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$\mathbb{T}^2$–cobordism of quasitoric 4–manifolds

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We show the $\mathbb{T}^2$–cobordism group of the category of 4–dimensional quasitoric manifolds is generated by the $\mathbb{T}^2$–cobordism classes of $\mathbb{CP}^2$. We construct nice oriented $\mathbb{T}^2$ manifolds with boundary whose boundaries are the Hirzebruch surfaces. The main tool is the theory of quasitoric manifolds.

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1 Introduction

Cobordism was explicitly introduced by Pontryagin in geometric work on manifolds. In the early 1950's Thom [7] showed cobordism groups could be computed by results of homotopy theory using the Thom complex construction. The nonoriented, oriented and complex cobordism rings are completely determined. Since the Thom transversality theorem does not hold in the equivariant category, the results (like the nonequivariant case) can not be reduced to homotopy theory. The equivariant cobordism has many developments, but the equivariant cobordism ring is not determined for any group. We consider the following category: the objects are all quasitoric manifolds and morphisms are torus equivariant maps between quasitoric manifolds. Here by torus we mean compact torus $\mathbb{T}^n := U(1)^n = (\mathbb{Z}^n \otimes \mathbb{R})/\mathbb{Z}^n$ of dimension n. We compute the $\mathbb{T}^2$–cobordism group of 4–dimensional manifolds in this category. We show that the $\mathbb{T}^2$–cobordism group of the category of 4–dimensional quasitoric manifolds is generated by the $\mathbb{T}^2$–cobordism classes of $\mathbb{CP}^2$. The main tool is the theory of quasitoric manifolds.

Quasitoric manifolds and small covers were introduced by Davis and Januskiewicz in [3]. A manifold with quasitoric (small cover) boundary is a manifold with boundary where the boundary is a disjoint union of some quasitoric manifolds (respectively small covers).

Following Orlik and Raymond [6] we discuss the definition of quasitoric manifolds and the classification of 4–dimensional quasitoric manifolds in Section 2. This classification is needed to prove Lemma 6.3. In Section 3 we introduce *edge-simple polytopes* and study their properties. We give the brief definition of some manifolds with quasitoric

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and small cover boundary in a constructive way in Section 4. There is a natural torus action on these manifolds with quasitoric boundary having a simple convex polytope as the orbit space. The fixed point set of the torus action on the manifold with quasitoric boundary corresponds to the disjoint union of closed intervals of positive length. Interestingly, we show that such a manifold with quasitoric boundary could be viewed as the quotient space of a quasitoric manifold corresponding to a certain circle action on it. This is done in Section 4.3.

In Section 5 we show these manifolds with quasitoric boundary are orientable and compute their Euler characteristic. In Section 6 we show the $\mathbb{T}^2$–cobordism group of 4–dimensional quasitoric manifolds is generated by the $\mathbb{T}^2$–cobordism classes of the complex projective space $\mathbb{CP}^2$; see Lemma 6.3. We construct nice oriented $\mathbb{T}^2$ manifolds with boundary whose boundaries are the Hirzebruch surfaces. In particular, the $\mathbb{T}^2$–cobordism class of a Hirzebruch surface is trivial; see Lemma 6.1. In Theorem 6.6 we compute a set of generators of the $\mathbb{T}^2$–cobordism group of 4–dimensional quasitoric manifolds.

2 Quasitoric manifolds

An n–dimensional simple polytope in $\mathbb{R}^n$ is a convex polytope where exactly n bounding hyperplanes meet at each vertex. The codimension one faces of a convex polytope are called *facets*. Let $\mathcal{F}(P)$ be the set of facets of an n–dimensional simple polytope P. Following Buchstaber and Panov [1] we give definitions of quasitoric manifold, characteristic function and classification.

Definition 2.1 A smooth action of $\mathbb{T}^n$ on a $2n$–dimensional smooth manifold M is said to be locally standard if every point $y \in M$ has a $\mathbb{T}^n$–stable open neighborhood U_y and a diffeomorphism $\psi\colon U_y \to V$, where V is a $\mathbb{T}^n$–stable open subset of $\mathbb{C}^n$, and an isomorphism $\delta_y\colon \mathbb{T}^n \to \mathbb{T}^n$ such that $\psi(t \cdot x) = \delta_y(t) \cdot \psi(x)$ for all $(t, x) \in \mathbb{T}^n \times U_y$.

Definition 2.2 A closed smooth $2n$–dimensional $\mathbb{T}^n$–manifold M is called a quasitoric manifold over P if the following conditions are satisfied:

(1) The $\mathbb{T}^n$ action is locally standard.

(2) There is a projection map $\mathfrak{q}\colon M \to P$ constant on $\mathbb{T}^n$ orbits which maps every l–dimensional orbit to a point in the interior of a codimension-l face of P.

All complex projective spaces $\mathbb{CP}^n$ and their equivariant connected sums and products are quasitoric manifolds.

Lemma 2.1 [3, Lemma 1.4] *Let $\mathfrak{q}\colon M \to P$ be a $2n$–dimensional quasitoric manifold over P. There is a projection map $f\colon \mathbb{T}^n \times P \to M$ so that for each $q \in P$, f maps $\mathbb{T}^n \times q$ onto $\mathfrak{q}^{-1}(q)$.*

Define an equivalence relation $\sim_2$ on $\mathbb{Z}^n$ by $x \sim_2 y$ if and only if $y = \pm x$. Denote the equivalence class of x in the quotient space $\mathbb{Z}^n/\mathbb{Z}_2$ by $[x]$.

Definition 2.3 A function $\eta\colon \mathcal{F}(P) \to \mathbb{Z}^n/\mathbb{Z}_2$ is called a characteristic function if the submodule generated by $\{\eta(F_{j_1}), \dots, \eta(F_{j_l})\}$ is an l–dimensional direct summand of $\mathbb{Z}^n$ whenever the intersection of the facets $F_{j_1}, \dots, F_{j_l}$ is nonempty.

The vectors $\eta(F_j)$ are called characteristic vectors and the pair (P, η) is called a characteristic pair.

In [3] the authors show we can construct a quasitoric manifold from the pair (P, η). Also, given a quasitoric manifold we can associate a characteristic pair to it up to choice of signs of characteristic vectors. For simplicity of notation we may write the images of characteristic and isotropy functions by their class representative. The isotropy function is defined in Section 4.

Definition 2.4 Two actions of $\mathbb{T}^n$ on $2n$–dimensional quasitoric manifolds M_1 and M_2 are called equivalent if there is a homeomorphism $f\colon M_1 \to M_2$ such that $f(t \cdot x) = t \cdot f(x)$ for all $(t, x) \in \mathbb{T}^n \times M_1$.

Definition 2.5 Let $\delta\colon \mathbb{T}^n \to \mathbb{T}^n$ be an automorphism. Two quasitoric manifolds M_1 and M_2 over the same polytope P are called δ–equivariantly homeomorphic if there is a homeomorphism $f\colon M_1 \to M_2$ such that $f(t \cdot x) = \delta(t) \cdot f(x)$ for all $(t, x) \in \mathbb{T}^n \times M_1$.

When δ is the identity automorphism, f is called an equivariant homeomorphism.

Lemma 2.2 [3, Proposition 1.8] *Let $\mathfrak{q}\colon M \to P$ be a $2n$–dimensional quasitoric manifold over P and $\eta\colon \mathcal{F}(P) \to \mathbb{Z}^n/\mathbb{Z}_2$ be its associated characteristic function. Let $\mathfrak{q}_M\colon M(P, \eta) \to P$ be the quasitoric manifold constructed from the pair (P, η). Then the map $f\colon \mathbb{T}^n \times P \to M$ of Lemma 2.1 descends to an equivariant homeomorphism $M(P, \eta) \to M$ covering the identity on P.*

The automorphism δ of Definition 2.5 induces an automorphism δ_* of the poset of subtori of $\mathbb{T}^n$ or equivalently, an automorphism δ_* of the poset of submodules of $\mathbb{Z}^n$. This automorphism descends to a *δ–translation* of characteristic pairs, in which the two characteristic functions differ by δ_*. Using Lemmas 2.1 and 2.2 we can prove the following proposition.

Proposition 2.3 [1, Proposition 5.14] *There is a bijection between δ–equivariant homeomorphism classes of quasitoric manifolds and δ–translations of characteristic pairs (P,η).*

Remark 2.4 Suppose δ is the identity automorphism of $\mathbb{T}^n$. From Proposition 2.3 we have two quasitoric manifolds are equivariantly homeomorphic if and only if their characteristic functions are the same.

Remark 2.5 A quasitoric manifold M over P is simply connected. So M is orientable. A choice of orientation on $\mathbb{T}^n$ and P gives an orientation on M. In this article we fix the positive orientation on $\mathbb{T}^n$. The orientation on the circle subgroup determined by the vectors $\eta(F_j)$ is the induced orientation of $\mathbb{T}^n$. So an orientation of P determines an orientation of the corresponding quasitoric manifolds.

