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The Atiyah–Segal completion theorem in twisted K–theory

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A basic result in equivariant K–theory, the Atiyah–Segal completion theorem relates the G–equivariant K–theory of a finite G–CW complex to the non-equivariant K–theory of its Borel construction. We prove the analogous result for twisted equivariant K–theory.

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1 Introduction

The aim of this note is to prove the following twisted analogue of the Atiyah–Segal completion theorem [3].

Theorem 1 *Let X be a finite G–CW complex, where G is a compact Lie group. Then the projection map $\pi\colon EG \times X \to X$ induces an isomorphism*

$$K_G^{\tau+*}(X)^{\wedge}_{I_G} \xrightarrow{\approx} K_G^{\pi^*(\tau)+*}(EG \times X)$$

for any twisting τ corresponding to an element of $H^1_G(X;\mathbb{Z}/2) \oplus H^3_G(X;\mathbb{Z})$.

Here $I_G \subset R(G)$ is the augmentation ideal of the representation ring $R(G)$ and $(-)^{\wedge}_{I_G}$ indicates completion. The classical theorem is the case $\tau = 0$. As in the untwisted case, Theorem 1 implies a comparison between the equivariant K–theory of a finite G–CW complex X and the non-equivariant K–theory of its Borel construction.

Corollary 2 *Let X be a finite G–CW complex. Then there exists an isomorphism*

$$K_G^{\tau+*}(X)^{\wedge}_{I_G} \approx K^{\tau+*}(EG \times_G X)$$

for any twisting τ corresponding to an element of

$$H^1_G(X;\mathbb{Z}/2) \oplus H^3_G(X;\mathbb{Z}) = H^1(EG \times_G X;\mathbb{Z}/2) \oplus H^3(EG \times_G X;\mathbb{Z}). \qquad \square$$

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Theorem 1 generalizes a result by C Dwyer, who has proved the theorem in the case where G is finite and the twisting τ is given by a cocycle of G [6]. While versions of the theorem for compact Lie groups have been known to experts (for example, such a theorem is used in the proof of Freed, Hopkins and Teleman [7, Proposition 3.11]), to our knowledge no proof of the general theorem appears in the current literature. Our goal is to fill in this gap.

We shall prove Theorem 1 in two stages. First we prove the theorem in the special case of a twisting arising from a graded central extension

$$1 \to \mathbb{T} \to G^\tau \to G \to 1, \qquad \epsilon\colon G \to \mathbb{Z}/2$$

of G by the circle group $\mathbb{T}$. For such twistings, twisted G–equivariant K–groups correspond to certain direct summands of untwisted G^τ–equivariant K–groups, and the Adams–Haeberly–Jackowski–May argument contained in [1] goes through with these summands to prove the theorem in this case. It follows that the theorem holds when X is a point, and the general theorem then follows by a Mayer–Vietoris argument.

As our definition of twisted K–theory, we use Freed, Hopkins and Teleman's elaboration [8] of the Atiyah–Segal model developed in [4]. Thus for a G–space X, the notation $K_G^{\tau+*}(X)$ is a shorthand for $K^{\tau+*}(X /\!\!/ G)$, where $X /\!\!/ G$ is the quotient groupoid of X. Of course, the completion theorem should remain true in any reasonable alternative model for twisted equivariant K–theory as well.

This note is structured as follows. In Section 2 we describe a pro-group valued variant of K–theory which we shall employ in Section 3 to handle the case of a twisting given by a central extension. Section 4 then contains a proof of the general theorem.

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2 A convenient cohomology theory

We shall now describe a cohomology theory which will be used in the next section to prove the completion theorem for twistings arising from a graded central extension. Let

$$1 \to C \to \widetilde{G} \to G \to 1$$

be a central extension of a compact Lie group G by a commutative compact Lie group C, and let X be a finite G–CW complex. Via the map $\widetilde{G} \to G$ we can view X as a $\widetilde{G}$–space on which C acts trivially. The semigroup $\mathrm{Vect}_{\widetilde{G}}(X)$ of isomorphism classes of $\widetilde{G}$–equivariant vector bundles over X decomposes as a direct sum

$$\mathrm{Vect}_{\widetilde{G}}(X) = \bigoplus_{\pi \in \widehat{C}} \mathrm{Vect}_{\widetilde{G}}(X)(\pi) \tag{1}$$

where $\widehat{C}$ denotes the set of isomorphism classes of irreducible representations of C, and where $\mathrm{Vect}_{\widetilde{G}}(X)(\pi)$ is the semigroup of isomorphism classes of those $\widetilde{G}$–vector bundles over X whose fibers are π–isotypical as representations of C, that is, isomorphic to sums of copies of π. The decomposition (1) leads to a decomposition

$$K^*_{\widetilde{G}}(X) = \bigoplus_{\pi \in \widehat{C}} K^*_{\widetilde{G}}(X)(\pi) \tag{2}$$

and similarly for reduced K–groups.[1] Here $K^0_{\widetilde{G}}(X)(\pi)$ is the Grothendieck group of $\mathrm{Vect}_{\widetilde{G}}(X)(\pi)$, and $K^q_{\widetilde{G}}(X)(\pi)$ for non-zero q is defined by using the suspension isomorphism and the Bott periodicity map. By inspection and definition, the decomposition (2) is compatible with G–equivariant maps of spaces, with the suspension isomorphism, with the Thom isomorphism for G–equivariant vector bundles, and, as a special case, with the Bott periodicity map. Thus for each $\pi \in \widehat{C}$, we can view $K^*_{\widetilde{G}}(-)(\pi)$ as a $\mathbb{Z}/2$–graded cohomology theory defined on finite G–CW complexes and taking values in graded $R(G)$–modules.

