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Brunnian braids on surfaces

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We determine a set of generators for the Brunnian braids on a general surface M for $M \neq S^2$ or $\mathbb{R}\mathrm{P}^2$. For the case $M = S^2$ or $\mathbb{R}\mathrm{P}^2$, a set of generators for the Brunnian braids on M is given by our generating set together with the homotopy groups of a 2–sphere.

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1 Introduction

Let M be a compact connected surface, possibly with boundary, and let $B_n(M)$ denote the n–strand braid group on a surface M. From the point of view of braids, compactness of a surface is not essential: braids stay the same if you replace a boundary component by a puncture. However the number of punctures must be finite, so that the fundamental group and the braid groups will be finitely generated.

A *Brunnian braid* means a braid that becomes trivial after removing any one of its strands. The formal definition of Brunnian braids is given in Section 2. A typical example of a 3–strand Brunnian braid on a disk is the braid given by the expression $(\sigma_1^{-1}\sigma_2)^3$, where σ_1 and σ_2 are the standard generators of the 3–strand braid group $\langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$.

Let $\mathrm{Brun}_n(M)$ denote the set of the n–strand Brunnian braids. Then $\mathrm{Brun}_n(M)$ forms a subgroup of $B_n(M)$. A classical question proposed by G S Makanin [19] in 1980 is to determine a set of generators for Brunnian braids over the disk. Brunnian braids were called *smooth braids* by Makanin. This question was answered by D L Johnson [12] and G G Gurzo [11]. J Y Li and J Wu [16; 23] gave different approach to this question. In the 1970s, H Levinson [14; 15] defined a notion of k*–decomposable* braid, which becomes trivial after removal of any arbitrary k strings. In his terminology a *decomposable* braid means 1*–decomposable* and therefore, Brunnian.

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A J Berrick, F R Cohen, Y L Wong and J Wu [2] gave a connection between Brunnian braids and the homotopy groups of spheres. In particular, the exact sequence

$$1 \to \mathrm{Brun}_{n+1}(S^2) \to \mathrm{Brun}_n(D^2) \to \mathrm{Brun}_n(S^2) \to \pi_{n-1}(S^2) \to 1 \tag{1-1}$$

was proved for $n > 4$.

J Birman [3, Question 23, page 219] asked how to determine a free basis for the intersection $\mathrm{Brun}_n(D^2) \cap R_{n-1}$ where

$$R_{n-1} = \mathrm{Ker}(B_n(D^2) \to B_n(S^2)).$$

Her motivation was that the kernel of the Gassner representation is a subgroup of $\mathrm{Brun}_n(D^2) \cap R_{n-1}$. From the exact sequence (1-1) it follows that Birman's question, for $n > 5$, is about a free basis of Brunnian braids over the sphere S^2. As far as we know this question remains open.

The purpose of this article is to determine a set of generators for $\mathrm{Brun}_n(M)$ for a general surface M. We are able to determine a generating set for $\mathrm{Brun}_n(M)$ except in two special cases, where $M = S^2$ or $\mathbb{R}\mathrm{P}^2$. For the case $M = S^2$ or $\mathbb{R}\mathrm{P}^2$, we are able to determine a generating set for a (normal) subgroup of $\mathrm{Brun}_n(M)$, with the factor group given by $\pi_{n-1}(S^2)$.

Recall the notion of the symmetric commutator product (see Li and Wu [17] and Mikhailov, Passi and Wu [20]). Given a group G, and a set of normal subgroups $R_1, \dots, R_n$ $(n \geq 2)$, the symmetric commutator product of these subgroups is defined as

$$[R_1, \dots, R_n]_S := \prod_{\sigma \in \Sigma_n} [[R_{\sigma(1)}, R_{\sigma(2)}], \dots, R_{\sigma(n)}],$$

where Σ_n is the symmetric group of degree n.

Let $P_n(M)$ be the n–strand pure braid group on M. Let D^2 be a small disk in M. Then the inclusion $f\colon D^2 \hookrightarrow M$ induces a group homomorphism

$$f_*\colon P_n(D^2) \longrightarrow P_n(M).$$

Recall that the pure Artin braid group $P_n(D^2)$ is a subgroup of the braid group

$$B_n = \langle \sigma_1, \dots, \sigma_{n-1} \mid \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},\ i = 1, \dots, n-2, \\ \sigma_i\sigma_j = \sigma_j\sigma_i,\ |i-j| \geq 2\rangle.$$

generated by the elements

$$A_{i,j} = \sigma_{j-1}\sigma_{j-2}\cdots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\cdots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1},$$

for $1 \le i < j \le n$. Let $A_{i,j}[M] = f_*(A_{i,j})$ and let $\langle\langle A_{i,j}[M]\rangle\rangle^P$ be the normal closure of $A_{i,j}[M]$ in $P_n(M)$. Note that a set of generators for $\langle\langle A_{i,j}[M]\rangle\rangle^P$ is given by $\beta A_{i,j}[M]\beta^{-1}$ for $\beta \in P_n(M)$. Thus a set of generators for the iterated subgroup

$$[\langle\langle A_{1,n}[M]\rangle\rangle^P, \langle\langle A_{2,n}[M]\rangle\rangle^P, \dots, \langle\langle A_{n-1,n}[M]\rangle\rangle^P]_S$$

can be given.

Now we compute $\mathrm{Brun}_n(M)$ as follows.

Theorem 1.1 *Let M be a connected 2–manifold and let $n \geq 2$. Let*

$$R_n(M) = [\langle\langle A_{1,n}[M]\rangle\rangle^P, \langle\langle A_{2,n}[M]\rangle\rangle^P, \dots, \langle\langle A_{n-1,n}[M]\rangle\rangle^P]_S.$$

(1) *If $M \neq S^2$ or $\mathbb{R}\mathrm{P}^2$, then*

$$\mathrm{Brun}_n(M) = R_n(M).$$

(2) *If $M = S^2$ and $n \geq 5$, then there is a short exact sequence*

$$R_n(S^2) \hookrightarrow \mathrm{Brun}_n(S^2) \twoheadrightarrow \pi_{n-1}(S^2).$$

(3) *If $M = \mathbb{R}\mathrm{P}^2$ and $n \geq 4$ then there is a short exact sequence*

$$R_n(\mathbb{R}\mathrm{P}^2) \hookrightarrow \mathrm{Brun}_n(\mathbb{R}\mathrm{P}^2) \twoheadrightarrow \pi_{n-1}(S^2).$$

Remark (1) Assertion (2) fails for $n = 3, 4$. A free basis for $\mathrm{Brun}_4(S^2)$ was given in [2]. Assertion (3) fails for $n = 2, 3$. For the cases $n \leq 3$, our result is given in Propositions 3.3, 3.6 and 4.9 by explicit computations.

(2) In the classical case where $M = D^2$, assertion (1) gives a better format for answering Makanin's question as we describe Brunnian braids as an explicit iterated commutator subgroup. In this case the assertion was proved in [17]. Assertion (2) was essentially given in [2, Theorem 1.2]. Here we give an explicit determination for the kernel of $\mathrm{Brun}_n(S^2) \to \pi_{n-1}(S^2)$ for $n \geq 5$. Assertion (3) gives a new connection between Brunnian braids and homotopy groups. The first case in assertion (3) ($n = 4$) is that the Hopf map $S^3 \to S^2$ lifts to a 4–strand Brunnian braid on $\mathbb{R}\mathrm{P}^2$.

(3) For the classical case, the inclusion

$$R_n(D^2) \hookrightarrow \mathrm{Brun}_n(D^2)$$

was observed by Levinson [15, page 53].

By Corollary 2.5, $\mathrm{Brun}_n(M)$ is a normal subgroup of $B_n(M)$ for $n \geq 3$. As an abstract group, $\mathrm{Brun}_n(M)$ is a free group of infinite rank for $n \geq 3$ with $M \neq S^2$ or $\mathbb{R}\mathrm{P}^2$, for $n \geq 5$ with $M = S^2$ and for $n \geq 4$ with $M = \mathbb{R}\mathrm{P}^2$. A natural question is whether the factor group $B_n(M)/\mathrm{Brun}_n(M)$ is finitely presented. Our answer to this question is positive.

Theorem 1.2 *Let M be a connected compact 2–manifold. Then the factor groups $P_n(M)/\mathrm{Brun}_n(M)$ and $B_n(M)/\mathrm{Brun}_n(M)$ are finitely presented for each $n \geq 3$.*

The article is organized as follows. In Section 2, we give a review on Brunnian braids. The determination of a generating set for Brunnian braids is given in Section 3. In Section 4, we compute the 3–strand Brunnian braids on the projective plane. The proof of Theorem 1.2 is given in Section 5. In Section 6, we give an algorithm for determining a free basis for Brunnian Braids. In the Appendix we prove the technical results stated in Section 4.

2 Brunnian braids

2.1 Configuration spaces and the braid groups

Let M be a topological space and let M^n be the n–fold Cartesian product of M. The *n–th ordered configuration space*, $F(M,n)$ is defined by

$$F(M,n) = \{(x_1, \dots, x_n) \in M^n \mid x_i \neq x_j \text{ for } i \neq j\}$$

with the subspace topology on M^n. The symmetric group Σ_n acts on $F(M,n)$ by permuting coordinates. The orbit space

$$B(M,n) = F(M,n)/\Sigma_n$$

is called the *n–th unordered configuration space*. The *braid group* $B_n(M)$ is defined to be the fundamental group $\pi_1(B(M,n))$. The *pure braid group* $P_n(M)$ is defined to be the fundamental group $\pi_1(F(M,n))$. From the covering $F(M,n) \to F(M,n)/\Sigma_n$, there is a short exact sequence of groups

$$1 \to P_n(M) \to B_n(M) \to \Sigma_n \to 1.$$

A geometric description of the elements in $B_n(M)$ can be given as follows. Let $(q_1, \dots, q_n)$ be the basepoint of $F(M,n)$ and let

$$p\colon F(M,n) \to F(M,n)/\Sigma_n$$

be the quotient map. The basepoint of $F(M,n)/\Sigma_n$ is chosen to be $p(q_1,\dots,q_n)$. Let $[\lambda]$ be an element in $\pi_1(F(M,n)/\Sigma_n)$ represented by a loop $\lambda\colon S^1 \to F(M,n)/\Sigma_n$. Since

$$p\colon F(M,n) \to F(M,n)/\Sigma_n$$

is a covering, the loop λ lifts to a unique path $\tilde{\lambda}\colon [0,1] \to F(M,n)$ starting from $\tilde{\lambda}(0) = (q_1,\dots,q_n)$ and ending with $\tilde{\lambda}(1) = (q_{\sigma(1)},\dots,q_{\sigma(n)})$ for some $\sigma \in \Sigma_n$. Let

$$\tilde{\lambda}(t) = (\tilde{\lambda}_1(t),\dots,\tilde{\lambda}_n(t)) \in F(M,n) \subseteq M^n.$$

Then $\tilde{\lambda}_i(t) \neq \tilde{\lambda}_j(t)$ for $i \neq j$ and any $0 \le t \le 1$. The strands

$$\{(\tilde{\lambda}_i(t),t) \mid 1 \le i \le n\}$$

in the cylinder $M \times [0,1]$ give the intuitive braided description of λ. The precise definition of geometric braids is as follows.

Let $\{p_1, p_2,\dots, p_n\}$ be n distinct points in M. Consider the cylinder $M \times I$. A *geometric braid*

$$\rho = \{\rho_1,\dots,\rho_n\}$$

at the *basepoints* $\{p_1,\dots,p_n\}$ is a collection of n paths in the cylinder $M \times I$ such that $\rho_i(t) = (\lambda_i(t),t)$ and

(1) $\lambda_1(0) = p_1,\dots,\lambda_n(0) = p_n$;

(2) $\lambda_1(1) = p_{\sigma(1)},\dots,\lambda_n(1) = p_{\sigma(n)}$ for some $\sigma \in \Sigma_n$;

(3) $\lambda_i(t) \neq \lambda_j(t)$ for $0 \le t \le 1$ and $i \neq j$.

Let $\rho = \{\rho_1,\dots,\rho_n\}$ and $\rho' = \{\rho'_1,\dots,\rho'_n\}$ be two geometric braids. We say that ρ is equivalent to ρ', denoted by $\rho \sim \rho'$, if there exists a continuous sequence of geometric braids

$$\rho^s = (\lambda^s,t) = \{(\lambda^s_1(t),t),\ \dots,(\lambda^s_n(t),t)\},\quad 0 \le s \le 1,$$

such that

(1) $\lambda^s_1(0) = p_1,\dots,\lambda^s_n(0) = p_n$ for each $0 \le s \le 1$;

(2) $\lambda^s_1(1) = \lambda^0_1(1),\dots,\lambda^s_n(1) = \lambda^0_n(1)$ for each $0 \le s \le 1$;

(3) $\lambda^0 = \lambda$ and $\lambda^1 = \lambda'$.

In other words $\rho \sim \rho'$ if and only if they represent the same path homotopy class in the configuration space $F(M,n)$. We also use the term *geometric braid* to mean an equivalence class of geometric braids.

The product of two geometric braids β and β' is defined to be the composition of the strands. More precisely, let β be represented by $\rho = \{\rho_1, \dots, \rho_n\}$ with $\rho_1(1) = p_{\sigma(1)}, \dots, \rho_n(1) = p_{\sigma(n)}$ and let β' be represented by $\rho' = \{\rho'_1, \dots, \rho'_n)$. Then the product $\beta\beta'$ is represented by

$$\rho * \rho' = \{\rho_1 * \rho'_{\sigma(1)}, \dots, \rho_n * \rho'_{\sigma(n)}\},$$

where $\rho_i * \rho'_{\sigma(i)}$ is the path product.

2.2 Removing strands

A *simple (half-open) curve* in a space M is a continuous injection $\theta\colon \mathbb{R}^+ = [0, \infty) \to M$. The distinct points $\{p_1, \dots, p_n\}$ in M are said to be *well-ordered* with respect to a simple curve θ if there exist points $t_i \in [0, 1]$ with $0 \le t_1 < t_2 < \dots < t_n$ such that $p_i = \theta(t_i)$ for $1 \le i \le n$.

