy classes of parabolic subgroups, represented by subgroups $P^1, \dots, P^q$ ($q \geq 0$), each isomorphic to a rank–2 free abelian group. We often call these chosen subgroups *cusp representatives* of Γ. By a *slope-tuple* $\zeta = (\zeta^1, \dots, \zeta^q)$ with respect to $(\Gamma; P^1, \dots, P^q)$ (or ambiguously, w.r.t. Γ), we shall mean that for each $1 \leq j \leq q$, the j–th component ζ^j is either trivial or a primitive element in P^j. For any slope-tuple ζ of Γ, we denote the *Dehn filling* of Γ along ζ as

$$\Gamma_\zeta = \Gamma \,/\, \overline{\langle \zeta^1, \dots, \zeta^q \rangle},$$

ie Γ quotienting out the normal closure of the ζ^j's, and often denote the quotient epimorphism as

$$\iota_\zeta \colon \Gamma \twoheadrightarrow \Gamma_\zeta.$$

By Thurston's Hyperbolic Dehn Filling Theorem [16, Theorem 5.8.2], for generic slope-tuples ζ (avoiding finitely many primitive choices for each component), Γ_ζ is isomorphic to a torsion-free Kleinian group of finite covolume, indeed, strictly less than that of Γ if ζ has at least one nontrivial component. These Γ_ζ's are usually called *hyperbolic Dehn fillings* of Γ. In fact, one may choose faithful Kleinian representations $\rho_\zeta\colon \Gamma_\zeta \to \mathrm{PSL}_2(\mathbb{C})$, so that for any sequence $\{\zeta_n\}$ of slope-tuples, there is a subsequence for which the induced representations $\{\rho_{\zeta_n} \circ \iota_{\zeta_n}\}$ of Γ strongly converges. Moreover, if for each j–th component, $\{\zeta_n^j\}$ has no bounded subsequence of primitive elements in P^j, then $\{\rho_{\zeta_n} \circ \iota_{\zeta_n}\}$ strongly converges to the inclusion $\Gamma \subset \mathrm{PSL}_2(\mathbb{C})$.

The following rephrases [16, Theorem 5.12.1]:

Theorem 2.1 (Jørgensen–Thurston) *For any $V > 0$, there exist finitely many torsion-free Kleinian groups $\Gamma_1, \dots, \Gamma_k$ of covolume at most V, (together with chosen cusp representatives), such that any torsion-free Kleinian group of covolume at most V is isomorphic to the hyperbolic Dehn filling $(\Gamma_i)_\zeta$ of some Γ_i along some slope-tuple ζ of Γ_i.*

Remark 2.2 It is also implied from the proof that hyperbolic Dehn fillings decreases the covolume.

2.2 Dehn extensions

For aspherical orientable compact 3–manifolds, Dehn extensions have been investigated in [1] from a topological perspective. This kind of construction was introduced earlier to define knot invariants, known as generalized knot groups (see Kelly [9], Lin and Nelson [10] and Wada [17]). In this note, we rephrase Dehn extensions on the group level in terms of amalgamations. We shall restrict ourselves to torsion-free Kleinian groups of finite covolume, for simplicity, but we need a multicusp version, allowing slope-tuples and denominator-tuples.

Let $(\Gamma; P^1, \dots, P^q)$ be a torsion-free Kleinian group of finite covolume, together with chosen cusp representatives. Heuristically speaking, a Dehn extension is obtained by adjoining an "abelian root" of a slope in each cusp, or precisely as follows. Note that as each $P^j < \Gamma$ is isomorphic to a rank–2 free abelian group, we naturally identify P^j as the integral lattice of the $\mathbb{Q}$–vector space $P^j \otimes_{\mathbb{Z}} \mathbb{Q}$.