Connected sum 1 Equivariant connected sum of oriented quasitoric manifolds is discussed explicitly by Buchstaber and Ray [2, Section 6]. We discuss the equivariant connected sum of quasitoric manifolds briefly following [3; 2]. Let $\mathfrak{q}_1\colon M_1 \to P_1$ and $\mathfrak{q}_2\colon M_2 \to P_2$ be two $2n$–dimensional oriented quasitoric manifolds over P_1 and P_2 respectively. Let $x_1 \in M_1$ and $x_2 \in M_2$ be two fixed points. Changing the action (if necessary) of $\mathbb{T}^n$ on M_2 by an automorphism of $\mathbb{T}^n$, we can assume that $\mathbb{T}^n$ actions on a $\mathbb{T}^n$ invariant neighborhood U_1 of x_1 and U_2 of x_2 are equivalent. Let $B_1 \subseteq U_1$ and $B_2 \subseteq U_2$ be two invariant open ball around x_1 and x_2 respectively. Identifying the boundary spheres of $M_1 - B_1$ and $M_2 - B_2$ via an orientation reversing (with respect to the induced orientation) equivariant diffeomorphism we get a smooth manifold, denoted by $M_1 \# M_2$, with a natural locally standard $\mathbb{T}^n$ action. The orbit space $P_1 \# P_2$ of this action can be described as follows. Let $\mathfrak{q}_1(x_1) = v_1$ and $\mathfrak{q}_2(x_2) = v_2$ be the corresponding vertices in P_1 and P_2 respectively. Delete a neighborhood $\triangle_{v_1}$ of v_1 in P_1 such that the closer of $\triangle_{v_1}$ in P_1 is diffeomorphic to the n–simplex. Let P_1' be the resulting polytope. Then P_1' has a new facet $\triangle^{n-1}(v_1)$ which is an $(n-1)$–simplex. Similarly we construct the polytope P_2' from P_2. Let $F_1^i, F_2^i, \dots, F_n^i$ be the facets meeting at v_i of P_i. Since the actions of $\mathbb{T}^n$ in a neighborhood of x_1 and x_2 are equivalent, we may assume that the characteristic vector of F_j^1 and F_j^2 are same for $j = 1, 2, \dots, n$. We can obtain the space $P_1 \# P_2$ by gluing the polytopes P_1' and P_2' along $\triangle^{n-1}(v_1)$ and $\triangle^{n-1}(v_2)$ so that F_j^1 and F_j^2 make a new facet for $j = 1, \dots, n$. Then $M_1 \# M_2$ is an oriented quasitoric manifold over $P_1 \# P_2$. The manifold $M_1 \# M_2$ is called the equivariant connected sum of M_1 and M_2.

Example 2.6 Let Q be a triangle $\triangle^2$ in $\mathbb{R}^2$. The possible characteristic functions are indicated by Figure 1. The quasitoric manifold corresponding to the first characteristic

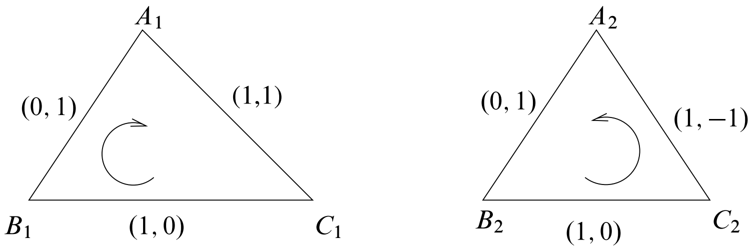


Figure 1: The characteristic functions corresponding to a triangle

pair is $\mathbb{CP}^2$ with the usual $\mathbb{T}^2$ action and standard orientation, denoted $\mathbb{CP}^2_s$. The second correspond to the same $\mathbb{T}^2$ action with the reverse orientation on $\mathbb{CP}^2$, we denote this quasitoric manifold by $\overline{\mathbb{CP}^2_s}$.

Note that there are many nonequivariant $\mathbb{T}^2$–actions on $\mathbb{CP}^2$. We discuss this classification in Section 6.

Example 2.7 Suppose that Q is combinatorially a square in $\mathbb{R}^2$. In this case there are many possible characteristic functions. Some examples are given by Figure 2.

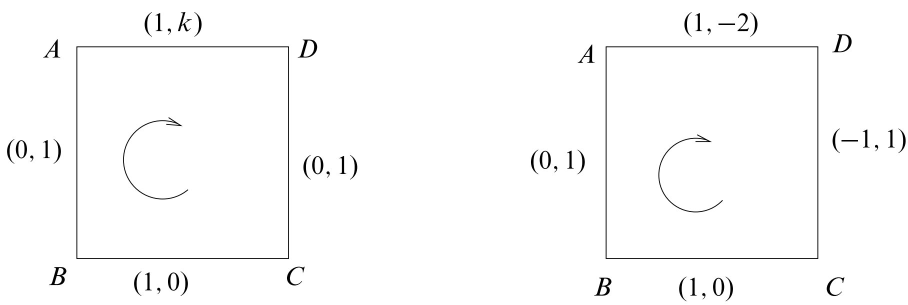


Figure 2: Some characteristic functions corresponding to a square

The first characteristic pairs may construct an infinite family of 4–dimensional quasitoric manifolds, denoted M^4_k for each $k \in \mathbb{Z}$. The manifolds $\{M^4_k : k \in \mathbb{Z}\}$ are equivariantly distinct. Let $L(k)$ be the complex line bundle over $\mathbb{CP}^1$ with the first Chern class k. The complex manifold $\mathbb{CP}(L(k) \oplus \mathbb{C})$ is the Hirzebruch surface for the integer k, where $\mathbb{CP}(\cdot)$ denotes the projectivisation of a complex bundle. So each Hirzebruch surface is the total space of the bundle $\mathbb{CP}(L(k) \oplus \mathbb{C}) \to \mathbb{CP}^1$ with fiber $\mathbb{CP}^1$. It is well-known that with the natural action of $\mathbb{T}^2$ on $\mathbb{CP}(L(k) \oplus \mathbb{C})$ it is equivariantly homeomorphic to M^4_k for each k; see Oda [5]. That is, with respect to the $\mathbb{T}^2$–action, Hirzebruch surfaces are quasitoric manifolds where the orbit space is a combinatorial square and the corresponding characteristic map is described in Figure 2.

On the other hand the second combinatorial model gives the quasitoric manifold $\mathbb{CP}^2 \# \mathbb{CP}^2$, the equivariant connected sum of $\mathbb{CP}^2$.

The following remark classifies all 4–dimensional quasitoric manifolds.

Remark 2.8 Orlik and Raymond [6, Page 553] show that any 4–dimensional quasitoric manifold M^4 over 2–dimensional simple polytope is an equivariant connected sum of several copies of $\mathbb{CP}^2$, $\overline{\mathbb{CP}}^2$ and M_k^4 for some $k \in \mathbb{Z}$.

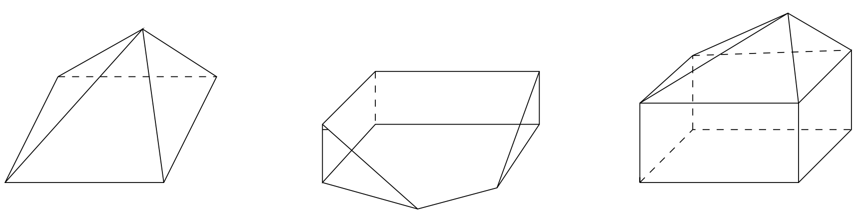
3 Edge-simple polytopes

In this section we introduce a particular type of polytope, which we call an edge-simple polytope. This polytopes are generalization of simple polytopes.

Definition 3.1 An n–dimensional convex polytope P is called an n–dimensional edge-simple polytope if each edge of P is the intersection of exactly $(n-1)$ facets of P.

Example 3.1 (1) An n–dimensional simple convex polytope is an n–dimensional edge-simple polytope.

(2) The following convex polytopes are edge-simple polytopes of dimension 3.



(3) The dual polytope of a 3–dimensional simple convex polytope is a 3–dimensional edge-simple polytope. This result is not true for higher dimensional polytopes, that is if P is a simple convex polytope of dimension $n \geq 4$ the dual polytope of P may not be an edge-simple polytope. For example the dual of the 4–dimensional standard cube in $\mathbb{R}^4$ is not an edge-simple polytope.

Proposition 3.2 (a) *If P is a 2–dimensional simple convex polytope then the suspension SP on P is an edge-simple polytope and SP is not a simple convex polytope.*

(b) *If P is an n–dimensional simple convex polytope then the cone CP on P is an $(n+1)$–dimensional edge-simple polytope.*

Proof (a) Let P be a 2–dimensional simple polytope with m edges $\{e_i : i \in I\}$ and m vertices $\{v_i : i \in I = \{1,2,\dots,m\}\}$. Let a and b be the other two vertices of SP. Then facets of SP are the cone $(Ce_i)_x$ on e_i at $x = a,b$. Edges of SP are $\{xv_i : x = a,b \text{ and } i \in I\} \cup \{e_i : i \in I\}$. The edge xv_i is the intersection of $(Ce_{i_1})_x$ and $(Ce_{i_2})_x$ if $v_i = e_{i_1} \cap e_{i_2}$ for $x = a,b$ and $e_i = (Ce_i)_a \cap (Ce_i)_b$. Hence SP is an edge-simple polytope. If v is a vertex of the polytope P, v is the intersection of 4 facets of SP. So SP is not a simple convex polytope.

(b) Let P be an n–dimensional simple convex polytope in $\mathbb{R}^n \times 0 \subseteq \mathbb{R}^{n+1}$ with m facets $\{F_i : i \in I = \{1,2,\dots,m\}\}$ and k vertices $\{v_1, v_2, \dots, v_k\}$. Assume that the cones are taken at a fixed point a in $\mathbb{R}^{n+1} - \mathbb{R}^n$ lying above the centroid of P. Then facets of CP are $\{(CF_i) : i = 1,2,\dots,m\} \cup \{P\}$. Edges of CP are $\{av_i = C(\{v_i\}) : i = 1,2,\dots,k\} \cup \{e_l : e_l \text{ is an edge of } P\}$. Since P is a simple convex polytope, each vertex v_i of P is the intersection of exactly n facets of P, namely $\{v_i\} = \bigcap_{j=1}^n F_{i_j}$ and each edge e_l is the intersection of unique collection of $(n-1)$ facets $\{F_{l_1}, \dots, F_{l_{n-1}}\}$. Then we have that $C\{v_i\} = \bigcap_{j=1}^n CF_{i_j}$ and $e_l = P \cap CF_{l_1} \cap CF_{l_2} \cap \cdots \cap CF_{l_{n-1}}$. That is $C\{v_i\}$ and $\{e_l\}$ are the intersection of exactly n facets of CP. Hence CP is an $(n+1)$–dimensional edge-simple polytope. □

Cut off a neighborhood of each vertex $v_i, i = 1,2,\dots,k$ of an n–dimensional edge-simple polytope $P \subset \mathbb{R}^n$ by an affine hyperplane $H_i, i = 1,2,\dots,k$ in $\mathbb{R}^n$ such that $H_i \cap H_j \cap P$ are empty sets for $i \neq j$. Then the remaining subset of the convex polytope P is a simple convex polytope of dimension n, denote it by Q_P. Suppose $P_{H_i} = P \cap H_i = H_i \cap Q_P$ for $i = 1,2,\dots,k$. Then P_{H_i} is a facet of Q_P called the facet corresponding to the vertex v_i for each $i = 1,\dots,k$. Since each vertex of P_{H_i} is an interior point of an edge of P and P is an edge-simple polytope, P_{H_i} is an $(n-1)$–dimensional simple convex polytope for each $i = 1,2,\dots,k$.