Let $\mathbf{p} = (p_1, \dots, p_n)$ and $\mathbf{p}' = (p'_1, \dots, p'_n)$ be two sets of n distinct well-ordered points with respect to θ with $p_i = \theta(t_i)$ and $p'_i = \theta(t'_i)$. Define

$$L(\mathbf{p}, \mathbf{p}')(s) = \{L(\mathbf{p}, \mathbf{p}')_i(s) = \theta((1-s)t_i + st'_i) \mid 1 \le i \le n\}$$

for $0 \le s \le 1$; $L(\mathbf{p}, \mathbf{p}')(s) \in M^n$. Observe that, for each $1 \le i < j \le n$ and $0 \le s \le 1$,

$$(1-s)t_i + st'_i < (1-s)t_j + st'_j$$

as $t_i < t_j$ and $t'_i < t'_j$. So $L(\mathbf{p}, \mathbf{p}')(s)$ is a set of n distinct well-ordered points with respect to θ for $0 \le s \le 1$.

Now let $\mathbf{p} = (p_1, \dots, p_n)$ and $\mathbf{p}' = (p'_1, \dots, p'_n)$ be two sets of n distinct points on the curve θ. There exist unique permutations $\sigma, \tau \in \Sigma_n$ such that

$$\mathbf{p}_\sigma = (p_{\sigma(1)}, \dots, p_{\sigma(n)}) \text{ and } \mathbf{p}'_\tau = (p'_{\tau(1)}, \dots, p'_{\tau(n)})$$

are well-ordered with respect to θ. We call

$$L(\mathbf{p}_\sigma, \mathbf{p}'_\tau)^{\sigma^{-1}}\{L(\mathbf{p}_\sigma, \mathbf{p}'_\tau)_{\sigma^{-1}(i)} \mid 1 \le i \le n\}$$

an *n–strand θ–linear braid* from $\mathbf{p}$ to a permutation of $\mathbf{p}'$.

Let M be a space with a simple curve θ and let the basepoints $(p_1, p_2, \dots, p_n)$ of the braids on M be well-ordered with respect to θ. The *system of removing strands* $d_i\colon B_n(M) \to B_{n-1}(M)$ is defined as follows:

Definition Let $\beta \in B_n(M)$ be a braid represented by $\lambda = \{\lambda_1, \dots, \lambda_n\}$ with

$$\lambda_1(1) = p_{\sigma(1)}, \dots, \lambda_n(1) = p_{\sigma(n)}.$$

Then the braid $d_i(\beta)$ is defined to be the equivalence class represented by the path product of the strands given by

$$L * \{\lambda_1, \dots, \lambda_{i-1}, \lambda_{i+1}, \dots, \lambda_n\} * L',$$

where L is the θ–linear braid from $(p_1, \dots, p_{n-1})$ to $(p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n)$, and L' is the θ–linear braid from $(p_{\sigma(1)}, \dots, p_{\sigma(i-1)}, p_{\sigma(i+1)}, \dots, p_{\sigma(n)})$ to a permutation of $(p_1, \dots, p_{n-1})$.

It follows from this definition that the operation d_i does not depend on the choice of λ in the class β. Intuitively, the operation $d_i\colon B_n(M) \to B_{n-1}(M)$ is obtained by forgetting the i–th strand and gluing back to the fixed choice of the basepoints using θ–linear braids.

From now on we always assume that the space M has a simple curve θ and that the basepoints of the braids on M are located on the curve θ starting with a set $\mathbf{p}$ of well-ordered points with respect to θ and ending with a permutation on $\mathbf{p}$. Recall that there is a short exact sequence

$$1 \to P_n(M) \to B_n(M) \to \Sigma_n \to 1.$$

The braid group $B_n(M)$ acts on the right on the letters $\{1, 2, \dots, n\}$ through the epimorphism $B_n(M) \to \Sigma_n$, which can be described as follows. Let β be represented by an n–strand geometric braid

$$\lambda = \{\lambda_i(t) \mid 1 \le i \le n\}$$

with $\lambda_i(0) = p_i$. Then $i \cdot \beta$ is given by the formula

$$\lambda_i(1) = p_{i\cdot\beta}$$

for $1 \le i \le n$.

Proposition 2.1 [2, Proposition 4.2.1(1)] *Let M be a space with a simple curve. Then the operations*

$$d_i\colon B_n(M) \to B_{n-1}(M), \quad 1 \le i \le n,$$

satisfy the following identities:

(1) $d_i d_j = d_j d_{i+1}$ for $i \ge j$.

(2) $d_i(\beta\beta') = d_i(\beta) d_{i\cdot\beta}(\beta')$. □

Corollary 2.2 *The map d_i is homomorphism when restricted to the pure braid group $P_n(M)$.* □

Note In [2], the removing-strand operations are labeled by $d_0, \dots, d_{n-1}$ to coincide with simplicial terminology. The above identities are directly translated from [2, Proposition 4.2.1(1)].

2.3 Brunnian braids

Definition 2.3 Let M be a space with a simple curve. A braid $\beta \in B_n(M)$ is called *Brunnian* if $d_i(\beta) = 1$ for each $1 \le i \le n$. The set of n–strand Brunnian braids is denoted by $\mathrm{Brun}_n(M)$. For convention, any 1–strand braid is regarded as a Brunnian braid.

Intuitively a Brunnian braid means a braid that becomes trivial after removing any one of its strands. If $\beta, \beta' \in \mathrm{Brun}_n(M)$, then

$$d_i(\beta\beta') = d_i(\beta) d_{i\cdot\beta}(\beta') = 1$$

for $1 \le i \le n$ and so the product $\beta\beta' \in \mathrm{Brun}_n(M)$. Similar β^{-1} is Brunnian provided β is. Thus $\mathrm{Brun}_n(M)$ is a subgroup of $B_n(M)$.

Proposition 2.4 *Suppose M is a space with a simple curve. Then the subgroup $\mathrm{Brun}_n(M) \cap P_n(M)$ is normal in $B_n(M)$ for each $n \ge 1$.*

Proof Let $\beta \in \mathrm{Brun}_n(M) \cap P_n(M)$ and let $\gamma \in B_n(M)$. Then

$$\begin{aligned} d_i(\gamma\beta\gamma^{-1}) &= d_i(\gamma\beta) d_{i\cdot(\gamma\beta)}(\gamma^{-1}) \\ &= d_i(\gamma) d_{i\cdot\gamma}(\beta) d_{i\cdot(\gamma\beta)}(\gamma^{-1}) \\ &= d_i(\gamma) d_{i\cdot(\gamma\beta)}(\gamma^{-1}) \end{aligned}$$

for $1 \le i \le n$. Since $\beta \in P_n(M)$, the elements γ and $\gamma\beta$ have the same image in $\Sigma_n = B_n(M)/P_n(M)$ and so $i\cdot(\gamma\beta) = i\cdot\gamma$. The assertion follows from the equation

$$1 = d_i(1) = d_i(\gamma\gamma^{-1}) = d_i(\gamma) d_{i\cdot\gamma}(\gamma^{-1}) = d_i(\gamma) d_{i\cdot(\gamma\beta)}(\gamma^{-1}). \qquad \square$$

Corollary 2.5 *Let M be a space with a simple curve. Then $\mathrm{Brun}_n(M)$ is a normal subgroup of $B_n(M)$ for $n \ge 3$.*

Proof According to [2, Proposition 4.2.2], $\mathrm{Brun}_n(M) \le P_n(M)$ for $n \ge 3$ and hence the result. □

The case $n = 2$ is exceptional, since Corollary 2.5 does not hold in this case.

Proposition 2.6 *Let M be a connected 2–manifold. Then $\mathrm{Brun}_2(M)$ is a normal subgroup of $B_2(M)$ if and only if $\pi_1(M)=\{1\}$.*

Proof If $\pi_1(M)=\{1\}$, then $B_2(M)=\mathrm{Brun}_2(M)$ as $B_1(M)=\pi_1(M)$.

Suppose that $\pi_1(M)\neq\{1\}$. Let D^2 be a small disk in $M\smallsetminus\partial M$. The inclusion $f\colon D^2\to M$ induces canonical maps

$$(f,f)\colon F(D^2,2)\rightarrowtail F(M,2) \quad\text{and}\quad (f,f)\colon F(D^2,2)/\Sigma_2\rightarrowtail F(M,2)/\Sigma_2.$$

Thus there is a commutative diagram of short exact sequences of groups

$$\begin{array}{ccccccccc}
1 & \longrightarrow & P_2(M) & \longrightarrow & B_2(M) & \longrightarrow & \Sigma_2 & \longrightarrow & 1\\
 & & \uparrow & & \uparrow{\scriptstyle (f,f)_*} & & \| & & \\
1 & \longrightarrow & P_2(D^2) & \longrightarrow & B_2(D^2) & \longrightarrow & \Sigma_2 & \longrightarrow & 1.
\end{array}$$

Let σ_1 be a generator for $B_2(D^2)=\mathbb{Z}$. Then $(f,f)_*(\sigma_1)\neq 1$ in $B_2(M)$ as it has nontrivial image in $\Sigma_2=B_2(M)/P_2(M)$. From the commutative diagram

$$\begin{array}{ccc}
B_2(D^2) & \xrightarrow{(f,f)_*} & B_2(M)\\
\downarrow{\scriptstyle d_i} & & \downarrow{\scriptstyle d_i}\\
B_1(D^2)=\{1\} & \xrightarrow{f_*} & B_1(M)
\end{array}$$

for $i=1,2$, the element $\beta=(f,f)_*(\sigma_1)$ is a Brunnian braid on M. Let p_1 be the basepoint of M. Choose a loop

$$\omega\colon [0,1]\to M$$

with $\omega(0)=\omega(1)=p_1$ representing a nontrivial element in $\pi_1(M)$. Take the second basepoint p_2 such that p_2 is not on the curve $\omega([0,1])$ and construct a 2–strand braid γ represented by

$$\rho(t)=\{\rho_1(t),\rho_2(t)\}$$

with $\rho_1(t)=(\omega(t),t)$ and $\rho_2(t)=(p_2,t)$ for $0\le t\le 1$ in the cylinder $M\times I$. Then $d_1(\gamma)=1$ as represented by the straight line-segment given by ρ_2, and $d_2(\gamma)=[\omega]\neq 1$ is the path homotopy class represented by ω. Observe that γ is a pure braid. We have

$d_i(\gamma^{-1}) = (d_i(\gamma))^{-1}$. From

$$\begin{aligned} d_1(\gamma\beta\gamma^{-1}) &= d_1(\gamma)d_{1\cdot\gamma}(\beta)d_{1\cdot(\gamma\beta)}(\gamma^{-1}) \\ &= d_1(\gamma)d_1(\beta)d_2(\gamma^{-1}) \\ &= d_1(\gamma)d_1(\beta)d_2(\gamma)^{-1} \\ &= 1\cdot 1\cdot [\omega]^{-1} \\ &\neq 1, \end{aligned}$$

the conjugate $\gamma\beta\gamma^{-1}$ is not Brunnian and so $\mathrm{Brun}_2(M)$ is not normal. This finishes the proof. □

3 Generating sets for Brunnian braids on surfaces

In this section, M is a connected compact 2–dimensional (oriented or nonoriented) manifold. The classical Fadell–Neuwirth Theorem will be useful in computations.

Theorem 3.1 [7] *The coordinate projection*

$$\delta^{(i)}\colon F(M,n) \to F(M,n-1),\ (x_1,\dots,x_n) \mapsto (x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)$$

is a fiber bundle with fiber $M \smallsetminus Q_{n-1}$, where Q_{n-1} is a set of $(n-1)$ distinct points in M. □

Proposition 3.2 *Up to a change of basepoint for the pure braid group $P_n(M)$ the homomorphism d_i coincides with the homomorphism of fundamental groups induced by $\delta^{(i)}$:*

$$d_i = h_i\delta^{(i)}_*\colon P_n(M) \to P_{n-1}(M),$$

where h_i is the automorphism of $\pi_1(F(M,n-1))$ induced by the change of basepoints

$$(F(M,n-1),(p_1,\dots,p_{i-1},p_{i+1},\dots,p_n)) \to (F(M,n-1),(p_1,\dots,p_{n-1})).$$ □

Let D^2 be a small disk in $M \smallsetminus \partial M$. The basepoints $\{p_1,p_2,\dots\}$ for the braids on M are chosen inside $D^2 \smallsetminus \partial D^2$. The embedding $f\colon D^2 \rightarrowtail M$ induces a map

$$f^n\colon F(D^2,n)/\Sigma_n \rightarrowtail F(M,n)/\Sigma_n$$

and so a group homomorphism

$$f^n_*\colon B_n(D^2) = \pi_1(F(D^2,n)/\Sigma_n) \longrightarrow B_n(M) = \pi_1(F(M,n)/\Sigma_n)$$

with a commutative diagram

$$\begin{array}{ccc} B_n(D^2) & \xrightarrow{f_*^n} & B_n(M) \\ \downarrow & & \downarrow \\ B_n(D^2)/P_n(D^2) = \Sigma_n & = & \Sigma_n = B_n(M)/P_n(M). \end{array}$$

For any braid $\beta \in B_n(D^2)$, we write $\beta[M]$ (or simply β if there are no confusions) for the braid $f_*^n(\beta)$ on M.

Recall that the Artin braid group $B_n(D^2)$ is generated by $\sigma_1, \dots, \sigma_{n-1}$ with defining relations

(1) $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j| \geq 2$;

(2) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for each i,

where as a geometric braid, σ_i is the canonical i–th elementary braid of n–strands that twists the positions i and $i+1$ once with the i th strand above the $(i+1)$ st and puts the trivial strands on the remaining positions. Also recall that the pure Artin braid group $P_n(D^2)$ is generated by

$$A_{i,j} = \sigma_{j-1}\sigma_{j-2}\cdots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\cdots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1}$$

for $1 \leq i < j \leq n$.

3.1 2–Strand Brunnian braids

Proposition 3.3 *Let M be any connected 2–manifold. Then the 2–strand Brunnian braids are determined as follows:*

(1) $\mathrm{Brun}_2(M) \cap P_2(M)$ *is the normal closure of the element $A_{1,2}$ in $B_2(M)$.*

(2) $\mathrm{Brun}_2(M)$ *is the subgroup of $B_2(M)$ generated by* $\mathrm{Brun}_2(M) \cap P_2(M)$ *and* σ_1, *that is* $\mathrm{Brun}_2(M) = \langle \mathrm{Brun}_2(M) \cap P_2(M), \sigma_1 \rangle$.