Definition 2.3 With the notation above, let $m = (m^1, \dots, m^q)$ be a q–tuple of positive integers, called a *denominator-tuple*, and let $\zeta = (\zeta^1, \dots, \zeta^q)$ be a slope-tuple, where each $\zeta^j \in P^j$ is either trivial or primitive. The *Dehn extension* of Γ along the slope-tuple ζ with the denominator-tuple m, denoted by $\Gamma^{e(\zeta,m)}$, is defined as the amalgamation of Γ with all the $(P^j + \mathbb{Z}\,(\zeta^j/m^j))$'s (as subgroups of $P^j \otimes_{\mathbb{Z}} \mathbb{Q}$'s) along all the P^j's respectively. In other words, it is the fundamental group of the graph of groups shown in Figure 1, which we write as

$$\Gamma^{e(\zeta,m)} = \Gamma \cdot \left(*_{P^1} \left(P^1 + \mathbb{Z}\,\frac{\zeta^1}{m^1} \right) \right) \cdots \left(*_{P^q} \left(P^q + \mathbb{Z}\,\frac{\zeta^q}{m^q} \right) \right).$$

We often briefly write Γ^e whenever m and ζ are clear from the context. The Dehn extension is said to be *trivial on the j–th cusp*, if either ζ^j is trivial or $m^j = 1$, and it is *trivial* if so it is on each cusp.

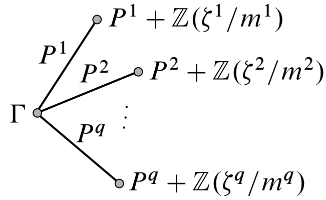

Figure 1. The Dehn extension $\Gamma^{e(\zeta,m)}$

Definition 2.4 Let ζ, ζ' be two slope-tuples, we say that they are *congruent* modulo a denominator-tuple m, if for each $1 \le j \le q$, the j–th components are congruent modulo m^j, namely, $\zeta^j \equiv (\zeta')^j \mod m^j$. We say ζ *dominates* ζ' modulo m, if for each $1 \le j \le q$, either $\zeta^j \equiv (\zeta')^j \mod m^j$, or that $\zeta^j = 0$.

It is clear that $\Gamma^{e(\zeta,m)}$ is naturally isomorphic to $\Gamma^{e(\zeta',m)}$, if ζ is congruent to ζ' modulo m. Thus, for any denominator-tuple m, there are only finitely many distinct Dehn extensions up to isomorphism.

Definition 2.5 Suppose Γ^e is the Dehn extension of Γ along the slope-tuple ζ with the denominator-tuple m, then there is a canonical *extended-filling* epimorphism

$$\iota^e \colon \Gamma^e \twoheadrightarrow \Gamma_\zeta,$$

quotienting out the normal closure of all the ζ^j/m^j's. Moreover, if ζ' is congruent to (resp. dominated by) ζ modulo m, there are also *congruent* (resp. *dominated*) extended-fillings

$$\iota^e_{\zeta'} \colon \Gamma^e \twoheadrightarrow \Gamma_{\zeta'},$$

quotienting out the normal closure of all the $(\zeta')^j/m^j$'s.

3 The finitely presented case

In this section, we prove the finitely presented case of Theorem 1.1, namely:

Proposition 3.1 *Suppose G is a finitely presented group, and suppose $V > 0$. Then the conclusion of Theorem 1.1 holds for G and V.*

We prove Proposition 3.1 in the rest of this section. It is a consequence of [1, Theorem 3.2] and the classical Jørgensen–Thurston Theorem.

Lemma 3.2 *Let G be a finitely presented group, and Γ be a torsion-free Kleinian group of finite covolume, with cusp representatives $P^1, \dots, P^q$. Then there are finite collections of (primitive) slopes $\mathcal{S}^j \subset P^j$ for $1 \le j \le q$, and there is some denominator-tuple $m = (m^1, \dots, m^q)$, such that for any slope-tuple $\zeta = (\zeta^1, \dots, \zeta^q)$ where each $\zeta^j \notin \mathcal{S}^j$, and for any homomorphism $\phi\colon G \to \Gamma_\zeta$, ϕ factors through the canonical extended-filling epimorphism $\iota^e\colon \Gamma^{e(\zeta,m)} \twoheadrightarrow \Gamma_\zeta$.*