Lemma 3.3 *Let F be a codimension $l < n$ face of P. Then F is the intersection of a unique set of l facets of P.*

Proof The intersection $F \cap Q_P$ is a codimension l face of Q_P not contained in $\bigcup_{i=0}^k \{P_{H_i}\}$. Since Q_P is a simple convex polytope, $F \cap Q_P = \bigcap_{j=1}^l F_{i_j}{}'$ for some facets $\{F'_{i_1}, \dots, F'_{i_l}\}$ of Q_P. Let F_{i_j} be the unique facet of P such that $F'_{i_j} \subseteq F_{i_j}$. Then $F = \bigcap_1^l F_{i_j}$. Hence each face of P of codimension $l < n$ is the intersection of unique set of l facets of P. □

Remark 3.4 If v_i is the intersection of facets $\{F_{i_1}, \dots, F_{i_l}\}$ of P for some positive integer l, the facets of P_{H_i} are $\{P_{H_i} \cap F_{i_1}, \dots, P_{H_i} \cap F_{i_l}\}$.

4 Construction of manifolds with boundary

Let P be an edge-simple polytope of dimension n with m facets $F_1, \dots, F_m$ and k vertices $v_1, \dots, v_k$. Let e be an edge of P. Then e is the intersection of a unique collection of $(n-1)$ facets $\{F_{i_j} : j = 1, \dots, (n-1)\}$. Let $\mathcal{F}(P) = \{F_1, \dots, F_m\}$ and $\mathbb{F}_2^{n-1}$ be the $(n-1)$–dimensional vector space over $\mathbb{F}_2$, the field of integers modulo 2.

Definition 4.1 The functions $\lambda\colon \mathcal{F}(P) \to \mathbb{Z}^{n-1}/\mathbb{Z}_2$ and $\lambda^s\colon \mathcal{F}(P) \to \mathbb{F}_2^{n-1}$ are called the isotropy function and $\mathbb{F}_2$–isotropy function respectively of the edge-simple polytope P if the set of vectors $\{\lambda(F_{i_1}), \dots, \lambda(F_{i_{n-1}})\}$ and $\{\lambda^s(F_{i_1}), \dots, \lambda^s(F_{i_{n-1}})\}$ form a basis of $\mathbb{Z}^{n-1}$ and $\mathbb{F}_2^{n-1}$ respectively whenever the intersection of the facets $\{F_{i_1}, \dots, F_{i_{n-1}}\}$ is an edge of P.

The vectors $\lambda_i := \lambda(F_i)$ and $\lambda_i^s := \lambda^s(F_i)$ are called isotropy vectors and $\mathbb{F}_2$–isotropy vectors respectively.

We define some isotropy functions of the edge-simple polytopes I^3 and P_0 in Figures 4.3 and 4.4 respectively.

Remark 4.1 It may not be possible to define an isotropy function on the set of facets of all edge-simple polytopes. For example there does not exist an isotropy function of the standard n–simplex $\triangle^n$ for each $n \geq 3$.

4.1 Manifolds with quasitoric boundary

Let F be a face of P of codimension $l < n$. Then F is the intersection of a unique collection of l facets $F_{i_1}, F_{i_2}, \dots, F_{i_l}$ of P. Let $\mathbb{T}_F$ be the torus subgroup of $\mathbb{T}^{n-1}$ corresponding to the submodule generated by $\lambda_{i_1}, \lambda_{i_2}, \dots, \lambda_{i_l}$ in $\mathbb{Z}^{n-1}$. Assume $\mathbb{T}_v = \mathbb{T}^{n-1}$ for each vertex v of P. We define an equivalence relation $\sim$ on the product $\mathbb{T}^{n-1} \times P$ as follows:

$$(t, p) \sim (u, q) \text{ if and only if } p = q \text{ and } tu^{-1} \in \mathbb{T}_F, \tag{4-1}$$

where $F \subset P$ is the unique face containing p in its relative interior. We denote the quotient space $(\mathbb{T}^{n-1} \times P)/\sim$ by $X(P, \lambda)$. The space $X(P, \lambda)$ is not a manifold except when P is a 2–dimensional polytope. If P is 2–dimensional polytope the space $X(P, \lambda)$ is homeomorphic to the 3–dimensional sphere.

But whenever $n > 2$ we can construct a manifold with boundary from the space $X(P, \lambda)$. We restrict the equivalence relation $\sim$ on the product $(\mathbb{T}^{n-1} \times Q_P)$ where $Q_P \subset P$

is a simple polytope as constructed in Section 3 corresponding to the edge-simple polytope P. Let $W(Q_P, \lambda) = (\mathbb{T}^{n-1} \times Q_P)/\sim\, \subset X(P, \lambda)$ be the quotient space. The natural action of $\mathbb{T}^{n-1}$ on $W(Q_P, \lambda)$ is induced by the group operation in $\mathbb{T}^{n-1}$.

Theorem 4.2 *The space $W(Q_P, \lambda)$ is a manifold with boundary. The boundary is a disjoint union of quasitoric manifolds.*

For each edge e of P, $e' = e \cap Q_P$ is an edge of the simple convex polytope Q_P. Let $U_{e'}$ be the open subset of Q_P obtained by deleting all facets of Q_P not containing e' as an edge. Then $U_{e'}$ is diffeomorphic to $I^0 \times \mathbb{R}^{n-1}_{\geqslant 0}$ where I^0 is the open interval $(0,1)$ in $\mathbb{R}$. The facets of $I^0 \times \mathbb{R}^{n-1}_{\geqslant 0}$ are $I^0 \times \{x_1 = 0\}, \dots, I^0 \times \{x_{n-1} = 0\}$ where we have that $\{x_j = 0,\ j = 1, 2, \dots, n-1\}$ are the coordinate hyperplanes in $\mathbb{R}^{n-1}$. Let $F'_{i_1}, \dots, F'_{i_{n-1}}$ be the facets of Q_P such that $\bigcap_{j=1}^{n-1} F'_{i_j} = e'$. Suppose the diffeomorphism $\phi\colon U_{e'} \to I^0 \times \mathbb{R}^{n-1}_{\geqslant 0}$ sends $F'_{i_j} \cap U_{e'}$ to $I^0 \times \{x_j = 0\}$ for all $j = 1, 2, \dots, n-1$. Define an isotropy function λ_e on the set of all facets of $I^0 \times \mathbb{R}^{n-1}_{\geqslant 0}$ by $\lambda_e(I^0 \times \{x_j = 0\}) = \lambda_{i_j}$ for all $j = 1, 2, \dots, n-1$. We define an equivalence relation $\sim_e$ on $(\mathbb{T}^{n-1} \times I^0 \times \mathbb{R}^{n-1}_{\geqslant 0})$ as follows:

$$(t, b, x) \sim_e (u, c, y) \text{ if and only if } (b, x) = (c, y) \text{ and } tu^{-1} \in \mathbb{T}_{\phi(F)}, \tag{4-2}$$

where $\phi(F)$ is the unique face of $I^0 \times \mathbb{R}^{n-1}_{\geqslant 0}$ containing (b, x) in its relative interior, for a unique face F of $U_{e'}$ and $\mathbb{T}_{\phi(F)} = \mathbb{T}_F$. So for each $a \in I^0$ the restriction of λ_e on $\{(\{a\} \times \{x_j = 0\}) : j = 1, 2, \dots, n-1\}$ defines a characteristic function (see Definition 2.3) on the set of facets of $\{a\} \times \mathbb{R}^{n-1}_{\geqslant 0}$. From the constructive definition of quasitoric manifold given in [3], the quotient space $\{a\} \times (\mathbb{T}^{n-1} \times \mathbb{R}^{n-1}_{\geqslant 0})/\sim_e$ is diffeomorphic to $\{a\} \times \mathbb{R}^{2(n-1)}$. Hence

$$(\mathbb{T}^{n-1} \times I^0 \times \mathbb{R}^{n-1}_{\geqslant 0})/\sim_e = I^0 \times (\mathbb{T}^{n-1} \times \mathbb{R}^{n-1}_{\geqslant 0})/\sim_e \cong I^0 \times \mathbb{R}^{2(n-1)}.$$

Since the maps $\pi\colon (\mathbb{T}^{n-1} \times U_{e'}) \to (\mathbb{T}^{n-1} \times U_{e'})/\sim$ and $\pi_e\colon (\mathbb{T}^{n-1} \times I^0 \times \mathbb{R}^{n-1}_{\geqslant 0}) \to (\mathbb{T}^{n-1} \times I^0 \times \mathbb{R}^{n-1}_{\geqslant 0})/\sim_e$ are quotient maps and ϕ is a diffeomorphism, the following commutative diagram ensures that the lower horizontal map ϕ_e is a homeomorphism:

$$\begin{tikzcd}
(\mathbb{T}^{n-1} \times U_{e'}) \arrow[r, "\mathrm{Id} \times \phi"] \arrow[d, "\pi"'] & (\mathbb{T}^{n-1} \times I^0 \times \mathbb{R}^{n-1}_{\geqslant 0}) \arrow[d, "\pi_e"'] & \\
(\mathbb{T}^{n-1} \times U_{e'})/\sim \arrow[r, "\phi_e"] & (\mathbb{T}^{n-1} \times I^0 \times \mathbb{R}^{n-1}_{\geqslant 0})/\sim_e \arrow[r, "\cong"] & I^0 \times \mathbb{R}^{2(n-1)}.
\end{tikzcd} \tag{4-3}$$