Proof (1) Let $\langle\langle A_{1,2} \rangle\rangle^B$ be the normal closure of $A_{1,2}$ in $B_2(M)$. By Proposition 2.4, $\mathrm{Brun}_2(M) \cap P_2(M)$ is normal in $B_2(M)$. Since $A_{1,2}$ is a pure Brunnian braid,

$$\langle\langle A_{1,2} \rangle\rangle^B \leq \mathrm{Brun}_2(M) \cap P_2(M).$$

To see the equality, consider the commutative diagram of fiber sequences

$$\begin{array}{ccccc}
F & \longrightarrow & M \smallsetminus \{p_2\} & \overset{i'}{\hookrightarrow} & M \\
\downarrow & & \downarrow i_1 & & \| \\
M \smallsetminus \{p_1\} & \overset{i_2}{\hookrightarrow} & F(M,2) & \xrightarrow{\delta^{(2)}} & M \\
\downarrow i & & \downarrow \delta^{(1)} & & \downarrow \\
M & = & M & \longrightarrow & *,
\end{array}$$

where $i_2(x) = (p_1, x)$ and $i_1(x) = (x, p_2)$ and F is a homotopy fiber of i, which is equivalent to a fiber of i'. From the middle row, there is an exact sequence

$$\begin{aligned}
\text{(3-1)} \quad \pi_2(M) \longrightarrow \pi_1(M \smallsetminus \{p_1\}) \xrightarrow{i_{2*}} \pi_1(F(M,2)) \\
= P_2(M) \overset{d_2}{\twoheadrightarrow} \pi_1(M) = P_1(M).
\end{aligned}$$

Note that

$$\mathrm{Brun}_2(M) \cap P_2(M) = \mathrm{Ker}(d_1\colon P_2(M) \to P_1(M)) \cap \mathrm{Ker}(d_2\colon P_2(M) \to P_1(M)).$$

Consider the diagram of short exact sequences of groups

$$\text{(3-2)} \qquad \begin{array}{ccc}
\langle\langle \omega \rangle\rangle & \overset{i_{2*}|}{\twoheadrightarrow} & \mathrm{Brun}_2(M) \cap P_2(M) \\
\downarrow & & \downarrow \\
\pi_1(M \smallsetminus \{p_1\}) & \overset{i_{2*}}{\twoheadrightarrow} & \mathrm{Ker}(d_2\colon P_2(M) \to P_1(M)) \\
\downarrow i_* & & \downarrow d_1 \\
P_1(M) & = & P_1(M),
\end{array}$$

where $\omega \in \pi_1(M \smallsetminus \{p_1\})$ is represented by a small circle around p_1. Its commutativity follows from construction and i_{2*} is an epimorphism by the exact sequence (3-1). It follow from diagram (3-2) that $\mathrm{Brun}_2(M) \cap P_2(M)$ is the normal closure of $i_{2*}(\omega)$ in $\mathrm{Ker}(d_2)$. From the commutative diagram

$$\begin{array}{ccc}
\pi_1(M \smallsetminus \{p_1\}) & \xrightarrow{i_{2*}} & \pi_1(F(M,2)) \\
\uparrow f_* & & \uparrow (f,f)_* \\
\pi_1(D^2 \smallsetminus \{p_1\}) = \mathbb{Z} & \overset{i_{2*}}{\underset{\cong}{\to}} & \pi_1(F(D,2)),
\end{array}$$

we get

$$i_{2*}(\omega) = A_{1,2}^{\pm 1}$$

and hence assertion (1) follows.

(2) Note that the braid σ_1 is Brunnian and represents the nontrivial element of $B_2(M)/P_2(M) \simeq \mathbb{Z}/2$. From the short exact sequence

$$1 \to P_2(M) \to B_2(M) \to \Sigma_2 \to 1,$$

we get the following commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{Brun}_2(M) \cap P_2(M) & \longrightarrow & \mathrm{Brun}_2(M) & \longrightarrow & \Sigma_2 & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & P_2(M) & \longrightarrow & B_2(M) & \longrightarrow & \Sigma_2 & \longrightarrow & 1, \end{array}$$

and the assertion follows. □

Corollary 3.4 *Let M be a connected 2–manifold. Then*

$$B_2(M)/(\mathrm{Brun}_2(M) \cap P_2(M))$$

is the quotient group of $B_2(M)$ obtained by adding the single relation

$$A_{1,2} = \sigma_1^2 = 1.$$

□

3.2 Homotopy properties of configuration spaces of surfaces

The following (well-known) fact will be useful for the computations in the next subsections.

Lemma 3.5 *Let M be a connected 2–manifold.*

(1) *If $M \neq S^2$ or $\mathbb{R}\mathrm{P}^2$, then $F(M,n)$ is a $K(\pi,1)$–space for $n \geq 1$. In particular, $\pi_2(F(M,n)) = 0$ for $n \geq 1$.*

(2) *$\pi_2(F(S^2,n)) = 0$ for $n \geq 3$.*

(3) *$\pi_2(F(\mathbb{R}\mathrm{P}^2,n)) = 0$ for $n \geq 2$.*

Proof Assertion (1) follows from the fact that M and $M \setminus Q_{n-1}$ are $K(\pi,1)$ spaces together with Fadell–Neuwirth fibration (Theorem 3.1).

Assertion (2) was proved by Fadell and Van Buskirk [8, Corollary, page 244].

Assertion (3) was proved by Van Buskirk in [22, Corollary, page 82]. □

3.3 3–Strand Brunnian braids

We will now determine the 3–strand Brunnian braids on M. By [2, Proposition 4.2.2],

$$\mathrm{Brun}_n(M) \subseteq P_n(M)$$

for $n \geq 3$. Thus the determination is given by

$$\mathrm{Brun}_n(M) = \mathrm{Brun}_n(M) \cap P_n(M) = \bigcap_{i=1}^{n} \mathrm{Ker}(d_i\colon P_n(M) \to P_{n-1}(M))$$

for $n \geq 3$.

For a subset S in $P_n(M)$, we write $\langle\langle S\rangle\rangle^P$ for the normal closure of S in $P_n(M)$, while we keep the notation $\langle\langle S\rangle\rangle$ for the normal closure of S in $B_n(M)$.

Proposition 3.6 *Let M be a connected 2–manifold. Then the 3–strand Brunnian braids on M are determined as follows:*

(1) $\mathrm{Brun}_3(S^2) = P_3(S^2) = \mathbb{Z}/2$.

(2) *If $M \neq S^2$ or $\mathbb{R}\mathrm{P}^2$, then*

$$\mathrm{Brun}_3(M) = [\langle\langle A_{1,3}\rangle\rangle^P, \langle\langle A_{2,3}\rangle\rangle^P],$$

the commutator subgroup of the normal closures in $P_3(M)$ of $A_{1,3}$ and $A_{2,3}$, respectively.

Proof Assertion (1) follows directly from the fact that $P_3(S^2) = \mathbb{Z}/2$ (which follows, for example, from [9, Theorem 3.1]) and $P_2(S^2) = \{1\}$. For assertion (2), observe that $d_k A_{i,j} = 1$ for $k = i, j$. Thus

$$\langle\langle A_{i,3}\rangle\rangle^P \leq \mathrm{Ker}(d_3\colon P_3(M) \to P_2(M)) \cap \mathrm{Ker}(d_i\colon P_3(M) \to P_2(M))$$

for $i = 1, 2$ and so the inclusion

$$[\langle\langle A_{1,3}\rangle\rangle^P, \langle\langle A_{2,3}\rangle\rangle^P] \leq \mathrm{Brun}_3(M)$$

is clear.

From the commutative diagram of the fiber sequences

$$\begin{array}{ccccc} M \smallsetminus \{p_1, p_2\} & \xrightarrow{i_3} & F(M,3) & \xrightarrow{\delta_3} & F(M,2) \\ \downarrow & & \downarrow \delta_2 & & \downarrow \delta_2 \\ M \smallsetminus \{p_1\} & \xrightarrow{i_2} & F(M,2) & \xrightarrow{\delta_2} & M, \end{array} \tag{3-3}$$

where $i_3(x)=(p_1,p_2,x)$ and $i_2(x)=(p_1,x)$, together with the facts that $\pi_2(M)=0$ and $\pi_2(F(M,2))=0$ (Lemma 3.5), there is a commutative diagram of short exact sequences

$$\begin{array}{ccccc} \pi_1(M\smallsetminus\{p_1,p_2\}) & \overset{i_{3*}}{\hookrightarrow} & P_3(M) & \overset{d_3}{\twoheadrightarrow} & P_2(M) \\ \downarrow{\scriptstyle d_2|} & & \downarrow{\scriptstyle d_2} & & \downarrow{\scriptstyle d_2} \\ \pi_1(M\smallsetminus\{p_1\}) & \overset{i_{2*}}{\hookrightarrow} & P_2(M) & \overset{d_2}{\twoheadrightarrow} & P_1(M). \end{array} \tag{3-4}$$

It follows from this diagram that

$$i_{3*}\colon \mathrm{Ker}(d_2|) \longrightarrow \mathrm{Ker}(d_3\colon P_3(M)\to P_2(M))\cap \mathrm{Ker}(d_2\colon P_3(M)\to P_2(M))$$

is an isomorphism. Since $d_2|\colon \pi_1(M\smallsetminus\{p_1,p_2\})\to\pi_1(M\smallsetminus\{p_1\})$ is induced by the inclusion

$$M\smallsetminus\{p_1,p_2\}\hookrightarrow M\smallsetminus\{p_1\},$$

$\mathrm{Ker}(d_2|)$ is the normal closure of $[\omega_2]$ in $\pi_1(M\smallsetminus\{p_1,p_2\})$, where ω_2 is a small circle around p_2. Similarly, the inclusion

$$M\smallsetminus\{p_1,p_2\}\hookrightarrow M\smallsetminus\{p_2\}$$

induces a homomorphism

$$d_1|\colon \pi_1(M\smallsetminus\{p_1,p_2\})\longrightarrow \pi_1(M\smallsetminus\{p_2\})$$

with the property that

$$i_{3*}\colon \mathrm{Ker}(d_1|) \longrightarrow \mathrm{Ker}(d_3\colon P_3(M)\to P_2(M))\cap \mathrm{Ker}(d_1\colon P_3(M)\to P_2(M))$$

is an isomorphism and $\mathrm{Ker}(d_1|)$ is the normal closure of the homotopy class $[\omega_1]$ in $\pi_1(M\smallsetminus\{p_1,p_2\})$, where ω_1 is a small circle around p_1. Thus

$$i_{3*}\colon \mathrm{Ker}(d_1|)\cap\mathrm{Ker}(d_2|)\longrightarrow \mathrm{Brun}_3(M) \tag{3-5}$$

is an isomorphism. By applying results of Brown [4] and Brown and Loday [5] to the homotopy pushout diagram of $K(\pi,1)$–spaces

$$\begin{array}{ccc} M\smallsetminus\{p_1,p_2\} & \hookrightarrow & M\smallsetminus\{p_1\} \\ \downarrow & & \downarrow \\ M\smallsetminus\{p_2\} & \hookrightarrow & M, \end{array}$$

we get an isomorphism

$$\frac{\mathrm{Ker}(d_1|)\cap \mathrm{Ker}(d_2|)}{[\mathrm{Ker}(d_1|),\mathrm{Ker}(d_2|)]}\cong \pi_2(M)=0,$$

and so,

$$\mathrm{Ker}(d_1|)\cap \mathrm{Ker}(d_2|)=[\mathrm{Ker}(d_1|),\mathrm{Ker}(d_2|)].$$

Together with the isomorphism (3-5) this gives

$$\mathrm{Brun}_3(M)=[\langle\langle i_{3*}([\omega_1])\rangle\rangle^P,\langle\langle i_{3*}([\omega_2])\rangle\rangle^P]. \tag{3-6}$$

Note that the basepoints $\{p_1,p_2\}$ are chosen in the interior of the small disk D^2. From the commutative diagram

$$\begin{array}{ccc} \pi_1(M\smallsetminus\{p_1,p_2\}) & \xrightarrow{i_{3*}} & P_3(M) \\ \uparrow{\scriptstyle f_*} & & \uparrow{\scriptstyle f^3_*} \\ \pi_1(D^2\smallsetminus\{p_1,p_2\}) & \xrightarrow{i_{3*}} & P_3(D^2), \end{array} \tag{3-7}$$

we have $i_{3*}([\omega_1])=A^{\pm1}_{1,3}$ and $i_{3*}([\omega_2])=A^{\pm1}_{2,3}$. Assertion (2) follows from replacing $i_{3*}([\omega_i])$ by $A_{i,3}$ in Equation (3-6). □

The projective plane case is dealt with separately in Section 4.

3.4 Colimits of classifying spaces

Given a group G and its normal subgroups $R_1,\dots,R_n$, let us define their *complete commutator subgroup* as follows

$$[\![R_1,R_2,\dots,R_n]\!]:=\prod_{\substack{I\cup J=\{1,2,\dots,n\}\\ I\cap J=\varnothing}}\Big[\bigcap_{i\in I}R_i,\bigcap_{j\in J}R_j\Big]. \tag{3-8}$$

It is clear that

$$[\![R_1,\dots,R_n]\!]\subseteq R_1\cap\dots\cap R_n$$

and that the quotient

$$\frac{R_1\cap\dots\cap R_n}{[\![R_1,R_2,\dots,R_n]\!]}$$

is an abelian group with a natural $\mathbb{Z}[G/R_1\dots R_n]$–module structure, where the action is defined via conjugation in G. An n–tuple of normal subgroups $(R_1,\dots,R_n)$ is called *connected* in G if either $n \le 2$, or $n \ge 3$ and for all subsets $I, J \subset \{1, \dots, n\}$ with $|I| \ge 2, |J| \ge 1$ (without the conditions of formula (3-8)) we have the equality

$$\left(\bigcap_{i\in I} R_i\right)\left(\prod_{j\in J} R_j\right) = \bigcap_{i\in I}\left(R_i\left(\prod_{j\in J} R_j\right)\right). \tag{3-9}$$

We will make use of the following result from Ellis and Mikhailov [6]:

Theorem 3.7 *Let G be a group, $n \ge 2$, and $(R_1, \dots, R_n)$ an n–tuple of normal subgroups in G such that the $(n-1)$–tuples $(R_1, \dots, \hat{R}_i, \dots, R_n)$ are connected for all $1 \le i \le n$. Let X be the topological space arising as the colimit of classifying spaces $K(G/\prod_{i\in I} R_i, 1)$, where I ranges over all subsets $I \subsetneq \{1, \dots, n\}$. Then there is an isomorphism of abelian groups*

$$\pi_n(X) \simeq \frac{R_1 \cap \cdots \cap R_n}{[\![R_1, \dots, R_n]\!]}.$$

3.5 n–Strand Brunnian braids for $n \ge 4$

Now we are going to determine $\mathrm{Brun}_n(M)$ for $n \ge 4$. The case $\mathrm{Brun}_4(S^2)$ was determined in [2, Proposition 7.2.2]. Our computation will exclude this special case.