Although the decomposition (2) fails for infinite X in general, it is possible to extend each one of the theories $K^*_{\widetilde{G}}(-)(\pi)$ to infinite G–CW complexes by means of suitable classifying spaces. However, since having the theories available for finite complexes suffices for most of our purposes, we will not elaborate this point. Instead, we point the reader to the proof of Freed, Hopkins and Teleman [8, Proposition 3.5] for a description of the appropriate classifying space when π is the defining representation of the circle group $\mathbb{T}$, which is the only case where we will need to apply $K^*_{\widetilde{G}}(-)(\pi)$ to an infinite complex in the sequel.

Our interest in the groups $K^*_{\widetilde{G}}(X)(\pi)$ is explained by the following proposition. Recall that a graded central extension of a group G is a central extension of G together with a homomorphism from G to $\mathbb{Z}/2$.

[1] In fact, tensor product makes $K^*_{\widetilde{G}}(X)$ into a $\widehat{C}$–graded algebra where the modules $K^*_{\widetilde{G}}(X)(\pi)$ are the homogeneous parts. However, we shall not need this graded algebra structure.

Proposition 3 (A reformulation of Freed, Hopkins and Teleman [8, Proposition 3.5]) *Let G be a compact Lie group, let X be a G–space, and let τ be the twisting given by a graded central extension*

$$1 \to \mathbb{T} \to G^\tau \to G \to 1, \qquad \epsilon\colon G \to \mathbb{Z}/2$$

of G by the circle group $\mathbb{T}$. Let $S^1(\epsilon)$ denote the one-point compactification of the 1-dimensional representation of G given by $(-1)^\epsilon$. Then there is a natural isomorphism

$$K_G^{\tau+n}(X) \approx \widetilde{K}_{G^\tau}^{n+1}(X_+ \wedge S^1(\epsilon))(1)$$

where "1" refers to the defining representation of $\mathbb{T}$. □

The groups $K^*_{\widetilde{G}}(X)(\pi)$ are not what we are going to use in the next section. Instead, we need pro-group valued versions completed at the augmentation ideal. (For background material on pro-groups, we refer the reader to Adams, Haeberly, Jackowski and May [2].) Given an arbitrary G–CW complex X and an irreducible representation π of C, we let $\mathbf{K}^*_{\widetilde{G}}(X)(\pi)$ denote the pro–$R(G)$–module

$$\mathbf{K}^*_{\widetilde{G}}(X)(\pi) = \{K^*_{\widetilde{G}}(X_\alpha)(\pi)\}_\alpha$$

where X_α runs over all finite G–CW subcomplexes of X and the structure maps of the pro-system are those induced by inclusions between subcomplexes. The groups of our interest are then the pro–$R(G)$–modules

$$\mathbf{K}^*_{\widetilde{G}}(X)(\pi)\widehat{_{I_G}} = \{K^*_{\widetilde{G}}(X_\alpha)(\pi) \,/\, I_G^n \cdot K^*_{\widetilde{G}}(X_\alpha)(\pi)\}_{\alpha,n},$$

where X_α again runs over the finite G–subcomplexes of X, n runs over the natural numbers, and the structure maps of the pro-system are the evident ones. We think of $\mathbf{K}^*_{\widetilde{G}}(X)(\pi)\widehat{_{I_G}}$ as the completion of $\mathbf{K}^*_{\widetilde{G}}(X)(\pi)$ with respect to the augmentation ideal I_G. Reduced variants $\widetilde{\mathbf{K}}^*_{\widetilde{G}}(X)(\pi)$ and $\widetilde{\mathbf{K}}^*_{\widetilde{G}}(X)(\pi)\widehat{_{I_G}}$ for a based G–CW complex X are defined in a similar way using the reduced groups $\widetilde{K}^*_{\widetilde{G}}(X_\alpha)(\pi)$, where X_α now runs through the finite G–CW subcomplexes of X containing the base point. The crucial feature of the groups $\mathbf{K}^*_{\widetilde{G}}(X)(\pi)\widehat{_{I_G}}$ for us is that they form a cohomology theory on the category of G–CW complexes (and therefore, by G–CW approximation, on the category of all G–spaces). Phrased in terms of the reduced groups, this means that the following axioms hold.

(1) (Homotopy invariance) If X and Y are based G–CW complexes and $f, g\colon X \to Y$ are homotopic through based G–equivariant maps, then the induced maps

$$f^*, g^*\colon \widetilde{\mathbf{K}}^*_{\widetilde{G}}(Y)(\pi)\widehat{_{I_G}} \to \widetilde{\mathbf{K}}^*_{\widetilde{G}}(X)(\pi)\widehat{_{I_G}}$$

are equal.