Let v'_1 and v'_2 be the vertices of the edge e' of Q_P. Suppose $H_1 \cap e' = \{v'_1\}$ and $H_2 \cap e' = \{v'_2\}$, where H_1 and H_2 are affine hyperplanes as considered in Section 3 corresponding to the vertices v_1 and v_2 of e respectively. Let $U_{v'_1}$ and $U_{v'_2}$ be the open subset of Q_P obtained by deleting all facets of Q_P not containing v_1' and v_2' respectively. Hence there are diffeomorphisms $\phi^1\colon U_{v_1'} \to [0,1)\times\mathbb{R}^{n-1}_{\geqslant 0}$ and $\phi^2\colon U_{v_2'} \to [0,1)\times\mathbb{R}^{n-1}_{\geqslant 0}$ which satisfy the same property as the map ϕ. We get the following commutative diagram and homeomorphisms ϕ_e^j for $j=1,2$:

$$\begin{tikzcd}
(\mathbb{T}^{n-1}\times U_{v_j'}) \arrow[r, "\mathrm{Id}\times\phi^j"] \arrow[d, "\pi"'] & (\mathbb{T}^{n-1}\times[0,1)\times\mathbb{R}^{n-1}_{\geqslant 0}) \arrow[d, "\pi_e"'] & \\
(\mathbb{T}^{n-1}\times U_{v_j'})/\sim \arrow[r, "\phi_e^j"] & (\mathbb{T}^{n-1}\times[0,1)\times\mathbb{R}^{n-1}_{\geqslant 0})/\sim_e \arrow[r, "\cong"] & {[0,1)\times\mathbb{R}^{2(n-1)}}.
\end{tikzcd} \tag{4-4}$$

Hence each point of $(\mathbb{T}^{n-1}\times Q_P)/\sim$ has a neighborhood homeomorphic to an open subset of $[0,1)\times\mathbb{R}^{2(n-1)}$. So $W(Q_P,\lambda)$ is a manifold with boundary. From the above discussion the interior of $W(Q_P,\lambda)$ is

$$\bigcup_{e'}(\mathbb{T}^{n-1}\times U_{e'})/\sim = W(Q_P,\lambda)\smallsetminus\{(\mathbb{T}^{n-1}\times\bigsqcup_{i=1}^{k}P_{H_i})/\sim\}$$

and the boundary is $\bigsqcup_{i=1}^{k}\{(\mathbb{T}^{n-1}\times P_{H_i})/\sim\}$. Let $F(H)_{i_j}$ be a facet of P_{H_i}. So there exists a unique facet F_j of P such that $F(H)_{i_j}=F_j\cap Q_P\cap H_i$. The restriction of the function λ on the set of all facets of P_{H_i} (namely $\lambda(F(H)_{i_j})=\lambda_j$) give a characteristic function of a quasitoric manifold over P_{H_i}. Hence restricting the equivalence relation $\sim$ on $(\mathbb{T}^{n-1}\times P_{H_i})$ we get that the quotient space $W_i=(\mathbb{T}^{n-1}\times P_{H_i})/\sim$ is a quasitoric manifold over P_{H_i}. Hence the boundary $\partial W(Q_P,\lambda)$ is the disjoint union $\bigsqcup_{i=1}^{k}W_i$, where W_i is a quasitoric manifold. So $W(Q_P,\lambda)$ is a manifold with quasitoric boundary.

In Section 5 we show that these manifolds with quasitoric boundary are orientable.

Example 4.3 An isotropy function of the standard cube I^3 is described in Figure 3. Here simple convex polytopes $P_{H_1},\dots,P_{H_8}$ are triangles. The restriction of the isotropy function on P_{H_i} gives that the space $(\mathbb{T}^2\times P_{H_i})/\sim$ is the complex projective space $\mathbb{CP}^2$ or $\overline{\mathbb{CP}}^2$. Since the antipodal map in $\mathbb{R}^3$ is an orientation reversing map we can show the disjoint union $\big(\bigsqcup_{i=1}^{4}\mathbb{CP}^2\big)\sqcup\big(\bigsqcup_{i=1}^{4}\overline{\mathbb{CP}}^2\big)$ is the boundary of $(\mathbb{T}^2\times Q_{I^3})/\sim$.

Example 4.4 In Figure 4 we define an isotropy function of the edge-simple polytope P_0. Here simple convex polytopes $P_{H_1},P_{H_2},P_{H_3},P_{H_4}$ are triangles and the simple convex polytope P_{H_5} is a rectangle. The restriction of the isotropy function on P_{H_i} gives that the space $(\mathbb{T}^2\times P_{H_i})/\sim$ is $\mathbb{CP}^2$ or $\overline{\mathbb{CP}}^2$ for each $i\in\{1,2,3,4\}$ and

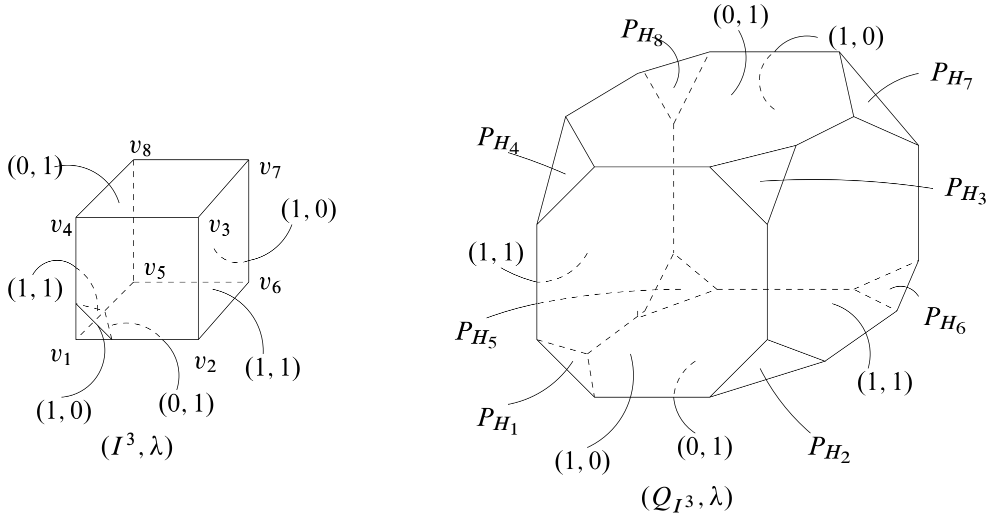


Figure 3: An isotropy function λ of the edge-simple polytope I^3

$(\mathbb{T}^2 \times P_{H_5})/\sim$ is $\mathbb{CP}^1 \times \mathbb{CP}^1$. Hence $\left(\bigsqcup_{i=1}^2 \mathbb{CP}^2\right) \sqcup \left(\bigsqcup_{i=1}^2 \overline{\mathbb{CP}^2}\right) \sqcup (\mathbb{CP}^1 \times \mathbb{CP}^1)$ is the boundary of $W(Q_{P_0}, \lambda) := (\mathbb{T}^2 \times Q_{P_0})/\sim$; see Section 6.

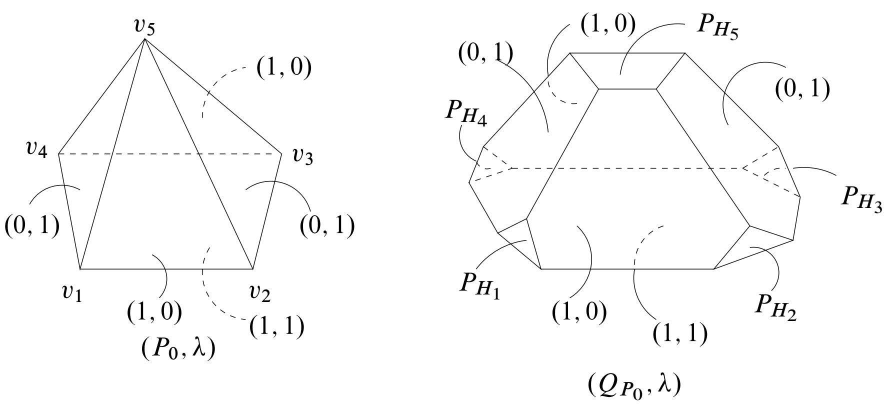


Figure 4: An isotropy function λ of the edge-simple polytope P_0

4.2 Manifolds with small cover boundary

We assign each face F to the subgroup G_F of $\mathbb{F}_2^{n-1}$ determined by the vectors $\lambda_{i_1}^s, \dots, \lambda_{i_l}^s$ where F is the intersection of the facets $F_{i_1}, \dots, F_{i_l}$. Let $\sim_s$ be an equivalence relation on $(\mathbb{F}_2^{n-1} \times P)$ defined by

(4-5) $$(t, p) \sim_s (u, q) \text{ if and only if } p = q \text{ and } t - u \in G_F,$$

where $F \subset P$ is the unique face containing p in its relative interior. The quotient space $(\mathbb{F}_2^{n-1} \times Q_P)/\sim_s \subset (\mathbb{F}_2^{n-1} \times P)/\sim_s$, denoted by $S(Q_P, \lambda^s)$, is a manifold with boundary. This can be shown by the same arguments given in Section 4.1. The boundary of this manifold is $\{(\mathbb{F}_2^{n-1} \times \bigsqcup_{i=1}^k P_{H_i})/\sim_s\} = \bigsqcup_{i=1}^k \{(\mathbb{F}_2^{n-1} \times P_{H_i})/\sim_s\}$. Clearly the restriction of the $\mathbb{F}_2$–isotropy function λ^s on the set of all facets of P_{H_i} gives the characteristic function of a small cover over P_{H_i}. So $(\mathbb{F}_2^{n-1} \times P_{H_i})/\sim_s$ is a small cover for each $i = 0, \dots, k$. Hence $S(Q_P, \lambda^s)$ is a manifold with small cover boundary.