Lemma 3.8 *Let M be a connected 2–manifold. Let*

$$d_k\colon P_n(M) \to P_{n-1}(M)$$

be the operation that removes the k–th strand.

(1) *Suppose that $M \ne S^2$ or $\mathbb{R}\mathrm{P}^2$. Then, for $n \ge 2$,*

$$\mathrm{Ker}(d_n) \cap \mathrm{Ker}(d_k) = \langle\!\langle A_{k,n} \rangle\!\rangle^P$$

for $1 \le k \le n-1$ and therefore

$$\mathrm{Brun}_n(M) = \bigcap_{k=1}^{n-1} \langle\!\langle A_{k,n} \rangle\!\rangle^P.$$

Moreover $i_{n*}\colon \pi_1(M \smallsetminus \{p_1, p_2, \dots, p_{n-1}\}) \to P_n(M)$ is a monomorphism with

$$i_{n*}(\mathrm{Ker}(d_k|)) = \langle\!\langle A_{k,n} \rangle\!\rangle^P,$$

where i_n is given as in (3-3) and

$$d_k|\colon \pi_1(M \smallsetminus \{p_1, \dots, p_{n-1}\}) \longrightarrow \pi_1(M \smallsetminus \{p_1, \dots, p_{k-1}, p_{k+1}, \dots, p_{n-1}\})$$

is the group homomorphism induced by inclusion.

(2) *If $M = S^2$, then the above statement holds for $n \ge 5$.*

(3) *If $M = \mathbb{R}\mathrm{P}^2$, then the above statement holds for $n \ge 4$.*

Proof Diagram (3-3) can be extended to the general case, and so we have the starting commutative diagram for $n \geq 2$ and $1 \leq k \leq n-1$

$$\begin{array}{ccccc} \pi_1(M \smallsetminus \{p_1, p_2, \dots, p_{n-1}\}) & \xhookrightarrow{i_{n*}} & P_n(M) & \xrightarrow{d_n} & P_{n-1}(M) \\ \downarrow d_k| & & \downarrow d_k & & \downarrow d_k \\ \pi_1(M \smallsetminus \{p_1, \dots, p_{k-1}, p_{k+1}, \dots, p_{n-1}\}) & \xrightarrow{i_{k*}} & P_{n-1}(M) & \xrightarrow{d_{n-1}} & P_{n-2}(M) \end{array} \tag{3-10}$$

where $i_k(x) = (p_1, \dots p_{k-1}, x, p_{k+1}, \dots, p_{n-1})$. Let us consider the homomorphism

$$i_{k*}\colon \pi_1(M \smallsetminus \{p_1, \dots, p_{k-1}, p_{k+1}, \dots, p_{n-1}\}) \to P_{n-1}(M).$$

It is a monomorphism except 2 cases:

$$M = S^2 \quad \text{and } n = 4, \tag{3-11}$$

$$M = \mathbb{R}P^2 \text{ and } n = 3. \tag{3-12}$$

For $n = 2$ this is identical isomorphism. For $n = 3$ and $M = S^2$ $\pi_1(M \smallsetminus \{p_i\})$, $i = 1, 2$, is the trivial group, so i_{k*} is a monomorphism. For the other cases it follows from the exact sequence of the fibration and since

$$\pi_2(F(M, n-2)) = 0 \quad \text{for} \begin{cases} n \geq 3 & \text{if } M \neq \mathbb{R}\mathrm{P}^2, S^2, \\ n \geq 4 & \text{if } M = \mathbb{R}P^2, \\ n \geq 5 & \text{if } M = S^2 \end{cases}$$

(by Lemma 3.5). For the exceptional case (3-11) $\pi_1(S^2 \smallsetminus \{p_1, p_k, p_3\})$ is infinite cyclic and $P_3(S^2)$ is the cyclic group of order 2. For the exceptional case (3-12) $\pi_1(\mathbb{R}P^2 \smallsetminus \{p_i\})$, $i = 1, 2$, is infinite cyclic and $P_2(\mathbb{R}P^2)$ is isomorphic to the finite quaternionic group $\mathbf{Q}_8$ [22] (see also Section 4). Thus

$$i_{n*}\colon \mathrm{Ker}(d_k|) \longrightarrow \mathrm{Ker}(d_n) \cap \mathrm{Ker}(d_k)$$

is an isomorphism for the cases

$$n \geq 2 \quad \text{if } M \neq S^2, \mathbb{R}\mathrm{P}^2, \tag{3-13}$$

$$n > 3 \quad \text{if } M = \mathbb{R}P^2, \tag{3-14}$$

$$n > 4 \quad \text{if } M = S^2. \tag{3-15}$$

Note that $\mathrm{Ker}(d_k|)$ is the normal closure in $\pi_1(M \smallsetminus \{p_1, p_2, \dots, p_{n-1}\})$ of the homotopy class $[\omega_k]$, where ω_k is a small circle around p_k. For the same reasons as in diagram (3-7), we have $i_{n*}([\omega_k]) = A_{k,n}^{\pm 1}$ and so

$$\mathrm{Ker}(d_n) \cap \mathrm{Ker}(d_k) = i_{n*}(\mathrm{Ker}(d_k|)) \le \langle\langle A_{k,n} \rangle\rangle^P.$$

On the other hand, $\langle\langle A_{k,n} \rangle\rangle^P \le \mathrm{Ker}(d_n) \cap \mathrm{Ker}(d_k)$ because $A_{k,n}$ lies in the normal subgroup $\mathrm{Ker}(d_n) \cap \mathrm{Ker}(d_k)$. Thus, for all cases (3-13)–(3-15),

$$\mathrm{Ker}(d_n) \cap \mathrm{Ker}(d_k) = \langle\langle A_{k,n} \rangle\rangle^P.$$

Hence, the result. □

The remaining question is of course how to determine the intersection of the normal subgroups $\langle\langle A_{k,n} \rangle\rangle^P$ for general n. The following result is also given by Li and Wu [17, Equation (4.1)]. (Note: In [17], the proof was given by checking $K(\pi, 1)$–hypothesis. Our proof is given by checking the connectivity hypothesis in Theorem 3.7.)

Theorem 3.9 *Let M be a connected 2–manifold and let $\{p_1, \dots, p_n\}$ be the set of n distinct points in $M \smallsetminus \partial M$. Let*

$$d_i|\colon \pi_1(M \smallsetminus \{p_1, \dots, p_n\}) \longrightarrow \pi_1(M \smallsetminus \{p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n\})$$

be the group homomorphism induced from the inclusion by filling in the missing point p_i. Then

$$\left(\bigcap_{i=1}^{n} \mathrm{Ker}(d_i|) \right) \Big/ [\mathrm{Ker}(d_1|), \mathrm{Ker}(d_2|), \dots, \mathrm{Ker}(d_n|)]_S \cong \pi_n(M)$$

for each $n \ge 2$.

Proof The surface M can be viewed as a colimit of the spaces $M \smallsetminus \bigsqcup_{i \in I} p_i$, where I ranges over all subsets $I \subsetneq \{1, \dots, n\}$. Denote $G := \pi_1(M \smallsetminus \{p_1, \dots, p_n\})$ and $R_i := \mathrm{Ker}(d_i|)$. Since punctured surfaces are aspherical, the spaces $M \smallsetminus \bigsqcup_{i \in I} p_i$ are classifying spaces for groups $G / \prod_{i \in I} R_i$. Let us check that the connectivity condition (3-9) holds for every $(n-1)$–tuple of subgroups $(R_1, \dots, \widehat{R}_m, \dots, R_n)$, $1 \le m \le n$. For $n = 2, 3$, the connectivity condition holds by definition. We prove the statement by induction on n. We fix the number m: $1 \le m \le n$, and prove the connectivity (3-9) of the $(n-1)$–tuple $(R_1, \dots, \widehat{R}_m, \dots, R_n)$. Let $I, J \subseteq \{1, \dots, \hat{m}, \dots, n\}$. Suppose that $I \cap J \ne \varnothing$. Then the left and right-hand sides of (3-9) are equal to $\prod_{j \in J} R_j$ and the condition is proved. So, we can assume that $I \cap J = \varnothing$. Consider the epimorphism

$$f_J\colon G \to G \Big/ \prod_{j \in J} R_j.$$

The condition (3-9) is equivalent to the condition

$$f_J\Big(\bigcap_{i\in I} R_i\Big) = \bigcap_{i\in I} f_J(R_i). \tag{3-16}$$

Any punctured surface has a free fundamental group and

$$f_J(R_i) = \mathrm{Ker}\{\pi_1\Big(M \smallsetminus \bigsqcup_{k\in I} p_k\Big) \to \pi_1\Big(M \smallsetminus \bigsqcup_{k\in I,\, k\neq i} p_k\Big)\}.$$

By induction we have

$$\bigcap_{i\in I} R_i = [\![R_{i_1},\dots,R_{i_{|I|}}]\!]$$

for $I=\{i_1,\dots,i_{|I|}\}$ due to Theorem 3.7 and the fact that punctured surface is aspherical. The same argument shows that

$$\bigcap_{i\in I} f_J(R_i) = [\![f_J(R_{i_1}),\dots,f_J(R_{i_{|I|}})]\!]$$

(we repeat argument for the punctured surface with discs glued to $|J|$ boundary components, the surface remains punctured since $M \smallsetminus \{p_1,\dots,p_n\}$ has at least n boundary components). The same argument shows that

$$[\![R_{i_1},\dots,R_{i_{|I|}}]\!] = [R_{i_1},\dots,R_{i_{|I|}}]_S,$$
$$[\![f_J(R_{i_1}),\dots,f_J(R_{i_{|I|}})]\!] = [f_J(R_{i_1}),\dots,f_J(R_{i_{|I|}})]_S.$$

Since f_J is a homomorphism, the condition (3-16) and hence (3-9) follow. Again observe that

$$[\![R_1,R_2,\dots,R_n]\!] = [R_1,\dots,R_n]_S,$$

hence the needed statement follows from Theorem 3.7. □

Proof of Theorem 1.1 By Lemma 3.8,

$$\mathrm{Brun}_n(M) = \bigcap_{i=1}^{n-1} \langle\langle A_{i,n}\rangle\rangle^P$$

and $\langle\langle A_{k,n}\rangle\rangle^P = i_{n*}(\mathrm{Ker}(d_k|))$. The assertion follows by Theorem 3.9. □

4 3–Strand Brunnian braids on the projective plane

4.1 Braid group of the projective plane

There exist several presentations of the group $B_n(\mathbb{R}\mathrm{P}^2)$. See, for example, van Buskirk [22] or Gonçalves and Guaschi [10]. We will use a presentation similar to presentations of the surface braid group from [22].

Theorem 4.1 *The group $B_n(\mathbb{R}\mathrm{P}^2)$ can be presented as having the set of generators*

$$\sigma_1, \sigma_2, \dots, \sigma_{n-1}, \rho,$$

where in the braid ρ the first string represents a nontrivial element of the fundamental group and the rest of the braid is trivial; the generators $\sigma_1, \sigma_2, \dots, \sigma_{n-1}$ are the images of classical braid generators of the disk; the set of defining relations is the following:

$$\begin{aligned} \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1}, & i &= 1, 2, \dots, n-2, \\ \sigma_i\sigma_j &= \sigma_j\sigma_i, & |i-j| &> 1, \\ \rho\sigma_i &= \sigma_i\rho, & i &\neq 1, \\ \sigma_1^{-1}\rho\sigma_1^{-1}\rho &= \rho\sigma_1^{-1}\rho\sigma_1, \\ \rho^2 &= \sigma_1\sigma_2\cdots\sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_2\sigma_1. \end{aligned}$$

The proof of Theorem 4.1 is given in the Appendix.

Remark 4.2 Geometrically, the element ρ can be depicted similarly to that of [1, Figure 10].

There is a canonical homomorphism $\tau\colon B_n(\mathbb{R}\mathrm{P}^2) \longrightarrow \Sigma_n$, $\tau(\sigma_i) = (i, i+1)$, $\tau(\rho) = e$. The kernel, $\mathrm{Ker}(\tau)$, is the pure braid group $P_n(\mathbb{R}\mathrm{P}^2)$. This group was studied in [10]. We will find a presentation for $P_3(\mathbb{R}\mathrm{P}^2)$ which we shall use later. Consider at first the group $B_2(\mathbb{R}\mathrm{P}^2)$. We have

$$B_2(\mathbb{R}\mathrm{P}^2) = \langle \rho, \sigma_1 \mid \sigma_1^{-1}\rho\sigma_1^{-1}\rho = \rho\sigma_1^{-1}\rho\sigma_1,\ \rho^2 = \sigma_1^2 \rangle.$$

This group has order 16 and $P_2(\mathbb{R}\mathrm{P}^2)$ is isomorphic to the quaternion group $\mathbf{Q}_8$ of order 8 [22]. The relation $\rho^2 = \sigma_1^2$ gives that $P_2(\mathbb{R}\mathrm{P}^2)$ is normally generated by ρ. Let us define the following element of $P_2(\mathbb{R}\mathrm{P}^2)$:

$$u = \sigma_1\rho\sigma_1^{-1}.$$

The Reidemeister–Schreier method (see [18, Theorem 2.9]) gives the presentation

$$P_2(\mathbb{R}\mathrm{P}^2) = \langle \rho, u \mid \rho u \rho = u,\ \rho^2 = u^2 \rangle. \tag{4-1}$$

This presentation is equivalent to

$$P_2(\mathbb{R}\mathrm{P}^2) = \langle \rho, u \mid \rho u \rho = u^{-1},\ \rho^2 = u^2 \rangle,$$

which appears in Lemma 4.6.