(2) (Exactness) If X is a based G–CW complex and A is a subcomplex of X containing the base point, then the sequence

$$\widetilde{\mathbf{K}}^*_{\widetilde{G}}(X/A)(\pi)\widehat{_{I_G}} \to \widetilde{\mathbf{K}}^*_{\widetilde{G}}(X)(\pi)\widehat{_{I_G}} \to \widetilde{\mathbf{K}}^*_{\widetilde{G}}(A)(\pi)\widehat{_{I_G}}$$

is pro-exact.

(3) (Suspension) For each q, there exists a natural isomorphism

$$\Sigma\colon \widetilde{\mathbf{K}}^q_{\widetilde{G}}(X)(\pi)\widehat{_{I_G}} \approx \widetilde{\mathbf{K}}^{q+1}_{\widetilde{G}}(\Sigma X)(\pi)\widehat{_{I_G}}$$

(4) (Additivity) If X is the wedge sum of a family $\{X_i\}_{i\in I}$ of based G–CW complexes, the inclusions $X_i \hookrightarrow X$ induce an isomorphism

$$\widetilde{\mathbf{K}}^*_{\widetilde{G}}(X)(\pi)\widehat{_{I_G}} \xrightarrow{\approx} \prod_{i\in I} \widetilde{\mathbf{K}}^*_{\widetilde{G}}(X_i)(\pi)\widehat{_{I_G}}$$

The only difficulties in verifying these properties arise from the exactness axiom.

Proposition 4 *The functor $\widetilde{\mathbf{K}}^*_{\widetilde{G}}(-)(\pi)\widehat{_{I_G}}$ satisfies the exactness axiom.*

Sketch of proof As in Adams–Haeberly–Jackowski–May [2], because the ring $R(G)$ is Noetherian (see Segal [9, Corollary 3.3]), the result follows from the Artin–Rees lemma once $\widetilde{K}^*_{\widetilde{G}}(Z)(\pi)$ is known to be finitely generated as an $R(G)$–module for any finite based G–CW complex Z. We shall prove that $\widetilde{K}^*_{\widetilde{G}}(Z)(\pi)$ is finitely generated by reduction to successively simpler cases. Filtering Z by skeleta and using the wedge and suspension axioms, we see that it is enough to consider the case where $Z = G/H_+$ for some closed subgroup H of G. Let $\widetilde{H}$ denote the inverse image of H in $\widetilde{G}$. Then $\widetilde{H}$ is a central extension of H by C, and we have $\widetilde{G}$–equivariant isomorphisms

$$G/H \approx (\widetilde{G}/C)/(\widetilde{H}/C) \approx \widetilde{G}/\widetilde{H}.$$

The $R(G)$–module isomorphisms

$$K^*_{\widetilde{G}}(G/H) \approx K^*_{\widetilde{G}}(\widetilde{G}/\widetilde{H}) \approx K^*_{\widetilde{H}}(\mathrm{pt})$$

preserve the direct sum decomposition (2), whence we obtain an isomorphism

$$K^*_{\widetilde{G}}(G/H)(\pi) \approx K^*_{\widetilde{H}}(\mathrm{pt})(\pi).$$

Here the latter group can be identified with the summand $R(\widetilde{H})(\pi)$ of $R(\widetilde{H})$ generated by those representations of $\widetilde{H}$ which restrict to π–isotypical representations of C. The $R(G)$–module structure on $R(\widetilde{H})(\pi)$ arises from its $R(H)$–module structure via the map $R(G) \to R(H)$, and since $R(H)$ is finite over $R(G)$ (see Segal [9, Proposition 3.2]), we are reduced to showing that $R(\widetilde{H})(\pi)$ is finite as an $R(H)$–module.

Now consider the restriction

$$R(\widetilde{H}) \to \prod_S R(S) \tag{3}$$

where the S runs through the conjugacy classes of Cartan subgroups of $\widetilde{H}$ (conjugacy classes of such subgroups are finite in number and each one of the subgroups is closed, Abelian and contains the central subgroup C). This map is injective [9, Proposition 1.2], whence $R(\widetilde{H})(\pi)$ is a subgroup of $\prod_S R(S)(\pi)$. Therefore it is enough to show that $R(S)(\pi)$ is finite as an $R(H)$–module for each S. The $R(H)$–module structure on $R(S)(\pi)$ arises from its structure of an $R(S/C)$–module via the map of representation rings induced by the inclusion $S/C \hookrightarrow H$, and as $R(S/C)$ is finite over $R(H)$, it is enough to prove that $R(S)(\pi)$ is finite over $R(S/C)$.

We shall now show that $R(S)(\pi)$ is in fact a free $R(S/C)$–module with one generator. To prove this, recall that for a compact Abelian Lie group A, tensor product gives the set $\widehat{A}$ of irreducible representations the structure of a finitely generated Abelian group, and that the representation ring of A is given by the group ring $\mathbb{Z}[\widehat{A}]$. Moreover, our exact sequence of compact Abelian groups

$$1 \to C \to S \to S/C \to 1$$

gives rise to an exact sequence

$$1 \to \widehat{S/C} \to \widehat{S} \to \widehat{C} \to 1.$$

From this it is clear that the summand $R(S)(\pi)$ of $R(S) = \mathbb{Z}[\widehat{S}]$ is the subgroup freely generated by members of the coset of $\widehat{S/C}$ in $\widehat{S}$ mapping to π in $\widehat{C}$, with the $R(S/C)$–module structure arising from the action of $\widehat{S/C}$ on the coset. Thus any representative of the coset will form an $R(S/C)$–basis for $R(S)(\pi)$, and we are done. □

The next two lemmas point out further useful properties of the theories $\mathbf{K}^*_{\widetilde{G}}(-)(\pi)^\wedge_{I_G}$.