Proof By Thurston's Hyperbolic Dehn Filling Theorem [16, Theorem 5.8.2], we regard Γ and all the hyperbolic Dehn fillings Γ_ζ's as subgroups of $\mathrm{PSL}_2(\mathbb{C})$, so that the Dehn filling epimorphisms $\iota_\zeta\colon \Gamma \twoheadrightarrow \Gamma_\zeta$ is a strongly convergent family, namely, that for any sequence of slope-tuples $\{\zeta_n\}$ that converges to some slope-tuple, the corresponding sequence $\{\iota_{\zeta_n}\}$ converges algebraically and the sequence of images $\{\iota_{\zeta_n}(\Gamma)\}$ converges geometrically (cf Section 2.1). Choose a base-point O of the 3–dimensional hyperbolic space $\mathbb{H}^3$ and an orthonormal basis of the tangent space to identify $\mathrm{PSL}_2(\mathbb{C})$ with $\mathrm{Isom}_+(\mathbb{H}^3)$. The quotient manifolds $N_\zeta = \mathbb{H}^3 / \Gamma_\zeta$ are all complete finite-volume hyperbolic manifolds with naturally induced base points, forming a sequentially compact family converging to $N_\star = \mathbb{H}^3 / \Gamma$ in the pointed Gromov–Hausdorff topology. Let $\epsilon > 0$ be a Margulis number of $\mathbb{H}^3$. We may assume ϵ to be small enough so that the ϵ–thin part of $N_\star$ consists only of properly embedded disjoint horocusps. Then there are finite collections $\mathcal{S}^j \subset P^j$ of primitive slopes, such that whenever ζ is a slope-tuple whose components avoid elements of the $\mathcal{S}^j$'s, the ϵ–thin part of N_ζ is the disjoint union of $V^1_\zeta, \dots V^q_\zeta$, each homeomorphic to either a solid torus $S^1 \times D^2$ or $T^2 \times [0, +\infty)$, corresponding to (possibly null) Dehn fillings of the cusps. Furthermore, one may pick $\mathcal{S}^j$ sufficiently large so that to ensure that the meridional area of every solid torus V^j_ζ is sufficiently large, for instance, greater than $A(\ell(G))$ where $\ell(G)$ is the presentation length of G (cf [1, Definition 3.1]) and $A(n) = 27^n(9n^2 + 4n)\pi$.

Note that the argument of [1, Theorem 3.2] only used the local geometry of the hyperbolic piece near a deep Margulis tube. We may apply the same argument simultaneously to all the tubes $V^1_\zeta, \dots, V^q_\zeta$ for each N_ζ to see that there is some uniform denominator-tuple $m = (m^1, \dots, m^j)$, such that every map $f\colon K \to N_\zeta$ factors through the (topological) Dehn extension $N^{e(\zeta,m)}$ via the extended-filling map $N^{e(\zeta,m)} \to N_\zeta$, up to homotopy. Here K is a the presentation complex of G with $\pi_1(K) \cong G$ (cf [1, Section 3.1]), and $N^{e(\zeta,m)}$ is a natural topological space with $\pi_1(N^{e(\zeta,m)}) \cong \Gamma^{e(\zeta,m)}$, obtained by attaching "ridge pieces" to the cusp tori of $N_\star$ (cf [1, Section 2.2]). In fact, one may require each m^j satisfy $0 < m^j \le T(\ell(G))$, where $T(n) = 2 \cdot 3^n$. In language of groups, this is the conclusion of Lemma 3.2. □

The lemma below may be regarded as a homomorphism analogue to Thurston's Hyperbolic Dehn Filling Theorem.

Lemma 3.3 *Suppose G is finitely presented, and Γ is a torsion-free Kleinian group of finite covolume. Then there are finitely many groups $R_1, \dots, R_l$, each being a Dehn extension of a hyperbolic Dehn filling of Γ, such that for every hyperbolic Dehn filling Γ_ζ of Γ, there is an R_i and an extended filling epimorphism $\iota_{i,\zeta}\colon R_i \twoheadrightarrow \Gamma_\zeta$, so that every homomorphism $\phi\colon G \to \Gamma_\zeta$ factors through $\iota_{i,\zeta}$.*

Proof We make an induction argument on the number of cusps of Γ. When Γ has no cusp, there is no further Dehn filling, so there is nothing to prove. Suppose that we have proved the statement when Γ has $q-1$ cusps, where $q > 0$.