4.3 Some observations

The set of all facets of the simple convex polytope Q_P is given by $\mathcal{F}(Q_P) = \{P_{H_j} : j = 1, 2, \dots, k\} \cup \{F_i' : i = 1, 2, \dots, m\}$, where $F_i' = F_i \cap Q_P$ for unique facets F_i of P. We define the function $\eta \colon \mathcal{F}(Q_P) \to \mathbb{Z}^n/\mathbb{Z}_2$ as follows:

$$\eta(F) = \begin{cases} [(0, \dots, 0, 1)] \in \mathbb{Z}^n/\mathbb{Z}_2 & \text{if } F = P_{H_j}, j \in \{1, \dots, k\}, \\ [\lambda_i, 0] \in \mathbb{Z}^{n-1}/\mathbb{Z}_2 \times \{0\} \subset \mathbb{Z}^n/\mathbb{Z}_2 & \text{if } F = F_i, i \in \{1, 2, \dots, m\}. \end{cases} \tag{4-6}$$

So the function η satisfies the condition for the characteristic function (see Definition 2.3) of a quasitoric manifold over the n–dimensional simple convex polytope Q_P. Hence from the characteristic pair (Q_P, η) we can construct the quasitoric manifold $M(Q_P, \eta)$ over Q_P. There is a natural $\mathbb{T}^n$ action on $M(Q_P, \eta)$. Let $\mathbb{T}_H$ be the circle subgroup of $\mathbb{T}^n$ determined by the submodule $\{0\} \times \{0\} \times \cdots \times \{0\} \times \mathbb{Z}$ of $\mathbb{Z}^n$. Hence $W(Q_P, \lambda)$ is the orbit space of the circle $\mathbb{T}_H$ action on $M(Q_P, \eta)$. The quotient map $\phi_H \colon M(Q_P, \eta) \to W(Q_P, \lambda)$ is not a fiber bundle map.

Remark 4.5 The manifold $S(Q_p, \lambda_s)$ with small cover boundary constructed in Section 4.2 is the orbit space of $\mathbb{Z}_2$ action on a small cover.

5 Orientability of $W(Q_P, \lambda)$

Suppose $W = W(Q_P, \lambda)$. The boundary ∂W has a collar neighborhood in W. Hence by Hatcher [4, Proposition 2.22] we get $H_i(W, \partial W) = \widetilde{H}_i(W/\partial W)$ for all i. We show the space $W/\partial W$ has a CW–structure. Actually we show that corresponding to each edge of P there exists an odd-dimensional cell of $W/\partial W$. Realize Q_P as a simple convex polytope in $\mathbb{R}^n$ and choose a linear functional $\phi \colon \mathbb{R}^n \to \mathbb{R}$ which distinguishes the vertices of Q_P, as in the proof of Theorem 3.1 in [3]. The vertices are linearly ordered according to ascending value of ϕ. We make the 1–skeleton of Q_P into a directed graph by orienting each edge such that ϕ increases along edges. For each vertex v of Q_P define its index, $\mathrm{ind}(v)$, as the number of incident edges that point towards v. Suppose $\mathcal{V}(Q_P)$ is the set of all vertices and $\mathcal{E}(Q_P)$ is the set of edges of Q_P. For each $j \in \{1, 2, \dots, n\}$, let

$$I_j = \{(v, e_v) \in \mathcal{V}(Q_P) \times \mathcal{E}(Q_P) : \mathrm{ind}(v) = j \text{ and } e_v \text{ is the incident edge that points towards } v \text{ such that } e_v = e \cap Q_P \text{ for an edge } e \text{ of } P\}.$$

Suppose $(v, e_v) \in I_j$. Let $F_{e_v} \subset Q_P$ denote the smallest face which contains the inward pointing edges incident to v. Then F_{e_v} is a unique face not contained in any P_{H_i}. Let U_{e_v} be the open subset of F_{e_v} obtained by deleting all faces of F_{e_v} not containing the edge e_v. The restriction of the equivalence relation $\sim$ on $(\mathbb{T}^{n-1} \times U_{e_v})$ gives that the quotient space $(\mathbb{T}^{n-1} \times U_{e_v})/\sim$ is homeomorphic to the open disk B^{2j-1}. Hence the quotient space $(W/\partial W)$ has a CW–complex structure with odd dimensional cells and one zero dimensional cell only. The number of $(2j-1)$–dimensional cell is $|I_j|$, the cardinality of I_j for $j = 1, 2, \dots, n$. So we get the following theorem.

Theorem 5.1 $$H_i(W, \partial W) = \begin{cases} \bigoplus_{|I_j|} \mathbb{Z} & \text{if } i = 2j-1 \text{ and } j \in \{1, \dots, n\}, \\ \mathbb{Z} & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

When $j = n$ the cardinality of I_j is one. So $H_{2n-1}(W, \partial W) = \mathbb{Z}$. Hence W is an orientable manifold with boundary.

Example 5.2 We adhere to the notation of Example 4.4. Observe $I_3 = \{(v_{14}, e_{v_{14}})\}$, $I_2 = \{(v_8, e_{v_8}), (v_{13}, e_{v_{13}}), (v_{15}, e_{v_{15}})\}$ and $I_1 = \{(v_3, e_{v_3}), (v_6, e_{v_6}), (v_9, e_{v_9})\}$. The face $F_{e_{v_{13}}}$ corresponding to the point $(v_{13}, e_{v_{13}})$ is $v_0 v_3 v_5 v_{13} v_{12} v_1$. Thus we can give a CW–structure of $W(Q_{P_0}, \lambda)/\partial W(Q_{P_0}, \lambda)$ with one 0–cell, two 1–cells, three 3–cells and one 5–cell.

In [3] the authors showed that the odd dimensional homology of quasitoric manifolds are zero. So $H_{2i-1}(\partial W) = 0$ for all i. Hence we get the following exact sequences for the collared pair $(W, \partial W)$.

(5-1)
$$\begin{array}{ccccccccc}
0 \to H_{2n-1}(W) & \xrightarrow{j_*} & H_{2n-1}(W, \partial W) & \xrightarrow{\partial} & H_{2n-2}(\partial W) & \xrightarrow{i_*} & H_{2n-2}(W) \longrightarrow 0 \\
\vdots & & \vdots & & \vdots & & \vdots \\
0 \longrightarrow H_3(W) & \xrightarrow{j_*} & H_3(W, \partial W) & \xrightarrow{\partial} & H_2(\partial W) & \xrightarrow{i_*} & H_2(W) \longrightarrow 0 \\
0 \longrightarrow H_1(W) & \xrightarrow{j_*} & H_1(W, \partial W) & \xrightarrow{\partial} & H_0(\partial W) & \xrightarrow{i_*} & H_0(W) \twoheadrightarrow \mathbb{Z},
\end{array}$$

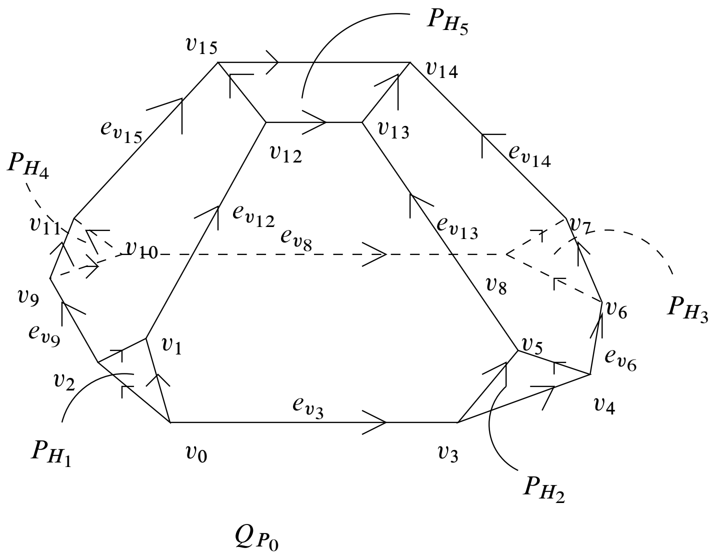


Figure 5: The index function of Q_{P_0}

where $\mathbb{Z} \cong H_0(W, \partial W)$. Let $(h_{i_0}, \ldots, h_{i_{n-1}})$ be the *h–vector* of P_{H_i} for $i = 1, 2, \ldots, k$. The definition of *h–vector* of a simple convex polytope is given in [3]. Hence we have that the Euler characteristic of the manifold W with quasitoric boundary is $\sum_{i=1}^{k} \sum_{j=0}^{n-1} h_{i_j} - \sum_{j=1}^{n-1} |I_j|$.

Fix the standard orientation on $\mathbb{T}^{n-1}$. Let $I_n = \{(v', e_{v'})\}$. Then the $(2n-1)$–dimensional cell $(\mathbb{T}^{n-1} \times U_{e_{v'}})/\sim \subset W$ represents a fundamental class of $W/\partial W$ with coefficient in $\mathbb{Z}$. Thus an orientation of $U_{e_{v'}}$ (hence of Q_P) determines an orientation of W. Note that an orientation of Q_P is induced by orienting the ambient space $\mathbb{R}^n$.

So the boundary orientation on P_{H_i} induced from the orientation of Q_P gives the orientation on the quasitoric manifold $W_i \subset \partial W$. In the next section we consider the orientation of Q's and Q_P's induced from the standard orientation of their ambient spaces.

6 Torus cobordism of quasitoric manifolds

Let $\mathfrak{C}$ be the following category: the objects are all quasitoric manifolds and morphisms are torus equivariant maps between quasitoric manifolds. We are considering torus cobordism in this category only.

Definition 6.1 Two $2n$–dimensional quasitoric manifolds M_1 and M_2 are said to be $\mathbb{T}^n$–cobordant if there exists an oriented $\mathbb{T}^n$ manifold W with boundary ∂W such that ∂W is $\mathbb{T}^n$ equivariantly diffeomorphic to $M_1 \sqcup (-M_2)$ under an orientation preserving diffeomorphism. Here $-M_2$ represents the reverse orientation of M_2.

