JSJ decompositions of quadratic Baumslag–Solitar groups

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Generalized Baumslag–Solitar groups are defined as fundamental groups of graphs of groups with infinite cyclic vertex and edge groups. Forester [6] proved that in most cases the defining graphs are cyclic JSJ decompositions, in the sense of Rips and Sela. Here we extend Forester's results to graphs of groups with vertex groups that can be either infinite cyclic or quadratically hanging surface groups.

1 Introduction

To understand a group G, it is often useful to decompose it as an amalgamated free product or an HNN extension over a subgroup that belongs to a well understood class of groups, such as trivial groups, finite groups or cyclic groups. More generally, consider all possible factorizations of G as a graph of groups with edge stabilizers in some single class of groups.

It is often possible to show the existence of a single graph of groups decomposition, from which all of these factorizations can be obtained. This is called a *JSJ decomposition* of G (over subgroups in the given class), although the notion is imprecise on how the other factorizations of G are to be obtained from the JSJ decomposition. An example is Grushko's theorem, which gives all the maximal decompositions of a finitely generated group over the class of trivial groups (ie the free factorizations).

The letters JSJ stand for Jaco, Shalen and Johannson. Their results in [11; 12] can be interpreted as the existence of a JSJ decomposition for 3–manifold groups over subgroups isomorphic to $\mathbb{Z} \times \mathbb{Z}$. It was these works that motivated the study of JSJ decompositions over nontrivial subgroups (ie aside from the Grushko decomposition). Various existence theorems were obtained by Kropholler [13], Rips and Sela [16; 14], Bowditch [2], Dunwoody and Sageev [3], Fujiwara and Papasoglu [7], Dunwoody and Swenson [4] and Scott and Swarup [15]. In [10; 8] Guirardel and Levitt propose a precise definition of JSJ decomposition, which is verified by the graphs of groups constructed in most of the mentioned works. The constructions in [4] and [15] constitute other

notions of JSJ decomposition, as does the compatibility JSJ decomposition introduced by Guirardel and Levitt [9].

In this paper we will focus on the JSJ decomposition due to Rips and Sela [14] for finitely presented one-ended groups with infinite cyclic edge stabilizers (stated here as Theorem 2.14).

It is not always clear how to recognize whether a given graph of groups is a JSJ decomposition of its fundamental group. In [6], Forester studied the *Generalized Baumslag–Solitar (GBS) groups*, which are defined by graphs of groups whose vertex and edge stabilizers are infinite cyclic. He showed that the defining graph of a GBS group is a JSJ decomposition, under mild hypotheses.

In this paper we introduce the *quadratic Baumslag–Solitar (QBS) groups*. They are defined by graphs of groups whose edge groups are infinite cyclic, and whose vertex groups can be either infinite cyclic or *quadratically hanging* (see Definition 2.7) surface groups. It is clear that the GBS groups are a subclass of the QBS groups. We extend Forester's result to the class of QBS groups, ie we show that the defining graph of a QBS group is a JSJ decomposition, under some conditions. Specifically, the main theorem in this paper is the following.

Theorem 1.1 Let Γ be a QBS graph, $G = \pi_1(\Gamma)$. Suppose that Γ is reduced, has no leaves, and satisfies the following conditions:

(1) Each edge e of Γ has labels $m_e^+, m_e^- > 1$.

(2) Each GBS component Γ_i of Γ is reduced, and T_{Γ_i} is not a point or a line.

Then Γ is a Rips–Sela JSJ decomposition for G.

A *QBS graph* is the defining graph of a QBS group. See Section 2.2 for the definition of a reduced graph of groups. For the edge labels, see Definition 5.1. The GBS components and the leaves of a QBS graph are defined at the beginning of Section 6.