Lemma 5 *Let H be a closed subgroup of G, and let X be a based H–CW complex. Then there is a natural isomorphism of pro–$R(G)$–modules*

$$\widetilde{\mathbf{K}}^*_{\widetilde{G}}(G_+ \wedge_H X)(\pi)^\wedge_{I_G} \approx \widetilde{\mathbf{K}}^*_{\widetilde{H}}(X)(\pi)^\wedge_{I_H},$$

where $\widetilde{H}$ denotes the inverse image of H in $\widetilde{G}$.

Proof Observe that the H–CW structure on X gives rise to a G–CW structure on $G_+ \wedge_H X$, and that as X_α runs over the finite H–CW subcomplexes of X, $G_+ \wedge_H X_\alpha$ runs over the finite G–CW subcomplexes of $G_+ \wedge_H X$. Now the lemma follows from the $\widetilde{G}$–equivariant isomorphism

$$\widetilde{G} \wedge_{\widetilde{H}} X_\alpha \approx G \wedge_H X_\alpha;$$

from the change of groups isomorphism

$$\widetilde{K}^*_{\widetilde{G}}(\widetilde{G} \wedge_{\widetilde{H}} X_\alpha) \approx \widetilde{K}^*_{\widetilde{H}}(X_\alpha);$$

from the compatibility of this isomorphism with the decomposition (2); and from the fact that the I_G–adic and I_H–adic topologies on an $R(H)$–module coincide [9, Corollary 3.9]. □

Lemma 6 *Let X be a free G–CW complex. Then there is a natural isomorphism $\mathbf{K}^*_{\widetilde{G}}(X)(\pi)\widehat{_{I_G}} \approx \mathbf{K}^*_{\widetilde{G}}(X)(\pi)$.*

Proof (Compare with the proof of Atiyah and Segal [3, Proposition 4.3].) Let X_α be a finite G–CW subcomplex of X, and let $X_{\alpha,1}, \dots, X_{\alpha,k}$ be the G–CW subcomplexes of X_α such that $X_{\alpha,1}/G, \dots, X_{\alpha,k}/G$ are the connected components of X_α/G. Since the action of G on X is free, for each $i = 1, \dots, k$ we have an isomorphism

$$K_G(X_{\alpha,i}) \xrightarrow{\approx} K(X_{\alpha,i}/G).$$

Pick a base point for $X_{\alpha,i}/G$. Then the diagram

$$\begin{tikzcd} R(G) \arrow[r] \arrow[d] & K_G(X_{\alpha,i}) \arrow[r, "\approx"] & K(X_{\alpha,i}/G) \arrow[d] \\ \mathbb{Z} \arrow[rr, equal] & & \mathbb{Z} \end{tikzcd}$$

commutes, whence the composite of the maps in the top row sends I_G into $\widetilde{K}(X_{\alpha,i}/G)$. However, since $X_{\alpha,i}/G$ is a connected finite CW-complex, the elements of $\widetilde{K}(X_{\alpha,i}/G)$ are nilpotent. Because $R(G)$ is Noetherian, the ideal I_G is finitely generated, and it follows that for large enough n, the image of I_G^n in $K_G(X_{\alpha,i})$ vanishes. As this happens for all $i = 1, \dots, k$, the same is true of the image of I_G^n in $K_G(X_\alpha) \approx \prod_{i=1}^k K_G(X_{\alpha,i})$. Thus

$$I_G^n \cdot K^*_{\widetilde{G}}(X_\alpha)(\pi) = 0$$

for large n, and therefore

$$\begin{aligned}\mathbf{K}^*_{\widetilde{G}}(X)(\pi)^\wedge_{I_G} &= \{K^*_{\widetilde{G}}(X_\alpha)(\pi)/I^n_G \cdot K^*_{\widetilde{G}}(X_\alpha)(\pi)\}_{\alpha,n} \\ &\approx \{K^*_{\widetilde{G}}(X_\alpha)(\pi)\}_\alpha \\ &= \mathbf{K}^*_{\widetilde{G}}(X)(\pi)\end{aligned}$$

as claimed. □

Remark 7 The main technical benefit of introducing the pro-group-valued theories $\mathbf{K}^*_{\widetilde{G}}(-)(\pi)$ and $\mathbf{K}^*_{\widetilde{G}}(-)(\pi)^\wedge_{I_G}$ is that they allow us to sidestep problems with exactness that would otherwise complicate the proof of Theorem 1. The source of these problems is the failure of inverse limits to preserve exactness, as well as the failure of completion to be exact for non-finitely generated modules. The idea of using pro-groups to prove the completion theorem goes back to the original paper of Atiyah and Segal [3].

3 The case of a twisting arising from a graded central extension

In this section we will prove Theorem 1 in the case where the twisting τ arises from a central extension in the way explained by Freed, Hopkins and Teleman [8]. That is, we will prove the following.