When Γ has q cusps, represented by $P^1, \cdots, P^q$, by Lemma 3.2, there are finite collections of primitive slopes, $\mathcal{S}^j \subset P^j$ for $1 \le j \le q$, and there is a denominator-tuple $m = (m^1, \dots, m^q)$, such that for any slope-tuple ζ whose components avoid elements of $\mathcal{S}^1, \dots, \mathcal{S}^q$, any $\phi\colon G \to \Gamma_\zeta$ factors through the extended filling epimorphism $\iota^e_\zeta\colon \Gamma^{e(\zeta,m)} \twoheadrightarrow \Gamma_\zeta$. Note that since m is fixed, there only finitely many isomorphism classes of $\Gamma^{e(\zeta,m)}$'s, and two extended fillings ι^e_ζ and $\iota^e_{\zeta'}$ are congruent when ζ is congruent to ζ' modulo m (cf Section 2.2). We denote these isomorphically distinct Dehn extensions as $R_1, \dots, R_{l_0}$.

If ζ does not satisfy the condition above, it is dominated by a slope-tuple ζ' of the form $(0, \dots, 0, \zeta^j, 0, \dots, 0)$ where $\zeta^j \in \mathcal{S}^j$ (cf Definition 2.4). We enumerate these ζ''s as $\zeta'_1, \dots, \zeta'_s$. For each $1 \le t \le s$, since $\Gamma_{\zeta'_t}$ has only $q-1$ cusps, we apply the induction assumption so there are Dehn extensions of hyperbolic Dehn fillings of $\Gamma_{\zeta'_t}$, say $R^t_1, \dots, R^t_{l_t}$, such that if ζ is dominated by ζ'_t, any $\phi\colon G \to \Gamma_{\zeta'_t}$ factors through the extended filling epimorphism from some R^t_i to $\Gamma_{\zeta'_t}$.

Now we take $R_1, \dots, R_l$ as $R_1, \dots, R_{l_0}$ together with all the R^t_i's, where $1 \le i \le l_t$ and $1 \le t \le s$. Then the conclusion of the statement holds for Γ. This completes the induction. □

Proof of Proposition 3.1 Let G be a finitely presented, and $V > 0$ as assumed. By the Jørgensen–Thurston Theorem (cf Theorem 2.1), there are finitely many torsion-free Kleinian groups $\Gamma_1, \dots, \Gamma_s$ of covolume at most V, such that any torsion-free Kleinian group H of covolume at most V is isomorphic to a Dehn filling $(\Gamma_i)_\zeta$ of some Γ_i along some slope-tuple ζ (with respect to chosen cusp representatives). For each Γ_i, we apply Lemma 3.3 to obtain finitely many groups $R_{i,1}, \dots, R_{i,k_i}$, each being a Dehn extension of a hyperbolic Dehn filling of Γ_i. Hence all the $R_{i,t}$'s (for all i and t) are Dehn extensions of torsion-free Kleinian groups of covolume at most V. We rewrite them as $R_1, \dots, R_l$, then these are the groups as claimed by Proposition 3.1. □

4 The finitely generated case

In this section, we deduce Theorem 1.1 from the finitely presented case (Proposition 3.1). We show that Dehn extensions of finite-covolume torsion-free Kleinian groups are stable limits of congruent extended fillings. With some general trick of representation varieties, which are certainly well-known to experts (see Sela [14] and Soma [15]), this will imply Theorem 1.1 using the finitely presented case.