The paper is organized as follows: In Section 2 we review the basics of the Rips–Sela JSJ decomposition. There we discuss *universality* and *unfoldedness*, the two main conditions for a graph of groups (with cyclic edge groups) to be a JSJ decomposition (in the sense of Rips and Sela). In Section 3 we recall the results of Forester about GBS groups, which we will need when dealing with QBS groups. In Section 4 we prove a general criterion for unfoldedness, Theorem 4.2, in the same fashion of the one by Forester [6, Proposition 2.17]. Section 5 is devoted to Theorem 5.5. This theorem applies to general graphs of groups with cyclic edge stabilizers, and it allows us to

show the universality of the whole graph from the universality of certain subgraphs. Finally, in Section 6 we introduce the QBS groups, and prove they are one-ended with the exception of $\mathbb{Z}$ (see Proposition 6.1). We also give the proof of Theorem 1.1. In this proof we combine Theorem 5.5 with the results of Forester (outlined in Section 3) to show universality, and we use Theorem 4.2 to prove unfoldedness.

2 Review of the JSJ decomposition

2.1 Graphs of groups and Bass–Serre theory

Bass–Serre theory is the most fundamental prerequisite for understanding the JSJ decomposition. This is a very quick review, a comprehensive treatment is given in the book by Serre [17].

For a *graph* we understand a pair of sets $\Gamma = (V, E)$, the vertex and edge set of Γ respectively, together with two maps $s, t\colon E \to V$, which give the *source* and *target* of an edge. If $e \in E$ is such an edge, the vertices $s(e)$, $t(e)$ will be called the *endpoints* of e. Thus our graphs have oriented edges, and admit loops (edges e with $s(e) = t(e)$) and multiple edges (different edges having the same endpoints). We will usually drop the maps s and t from the notation, thus we will say an edge e has endpoints v^+, v^- to mean that $s(e) = v^-$ and $t(e) = v^+$.

Definition 2.1 A *graph of groups* consists of the following:

(1) a connected graph Γ;

(2) a group G_v for each vertex v of Γ;

(3) a group G_e for each edge e of Γ, and two injective homomorphisms

$$\partial_e^+\colon G_e \to G_{v+}$$
$$\partial_e^-\colon G_e \to G_{v-}$$

where v^+, v^- are the endpoints of e.

This is denoted by $(\Gamma, G, \partial^+, \partial^-)$, or simply by Γ. Note that even if the endpoints of an edge e agree, ie $v^+ = v^-$, there are two different maps ∂_e^+ and ∂_e^-, one for the source and one for the target of e.

If T is a spanning tree for Γ, let $\pi_1(\Gamma, T)$ be defined by the following presentation.

- Generators: the elements of G_v for the vertices $v \in V(\Gamma)$, and an element t_e for each edge $e \in E(\Gamma)$, $e \notin T$.

- Relations: the relations in G_v for each vertex v, and

$$\partial_e^+(g) = \partial_e^-(g) \quad \text{for } e \in T, \ g \in G_e,$$

$$t_e \partial_e^+(g) t_e^{-1} = \partial_e^-(g) \quad \text{for } e \in E(\Gamma), \ e \notin T, \ g \in G_e.$$

This group is called the *fundamental group* of Γ. It does not depend on the spanning tree T.

Proposition 2.2 *If T, S are two spanning trees for Γ, then $\pi_1(\Gamma, T) \cong \pi_1(\Gamma, S)$.*

Thus we often drop T from the notation. When G is a group and $G \cong \pi_1(\Gamma)$ for a graph of groups Γ, we say that Γ is a *splitting* of G. Note that one-edge splittings correspond to decompositions of G as an amalgamated product or an HNN extension.

If Γ is a graph of groups and $A \subset \Gamma$ is a connected subgraph, let $\Gamma^* = \Gamma/A$ be the graph obtained by collapsing A to a vertex, which we call w. Put $G_w = \pi_1(A)$ and leave the same groups in the noncollapsed vertices and edges. This defines a graph of groups in Γ^*.

Proposition 2.3 *If $A \subset \Gamma$ is a connected subgraph and $\Gamma^* = \Gamma/A$ as above, then $\pi_1(\Gamma) \cong \pi_1(\Gamma^*)$.*

We say that Γ is a *refinement* of the splitting Γ^*. Through this proposition, we can see general splittings as iteration of amalgamated products and HNN extensions.