We denote the $\mathbb{T}^n$–cobordism class of quasitoric $2n$–manifold M by $[M]$.

Definition 6.2 The n–th torus cobordism group is the group of all cobordism classes of $2n$–dimensional quasitoric manifolds with the operation of disjoint union. We denote this group by CG_n.

Let $M \to Q$ be a 4–dimensional quasitoric manifold over the square Q with the characteristic function $\eta\colon \mathcal{F}(Q) \to \mathbb{Z}^2/\mathbb{Z}_2$. We construct an oriented $\mathbb{T}^2$ manifold W with boundary ∂W, where ∂W is equivariantly homeomorphic to either $-M \sqcup \left(\bigsqcup_{k_1} \mathbb{CP}^2\right) \sqcup \left(\bigsqcup_{k_2} \overline{\mathbb{CP}}^2\right)$ or $M \sqcup \left(\bigsqcup_{k_1} \mathbb{CP}^2\right) \sqcup \left(\bigsqcup_{k_2} \overline{\mathbb{CP}}^2\right)$ for some integers k_1, k_2. In order to show this we construct a 3–dimensional edge-simple polytope $P_{\mathcal{E}}$ such that $P_{\mathcal{E}}$ has exactly one vertex O which is the intersection of 4 facets with $P_{\mathcal{E}} \cap H_O = Q$ and other vertices of $P_{\mathcal{E}}$ are intersection of 3 facets. We define an isotropy function λ, extending the characteristic function η of M, from the set of facets of $P_{\mathcal{E}}$ to $\mathbb{Z}^2/\mathbb{Z}_2$. Then $W(Q_{P_{\mathcal{E}}}, \lambda)$ is the required oriented $\mathbb{T}^2$ manifold with quasitoric boundary. We have done an explicit calculation in the following.

Let $Q = ABCD$ be a rectangle which belongs to $\{(x,y,z) \in \mathbb{R}^3_{\geq 0} : x+y+z = 1\}$; see Figure 6. Let $\eta\colon \{AB, BC, CD, DA\} \to \mathbb{Z}^2/\mathbb{Z}_2$ be the characteristic function for a quasitoric manifold M over $ABCD$ such that the characteristic vectors are

$$\eta(AB) = \eta_1, \quad \eta(BC) = \eta_2, \quad \eta(CD) = \eta_3, \quad \eta(DA) = \eta_4.$$

We may assume that $\eta_1 = (0,1)$ and $\eta_2 = (1,0)$. From the classification results given in Section 2, it is enough to consider only the following cases:

(6-1) $\quad \eta_3 = (0,1) \quad$ and $\quad \eta_4 = (1,0)$,

(6-2) $\quad \eta_3 = (0,1) \quad$ and $\quad \eta_4 = (1,k), \quad k = 1$ or -1,

(6-3) $\quad \eta_3 = (0,1) \quad$ and $\quad \eta_4 = (1,k), \quad k \in \mathbb{Z} - \{-1,0,1\}$,

(6-4) $\quad \eta_3 = (-1,1) \quad$ and $\quad \eta_4 = (1,-2)$.

Case (6-1) In this case the edge-simple polytope $\widetilde{P}_1$, given in Figure 6, is the required edge-simple polytope. The isotropy vectors of $\widetilde{P}_1$ are given by

$$\lambda(OGH) = \eta_1, \qquad \lambda(OHI) = \eta_2, \qquad \lambda(OIJ) = \eta_3,$$
$$\lambda(OGJ) = \eta_4, \qquad \lambda(GHIJ) = \eta_1 + \eta_2.$$

So we get an oriented $\mathbb{T}^2$ manifold $W(Q_{\widetilde{P}_1}, \lambda)$ with quasitoric boundary where the boundary is the quasitoric manifold $-M \sqcup \bigsqcup_{k_1} \mathbb{CP}^2 \sqcup \bigsqcup_{k_2} \overline{\mathbb{CP}}^2$ for some integers k_1, k_2. Note that orientation on $\widetilde{P}_1 \subset \mathbb{R}^3_{\geq 0}$ comes from the standard orientation of $\mathbb{R}^3$. Let A' and B' be the midpoints of GJ and HI respectively. Let $\mathcal{H}$ be the plane passing through O, A' and B' in $\mathbb{R}^3$. Since a reflection in $\mathbb{R}^3$ is an orientation reversing homeomorphism, it is easy to observe that the reflection on $\mathcal{H}$ induces the following orientation reversing equivariant homeomorphisms:

$$\begin{aligned}(\mathbb{T}^2 \times \widetilde{P}_{1_I})/\sim &\to (\mathbb{T}^2 \times \widetilde{P}_{1_H})/\sim,\\ (\mathbb{T}^2 \times \widetilde{P}_{1_J})/\sim &\to (\mathbb{T}^2 \times \widetilde{P}_{1_G})/\sim.\end{aligned} \tag{6-5}$$

So $k_1 = k_2$, since $[\overline{\mathbb{CP}}^2] = -[\mathbb{CP}^2]$, $[M] = 0[\mathbb{CP}^2]$. Identifying the corresponding boundaries of $W(Q_{\widetilde{P}_1}, \lambda)$ via the equivariant homeomorphisms of Equation (6-5) we get that M is the boundary of a nice oriented $\mathbb{T}^2$ manifold. By "nice manifold" we mean it has good CW–complex structures.

Case (6-2) In this case $|\det(\eta_2, \eta_4)| = 1$. Let O be the origin of $\mathbb{R}^3$. Let C_Q be the open cone on rectangle $ABCD$ at the origin O. Let G, H, I, J be points on extended OA, OB, OC, OD respectively. Let E and F be two points in the interior of the open cones on AB and CD at O respectively such that $|OG| < |OE|$, $|OH| < |OE|$ and $|OI| < |OF|$, $|OJ| < |OF|$. We may assume $OH = OI$, $OG = OJ$, $HE = EG$ and $IF = FJ$. Then the convex polytope $P_1 \subset C_Q$ on the set of vertices $\{O, G, E, H, I, F, J\}$ is an edge-simple polytope (see Figure 6) of dimension 3. Define a function, denote by λ, on the set of facets of P_1 by

$$\begin{aligned}\lambda(OGEH) = \eta_1, \quad \lambda(OHI) = \eta_2, \quad \lambda(OJFI) = \eta_3,\\ \lambda(OJG) = \eta_4, \quad \lambda(HIFE) = \eta_4, \quad \lambda(GJFE) = \eta_2.\end{aligned} \tag{6-6}$$

Hence λ is an isotropy function on the edge-simple polytope P_1. The boundary of the oriented $\mathbb{T}^2$ manifold $W(Q_{P_1}, \lambda)$ is the quasitoric manifold $-M \sqcup \bigsqcup_{k_1} \mathbb{CP}^2 \sqcup \bigsqcup_{k_2} \overline{\mathbb{CP}}^2$ for some integers k_1, k_2. Similarly to the previous case we can show that suitable reflections induce the following orientation reversing equivariant homeomorphisms:

$$\begin{aligned}(\mathbb{T}^2 \times P_{1_H})/\sim &\to (\mathbb{T}^2 \times P_{1_I})/\sim,\\ (\mathbb{T}^2 \times P_{1_E})/\sim &\to (\mathbb{T}^2 \times P_{1_F})/\sim,\\ (\mathbb{T}^2 \times P_{1_G})/\sim &\to (\mathbb{T}^2 \times P_{1_J})/\sim.\end{aligned} \tag{6-7}$$

So $k_1 = k_2$. Hence $[M] = 0[\mathbb{CP}^2]$. Identifying the corresponding boundaries of $W(Q_{P_1}, \lambda)$ via the equivariant homeomorphisms of Equation (6-7) we get that M is the boundary of a nice oriented $\mathbb{T}^2$ manifold.

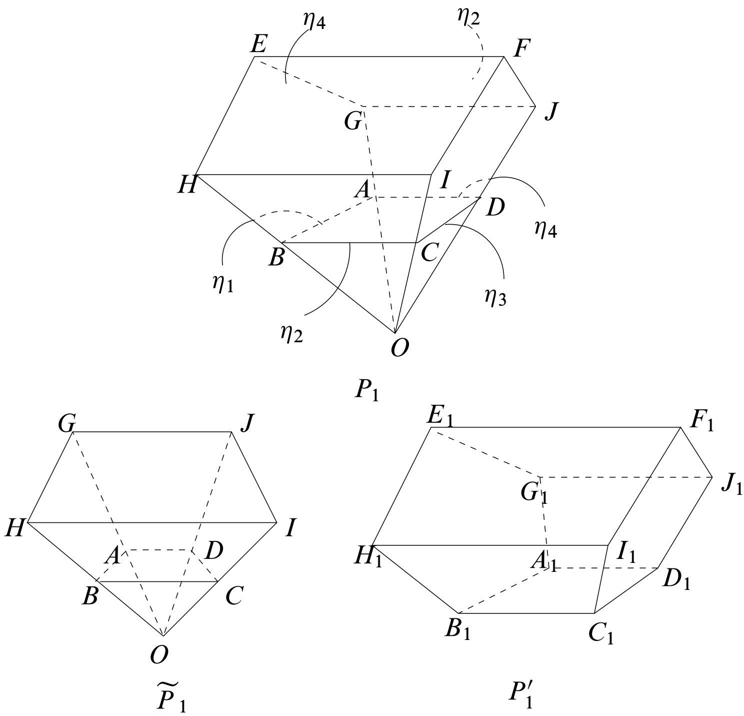


Figure 6: The edge-simple polytope P_1, $\widetilde{P}_1$ and the convex polytope P'_1, respectively