Consider now the case $n = 3$. We have

$$B_3(\mathbb{R}\mathrm{P}^2) = \langle \rho, \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2,\ \rho\sigma_2 = \sigma_2\rho, \\ \sigma_1^{-1}\rho\sigma_1^{-1}\rho = \rho\sigma_1^{-1}\rho\sigma_1, \rho^2 = \sigma_1\sigma_2^2\sigma_1 \rangle.$$

To construct a presentation for $P_3(\mathbb{R}\mathrm{P}^2)$ we use the Reidemeister–Schreier method. As representatives of cosets of the normal subgroup $P_3(\mathbb{R}\mathrm{P}^2)$ in the group $B_3(\mathbb{R}\mathrm{P}^2)$ we take the elements: e, σ_1, σ_2, $\sigma_2\sigma_1$ $\sigma_1\sigma_2$, $\sigma_1\sigma_2\sigma_1$. Then by [18, Theorem 2.7] the group $P_3(\mathbb{R}\mathrm{P}^2)$ is generated by elements

$$ka\overline{(ka)}^{-1},$$

where $a \in \{\rho, \sigma_1, \sigma_2\}$, $k \in \{e, \sigma_1, \sigma_2, \sigma_2\sigma_1, \sigma_1\sigma_2, \sigma_1\sigma_2\sigma_1\}$ and the bar denotes the mapping from words to their coset representatives [18, page 88]. Having in mind that $\sigma_2\rho\sigma_2^{-1} = \rho$, we obtain that the group $P_3(\mathbb{R}\mathrm{P}^2)$ is generated by

$$\rho, \quad u = \sigma_1\rho\sigma_1^{-1}, \quad w = \sigma_2\sigma_1\rho\sigma_1^{-1}\sigma_2^{-1}, \quad A_{12}, \quad A_{23} = \sigma_2^2, \quad A_{13} = \sigma_2\sigma_1^2\sigma_2^{-1}.$$

The following set of defining relations is obtained by application of Reidemeister–Schreier method [18, Theorem 2.9]:

$$A_{12}A_{13}A_{12}^{-1} = A_{23}^{-1}A_{13}A_{23}, \quad A_{12} = \sigma_1^2(A_{13}A_{23})A_{12}^{-1} = A_{13}A_{23},$$

$$\rho A_{23}\rho^{-1} = A_{23}, \quad u(A_{23}^{-1}A_{13}A_{23})u^{-1} = A_{23}^{-1}A_{13}A_{23}, \tag{4-2}$$

$$\rho(A_{13}^{-1}w^{-1}A_{13})\rho^{-1} = w^{-1}A_{13}, \quad \rho(A_{13}^{-1}w)\rho^{-1} = w, \tag{4-3}$$

$$\rho(A_{12}^{-1}u)\rho^{-1} = u,$$

$$u(A_{23}^{-1}w^{-1}A_{23})u^{-1} = w^{-1}A_{23}, \quad u(A_{23}^{-1}w)u^{-1} = w, \tag{4-4}$$

$$A_{23}^{-1}A_{13}A_{23}A_{12} = \rho^2, \quad A_{12}A_{13} = \rho^2, \quad A_{12}A_{23} = u^2, \tag{4-5}$$

$$A_{13}A_{23} = w^2. \tag{4-6}$$

From these relations we have the following formulas for conjugation by A_{12}, ρ, u:

$$A_{12}A_{13}A_{12}^{-1} = A_{23}^{-1}A_{13}A_{23}, \quad A_{12}A_{23}A_{12}^{-1} = A_{23}^{-1}A_{13}^{-1}A_{23}A_{13}A_{23},$$

(4-7) $$A_{12}wA_{12}^{-1} = w,$$

(4-8) $$\rho w\rho^{-1} = w^{-1}A_{13}^{-1}w^2, \quad \rho A_{13}\rho^{-1} = w^{-1}A_{13}^{-1}w,$$
$$\rho A_{23}\rho^{-1} = A_{23},$$

(4-9) $$uwu^{-1} = w^{-1}A_{23}^{-1}w^2, \quad uA_{23}u^{-1} = w^{-1}A_{23}^{-1}w,$$

(4-10) $$uA_{13}u^{-1} = w^{-1}A_{23}^{-1}wA_{23}^{-1}wA_{23}w.$$

Remark 4.3 Relation (4-7) can be more easily seen directly from the relations in $B_3(\mathbb{R}\mathrm{P}^2)$. Relations (4-8) are obtained from relations (4-3). Relations (4-9) are obtained from relations (4-4). Relation (4-10) is obtained from relations (4-6) and (4-9).

We see from these formulas that the subgroup

$$U_3(\mathbb{R}\mathrm{P}^2) = \langle w, A_{13}, A_{23} \mid A_{13}A_{23} = w^2\rangle$$

is normal in $P_3(\mathbb{R}\mathrm{P}^2)$. Geometrically, it can be identified with $\pi_1(\mathbb{R}\mathrm{P}^2 \smallsetminus \{p_1, p_2\})$ which is included in the short exact sequence (see diagram (3-4))

$$\pi_1(\mathbb{R}\mathrm{P}^2 \smallsetminus \{p_1, p_2\}) \xhookrightarrow{i_{3*}} P_3(\mathbb{R}\mathrm{P}^2) \xrightarrow{d_3} P_2(\mathbb{R}\mathrm{P}^2)$$

and so $U_3(\mathbb{R}\mathrm{P}^2)$ is the free group of rank 2 and $P_3(\mathbb{R}\mathrm{P}^2)/U_3(\mathbb{R}\mathrm{P}^2) \simeq P_2(\mathbb{R}\mathrm{P}^2)$.

We can exclude the generators A_{12}, A_{13} from the list of generators for $P_3(\mathbb{R}\mathrm{P}^2)$, using the formulas

(4-11) $$A_{12} = u\rho^{-1}u^{-1}\rho, \quad A_{13} = w^2A_{23}^{-1}.$$

The proof of the following statement is given in the Appendix.

Lemma 4.4 *The group $P_3(\mathbb{R}\mathrm{P}^2)$ is generated by elements*

$$\rho, \quad u, \quad w, \quad A_{23}$$

and has the following relations:

(1) $\rho w\rho^{-1} = w^{-1}A_{23}$, $\quad \rho A_{23}\rho^{-1} = A_{23}$,

(1′) $\rho^{-1}w\rho = A_{23}w^{-1}$, $\quad \rho^{-1}A_{23}\rho = A_{23}$,

(2) $uwu^{-1} = w^{-1}A_{23}^{-1}w^2$, $\quad uA_{23}u^{-1} = w^{-1}A_{23}^{-1}w$,

(2′) $u^{-1}wu = A_{23}^{-1}w$, $\quad u^{-1}A_{23}u = A_{23}^{-1}wA_{23}^{-1}w^{-1}A_{23}$,

(3) $\rho^{-1}u\rho^{-1}u^{-1} = wA_{23}^{-1}w$, $\quad u^{-1}\rho^{-1}u^{-1}\rho = A_{23}^{-1}$.

Remark 4.5 A similar presentation was constructed in [10, page 765], but in the list of relations there, in the fourth relation of formula (3) instead of

$$\rho_2^{-1} B_{2,3} \rho_2 = B_{2,3}^{-1} \rho_3 B_{2,3} \rho_3^{-1} B_{2,3},$$

it should be

$$\rho_2^{-1} B_{2,3} \rho_2 = B_{2,3}^{-1} \rho_3 B_{2,3}^{-1} \rho_3^{-1} B_{2,3}.$$

Let us introduce new generators $a = \rho w$, $b = wu$. Then we have from Lemma 4.4 the following statement.

Lemma 4.6 *The group $P_3(\mathbb{R}\mathrm{P}^2)$ can be generated by elements*

$$a, \quad b, \quad w, \quad A_{23}$$

and has the following relations:

(4) $$awa^{-1} = w^{-1} A_{23}, \quad aA_{23}a^{-1} = w^{-1} A_{23} w,$$

(4′) $$a^{-1}wa = w^{-1} A_{23}, \quad a^{-1} A_{23} a = w^{-1} A_{23} w,$$

(5) $$bwb^{-1} = A_{23}^{-1} w, \qquad bA_{23}b^{-1} = A_{23}^{-1},$$

(5′) $$b^{-1}wb = A_{23}^{-1} w, \qquad b^{-1} A_{23} b = A_{23}^{-1},$$

(6) $$bab^{-1} = a^{-1}, \qquad a^2 = b^2.$$

In particular, $\langle a, b \rangle \simeq P_2(\mathbb{R}\mathrm{P}^2) \leq P_3(\mathbb{R}\mathrm{P}^2)$. □

From this lemma we have the following statement.

Proposition 4.7 *There exists the split short exact sequence*

$$1 \longrightarrow U_3(\mathbb{R}\mathrm{P}^2) \longrightarrow P_3(\mathbb{R}\mathrm{P}^2) \xrightarrow{d_3} P_2(\mathbb{R}\mathrm{P}^2) \longrightarrow 1,$$

and hence $P_3(\mathbb{R}\mathrm{P}^2) = U_3(\mathbb{R}\mathrm{P}^2) \rtimes P_2(\mathbb{R}\mathrm{P}^2)$. □

This proposition was proved by Gonçalves and Guaschi [10]. It was also proved there that for $n = 2, 3$ and for all $m \geq 4$ the short exact sequence

$$1 \longrightarrow P_{m-n}(\mathbb{R}\mathrm{P}^2 \setminus \{x_1, x_2, \ldots, x_n\}) \longrightarrow P_m(\mathbb{R}\mathrm{P}^2) \longrightarrow P_n(\mathbb{R}\mathrm{P}^2) \longrightarrow 1$$

does not split.

4.2 3–Strand Brunnian braids on the projective plane

In order to pass to Brunnian braids recall the geometric interpretations for the generators ρ, u, w. We represent $\mathbb{R}\mathrm{P}^2$ as a 2–gon L where opposite points on the two edges are identified in the standard manner. In the braid ρ, the second and the third strings are just two parallel lines. Its first strand passes through the edge of L. The braids u and w are defined in a similar manner. In u, the second strand passes through the edge and, in w, the third one. The braid A_{23} is defined as in the braid group of a disk. Remember that the presentation for $P_2(\mathbb{R}\mathrm{P}^2)$ is given by formula (4-1). Hence the maps

$$d_1, d_2, d_3\colon P_3(\mathbb{R}\mathrm{P}^2) \longrightarrow P_2(\mathbb{R}\mathrm{P}^2)$$

act on the generators by the rules

$$d_1\colon \begin{cases} a & \longrightarrow u, \\ b & \longrightarrow u\rho, \\ A_{23} & \longrightarrow A_{12}, \\ w & \longrightarrow u, \end{cases} \qquad d_2\colon \begin{cases} a & \longrightarrow \rho u, \\ b & \longrightarrow u, \\ A_{23} & \longrightarrow 1, \\ w & \longrightarrow u, \end{cases} \qquad d_3\colon \begin{cases} a & \longrightarrow \rho, \\ b & \longrightarrow u, \\ A_{23} & \longrightarrow 1, \\ w & \longrightarrow 1. \end{cases}$$

From the exact sequence of Proposition 4.7 we see that $\mathrm{Brun}_3(\mathbb{R}\mathrm{P}^2)$ is a subgroup of $U_3(\mathbb{R}\mathrm{P}^2)$ and so in our study of Brunnian braids on $\mathbb{R}\mathrm{P}^2$ we can restrict ourselves looking at $U_3(\mathbb{R}\mathrm{P}^2)$ and the action of d_1 and d_2 on it. We write the action of d_3 as supplementary information.

We have

$$d_1(w^4) = d_2(w^4) = u^4, \quad d_3(w^4) = 1,$$

and since $u^4 = 1$ in $P_2(\mathbb{R}\mathrm{P}^2)$ then $w^4 \in \mathrm{Brun}_3(\mathbb{R}\mathrm{P}^2)$. Similarly

$$d_1(A_{23}^2) = A_{12}^2, \quad d_2(A_{23}^2) = d_3(A_{23}^2) = 1,$$

and since $A_{12}^2 = \sigma_1^4 = 1$ in $P_2(\mathbb{R}\mathrm{P}^2)$ (see formula (4-1)), then $A_{23}^2 \in \mathrm{Brun}_3(\mathbb{R}\mathrm{P}^2)$. For the commutator $[w, A_{23}]$ we have

$$d_1([w, A_{23}]) = [u, A_{12}], \quad d_2([w, A_{23}]) = d_3([w, A_{23}]) = 1,$$

and A_{12} lies in the center of $P_2(\mathbb{R}\mathrm{P}^2)$, so $d_1([w, A_{23}]) = 1$ and $[w, A_{23}] \in \mathrm{Brun}_3(\mathbb{R}\mathrm{P}^2)$.

Now we are going to determine a free basis for $\mathrm{Brun}_3(\mathbb{R}\mathrm{P}^2)$.

Lemma 4.8 *Let $F(S)$ be the free group (freely) generated by the set S. Given $x \in S$, let $C_q(\bar{x}) \cong \mathbb{Z}/q$ be the cyclic group of order q generated by a formal generator $\bar{x}$. Let $p_x\colon F(S) \to C_q(\bar{x})$ be the group homomorphism with $p(y) = 1$ for $y \neq x \in S$ and $p_x(x) = \bar{x}$. Then $\mathrm{Ker}(p_x)$ has a free basis*

$$\{x^q, y, [y, x^j] \mid y \in S,\ y \neq x,\ 1 \leq j \leq q-1\}.$$

Proof By using Schreier method, $\mathrm{Ker}(p_x)$ has a free basis

$$\{x^q, x^{-j} y x^j \mid y \in S,\ y \neq x, 0 \le j \le q-1\}$$

which is equivalent to the generating set in the statement as

$$[y, x^j] = y^{-1}(x^{-j} y x^j)$$

and hence the assertion. □

Proposition 4.9 *As a subgroup of* $B_3(\mathbb{R}\mathrm{P}^2)$, $\mathrm{Brun}_3(\mathbb{R}\mathrm{P}^2)$ *has a free basis given by*

$$\begin{array}{ccc} x_2^2, & x_1^4, & [x_1^4, x_2], \\ & [x_2, x_1], & [[x_2, x_1], x_2], \\ & [x_2, x_1^2], & [[x_2, x_1^2], x_2], \\ & [x_2, x_1^3], & [[x_2, x_1^3], x_2], \end{array}$$

where $x_1 = w$ *and* $x_2 = A_{2,3}$.