Theorem 8 *Let X be a finite G–CW complex, where G is a compact Lie group. Then the projection $\pi\colon EG \times X \to X$ induces an isomorphism*

$$K^{\tau+*}_G(X)^\wedge_{I_G} \xrightarrow{\approx} K^{\pi^*(\tau)+*}_G(EG \times X)$$

for any twisting τ arising from a graded central extension

$$1 \to \mathbb{T} \to G^\tau \to G \to 1, \qquad \epsilon\colon G \to \mathbb{Z}/2.$$

Our argument for proving Theorem 8 is closely based on the one Adams, Haeberly, Jackowski and May present for proving a generalization of the Atiyah–Segal completion theorem in the untwisted case [1]. Their argument in turn builds on ideas due to Carlsson [5]. As before, let

$$1 \to C \to \widetilde{G} \to G \to 1$$

be a central extension of a compact Lie group G by a compact commutative Lie group C, and let π be an irreducible representation of C. We shall derive Theorem 8 from the following result.

Theorem 9 *Let X_1 and X_2 be G–spaces, and let $f\colon X_1 \to X_2$ be a G–equivariant map which is a non-equivariant homotopy equivalence. Then the map*

$$f^*\colon \mathbf{K}^*_{\widetilde{G}}(X_2)(\pi)\widehat{_{I_G}} \to \mathbf{K}^*_{\widetilde{G}}(X_1)(\pi)\widehat{_{I_G}}$$

is an isomorphism.

Before proving Theorem 9, we explain how it implies Theorem 8.

Proof of Theorem 8 assuming Theorem 9 Let Z be a finite G–CW complex. By Theorem 9, the projection map $\pi\colon EG \times Z \to Z$ induces an isomorphism

$$\mathbf{K}^*_{G^\tau}(Z)(1)\widehat{_{I_G}} \xrightarrow[\approx]{\pi^*} \mathbf{K}^*_{G^\tau}(EG \times Z)(1)\widehat{_{I_G}}. \tag{4}$$

Since Z is finite, we have

$$\begin{aligned}\mathbf{K}^*_{G^\tau}(Z)(1)\widehat{_{I_G}} &= \{K^*_{G^\tau}(Z_\alpha)(1)/I^n_G \cdot K^*_{G^\tau}(Z_\alpha)(1)\}_{\alpha,n}\\ &= \{K^*_{G^\tau}(Z)(1)/I^n_G \cdot K^*_{G^\tau}(Z)(1)\}_n.\end{aligned} \tag{5}$$

Fix a model for EG which is a countable ascending union of finite G–CW subcomplexes EG_k, $k \geq 1$; for example, we could take EG to be the iterated join construction of Milnor and take EG_k to be the k–fold join of G with itself. Then Lemma 6 and the finiteness of Z imply that

$$\begin{aligned}\mathbf{K}^*_{G^\tau}(EG \times Z)(1)\widehat{_{I_G}} &= \mathbf{K}^*_{G^\tau}(EG \times Z)(1)\\ &= \{K^*_{G^\tau}(EG_k \times Z)(1)\}_k.\end{aligned} \tag{6}$$

Thus applying the limit functor taking pro–$R(G)$–modules to $R(G)$–modules to the isomorphism (4) gives us an isomorphism

$$K^*_{G^\tau}(Z)(1)\widehat{_{I_G}} \xrightarrow[\approx]{\pi^*} \varprojlim_k K^*_{G^\tau}(EG_k \times Z)(1). \tag{7}$$

Using (6), (4) and (5), we see that inverse system $\{K^*_{G^\tau}(EG_k \times Z)(1)\}_k$ is equivalent to one that satisfies the Mittag–Leffler condition, whence the $\lim^1$ error terms vanish and the codomain in (7) is isomorphic to $K^*_{G^\tau}(EG \times Z)(1)$. Thus for any finite G–CW complex Z, we have a natural isomorphism

$$K^*_{G^\tau}(Z)(1)\widehat{_{I_G}} \xrightarrow[\approx]{\pi^*} K^*_{G^\tau}(EG \times Z)(1).$$

Suppose now Z is a based finite G–CW complex. Then from the diagram

$$\begin{tikzcd} 0 \arrow[r] & \widetilde{K}^*_{G^\tau}(Z)(1)^\wedge_{I_G} \arrow[r] \arrow[d, dotted] & K^*_{G^\tau}(Z)(1)^\wedge_{I_G} \arrow[r] \arrow[d, "\approx"] & K^*_{G^\tau}(\mathrm{pt})(1)^\wedge_{I_G} \arrow[r] \arrow[d, "\approx"] & 0 \\ 0 \arrow[r] & \widetilde{K}^*_{G^\tau}(EG_+ \wedge Z)(1) \arrow[r] & K^*_{G^\tau}(EG \times Z)(1) \arrow[r] & K^*_{G^\tau}(EG)(1) \arrow[r] & 0 \end{tikzcd}$$

we see that there is an induced isomorphism

$$\widetilde{K}^*_{G^\tau}(Z)(1)^\wedge_{I_G} \xrightarrow[\approx]{\pi^*} \widetilde{K}^*_{G^\tau}(EG_+ \wedge Z)(1).$$

The claim now follows by taking Z to be the space $X_+ \wedge S^1(\epsilon)$ and applying Proposition 3. □

The rest of this section is dedicated to the proof of Theorem 9. Let $\{V_i\}_{i\in I}$ be a set of representatives for the isomorphism classes of the non-trivial irreducible complex representations of G. Then I is countable, the fixed-point subspace V_i^G is zero for each $i \in I$, and for every proper closed subgroup H of G, there is some $i \in I$ such that $V_i^H \neq 0$. Let U be the direct sum of countably infinite number of copies of $\bigoplus_{i\in I} V_i$, and let

$$Y = \mathrm{colim}_{V\subset U}\, S^V$$

where the colimit is over all finite-dimensional G–subspaces of U and S^V denotes the one-point compactification of V. Pick a G–invariant inner product on U, and observe that Y^G is S^0.