To be precise, let Γ be a torsion-free Kleinian group of finite covolume, with cusp representatives $P^1, \dots, P^q$. We say a slope-tuple $\zeta = (\zeta^1, \dots, \zeta^q)$ is *all-primitive* if every component ζ^j is primitive in P^j. An infinite sequence $\{\zeta_n\}$ of all-primitive slope-tuples is said to *converge to all-cusps*, if each component sequence $\{\zeta_n^j\}$ has no bounded infinite subsequence.

We need some further terminology of limit groups. Let G be a finitely generated group, and $\{H_n\}$ be a sequence of groups. Suppose $\{\phi_n\colon G \to H_n\}$ is a sequence of homomorphisms. We say $\{\phi_n\}$ is a *stable sequence*, if it satisfies that for any $g \in G$, either $\phi_n(g)$ is trivial for all but finitely many n's, or that $\phi_n(g)$ is nontrivial for all but finitely many n's. Moreover, for such a sequence, we define the *stable kernel* of $\{\phi_n\}$ to be the (possibly trivial) normal subgroup K_∞ of G, consisting of all the elements $g \in G$ each of which is trivial under all but finitely many ϕ_n's. The quotient epimorphism $\pi\colon G \twoheadrightarrow G / K_\infty$, (or ambiguously, the quotient group $L_\infty = G / K_\infty$), will be called the *stable limit* of the stable sequence $\{\phi_n\}$. The notion of stable limits is related to the study of limit groups in geometric group theory (see Sela [14] and Bestvina and Feighn [3]). Note that our definition here allows variations on the target groups, but in our application all the target groups will be subgroups of $\mathrm{SL}_2(\mathbb{C})$, so one may also regard the stable limits here as $\mathrm{SL}_2(\mathbb{C})$–limit groups, in the sense of, eg Ould Houcine [13].

Lemma 4.1 *Let Γ be a torsion-free Kleinian group of finite covolume, with cusp representatives $P^1, \dots, P^q$. Let $\Gamma^{e(\zeta,m)}$ be the Dehn extension of Γ along some slope-tuple ζ with the denominator-tuple m, and let $\{\zeta_n\}$ be a sequence of all-primitive slope-tuples, each congruent to ζ modulo m. Suppose $\{\zeta_n\}$ converges to all-cusps. Then $\Gamma^{e(\zeta,m)}$ is the stable limit of the stable sequence of congruent extended-filling epimorphisms*

$$\iota^e_{\zeta_n}\colon \Gamma^{e(\zeta,m)} \twoheadrightarrow \Gamma_{\zeta_n}.$$

Proof This is a special case of some more general result for Dehn fillings of relatively hyperbolic groups. In fact, by a combination theorem for amalgamations of relatively hyperbolic groups (see Dahmani [5, Theorem 0.1], or more generally, Osin [11, Theorem 1.3]), and by the fact that finitely generated torsion-free Kleinian groups are hyperbolic relative to the cusp subgroups (see Farb [6, Theorem 4.11]), Dehn extensions of finite-covolume torsion-free Kleinian groups are all hyperbolic relative to the extended cusp subgroups, namely, the $(P^j + \mathbb{Z}\,(\zeta^j/m^j))$'s. Then the statement of Lemma 4.1 follows immediately from a hyperbolic Dehn filling theorem for relatively hyperbolic groups due to Groves and Manning, [8, Corollary 9.7] (cf Osin [12, Theorem 1.1]). $\square$

Recall that the representation variety $R(G)$ of a finitely generated group G in $\mathrm{SL}_2(\mathbb{C})$ is the set $\mathrm{Hom}(G, \mathrm{SL}_2(\mathbb{C}))$ with the canonical Zariski closed affine algebraic structure (see Culler and Shalen [4]). For any quotient $\pi\colon G \twoheadrightarrow Q$, there is an induced embedding of algebraic set

$$\pi^*\colon R(Q) \hookrightarrow R(G),$$

defined by $\pi^*(\rho) = \rho \circ \pi$, for $\rho \in R(Q)$. In particular, for every $\rho \in R(G)$, the subvariety $R(\rho) \subset R(G)$, is defined as $\rho^* R(\mathrm{Im}(\rho))$. Note that there is another natural topology, introduced by Baumslag, Myasnikov and Remeslennikov [2], which is analogous to but in general weaker than the classical Zariski topology. Since we will not explicitly use these topologies in the rest of our argument, we refer the reader to Culler and Shalen [4] and for details of the notions.