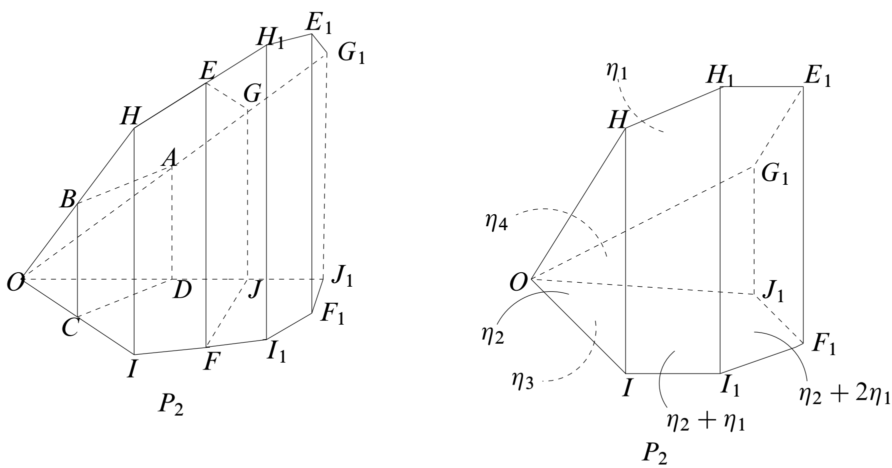


Figure 7: The edge-simple polytope P_2 with the function $\lambda^{(2)}$

Case (6-3) Suppose $\det(\eta_2,\eta_4)=k>1$. Define a function $\lambda^{(1)}$ on the set of facets of P_1 except $GEFJ$ by

$$\begin{aligned}\lambda^{(1)}(OGEH)=\eta_1,\quad \lambda^{(1)}(OHI)=\eta_2,\quad \lambda^{(1)}(OIFJ)=\eta_3,\\ \lambda^{(1)}(OGJ)=\eta_4,\quad \lambda^{(1)}(EHIF)=\eta_2+\eta_1.\end{aligned}\tag{6-8}$$

So the function $\lambda^{(1)}$ satisfies the condition of an isotropy function of the edge-simple polytope P_1 along each edge except the edges of the rectangle $GEFJ$. The restriction of the function $\lambda^{(1)}$ on the edges GE, EF, FJ, GJ of the rectangle $GEFJ$ gives the following equations:

$$\begin{aligned}|\det[\lambda^{(1)}(GE),\lambda^{(1)}(EF)]|=1,\quad |\det[\lambda^{(1)}(EF),\lambda^{(1)}(FJ)]|=1,\\ |\det[\lambda^{(1)}(FJ),\lambda^{(1)}(GJ)]|=1,\quad |\det[\lambda^{(1)}(GJ),\lambda^{(1)}(GE)]|=1,\\ \det[\lambda^{(1)}(EF),\lambda^{(1)}(GJ)]=k-1<k.\end{aligned}\tag{6-9}$$

Let P_1' be a 3–dimensional convex polytope as in Figure 6. Identifying the facet $GEFJ$ of P_1 and $A_1B_1C_1D_1$ of P_1' through a suitable diffeomorphism of manifold with corners such that the vertices G, E, F, J maps to the vertices A_1, B_1, C_1, D_1 respectively, we can form a new convex polytope P_2; see Figure 7. After the identification the following holds.

(1) The facet of P_1 containing GE and the facet of P_1' containing A_1B_1 make the facet $OHH_1E_1G_1$ of P_2.

(2) The facet of P_1 containing EF and the facet of P_1' containing B_1C_1 make the facet HH_1I_1I of P_2.

(3) The facet of P_1 containing FJ and the facet of P_1' containing C_1D_1 make the facet $OII_1F_1J_1$ of P_2.

(4) The facet of P_1 containing JG and the facet of P_1' containing D_1A_1 make the facet OJ_1G_1 of P_2.

The polytope P_2 is an edge-simple polytope. We define a function $\lambda^{(2)}$ on the set of facets of P_2 except $G_1E_1F_1J_1$ by

$$\begin{aligned}\lambda^{(2)}(OHH_1E_1G_1)=\eta_1,\quad \lambda^{(2)}(OJ_1G_1)=\eta_4,\\ \lambda^{(2)}(OIH)=\eta_2,\quad \lambda^{(2)}(HH_1I_1I)=\eta_2+\eta_1,\\ \lambda^{(2)}(OII_1F_1J_1)=\eta_3,\quad \lambda^{(2)}(H_1I_1F_1E_1)=\eta_2+2\eta_1.\end{aligned}\tag{6-10}$$

So the function $\lambda^{(2)}$ satisfies the condition of an isotropy function of the edge-simple polytope P_2 along each edge except the edges of the rectangle $G_1E_1F_1J_1$. The restriction of the function $\lambda^{(2)}$ on the edges namely $G_1E_1, E_1F_1, F_1J_1, G_1J_1$ of the rectangle $G_1E_1F_1J_1$ gives the following equations:

$$\begin{aligned} &|\det[\lambda^2(G_1E_1), \lambda^2(E_1F_1)]| = 1, \quad |\det[\lambda^2(G_1J_1), \lambda^2(G_1E_1)]| = 1, \\ &|\det[\lambda^2(F_1J_1), \lambda^2(G_1J_1)]| = 1, \quad \det[\lambda^2(E_1F_1), \lambda^2(G_1J_1)] = k-2 < k-1. \\ &|\det[\lambda^2(E_1F_1), \lambda^2(F_1J_1)]| = 1, \end{aligned} \tag{6-11}$$

Proceeding in this way, at the k–th step we construct an edge-simple polytope P_k with the function $\lambda^{(k)}$, extending the function $\lambda^{(k-1)}$, on the set of facets of P_k such that

$$\begin{aligned} \lambda^{(k)}(H_{k-2}H_{k-1}I_{k-1}I_{k-2}) &= \eta_2 + (k-1)\eta_1 \\ &= \lambda^{(k-1)}(H_{k-2}I_{k-2}F_{k-2}E_{k-2}), \\ \lambda^{(k)}(OG_{k-1}J_{k-1}) &= \eta_4 \\ &= \lambda^{(k-1)}(OG_{k-2}J_{k-2}), \\ \lambda^{(k)}(H_{k-1}I_{k-1}F_{k-1}E_{k-1}) &= \eta_4, \\ \lambda^{(k)}(G_{k-1}E_{k-1}F_{k-1}J_{k-1}) &= \eta_2 + (k-1)\eta_1. \end{aligned} \tag{6-12}$$

We observe that the function $\lambda := \lambda^{(k)}$ is an isotropy function of the edge-simple polytope P_k. So we get an oriented $\mathbb{T}^2$–manifold with boundary $W(Q_{P_k}, \lambda)$ where the boundary is the quasitoric manifold $-M \sqcup \bigsqcup_{k_1} \mathbb{CP}^2 \sqcup \bigsqcup_{k_2} \overline{\mathbb{CP}^2}$ for some integers k_1, k_2. Similarly to the previous cases we can construct the following orientation reversing equivariant homeomorphisms:

$$\begin{aligned} (\mathbb{T}^2 \times P_{k_H})/\sim &\to (\mathbb{T}^2 \times P_{k_I})/\sim, \\ (\mathbb{T}^2 \times P_{1_{G_{k-1}}})/\sim &\to (\mathbb{T}^2 \times P_{1_{J_{k-1}}})/\sim, \\ (\mathbb{T}^2 \times P_{k_{E_{k-1}}})/\sim &\to (\mathbb{T}^2 \times P_{k_{F_{k-1}}})/\sim, \\ (\mathbb{T}^2 \times P_{k_{H_i}})/\sim &\to (\mathbb{T}^2 \times P_{k_{I_i}})/\sim, \end{aligned} \tag{6-13}$$

for $i = 1, \dots, k-1$. So $k_1 = k_2$. Hence $[M] = 0[\mathbb{CP}^2]$. Identifying the corresponding boundaries of $W(Q_{P_k}, \lambda)$ via the equivariant homeomorphisms of Equation (6-13) we get that M is the boundary of a nice oriented $\mathbb{T}^2$ manifold.

If $k < -1$, similarly we can show $[M] = 0[\mathbb{CP}^2]$ and we can construct nice oriented $\mathbb{T}^2$ manifold with boundary W where the boundary is M.

Hence given a Hirzebruch surface M with natural $\mathbb{T}^2$ action we construct a nice 5–dimensional oriented $\mathbb{T}^2$ manifold with boundary where the boundary is M. Thus we get the following interesting lemma.

Lemma 6.1 *The $\mathbb{T}^2$–cobordism class of a Hirzebruch surface is trivial. In particular, the oriented cobordism class of a Hirzebruch surface is also trivial.*

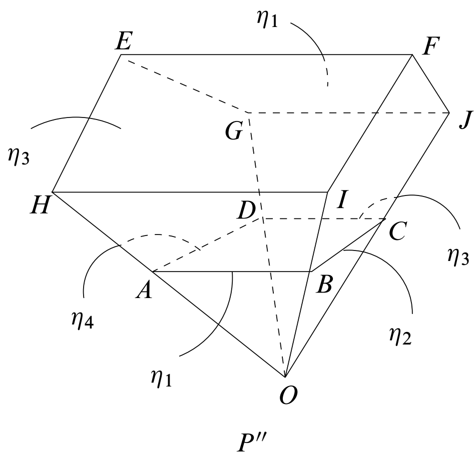


Figure 8: The edge-simple polytope P'' and an isotropy function λ associated to case (6-4)

Case (6-4) In this case $|\det[\eta_1, \eta_3]| = 1$. Following case (6-2), we can construct an edge simple polytope P'' and an isotropy function λ over this edge-simple polytope; see Figure 8. Hence we can construct an oriented $\mathbb{T}^2$ manifold with quasitoric boundary $W(Q_{P''}, \lambda)$ where the boundary is $-M \sqcup \bigsqcup_{k_1} \mathbb{CP}^2 \sqcup \bigsqcup_{k_2} \overline{\mathbb{CP}^2}$ for some integers k_1, k_2. We may assume that "the angles between the planes OHI and $HIFE$" and "the angles between the planes $EFJG$ and $HIFE$" are equal. Clearly a suitable reflection induces the following orientation reversing equivariant homeomorphisms:

$$(\mathbb{T}^2 \times P''_H)/\sim \;\to (\mathbb{T}^2 \times P''_E)/\sim,$$
$$(\mathbb{T}^2 \times P''_I)/\sim \;\to (\mathbb{T}^2 \times P''_F)/\sim. \tag{6-14}$$

Let $\mathbb{CP}^2_J = (\mathbb{T}^2 \times P''_J)/\sim$ and $\mathbb{CP}^2_G = (\mathbb{T}^2 \times P''_G)/\sim$. Observe that the characteristic functions of the triangles P''_J and P''_G differ by a nontrivial automorphism of $\mathbb{T}^2$ (or $\mathbb{Z}^2$). So $\mathbb{CP}^2_J$ and $\mathbb{CP}^2_G$ are complex projective space $\mathbb{CP}^2$ with two nonequivariant $\mathbb{T}^2$–actions. Hence $[M] = [\mathbb{CP}^2_J] + [\mathbb{CP}^2_G]$.