Lemma 10 *The space Y is H–equivariantly contractible for any proper closed subgroup H of G.*

Proof Since Y has the structure of an H–CW complex, it is enough to show that the fixed point set Y^K is weakly equivalent to a point for any subgroup K of H. Given any finite-dimensional G–subspace $V \subset U$, we can find a finite-dimensional G–subspace $W \subset U$ such that $V \subset W$ and $(W-V)^K \neq 0$, where $W - V$ denotes the orthogonal complement of V in W. But then the inclusion $S^V \hookrightarrow S^W$ is K–equivariantly null-homotopic, whence the map $(S^V)^K \hookrightarrow (S^W)^K$ is null-homotopic. Since Y^K is given by the union

$$Y^K = \mathrm{colim}_{V\subset U}(S^V)^K,$$

the claim follows. □

Lemma 11 *The pro–$R(G)$–module $\widetilde{\mathbf{K}}^*_{\widetilde{G}}(Y)(\pi)^\wedge_{I_G}$ is pro-zero.*

Proof For a finite-dimensional G–subspace $V \subset U$, let

$$\lambda_V \in \widetilde{K}_G(S^V) = \widetilde{K}_{\widetilde{G}}(S^V)(0) \subset \widetilde{K}_{\widetilde{G}}(S^V)$$

denote the equivariant Bott class, where "0" refers to the trivial representation of C. Then by Bott periodicity, each element of $\widetilde{K}^*_{\widetilde{G}}(S^V)(\pi)$ is uniquely expressible as a product $x\lambda_V$, where $x \in \widetilde{K}^*_{\widetilde{G}}(S^0)(\pi)$. Suppose $W \supset V$. From the diagram

$$\begin{array}{ccc}
\widetilde{K}^*_{\widetilde{G}}(S^{W-V})(\pi) & \longrightarrow & \widetilde{K}^*_{\widetilde{G}}(S^0)(\pi) \\
\approx \Big\downarrow \wedge\lambda_V & & \approx \Big\downarrow \wedge\lambda_V \\
\widetilde{K}^*_{\widetilde{G}}(S^W)(\pi) & \longrightarrow & \widetilde{K}^*_{\widetilde{G}}(S^V)(\pi)
\end{array}$$

it follows that the map

$$\widetilde{K}^*_{\widetilde{G}}(S^W)(\pi) \to \widetilde{K}^*_{\widetilde{G}}(S^V)(\pi)$$

sends the element $x\lambda_W = x\lambda_{W-V}\lambda_V$ to $x\chi_{W-V}\lambda_V$, where χ_{W-V} denotes the image of λ_{W-V} under the map

$$\widetilde{K}_G(S^{W-V}) \to \widetilde{K}_G(S^0)$$

induced by the inclusion $S^0 \hookrightarrow S^{W-V}$. Since this map is non-equivariantly null-homotopic, it follows from the diagram

$$\begin{array}{ccccc}
\widetilde{K}_G(S^{W-V}) & \longrightarrow & \widetilde{K}_G(S^0) & = & R(G) \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
\widetilde{K}(S^{W-V}) & \longrightarrow & \widetilde{K}(S^0) & = & \mathbb{Z}
\end{array}$$

that $\chi_{W-V} \in I_G$. Thus if we choose $W \subset U$ so that it is the direct sum of V with n G–invariant subspaces of U, then the map

$$\widetilde{K}^*_G(S^W)(\pi)/I^n_G \cdot \widetilde{K}^*_G(S^W)(\pi) \to \widetilde{K}^*_G(S^V)(\pi)/I^n_G \cdot \widetilde{K}^*_G(S^V)(\pi)$$

is zero. It follows that for any fixed n the pro–$R(G)$–module

$$\{\widetilde{K}_{\widetilde{G}}(S^V)/I^n_G \cdot \widetilde{K}_{\widetilde{G}}(S^V)\}_V$$

is pro-zero, and therefore so is

$$\begin{aligned}
\widetilde{\mathbf{K}}_{\widetilde{G}}(Y)(\pi)\widehat{}_{I_G} &= \{\widetilde{K}_{\widetilde{G}}(S^V)/I^n_G \cdot \widetilde{K}_{\widetilde{G}}(S^V)\}_{n,V} \\
&= \varprojlim_n \{\widetilde{K}_{\widetilde{G}}(S^V)/I^n_G \cdot \widetilde{K}_{\widetilde{G}}(S^V)\}_V
\end{aligned}$$

where the inverse limit is taken in the category of pro–$R(G)$–modules. □

We are now ready to prove Theorem 9.

Proof of Theorem 9 It is enough to prove that $\widetilde{\mathbf{K}}^*_{\widetilde{G}}(Z)(\pi)\widehat{_{I_G}}$ is pro-zero when Z is a non-equivariantly contractible G–space; the claim then follows by taking Z to be the mapping cone of f. We shall show that $\widetilde{\mathbf{K}}^*_{\widetilde{G}}(Z)(\pi)\widehat{_{I_G}} = 0$ for such Z by induction on the subgroups of G, making use of the fact that any strictly descending chain of closed subgroups of a Lie group is of finite length.