Proof Consider the projection $p_{x_1}\colon F(x_1, x_2) \to C_4(x_1)$. (It is d_2 in our case.) By the above lemma, $\mathrm{Ker}(p_{x_1})$ has a free basis given by

$$S = \{x_1^4, x_2, [x_2, x_1], [x_2, x_1^2], [x_2, x_1^3]\}.$$

The assertion follows by applying the above lemma to the projection $p_{x_2}\colon F(x_1, x_2) \to C_2(x_2)$ (d_1 in our case) restricted to the subgroup $F(S) = \mathrm{Ker}(p_{x_1})$. □

Let us describe the quotient groups $P_3(\mathbb{R}\mathrm{P}^2)/\mathrm{Brun}_3(\mathbb{R}\mathrm{P}^2)$ and $B_3(\mathbb{R}\mathrm{P}^2)/\mathrm{Brun}_3(\mathbb{R}\mathrm{P}^2)$.

Proposition 4.10 (1) *Let* $\overline{w}$ *and* $\overline{A}$ *be the images of* w *and* A_{23} *respectively after applying the natural projection*

$$U_3(\mathbb{R}\mathrm{P}^2) \longrightarrow U_3(\mathbb{R}\mathrm{P}^2)/\,\mathrm{Brun}_3(\mathbb{R}\mathrm{P}^2).$$

Then

$$U_3(\mathbb{R}\mathrm{P}^2)/\,\mathrm{Brun}_3(\mathbb{R}\mathrm{P}^2) = \langle \overline{w}, \overline{A} \mid \overline{w}^4 = \overline{A}^2 = 1,\ \overline{A}\overline{w} = \overline{w}\overline{A} \rangle \simeq \mathbb{Z}_4 \oplus \mathbb{Z}_2.$$

(2) *The quotient* $P_3(\mathbb{R}\mathrm{P}^2)/\,\mathrm{Brun}_3(\mathbb{R}\mathrm{P}^2)$ *has order* 64 *and is the semidirect product*

$$P_3(\mathbb{R}\mathrm{P}^2)/\,\mathrm{Brun}_3(\mathbb{R}\mathrm{P}^2) = (U_3(\mathbb{R}\mathrm{P}^2)/\,\mathrm{Brun}_3(\mathbb{R}\mathrm{P}^2)) \rtimes P_2(\mathbb{R}\mathrm{P}^2).$$

More precisely $P_3(\mathbb{R}\mathrm{P}^2)/\,\mathrm{Brun}_3(\mathbb{R}\mathrm{P}^2)$ *is generated by*

$$\overline{w}, \quad \overline{A}, \quad a, \quad b$$

and has defining relations:

$$\bar{w}^4 = \bar{A}^2 = 1, \quad \bar{A}\bar{w} = \bar{w}\bar{A}, \qquad bab^{-1} = a^{-1}, \quad a^2 = b^2,$$

(1) $$a^{-1}\bar{w}a = \bar{w}^{-1}\bar{A}, \quad a^{-1}\bar{A}a = \bar{A},$$

(1′) $$a\bar{w}a^{-1} = \bar{w}^{-1}\bar{A}, \quad a\bar{A}a^{-1} = \bar{A},$$

(2) $$b^{-1}\bar{w}b = \bar{w}\bar{A}, \qquad b^{-1}\bar{A}b = \bar{A},$$

(2′) $$b\bar{w}b^{-1} = \bar{w}\bar{A}, \qquad b\bar{A}b^{-1} = \bar{A}.$$

Proof The first statement follows from Proposition 4.9 and the second statement follows from Proposition 4.9 and Lemma 4.6. □

Remark 4.11 The relations without primes are equivalent to those with primes.

Using the short exact sequence

(4-12) $$1 \longrightarrow P_3(\mathbb{R}\mathrm{P}^2) \longrightarrow B_3(\mathbb{R}\mathrm{P}^2) \longrightarrow \Sigma_3 \longrightarrow 1,$$

we want to describe $B_3(\mathbb{R}\mathrm{P}^2)$ as an extension of $P_3(\mathbb{R}\mathrm{P}^2)$ by Σ_3.

Proposition 4.12 *The group $B_3(\mathbb{R}\mathrm{P}^2)$ can be presented as having generators*

$$a, \quad b, \quad w, \quad A_{23}, \quad \sigma_1, \quad \sigma_2,$$

satisfying relations (4)–(6) *from* Lemma 4.6 *and the following relations:*

$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$$

(4-13) $$\sigma_1^2 = a^2w^{-2}, \quad \sigma_2^2 = A_{23}.$$

(4-14) $$\sigma_1^{-1}a\sigma_1 = bA_{23}^{-1},$$

(4-15) $$\sigma_1^{-1}b\sigma_1 = aw^{-1}A_{23}w^{-1},$$

(4-16) $$\sigma_1^{-1}w\sigma_1 = w,$$

(4-17) $$\sigma_1^{-1}A_{23}\sigma_1 = w^2A_{23}^{-1},$$

(4-18) $$\sigma_2^{-1}a\sigma_2 = ab(w^{-1}A_{23})^2,$$

(4-19) $$\sigma_2^{-1}b\sigma_2 = bA_{23},$$

(4-20) $$\sigma_2^{-1}w\sigma_2 = bw^{-1}A_{23},$$

(4-21) $$\sigma_2^{-1}A_{23}\sigma_2 = A_{23}.$$

The proof is given in the Appendix.

Proposition 4.13 *The quotient $B_3(\mathbb{R}\mathrm{P}^2)/\operatorname{Brun}_3(\mathbb{R}\mathrm{P}^2)$ has order 384 and is an extension of $P_3(\mathbb{R}\mathrm{P}^2)/\operatorname{Brun}_3(\mathbb{R}\mathrm{P}^2)$ by Σ_3:*

$$1 \longrightarrow P_3(\mathbb{R}\mathrm{P}^2)/\operatorname{Brun}_3(\mathbb{R}\mathrm{P}^2) \longrightarrow B_3(\mathbb{R}\mathrm{P}^2)/\operatorname{Brun}_3(\mathbb{R}\mathrm{P}^2) \longrightarrow \Sigma_3 \longrightarrow 1.$$

The quotient $B_3(\mathbb{R}\mathrm{P}^2)/\operatorname{Brun}_3(\mathbb{R}\mathrm{P}^2)$ is generated by

$$\bar{w}, \quad \bar{A}, \quad a, \quad b, \quad \sigma_1, \quad \sigma_2,$$

and has defining relations:

$$\bar{w}^4 = \bar{A}^2 = 1, \quad \bar{A}\bar{w} = \bar{w}\bar{A}, \quad bab^{-1} = a^{-1}, \quad a^2 = b^2,$$
$$a^{-1}\bar{w}a = \bar{w}^{-1}\bar{A}, \quad a^{-1}\bar{A}a = \bar{A},$$
$$b^{-1}\bar{w}b = \bar{w}\bar{A}, \qquad b^{-1}\bar{A}b = \bar{A},$$
$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$$
$$\sigma_1^2 = a^2\bar{w}^2, \quad \sigma_2^2 = \bar{A},$$
$$\sigma_1^{-1}a\sigma_1 = b\bar{A}, \qquad \sigma_1^{-1}b\sigma_1 = a\bar{A}\bar{w}^2, \quad \sigma_1^{-1}\bar{w}\sigma_1 = \bar{w}, \qquad \sigma_1^{-1}\bar{A}\sigma_1 = \bar{A}\bar{w}^2,$$
$$\sigma_2^{-1}a\sigma_2 = ab\bar{w}^2, \quad \sigma_2^{-1}b\sigma_2 = b\bar{A}, \qquad \sigma_2^{-1}\bar{w}\sigma_2 = b\bar{A}\bar{w}^{-1}, \quad \sigma_2^{-1}\bar{A}\sigma_2 = \bar{A}.$$

Proof This follows directly from Proposition 4.10(2) and Proposition 4.12. □

5 Proof of Theorem 1.2

5.1 Some lemmas on free groups

Let S be a set and let $F(S)$ be the free group freely generated by S. Let S_0 be a set and let $x_1, x_2, \dots$ be additional letters. Let $S_n = S_0 \cup \{x_1, \dots, x_n\}$ be the disjoint union. Consider the group homomorphism

$$d_i\colon F(S_n) \to F(S_{n-1}), \quad 1 \le i \le n,$$

defined by

$$d_i(x) = \begin{cases} x & \text{if } x \in S_0 \text{ or } x = x_j \text{ with } j < i, \\ 1 & \text{if } x = x_i, \\ x_{j-1} & \text{if } x = x_j \text{ with } j > i. \end{cases} \tag{5-1}$$

Roughly speaking, d_i is obtained by sending x_i to 1 and keeping other generators. The following lemma is a special case of [17, Theorem 4.3].

Lemma 5.1 *Let $d_i\colon F(S_n) \to F(S_{n-1})$ be defined by the formula (5-1). Then*

$$\bigcap_{j=1}^{k} \mathrm{Ker}(d_i) = [\mathrm{Ker}(d_1), \mathrm{Ker}(d_2), \dots, \mathrm{Ker}(d_k)]_S$$

for $2 \le k \le n$. □

Let H be a normal subgroup of G. A set X of elements of H is called a set of *normal generators* for H in G if H is the normal closure of X in G. We say that H has *finitely many normal generators* in G if there is a finite set X such that H is the normal closure of X in G.

Lemma 5.2 *Let R_1 and R_2 be normal subgroups of G. Suppose that*

(1) *R_1 has finitely many normal generators;*

(2) *R_2 has finitely many generators (in the usual sense).*

Then the commutator subgroup $[R_1, R_2]$ has finitely many normal generators.

Proof Let $\{a_1, \dots, a_m\}$ be a set of normal generators for R_1. The set of generators for R_1 can be given as $\{g^{-1}a_i g \mid 1 \le i \le m,\ g \in G\}$. Let $\{b_1, \dots, b_n\}$ be a set of generators for R_2. Let H be the normal closure of

$$\{[a_i, b_j] \mid 1 \le i \le m,\ 1 \le j \le n\}.$$

Now take any $r \in R_2$, $r = b_{i_1} \dots b_{i_k}$. Then

$$[a_i, r] = [a_i, b_{i_1}]\, g_1[a_i, b_{i_2}]g_1^{-1} \cdots g_j[a_i, b_{i_{j+1}}]g_j^{-1} \cdots g_{k-1}[a_i, b_{i_k}]g_{k-1}^{-1},$$

where $g_j = b_{i_1} \cdots b_{i_j}$. So $[a_i, r] \in H$ for any $r \in R_2$. Now

$$[g^{-1}a_i g, b_j] = g^{-1}[a_i, g b_j g^{-1}]g \in H,$$

because $g b_j g^{-1} \in R_2$. This implies that $[R_1, R_2] = H$. □

Lemma 5.3 *Let M be a connected compact 2–manifold with nonempty boundary. Let $n \ge 2$. Then the subgroup*

$$\bigcap_{i=1}^{k} \mathrm{Ker}(d_i\colon P_n(M) \to P_{n-1}(M)) \cap \mathrm{Ker}(d_n\colon P_n(M) \to P_{n-1}(M))$$

has finitely many normal generators in $P_n(M)$ for each $1 \le k \le n-1$.

Proof The proof is given by induction on k. The assertion holds for $k=1$ by Lemma 3.8. Suppose that the assertion holds for $k-1$. Consider the short exact sequence of groups

$$\pi_1(M \smallsetminus \{p_1,\dots,p_{n-1}\}) \xhookrightarrow{i_*} P_n(M) \xrightarrow{d_n} P_{n-1}(M).$$

Let $[\omega_i] \in \pi_1(M \smallsetminus \{p_1,\dots,p_{n-1}\})$ represented by a small circle around p_i. By Lemma 3.8, for each $1 \le i \le n-1$, the subgroup $\mathrm{Ker}(d_i) \cap \mathrm{Ker}(d_n)$ is the normal closure of $[\omega_i]$ in $\pi_1(M \smallsetminus \{p_1,\dots,p_{n-1}\})$. Let $R_i = \mathrm{Ker}(d_i) \cap \mathrm{Ker}(d_n)$. Note that $\pi_1(M \smallsetminus \{p_1,\dots,p_{n-1}\})$ is a free group with a basis containing the elements $[\omega_i]$ for $1 \le i \le n-1$. By Lemma 5.1,

$$\begin{aligned}\bigcap_{i=1}^{k} R_i &= [R_1, R_2, \dots, R_k]_S \\ &= \prod_{j=1}^{k} \Big[\bigcap_{i\in\{1,\dots,\hat{j},\dots,k\}} R_i, R_j \Big]\end{aligned}$$

because R_i is the kernel of

$$d_i|\colon \pi_1(M \smallsetminus \{p_1,\dots,p_{n-1}\}) \longrightarrow \pi_1(M \smallsetminus \{p_1,\dots,p_{i-1},p_{i+1},\dots,p_{n-1}\})$$

for $1 \le i \le n-1$, and

$$\bigcap_{i\in\{1,\dots,\hat{j},\dots,k\}} R_i = [R_1, R_2, \dots, R_{j-1}, R_{j+1}, \dots, R_k]_S.$$

It should be noticed also that for normal subgroups H_1, H_2, H_3 of a group G

$$[H_1, H_3][H_2, H_3] = [H_1 H_2, H_3],$$

see, for example, Serre [21, identity (2'), Proposition 1.1]. It follows that

$$\begin{aligned}&\bigcap_{i=1}^{k} (\mathrm{Ker}(d_i) \cap \mathrm{Ker}(d_n)) \\ &= \prod_{j=1}^{k} \Big[\bigcap_{i\in\{1,\dots,\hat{j},\dots,k\}} (\mathrm{Ker}(d_i) \cap \mathrm{Ker}(d_n)), \mathrm{Ker}(d_j) \cap \mathrm{Ker}(d_n) \Big] \\ &\le \prod_{j=1}^{k} \Big[\bigcap_{i\in\{1,\dots,\hat{j},\dots,k\}} (\mathrm{Ker}(d_i) \cap \mathrm{Ker}(d_n)), \mathrm{Ker}(d_j) \Big].\end{aligned} \tag{5-2}$$

On the other hand, since

$$\Big[\bigcap_{i\in\{1,\dots,\hat{j},\dots,k\}}(\mathrm{Ker}(d_i)\cap\mathrm{Ker}(d_n)),\mathrm{Ker}(d_j)\Big]\le\bigcap_{i=1}^{k}(\mathrm{Ker}(d_i)\cap\mathrm{Ker}(d_n)),$$

for every $j=1,\dots,k$, we have

$$\bigcap_{i=1}^{k}(\mathrm{Ker}(d_i)\cap\mathrm{Ker}(d_n))=\prod_{j=1}^{k}\Big[\bigcap_{i\in\{1,\dots,\hat{j},\dots,k\}}(\mathrm{Ker}(d_i)\cap\mathrm{Ker}(d_n)),\mathrm{Ker}(d_j)\Big]. \tag{5-3}$$

By induction, the subgroup

$$\bigcap_{i\in\{1,\dots,\hat{j},\dots,k\}}(\mathrm{Ker}(d_i)\cap\mathrm{Ker}(d_n))$$

has finitely many normal generators for every $j=1,\dots,k$. From the short exact sequence of groups

$$\pi_1(M\smallsetminus\{p_1,\dots,p_{j-1},p_{j+1},\dots,p_n\})\hookrightarrow P_n(M)\xrightarrow{d_k}\!\!\!\!\to P_{n-1}(M),$$

the subgroup $\mathrm{Ker}(d_j)$ has finitely many generators. By Lemma 5.2, the commutator subgroup

$$\Big[\bigcap_{i\in\{1,\dots,\hat{j},\dots,k\}}(\mathrm{Ker}(d_i)\cap\mathrm{Ker}(d_n)),\mathrm{Ker}(d_j)\Big]$$

has finitely many normal generators for every $j=1,\dots,k$ and hence the group $\bigcap_{i=1}^{k}(\mathrm{Ker}(d_i)\cap\mathrm{Ker}(d_n))$ has finitely many normal generators. The induction is finished. □

5.2 Proof of Theorem 1.2

The proof is given by two different cases.