The property that we shall use about $\mathrm{SL}_2(\mathbb{C})$ is that it is *equationally noetherian* in the sense of [2]. The lemma below may be regarded as a representation-variety version of finitely presented approximation.

Lemma 4.2 *For any finitely generated group G, there is a finitely presented group G_0 of the same rank (ie the smallest number of generators), and a quotient epimorphism $p\colon G_0 \twoheadrightarrow G$ such that $p^*\colon R(G) \to R(G_0)$ is bijective.*

Proof Since $\mathrm{SL}_2(\mathbb{C})$ is linear, it is equationally noetherian by [2, Theorem B1]. It follows immediately from the definition that p^* is bijective. Indeed, one may take a presentation $\mathcal{P} = (x_1, \ldots, x_r\,;\, w_1, w_2 \cdots)$ of G, then $R(G)$ may be identified as the set of $\mathrm{SL}_2(\mathbb{C})$–points defined by the relators, namely, $V_{\mathrm{SL}_2(\mathbb{C})}(\{w_1, w_2 \cdots\}) \subset \mathrm{SL}_2(\mathbb{C})^r$ in terms of [2]. As $\mathrm{SL}_2(\mathbb{C})$ is equationally noetherian, G_0 may be obtained from the free group generated by $x_1, \ldots, x_r$ by killing finitely many relators $w_1, \ldots w_k$ which ensure $V_{\mathrm{SL}_2(\mathbb{C})}(\{w_1, w_2 \cdots\}) = V_{\mathrm{SL}_2(\mathbb{C})}(\{w_1, \ldots w_k\})$. We also remark that one can actually construct G_0 by taking sufficiently many relators so that p^* is isomorphic with respect to the (classical or analogous) Zariski topology of algebraic sets, and this follows from the proof of [2, Theorem B1]. $\square$

Lemma 4.3 *Suppose G is a finitely generated group, and L is a stable limit of a stable sequence of homomorphisms $\{\phi_n\colon G \to H_n\}$, where each H_n is isomorphic to a subgroup of $\mathrm{SL}_2(\mathbb{C})$. Then for all but finitely many n's, ϕ_n factors through the stable limit epimorphism $\pi\colon G \twoheadrightarrow L$. Moreover, L is isomorphic to a subgroup of $\mathrm{SL}_2(\mathbb{C})$.*

Proof As $\mathrm{SL}_2(\mathbb{C})$ is equationally noetherian, and as G is finitely generated, every $\mathrm{SL}_2(\mathbb{C})$–limit group is fully residually $\mathrm{SL}_2(\mathbb{C})$ by [13, Theorem 2.1(4)], namely, such that every finite collection of elements can be send injectively into $\mathrm{SL}_2(\mathbb{C})$ via some representation of the group. Then a standard argument of limit groups (see [3, Section 1.2]) can be adapted to our situation, as follows.

We first show L is isomorphic to a subgroup of $\mathrm{SL}_2(\mathbb{C})$. Suppose $g_1, g_2, \ldots$ is an enumeration of elements of L, we may pick a sequence of $\mathrm{SL}_2(\mathbb{C})$–representations $\rho_1, \rho_2, \ldots$ of G, such that ρ_n sends $g_1, \ldots, g_n$ injectively into $\mathrm{SL}_2(\mathbb{C})$, for each integer $n \geq 1$. As $R(L)$ consists only of finitely many maximal Zariski irreducible components, an infinite subsequence of ρ_i's lie on one and the same component Y of $R(L)$. Now the maximal irreducible component of $R(\rho_i)$ containing ρ_i must have strictly smaller dimension than that of Y, unless ρ_i is itself injective. Thus a Baire Category argument clearly implies that a generic element on Y is injective. This means L is isomorphic to a subgroup of $\mathrm{SL}_2(\mathbb{C})$.