To start the induction, we observe that in the case $G = \{e\}$ the claim follows from the assumption that Z is non-equivariantly contractible. Assume inductively that

$$\widetilde{\mathbf{K}}^*_{\widetilde{H}}(Z)(\pi)\widehat{_{I_H}} = 0$$

for all proper closed subgroups H of G; here as before $\widetilde{H}$ denotes the inverse image of H in $\widetilde{G}$. The inclusion of the fixed-point set $Y^G = S^0$ into Y gives a cofiber sequence

$$S^0 \to Y \to Y/S^0$$

whence we have a cofiber sequence

$$Z \to Z \wedge Y \to Z \wedge (Y/S^0).$$

Thus to show that $\widetilde{\mathbf{K}}^*_{\widetilde{G}}(Z)(\pi)\widehat{_{I_G}} = 0$, it is enough to show that

$$\widetilde{\mathbf{K}}^*_{\widetilde{G}}(Z \wedge Y)(\pi)\widehat{_{I_G}} = 0$$

and

$$\widetilde{\mathbf{K}}^*_{\widetilde{G}}(Z \wedge (Y/S^0))(\pi)\widehat{_{I_G}} = 0.$$

Let us first show that $\widetilde{\mathbf{K}}^*_{\widetilde{G}}(Z \wedge Y)(\pi)\widehat{_{I_G}} = 0$; we claim that in fact

$$\widetilde{\mathbf{K}}^*_{\widetilde{G}}(W \wedge Y)(\pi)\widehat{_{I_G}} = 0$$

for any based G–CW complex W. Observing that

$$\widetilde{\mathbf{K}}^*_{\widetilde{G}}(W \wedge Y)(\pi)\widehat{_{I_G}} = \varprojlim_{\alpha} \widetilde{\mathbf{K}}^*_{\widetilde{G}}(W_\alpha \wedge Y)(\pi)\widehat{_{I_G}}$$

where W_α runs through all finite G–CW complexes of W, we see that it is enough to consider the case where W is finite. Filtering W by skeleta and working inductively reduces us to the case where W is of the form $G/H_+ \wedge S^n$ for some n and some closed subgroup H of G, and using the suspension axiom further reduces us to the case $W = G/H_+$. But now in the case $H = G$ the claim follows from Lemma 11; and in the case $H \lneq G$, it follows from the change of groups isomorphism (Lemma 5)

$$\widetilde{\mathbf{K}}^*_{\widetilde{G}}(G/H_+ \wedge Y)(\pi)\widehat{_{I_G}} \approx \widetilde{\mathbf{K}}^*_{\widetilde{H}}(Y)(\pi)\widehat{_{I_H}}$$

together with Lemma 10.

It remains to show that $\widetilde{\mathbf{K}}^*_{\widetilde{G}}(Z \wedge (Y/S^0))(\pi)\widehat{_{I_G}} = 0$. We shall show that in fact

$$\widetilde{\mathbf{K}}^*_{\widetilde{G}}(Z \wedge W)(\pi)\widehat{_{I_G}} = 0$$

for any based G–CW complex W such that W^G is a point. Arguing as above, we see that it is enough to consider W of the form $W = G/H_+$, where H now has to be a proper closed subgroup of G. But in this case the claim follows from the change of groups isomorphism (Lemma 5)

$$\widetilde{\mathbf{K}}^*_{\widetilde{G}}(Z \wedge G/H_+)(\pi)\widehat{_{I_G}} = \widetilde{\mathbf{K}}^*_{\widetilde{H}}(Z)(\pi)\widehat{_{I_H}}$$

and the inductive assumption. □

4 The general case

In this section we finally prove Theorem 1 in full generality. We shall do so by considering successively more general spaces, starting with the case $X = \mathrm{pt}$ and proceeding by change of groups and Mayer–Vietoris arguments. Since in general completion is exact only for finitely generated modules, along the way we check that the twisted K–groups that enter the Mayer–Vietoris sequences are finitely generated over $R(G)$.

Lemma 12 *Theorem 1 holds and $K_G^{\tau+*}(X)$ is finitely generated over $R(G)$ when $X = \mathrm{pt}$.*

Proof By Freed, Hopkins and Teleman [8, Example 2.29], any twisting of a point arises from a graded central extension. Thus Theorem 8 shows that Theorem 1 holds in this case. The claim about finite generation follows from Proposition 3 and the proof of Proposition 4. □

Lemma 13 *Theorem 1 holds and $K_G^{\tau+*}(X)$ is finitely generated over $R(G)$ when $X = G/H$, where H is a closed subgroup of G.*

Proof Notice that $G/H = G \times_H \mathrm{pt}$ and that $EG \times G/H = G \times_H EG$. For any H–space Z, we have a natural local equivalence of topological groupoids

$$Z /\!\!/ H \to G \times_H Z /\!\!/ G$$

giving rise to a natural change of groups isomorphism

$$K_G^{\tau+*}(G \times_H Z) \xrightarrow{\approx} K_H^{\tau+*}(Z). \tag{8}$$

Consider the diagram

$$\begin{tikzcd} K_G^{\tau+*}(G/H)^{\wedge}_{I_G} \arrow[r] \arrow[d] & K_H^{\tau+*}(\mathrm{pt})^{\wedge}_{I_H} \arrow[d] \\ K_G^{\pi^*(\tau)+*}(EG \times G/H) \arrow[r] & K_H^{\pi^*(\tau)+*}(EG) \end{tikzcd}$$