To compute the group CG_2 we use induction on the number of facets of 2–dimensional simple convex polytope in $\mathbb{R}^2$. We rewrite the proof of the following well-known lemma, briefly.

Lemma 6.2 *The equivariant connected sum of two quasitoric manifolds is equivariantly cobordant to the disjoint union of these two quasitoric manifolds.*

Proof Let M_1 and M_2 be two quasitoric manifolds of dimension $2n$. Then $W_1 := [0,1]\times M_1$ and $W_2 := [0,1]\times M_2$ are oriented $\mathbb{T}^n$–manifolds with boundary such that

$$\partial W_1 = 0\times(-M_1)\sqcup 1\times M_1 \quad \text{and} \quad \partial W_2 = 0\times(-M_2)\sqcup 1\times M_2.$$

Let $x_1 \in M_1$ and $x_2 \in M_2$ be two fixed points. Let $U_1 \subset W_1$ and $U_2 \subset W_2$ be two $\mathbb{T}^n$ invariant open neighborhoods of $1\times x_1$ and $1\times x_2$ respectively. Identifying $\partial U_1 \subset (W_1 - U_1)$ and $\partial U_2 \subset (W_2 - U_2)$ via a suitable orientation preserving equivariant homeomorphism we get the lemma. □

Now consider the case of a quasitoric manifold M over a convex 2–polytope Q with m facets, where $m > 4$. By the classification result of 4–dimensional quasitoric manifold discussed in Remark 2.8, M is one of the following equivariant connected sums:

$$M = N_1 \# \mathbb{CP}^2,$$
$$M = N_2 \# \overline{\mathbb{CP}}^2,$$
$$M = N_3 \# M^4_k.$$

The quasitoric manifolds N_1, N_2 and N_3 are associated to the 2–polytopes Q_1, Q_2 and Q_3 respectively. The number of facets of Q_1, Q_2 and Q_3 are $m-1$, $m-1$ and $m-2$ respectively. The quasitoric manifold M^4_k is defined in Section 2. In previous calculations we have shown $[M^4_k] = 0[\mathbb{CP}^2]$. So by Lemma 6.2 we get either $[M] = [N_1] + [\mathbb{CP}^2]$ or $[M] = [N_2] - [\mathbb{CP}^2]$ or $[M] = [N_3]$. Thus using the induction on m, the number of facets of Q, we can prove the following.

Lemma 6.3 *Any 4–dimensional quasitoric manifold is equivariantly cobordant to some $\mathbb{T}^2$–cobordism classes of $\mathbb{CP}^2$.*

We classify the equivariant cobordism classes of all $\mathbb{T}^2$–actions on $\mathbb{CP}^2$. Let Q be a triangle and $\{F_1, F_2, F_3\}$ be the edges (facets) of Q. Let $\eta\colon \{F_1, F_2, F_3\} \to \mathbb{Z}^2/\mathbb{Z}_2$ be a characteristic function such that $\eta(F_1) = [(a_1, b_1)]$ and $\eta(F_2) = [(a_2, b_2)]$. We may assume that

$$\det(\eta(F_1), \eta(F_2)) = |(a_1, b_1; a_2, b_2)| = 1,$$

where $(a_1, b_1; a_2, b_2)$ is the 2×2 matrix in $\mathrm{SL}(2,\mathbb{Z})$ with row vectors $\eta(F_1)$ and $\eta(F_2)$. We also denote this matrix by η. Then we have that either $\eta(F_3) = [(a_1 + a_2, b_1 + b_2)]$ or $\eta(F_3) = [(a_1 - a_2, b_1 - b_2)]$. Let η' and η'' be two characteristic functions defined respectively by

$$\eta'(F_1) = [(a_1, b_1)], \eta'(F_2) = [(a_2, b_2)], \eta'(F_3) = [(a_1 + a_2, b_1 + b_2)],$$
$$\eta''(F_1) = [(a_1, b_1)], \eta''(F_2) = [(a_2, b_2)], \eta''(F_3) = [(a_1 - a_2, b_1 - b_2)].$$

Denote the quasitoric manifolds associated to the pairs (Q,η') and (Q,η'') by $\mathbb{CP}^2_{\eta'}$ and $\mathbb{CP}^2_{\eta''}$ respectively. Define an equivalence relation $\sim_{\text{eq}}$ on $\mathrm{SL}(2,\mathbb{Z})$ by

$$(a_1,b_1;a_2,b_2)\sim_{\text{eq}}(-a_1,-b_1;-a_2,-b_2).$$

Denote the equivalence class of $\eta\in\mathrm{SL}(2,\mathbb{Z})$ by $[\eta]_{\text{eq}}$. Observe that if $[\eta_1]_{\text{eq}}\neq[\eta_2]_{\text{eq}}$ then the corresponding characteristic functions are differ by δ_*, for some nontrivial automorphism $\delta\colon\mathbb{T}^2\to\mathbb{T}^2$. Using Lemma 2.2 we get the following classification.

Lemma 6.4 *A $\mathbb{T}^2$–actions on $\mathbb{CP}^2$ is equivariantly homeomorphic to either $\mathbb{CP}^2_{\eta'}$ or $\mathbb{CP}^2_{\eta''}$ for a unique $[\eta]_{\text{eq}}\in\mathrm{SL}(2,\mathbb{Z})/\sim_{\text{eq}}$.*

Note that the natural $\mathbb{T}^2$–actions on $\mathbb{CP}^2_{\eta'}$ and $\mathbb{CP}^2_{\eta''}$ are the same. Consider the linear map $L_\eta\colon\mathbb{Z}^2\to\mathbb{Z}^2$, defined by $L_\eta(1,0)=(a_1,b_1)$ and $L_\eta(0,1)=(a_2,b_2)$. The map L_η induces orientation preserving homeomorphisms $\mathbb{CP}^2_s\to\mathbb{CP}^2_{\eta'}$ and $\overline{\mathbb{CP}}^2_s\to\mathbb{CP}^2_{\eta''}$. Thus, we have the following lemma.

Lemma 6.5 *The oriented $\mathbb{T}^2$–cobordism class of a $\mathbb{T}^2$–action on $\mathbb{CP}^2$ is $[\mathbb{CP}^2_{\eta'}]$ for a unique $[\eta]_{\text{eq}}\in\mathrm{SL}(2,\mathbb{Z})/\sim_{\text{eq}}$.*

Since the order of oriented cobordism class of $\mathbb{CP}^2$ is infinite, we get the following theorem.

Theorem 6.6 *The oriented torus cobordism group CG_2 is an infinite abelian group with a set of generators $\{[\mathbb{CP}^2_{\eta'}]:[\eta]_{\text{eq}}\in\mathrm{SL}(2,\mathbb{Z})/\sim_{\text{eq}}\}$.*

We do not know whether these are the free generators.

Question 6.7 Describe all the relations among the generators given in Theorem 6.6.

We discuss the actions of $\mathbb{T}^n$ on $(2n+1)$–dimensional manifolds (possibly with boundary) where the actions are similar to the locally standard actions. We again call these actions *locally standard actions*. We discuss some properties of these actions explicitly in our next article, "Odd dimensional torus manifolds." Let ρ_s be the standard action of $\mathbb{T}^n$ on $\mathbb{C}^n$. Consider the action ρ of $\mathbb{T}^n$ on $\mathbb{C}^n\times\mathbb{R}$ defined by $\rho(t,(z,r))=(\rho_s(t,z),r)$.

Definition 6.3 A smooth action of $\mathbb{T}^n$ on a $(2n+1)$–dimensional smooth manifold (possibly with boundary) W is said to be locally standard if every point $a\in W$ has a $\mathbb{T}^n$–stable open neighborhood W_a and a diffeomorphism $\xi_a\colon W_a\to V_a$, where V_a is a $\mathbb{T}^n$–stable open subset of $\mathbb{C}^n\times\mathbb{R}_{\geq 0}$ under the action ρ, and an isomorphism $\delta_a\colon\mathbb{T}^n\to\mathbb{T}^n$ such that $\xi_a(t\cdot x)=\rho(\delta_a(t),\xi_a(x))$ for all $(t,x)\in\mathbb{T}^n\times W_a$.

Example 6.8 Consider $S^{2n+1} = \{(z_1, \dots, z_{n+1}) \in \mathbb{C}^{n+1} : \Sigma_i |z_i|^2 = 1\}$. The torus $\mathbb{T}^n$ acts on S^{2n+1} by $(t_1, \dots, t_n) \cdot (z_1, \dots, z_n, z_{n+1}) = (t_1 z_1, \dots, t_n z_n, z_{n+1})$. This action is a locally standard action in the sense of Definition 6.3. When $n = 1$, the orbit space of this action is a closed 2–dimensional disk.

Remark 6.9 If W is a $(2n+1)$–dimensional smooth manifold with boundary with a locally standard action of $\mathbb{T}^n$, the fixed point set $W^{\mathbb{T}^n}$ is a disjoint union of some circles and closed intervals.

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