Case 1 M is a connected compact manifold with nonempty boundary. It is a well-known fact that groups $P_n(M)$ and $B_n(M)$ are finitely presented; it can be seen directly using the fibration of Theorem 3.1 and the fact that an extension of finitely presented groups is finitely presented [13, Corollary 2, page 140].

By Lemma 5.3,

$$\mathrm{Brun}_n(M)=\bigcap_{i=1}^{n-1}\mathrm{Ker}(d_i)\cap\mathrm{Ker}(d_n)$$

has finitely many normal generators in $P_n(M)$. This implies the factor groups $P_n(M)/\operatorname{Brun}_n(M)$ and $B_n(M)/\operatorname{Brun}_n(M)$ are finitely presented.

Case 2 M is a compact closed manifold. Let $\widetilde{M} = M \setminus \{q_1\}$. Using the exact sequence of the fibration of Theorem 3.1 and induction on n we conclude that the inclusion $f\colon \widetilde{M} \to M$ induces an epimorphism

$$f_*^n\colon P_n(\widetilde{M}) \to P_n(M).$$

Since

$$\operatorname{Brun}_n(\widetilde{M}) = [\langle\langle A_{1,n}\rangle\rangle y^{P_n(\widetilde{M})}, \langle\langle A_{2,n}\rangle\rangle^{P_n(\widetilde{M})}, \dots, \langle\langle A_{n-1,n}\rangle\rangle^{P_n(\widetilde{M})}]_S,$$

we have

$$f_*^n(\operatorname{Brun}_n(\widetilde{M})) = [\langle\langle A_{1,n}\rangle\rangle y^{P_n(M)}, \langle\langle A_{2,n}\rangle\rangle^{P_n(M)}, \dots, \langle\langle A_{n-1,n}\rangle\rangle y^{P_n(M)}]_S.$$

From the fact that $\operatorname{Brun}_n(\widetilde{M})$ has finitely many normal generators in $P_n(\widetilde{M})$, the subgroup

$$[\langle\langle A_{1,n}\rangle\rangle^{P_n(M)}, \langle\langle A_{2,n}\rangle\rangle^{P_n(M)}, \dots, \langle\langle A_{n-1,n}\rangle\rangle^{P_n(M)}]_S$$

has finitely many normal generators in $P_n(M)$.

If $M \neq S^2$ or $\mathbb{R}\mathrm{P}^2$ with $n \geq 3$, then, by Theorem 1.1 and Proposition 3.6, the subgroup

$$\operatorname{Brun}_n(M) = [\langle\langle A_{1,n}\rangle\rangle y^{P_n(M)}, \langle\langle A_{2,n}\rangle\rangle^{P_n(M)}, \dots, \langle\langle A_{n-1,n}\rangle\rangle^{P_n(M)}]_S$$

has finitely many normal generators in $P_n(M)$. Therefore, $P_n(M)/\operatorname{Brun}_n(M)$ and $B_n(M)/\operatorname{Brun}_n(M)$ are finitely presented for $M \neq S^2$ or $\mathbb{R}\mathrm{P}^2$ with $n \geq 3$.

If $M = S^2$, then $P_3(S^2)/\operatorname{Brun}_3(S^2) = \{1\}$ and $B_3(S^2)/\operatorname{Brun}_3(S^2) = \mathbb{Z}/2$. For $n = 4$, the group $\operatorname{Brun}_4(S^2)$ has 5 generators according to [2, Proposition 7.2.1]. Thus $P_4(S^2)/\operatorname{Brun}_4(S^2)$ and $B_4(S^2)/\operatorname{Brun}_4(S^2)$ are finitely presented. For $n \geq 5$, by Theorem 1.1, $\operatorname{Brun}_n(S^2)$ is a finite extension of the subgroup

$$[\langle\langle A_{1,n}\rangle\rangle y^{P_n(S^2)}, \langle\langle A_{2,n}\rangle\rangle^{P_n(S^2)}, \dots, \langle\langle A_{n-1,n}\rangle\rangle^{P_n(S^2)}]_S$$

because $\pi_{n-1}(S^2)$ is finite. Thus $\operatorname{Brun}_n(S^2)$ has finitely many normal generators in $P_n(S^2)$ and so the assertion holds for the case $M = S^2$.

If $M = \mathbb{R}\mathrm{P}^2$, then $\operatorname{Brun}_3(\mathbb{R}\mathrm{P}^2)$ has 9 generators according to Proposition 4.9. Thus $P_3(\mathbb{R}\mathrm{P}^2)/\operatorname{Brun}_3(\mathbb{R}\mathrm{P}^2)$ and $B_3(\mathbb{R}\mathrm{P}^2)/\operatorname{Brun}_3(\mathbb{R}\mathrm{P}^2)$ are finitely presented. For $n \geq 4$, by (3) of Theorem 1.1 together with fact that $\pi_{n-1}(S^2)$ is finitely generated, the subgroup $\operatorname{Brun}_n(\mathbb{R}\mathrm{P}^2)$ has finitely many normal generators, and so the assertion holds for the case $M = \mathbb{R}\mathrm{P}^2$. □

6 An algorithm for determining a free basis for Brunnian braids

By Lemma 3.8, in order to get a free basis for $\mathrm{Brun}_{n+1}(M)$, it suffices to determine a free basis for

$$\bigcap_{i=1}^{n} \mathrm{Ker}(d_i|: \pi_1(M \smallsetminus \{p_1, \dots, p_n\}) \to \pi_1(M \smallsetminus \{p_1, \overset{i}{\wedge}, p_n\})). \tag{6-1}$$

Let M be connected a 2–manifold with nonempty boundary and let ω_i be a small circle around p_i. Then

$$\pi_1(M \smallsetminus \{p_1, \dots, p_n\}) = F(S_0 \sqcup \{[\omega_1], \dots, [\omega_n]\}),$$

where $\pi_1(M) = F(S_0)$. Let S be a set and let T be a subset of S. By a *projection homomorphism*

$$\pi\colon F(S) \to F(T)$$

we mean here a group homomorphism defined by

$$\pi(x) = \begin{cases} x & \text{if } x \in T, \\ 1 & \text{if } x \in S \setminus T. \end{cases}$$

In our case, the homomorphisms $d_i|$ are projection homomorphisms in the following sense:

Let $S = S_0 \sqcup \{[\omega_1], \dots, [\omega_n]\}$ and let

$$T_i = S_0 \sqcup \{[\omega_1], \dots, [\omega_{i-1}], [\omega_{i+1}], \dots, [\omega_n]\}$$

for $1 \le i \le n$. Then

$$d_i|\colon F(S) \to F(T_i)$$

is the projection homomorphism for each $1 \le i \le n$. The algorithm in [23, Section 3] provides a recursive formula to determine a free basis for the intersection subgroup $\bigcap_{i=1}^{n} \mathrm{Ker}(d_i|)$, as follows. For x a reduced word in the alphabet S, and y a reduced word in the alphabet T, define $\mu(x, y)$ by induction on the word length of y:

(1) $\mu(x, y) = x$ if y is the empty word;

(2) $\mu(x, y) = [\mu(x, y'), z^{\epsilon}]$ if $y = y'z^{\epsilon}$ with $z \in T$ and $\epsilon = \pm 1$.

Let V be a set of reduced words in the alphabet S, and let W be a set of reduced words in the alphabet T, a subalphabet of S. Define a set of words in the alphabet S:

$$\mathcal{A}(V)_W = \{\mu(x, y) \mid x \in V \text{ and } y \in W\}.$$

By [23, Proposition 3.3], $\mathcal{A}(\{S \setminus T\})_{F(T)}$ is a free basis for the kernel of the projection homomorphism $\pi\colon F(S) \to F(T)$. Now for the subsets $T_1, \dots, T_n$ of S, construct a subset $\mathcal{A}(T_1, \dots, T_k)$ of $F(S)$ by induction on k for $1 \le k \le n$:

(1) $\mathcal{A}(T_1) = \mathcal{A}(\{S \setminus T_1\})_{F(T_1)}$.

(2) Let

$$T_2^{(2)} = \{w \in \mathcal{A}(T_1) \mid w = [\dots[x, y_1^{\epsilon_1}], \dots], y_t^{\epsilon_t}] \text{ with } x, y_j \in T_2,\ \epsilon_j = \pm 1 \text{ for all } j\}$$

and define

$$\mathcal{A}(T_1, T_2) = \mathcal{A}(\mathcal{A}(T_1))_{F(T_2^{(2)})}.$$

(3) Suppose $\mathcal{A}(T_1, \dots, T_{k-1})$ is defined so every element in $\mathcal{A}(T_1, \dots, T_{k-1})$ are written in the form of iterated commutators in $F(S)$ with entries given by $\pm$ powers of elements in S. Let

$$T_k^{(k)} = \{w \in \mathcal{A}(T_1, \dots, T_{k-1}) \mid w = [x_1^{\epsilon_1}, \dots, x_\ell^{\epsilon_\ell}] \text{ with } x_j \in T_k \text{ for all } j\},$$

where $[x_1^{\epsilon_1}, \dots, x_\ell^{\epsilon_\ell}]$ are the elements in $\mathcal{A}(T_1, \dots, T_{k-1})$ that are written as iterated commutators. Define

$$\mathcal{A}(T_1, \dots, T_k) = \mathcal{A}(\mathcal{A}(T_1, \dots, T_{k-1}))_{F(T_k^{(k)})}.$$

By [23, Theorem 3.4], $\mathcal{A}(T_1, \dots, T_k)$ is a free basis for $\bigcap_{i=1}^k \operatorname{Ker}(d_i|)$ for $1 \le k \le n$. In particular, $\mathcal{A}(T_1, \dots, T_n)$ is a free basis for $\bigcap_{i=1}^n \operatorname{Ker}(d_i|)$.

Note In the construction of $\mathcal{A}(V)_W$, the words are obtained as iterated commutators with a fixed choice of commutator brackets from left to right. The above algorithm is given by iterating the process of $\mathcal{A}(V)_W$ and so the words in $\mathcal{A}(T_1, \dots, T_n)$ are given in the form of iterated commutators with commutator bracket operations given as compositions of left-to-right brackets.

7 Appendix: Proofs of statements of Section 4

Proof of Theorem 4.1 We start with the presentation of van Buskirk [22, page 83], also studied in [10]. It has the $2n-1$ generators $\sigma_1, \sigma_2, \dots, \sigma_{n-1}$, $\rho_1, \dots, \rho_n$, subject to the following relations:

(i) $$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad i = 1, 2, \dots, n-2,$$

(ii) $$\sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i-j| > 1,$$

(iii) $$\rho_j \sigma_i = \sigma_i \rho_j, \quad j \ne i, i+1,$$

(iv) $$\rho_i = \sigma_i \rho_{i+1} \sigma_i,$$

(v) $$\rho_{i+1}^{-1} \rho_i^{-1} \rho_{i+1} \rho_i = \sigma_i^2,$$

(vi) $$\rho_1^2 = \sigma_1 \sigma_2 \cdots \sigma_{n-2} \sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_2 \sigma_1.$$

Let us show at first that the system (i)–(vi) is equivalent to the system (i)–(iv), (vi) and the relations

$$\sigma_i^{-1} \rho_i \sigma_i^{-1} \rho_i = \rho_i \sigma_i^{-1} \rho_i \sigma_i, \quad i = 1, \dots, n-1. \tag{7-1}$$

We multiply the equality (7-1) by $\sigma_i \rho_i^{-1} \sigma_i \rho_i^{-1}$ on the left-hand side and we obtain

$$\sigma_i \rho_i^{-1} \sigma_i \rho_i^{-1} \sigma_i^{-1} \rho_i \sigma_i^{-1} \rho_i = \sigma_i^2, \qquad i = 1, \dots, n-1.$$

Then we use the expression

$$\rho_{i+1} = \sigma_i^{-1} \rho_i \sigma_i^{-1}$$

from (iv) and we obtain (v). Hence, the relations (7-1) hold in $B_n(\mathbb{R}\mathrm{P}^2)$.