Here the bottom row is a change of groups isomorphism as in (8); the top row is an isomorphism because of the isomorphism (8) and the fact that I_H–adic and I_G–adic completions of an $R(H)$–module agree (see Segal [9, Corollary 3.9]); and the vertical map on the right is an isomorphism by Lemma 12 and the observation that EG is a model for EH. Thus the map on the left is also an isomorphism, which shows that Theorem 1 holds in this case. To see that $K_G^{\tau+*}(G/H)$ is finitely generated as an $R(G)$–module, observe that the isomorphism (8) and Lemma 12 imply that it is finitely generated over $R(H)$. The claim now follows from the fact that $R(H)$ is finite over $R(G)$ (see Segal [9, Proposition 3.2]). □

Lemma 14 *Theorem 1 holds and $K_G^{\tau+*}(X)$ is finitely generated over $R(G)$ when X is of the form $X = G/H \times S^n$, $n \geq 0$.*

Proof The case where $n = 0$ follows from Lemma 13 and the axiom of disjoint unions. For $n > 0$, the claim follows inductively from the Mayer–Vietoris sequences arising from the decomposition of S^n into upper and lower hemispheres S^n_+ and S^n_-. Lemma 13 and the inductive assumption imply that all groups in the Mayer–Vietoris sequence

$$(9)\quad \cdots \longrightarrow K_G^{\tau+*}(G/H \times S^n) \longrightarrow K_G^{\tau+*}(G/H \times S^n_+) \oplus K_G^{\tau+*}(G/H \times S^n_+) \longrightarrow K_G^{\tau+*}(G/H \times (S^n_+ \cap S^n_-)) \longrightarrow \cdots$$

except $K_G^{\tau+*}(G/H \times S^n)$ are finitely generated over $R(G)$, whence the remaining group $K_G^{\tau+*}(G/H \times S^n)$ must also be finitely generated, as claimed. It follows that the sequence obtained from (9) by completion with respect to the augmentation ideal I_G is exact. Now the claim that Theorem 1 holds for the space $G/H \times S^n$ follows from Lemma 13 and the inductive assumption by comparing the completed sequence to the Mayer–Vietoris sequence of the pair

$$(EG \times G/H \times S^n_+, EG \times G/H \times S^n_-)$$

and applying the five lemma. □

Theorem 1 is now contained in the following theorem.

Theorem 15 *Theorem 1 holds and $K_G^{\tau+*}(X)$ is finitely generated over $R(G)$ for any finite G–CW complex X.*

Proof We proceed by induction on the number of cells in X. If X has no cells, that is, if X is the empty G–space, the claim holds trivially. Assume inductively that the claim holds for the space X, and consider the space $Y = X \cup_f (G/H \times D^n)$, where $f\colon G/H \times S^{n-1} \to X$ is an attaching map. Denote

$$D^n(r) = \{x \in \mathbb{R}^n : |x| \le r\},$$

and let

$$Y_1 = X \cup_f (G/H \times (D^n - D^n(1/3))) \subset Y$$

and

$$Y_2 = D^n(2/3) \subset Y.$$

By Lemma 13, Lemma 14 and the inductive assumption, in the Mayer–Vietoris sequence

$$\cdots \longrightarrow K_G^{\tau+*}(Y) \longrightarrow K_G^{\tau+*}(Y_1) \oplus K_G^{\tau+*}(Y_2) \longrightarrow K_G^{\tau+*}(Y_1 \cap Y_2) \longrightarrow \cdots$$

all groups except possibly $K_G^{\tau+*}(Y)$ are finitely generated over $R(G)$. It follows that $K_G^{\tau+*}(Y)$ is also finitely generated, as claimed. We conclude that the top row in the following diagram of Mayer–Vietoris sequences is exact.

$$\begin{tikzcd}
\cdots \arrow[r] & K_G^{\tau+*}(Y)^{\wedge}_{I_G} \arrow[r] \arrow[d] & K_G^{\tau+*}(Y_1)^{\wedge}_{I_G} \oplus K_G^{\tau+*}(Y_2)^{\wedge}_{I_G} \arrow[r] \arrow[d, "\approx"] & K_G^{\tau+*}(Y_1 \cap Y_2)^{\wedge}_{I_G} \arrow[r] \arrow[d, "\approx"] & \cdots \\
\cdots \arrow[r] & K_G^{\tau+*}(EG \times Y) \arrow[r] & K_G^{\tau+*}(EG \times Y_1) \oplus K_G^{\tau+*}(EG \times Y_2) \arrow[r] & K_G^{\tau+*}(EG \times (Y_1 \cap Y_2)) \arrow[r] & \cdots
\end{tikzcd}$$

In the diagram, the vertical map on the right is an isomorphism by Lemma 14 and the map in the middle is an isomorphism by Lemma 13 and the inductive assumption. Thus the map on the left is an isomorphism by the five lemma, showing that Theorem 1 holds for the space Y. □

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