The factorization part can be proved in a similar fashion, and it suffices to show that for any stable sequence of representations $\{\rho_n\}$ converging to $G \twoheadrightarrow L$, all but finitely many ρ_n's factors through the limit quotient. Suppose on the contrary that there were a stable sequence of $\mathrm{SL}_2(\mathbb{C})$–representations $\rho_1, \rho_2, \ldots$ of G converging to L, where no ρ_n factored through the quotient $G \twoheadrightarrow L$. Then a similar argument as above shows that there would be a subsequence of $\{\rho_n\}$, such that every element is contained in a maximal irreducible component Y of $R(G)$. Moreover, a generic element on Y descends via $G \twoheadrightarrow L$ to be a faithful representation of L. This means every representation in this subsequence would have factored through $G \twoheadrightarrow L$. This is a contradiction. □

We now give the proof of Theorem 1.1.

Proof of Theorem 1.1 Suppose G is a finitely generated group, and $V > 0$. By Lemma 4.2, there is a finitely presented group G_0 and an epimorphism $p\colon G_0 \twoheadrightarrow G$, inducing a bijection $p^*\colon R(G) \to R(G_0)$ on the $\mathrm{SL}_2(\mathbb{C})$–varieties. By Proposition 3.1, we may find finitely many Dehn extensions of torsion-free Kleinian groups of co-volume at most V, say $R_1, \ldots, R_k$, and for any torsion-free Kleinian group H, there is some R_i and an extended-filling epimorphism $\iota^e_{i,H}\colon R_i \twoheadrightarrow H$, such that any homomorphism $\phi\colon G_0 \to H$ factors through $\iota^e_{i,H}$.

We claim these R_i's (and $\iota_{i,H}^e$'s) also satisfy the conclusion of Theorem 1.1. A result of William Thurston asserts that finitely generated torsion-free Kleinian groups can be lifted to be embedded in $\mathrm{SL}_2(\mathbb{C})$ (see [4, Proposition 3.1.1]). Thus, by Lemma 4.1 and Lemma 4.3, there are faithful representations $\rho_i\colon R_i \hookrightarrow \mathrm{SL}_2(\mathbb{C})$. To see the claim, consider any homomorphism $\phi\colon G \to H$, where H is as above. Then $\phi \circ p\colon G_0 \to H$ factors through some $\iota_{i,H}^e$, say $\phi \circ p = \iota_{i,H}^e \circ \psi$, where $\psi\colon G_0 \to R_i$. Thus $\rho_i \circ \psi\colon G_0 \to \mathrm{SL}_2(\mathbb{C})$ is a representation of G_0 in $\mathrm{SL}_2(\mathbb{C})$. Because $p^*\colon R(G) \to R(G_0)$ is isomorphic, $\rho_i \circ \psi$ descends to a representation of G via p, or equivalently, $\psi = \overline{\psi} \circ p$ for some $\overline{\psi}\colon G \to \mathrm{Im}(\psi) \hookrightarrow R_i$. Hence $\phi \circ p = \iota_{i,H}^e \circ \overline{\psi} \circ p$, so $\phi = \iota_{i,H}^e \circ \overline{\psi}$ as p is surjective. This means ϕ factors through $\iota_{i,H}^e$ via $\overline{\psi}$. Then we have proved the claim. This also completes the proof of Theorem 1.1. □

5 Conclusions

In conclusion, Dehn extensions arise naturally from a limit process in the study of homomorphisms to torsion-free Kleinian groups with uniformly bounded covolume. From the perspective of geometric group theory, it seems reasonable to expect that there is a similar version of Theorem 1.1 describing the structure of homomorphisms from a finitely generated group to the Dehn-filling family of a torsion-free relatively hyperbolic group with abelian flats. However, as it relies severely on the fact that Kleinian groups are subgroups of $\mathrm{SL}_2(\mathbb{C})$, our approach in this note does not apply in general. We also wonder whether it is possible to describe the homomorphism structure if we drop the assumption of uniformly bounded covolume in Theorem 1.1.

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