Now we show by induction that we can eliminate all the equalities in (7-1) except the first one, ie for $i = 1$,

$$\sigma_1^{-1} \rho_1 \sigma_1^{-1} \rho_1 = \rho_1 \sigma_1^{-1} \rho_1 \sigma_1. \tag{7-2}$$

In other words we will show that relations (7-1) for $i = 2, \dots, n-1$ are consequences of relations (i)–(iv) and (7-2). For $i = 2$ we start with (7-2) and multiply it by $\sigma_1^{-1}\sigma_2^{-1}$ on the left-hand side and by $\sigma_2^{-1}\sigma_1^{-1}$ on the right-hand side. We get

$$\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\rho_1\sigma_1^{-1}\rho_1\sigma_2^{-1}\sigma_1^{-1} = \sigma_1^{-1}\sigma_2^{-1}\rho_1\sigma_1^{-1}\rho_1\sigma_1\sigma_2^{-1}\sigma_1^{-1}.$$

We apply relations (i) to this relation on the right-hand side and on the left-hand side, we obtain

$$\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}\rho_1\sigma_1^{-1}\rho_1\sigma_2^{-1}\sigma_1^{-1} = \sigma_1^{-1}\sigma_2^{-1}\rho_1\sigma_1^{-1}\rho_1\sigma_2^{-1}\sigma_1^{-1}\sigma_2.$$

Further we apply relation (iii) to permute ρ_1 and σ_2^{-1} in all four appearances of ρ_1 in the last relation, we get

$$\sigma_2^{-1}\sigma_1^{-1}\rho_1\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}\rho_1\sigma_1^{-1} = \sigma_1^{-1}\rho_1\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}\rho_1\sigma_1^{-1}\sigma_2.$$

Now apply relation (i) to the middle parts of both sides of the last relation, and obtain

$$\sigma_2^{-1}\sigma_1^{-1}\rho_1\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\rho_1\sigma_1^{-1} = \sigma_1^{-1}\rho_1\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\rho_1\sigma_1^{-1}\sigma_2.$$

Use relation (iv) in the form $\rho_2 = \sigma_1^{-1}\rho_1\sigma_1^{-1}$ and obtain

$$\sigma_2^{-1}\rho_2\sigma_2^{-1}\rho_2 = \rho_2\sigma_2^{-1}\rho_2\sigma_2.$$

This is relation (7-1) for $i=2$. Suppose now that for i our statement is true: the relation

$$\sigma_i^{-1}\rho_i\sigma_i^{-1}\rho_i=\rho_i\sigma_i^{-1}\rho_i\sigma_i$$

is a consequence of relations (i)–(iv) and (7-2). Multiplying this relation by $\sigma_i^{-1}\sigma_{i+1}^{-1}$ on the left-hand side and by $\sigma_{i+1}^{-1}\sigma_i^{-1}$ on the right-hand side and applying relations (i)–(iv) as before we obtain relation (7-1) for $i+1$. So all relations (v) can be replaced by one relation (7-2).

Let us consider, now, relations (iii) and show that all of them are consequences of relations (i), (ii), (iv) and relations

$$\rho_1\sigma_i=\sigma_i\rho_1,\quad i\neq 1. \tag{7-3}$$

Let $j>1$, then it follows from (iv) that

$$\rho_j=\sigma_{j-1}^{-1}\sigma_{j-2}^{-1}\cdots\sigma_1^{-1}\rho_1\sigma_1^{-1}\cdots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1}.$$

Consider $\sigma_i\rho_j$. Let $i<j-1$, then using relations (i), (ii) and (7-3) we have

$$\begin{aligned}\sigma_i\rho_j&=\sigma_i\sigma_{j-1}^{-1}\sigma_{j-2}^{-1}\cdots\sigma_1^{-1}\rho_1\sigma_1^{-1}\cdots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1}\\&=\sigma_{j-1}^{-1}\sigma_{j-2}^{-1}\cdots\sigma_1^{-1}\sigma_{i+1}\rho_1\sigma_1^{-1}\cdots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1}\\&=\sigma_{j-1}^{-1}\sigma_{j-2}^{-1}\cdots\sigma_1^{-1}\rho_1\sigma_{i+1}\sigma_1^{-1}\cdots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1}\\&=\sigma_{j-1}^{-1}\sigma_{j-2}^{-1}\cdots\sigma_1^{-1}\rho_1\sigma_1^{-1}\cdots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1}\sigma_i=\rho_j\sigma_i.\end{aligned}$$

If $i>j$, then using relations (i) and (7-3) we have

$$\begin{aligned}\sigma_i\rho_j&=\sigma_i\sigma_{j-1}^{-1}\sigma_{j-2}^{-1}\cdots\sigma_1^{-1}\rho_1\sigma_1^{-1}\cdots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1}\\&=\sigma_{j-1}^{-1}\sigma_{j-2}^{-1}\cdots\sigma_1^{-1}\sigma_i\rho_1\sigma_1^{-1}\cdots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1}\\&=\sigma_{j-1}^{-1}\sigma_{j-2}^{-1}\cdots\sigma_1^{-1}\rho_1\sigma_i\sigma_1^{-1}\cdots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1}\\&=\sigma_{j-1}^{-1}\sigma_{j-2}^{-1}\cdots\sigma_1^{-1}\rho_1\sigma_1^{-1}\cdots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1}\sigma_i=\rho_j\sigma_i.\end{aligned}$$

Hence all relations (iii) are consequences of relations (i), (ii), (iv) and (7-3). So, we can delete generators $\rho_2,\dots,\rho_n$, and relations (iv) from the presentation and replace relations (iii) and (v) by relations (7-3) and (7-2) respectively. □

Proof of Lemma 4.4 Relations $(1')$ and $(2')$ follow from relations (1) and (2) respectively. Relation (1) follows from (4-8), and relation (2) is (4-9). The first relation in (3) follows from (4-11), the second relation in (4-5), and the second relation in (4-8). The second relation in (3) follows from (4-11) and the third relation in (4-5). To prove that the statement of the lemma gives a presentation of $P_3(\mathbb{R}\mathrm{P}^2)$, denote by P the group which has a presentation given by these generators and relations. There exists an evident homomorphism

$$\phi\colon P \to P_3(\mathbb{R}\mathrm{P}^2).$$

The subgroup $U_3(\mathbb{R}\mathrm{P}^2)$ generated by w and $A_{2,3}$ is a free subgroup in P as it is free after the mapping by ϕ. It can be seen that the quotient $P/U_3(\mathbb{R}\mathrm{P}^2)$ is isomorphic to $P_2(\mathbb{R}\mathrm{P}^2)$ (relations (3)); so ϕ becomes an isomorphism after comparison of exact sequences:

$$\begin{array}{ccccc}
U_3(\mathbb{R}\mathrm{P}^2) & \hookrightarrow & P & \twoheadrightarrow & P_2(\mathbb{R}\mathrm{P}^2) \\
\downarrow & & \downarrow \phi & & \downarrow \\
U_3(\mathbb{R}\mathrm{P}^2) & \overset{i_{3*}}{\hookrightarrow} & P_3(\mathbb{R}\mathrm{P}^2)) & \overset{d_2}{\twoheadrightarrow} & P_2(\mathbb{R}\mathrm{P}^2).
\end{array}$$

This completes the proof. □

Proof of Proposition 4.12 The first relation in (4-13) follows from the definition of the elements a and w, and the relations of the presentation of $B_3(\mathbb{R}\mathrm{P}^2)$ with generators ρ, σ_1 and σ_2. The second relation in (4-13) is the definition of A_{23}.

To construct the formulas of conjugation we can take the corresponding relations from the paper of van Buskirk [22] and rewrite them in our generators of $P_3(\mathbb{R}\mathrm{P}^2)$. We can also prove these formulas using the relations that we already know to hold in $B_3(\mathbb{R}\mathrm{P}^2)$. Let us do it. At first let us prove (4-17). We start with the two equal expressions for A_{13}:

$$\sigma_1^{-1}\sigma_2^2\sigma_1 = \sigma_2\sigma_1^2\sigma_2^{-1}, \tag{7-4}$$

which is true in $B_3(\mathbb{R}\mathrm{P}^2)$. We insert $\sigma_2\sigma_2^{-1}$ in the right-hand part of (7-4):

$$\sigma_1^{-1}\sigma_2^2\sigma_1 = \sigma_2\sigma_1^2\sigma_2\sigma_2^{-2}.$$

Then we use the relation $\rho^2 = \sigma_1\sigma_2^2\sigma_1$ from the presentation of $B_3(\mathbb{R}\mathrm{P}^2)$:

$$\sigma_1^{-1}\sigma_2^2\sigma_1 = \sigma_2\sigma_1\rho^2\sigma_1^{-1}\sigma_2^{-1}A_{23}^{-1}.$$

Since $A_{23} = \sigma_2^2$ and $w = \sigma_2\sigma_1\rho\sigma_1^{-1}\sigma_2^{-1}$, we have

$$\sigma_1^{-1}A_{23}\sigma_1 = w^2A_{23}^{-1}.$$

To prove relation (4-16), we start with the definition of w:

$$\sigma_2\sigma_1\rho\sigma_1^{-1}\sigma_2^{-1} = w.$$

Since $\rho\sigma_2 = \sigma_2\rho$ we have

$$\sigma_2\sigma_1\rho\sigma_1^{-1}\sigma_2^{-1} = (\sigma_2\sigma_1\sigma_2^{-1})\rho(\sigma_2\sigma_1^{-1}\sigma_2^{-1}) = (\sigma_1^{-1}\sigma_2\sigma_1)\rho(\sigma_1^{-1}\sigma_2^{-1}\sigma_1) = \sigma_1^{-1}w\sigma_1,$$

and so

$$\sigma_1^{-1} w \sigma_1 = w.$$

To prove relation (4-15), we start with relation $(1')$ from Lemma 4.4

$$\rho^{-1} w \rho = A_{23} w^{-1},$$

which is equivalent to

$$w\rho = \rho A_{23} w^{-1}.$$

Since $\sigma_1^{-1} w \sigma_1 = w$ and $\sigma_1^{-1} u \sigma_1 = \rho$ we have

$$\sigma_1^{-1} w u \sigma_1 = (\rho w) w^{-1} A_{23} w^{-1}.$$

Using the definition of a and b, $a = \rho w$, $b = wu$, we obtain relation (4-15).

For relation (4-14), we start with the equality

$$w = (A_{23} w^{-1})(w A_{23}^{-1} w)$$

and apply the conjugation formulas $(1')$ and (3) from Lemma 4.4. This gives

$$w = (\rho^{-1} w \rho)(\rho^{-1} u \rho^{-1} u^{-1}),$$

which is equivalent to

$$w = \rho^{-1} w u \rho^{-1} u^{-1}. \tag{7-5}$$

We rewrite the first equation in (3) from Lemma 4.4 in the form

$$\rho(w A_{23}^{-1} w) u \rho u^{-1} = 1$$

and multiplying the right-hand side of (7-5) by $\rho(w A_{23}^{-1} w) u \rho u^{-1}$, we obtain

$$w = \rho^{-1} w u \rho^{-1} u^{-1} \rho(w A_{23}^{-1} w) u \rho u^{-1}.$$

We apply (1) of Lemma 4.4 and we get

$$w = \rho^{-1} w u \rho^{-1} u^{-1} \rho(\rho A_{23} w^{-2} \rho^{-1}) u \rho u^{-1}.$$

Using the formulas

$$A_{12} = u \rho^{-1} u^{-1} \rho, \quad A_{13}^{-1} = A_{23} w^{-2},$$

we obtain

$$w = \rho^{-1} w A_{12} \rho A_{13}^{-1} A_{12}^{-1} \quad \text{or} \quad \rho w = w A_{12} \rho A_{13}^{-1} A_{12}^{-1}.$$

Conjugating it by σ_1^{-1} we have

$$\sigma_1^{-1} (\rho w) \sigma_1 = w u A_{23}^{-1},$$

which is (4-14).

Formula (4-21) follows from $A_{23} = \sigma_2^2$.

To prove relation (4-20), we start with the first relation in $(2')$ of Lemma 4.4 and we rewrite it in equivalent forms

$$u^{-1}wu = A_{23}^{-1}w \Leftrightarrow 1 = u^{-1}wuw^{-1}A_{23} \Leftrightarrow u = wuw^{-1}A_{23}$$

or

$$\sigma_2^{-1}(\sigma_2\sigma_1\rho\sigma_1^{-1}\sigma_2^{-1})\sigma_2 = wuw^{-1}A_{23}$$

which is equivalent to (4-20):

$$\sigma_2^{-1}w\sigma_2 = bw^{-1}A_{23}.$$

For relation (4-19), we start with the identity

$$(bw^{-1}A_{23})(A_{23}^{-1}wA_{23}) = bA_{23}.$$

Using the formula

$$\sigma_2^{-1}u\sigma_2 = A_{23}^{-1}wA_{23},$$

and (4-20), we get

$$(\sigma_2^{-1}w\sigma_2)(\sigma_2^{-1}u\sigma_2) = bA_{23}.$$

This is equivalent to (4-19):

$$\sigma_2^{-1}(wu)\sigma_2 = bA_{23}.$$

Finally let us prove relation (4-18). We start with the identity

$$1 = (A_{23}^{-1}w)w^{-1}A_{23}.$$

Using $(2')$ of Lemma 4.4 we get

$$1 = (u^{-1}wu)w^{-1}A_{23}$$

and then

$$u = (wu)w^{-1}A_{23}.$$

We multiply this equality by ρw from the left-hand side

$$\rho wu = (\rho w)(wu)w^{-1}A_{23},$$

which is equivalent to

$$\rho b = abw^{-1}A_{23}.$$

Since ρ and σ_2 commute, this is the same as

$$(\sigma_2^{-1}\rho\sigma_2)b = abw^{-1}A_{23}.$$

Multiply this equality by $w^{-1}A_{23}$ from the right-hand side

$$(\sigma_2^{-1}\rho\sigma_2)(bw^{-1}A_{23}) = ab(w^{-1}A_{23})^2,$$

and use (4-20)

$$\sigma_2^{-1}\rho w\sigma_2 = ab(w^{-1}A_{23})^2,$$

which is equivalent to (4-18):

$$\sigma_2^{-1}a\sigma_2 = ab(w^{-1}A_{23})^2.$$

The proof that $B_3(\mathbb{R}\mathrm{P}^2)$ has a presentation as in the statement of the Proposition is the same as the proof of the presentation of Lemma 4.4 with the help of the exact sequence (4-12). □

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