The following is the main result in Bass–Serre theory. It relates the splittings of a group with its actions on simplicial trees, and it is crucial for both the theory of JSJ decompositions and the results on this paper.

Theorem 2.4 (Bass–Serre) *Given a group G, there is a correspondence between the splittings of G as a graph of groups and the actions of G on simplicial trees without edge inversions. If $G \curvearrowright X$ is such an action, then a corresponding splitting can be constructed as follows:*

- *The underlying graph is $\Gamma = X/G$.*

- *If $\tilde{x} \in X$ is a vertex or edge, and $x \in \Gamma$ is its projection, then G_x is isomorphic to $Stab_G(\tilde{x})$.*

Since the action on X is without edge inversions, the edges of $\Gamma = X/G$ can be given an orientation. We omitted the edge maps in the statement, but they are straightforward. Namely, if we identify G_x with $\operatorname{Stab}_G(\tilde{x})$ for some specific $\tilde{x}$, these maps are subgroup inclusions possibly composed with conjugations. The action corresponding to a splitting Γ under this theorem will be denoted by T_Γ, the *Bass–Serre tree* of Γ.

Given a simplicial G–tree X, a subgroup $H \leq G$ *acts elliptically* on X, or is an *elliptic subgroup* with respect to X, if there is a point in X that is fixed by every element of H. Note that if $X = T_\Gamma$ is the Bass–Serre tree of a graph of groups Γ, then a subgroup of $G = \pi_1(\Gamma)$ acts elliptically on X if and only if it is conjugate into one of the vertex groups of Γ.

Let X, Y be simplicial G–trees. A *morphism* $f\colon X \to Y$ is a G–equivariant map, which can be made simplicial by subdividing the edges of X. The following fact is widely known, and not hard to prove.

Proposition 2.5 *There is a morphism $X \to Y$ if and only if every elliptic subgroup of X is also elliptic in Y.*

2.2 Elementary deformations, foldings

Here we introduce some important transformations on graphs of groups.

Let Γ be a graph of groups. Let e be an edge of Γ and v^+, v^- its endpoints. First suppose that $v^+ \neq v^-$ and ∂_e^- is an isomorphism. That is, $G_e = G_{v^-} = C$ and $G_{v^+} = A$ with $C \subset A$. In this situation, the collapse of the edge e is called an *elementary collapse*. Note that v^+ and v^- are identified to a single vertex $\bar{v}$, and $G_{\bar{v}} = A$ (through the isomorphism $A *_C C \cong A$).

The inverse of an elementary collapse is called an *elementary expansion*, and these transformations are the *elementary deformations*, which were introduced by Forester [5].

Again, let e be an edge with different endpoints. Suppose $G_e = C \subset C_1 \subset A = G_{v^+}$ and $B = G_{v^-}$. Get Γ_1 from Γ by redefining $G_e = C_1$ and $G_{v^-} = C_1 *_C B$. We have $\pi_1(\Gamma) = \pi_1(\Gamma_1)$ by the isomorphism $A *_C B \cong A *_{C_1} (C_1 *_C B)$. In this case we say that Γ_1 is a *folding* of Γ, and that the folding occurs at the vertex v^+.

There is another case of folding when e is a loop, ie $v^+ = v^- = v$. Let $G_e = C$, $G_v = A$, and suppose that $\partial_e^+(C) \subset C_1 \subset A$. This time make Γ_1 with $G_v = A *_C t_e C_1 t_e^{-1}$ and $G_e = C_1$. The fundamental group is again preserved, and this transformation is also called *folding*. Making some abuse of notation, we say that the folding occurs at v^+ in the case just described, and at v^- if we use ∂_e^- instead.

Looking at the Bass–Serre trees, when there is a folding we have a map $T_\Gamma \to T_{\Gamma_1}$, simplicial and equivariant. If $x \in T_\Gamma$ is a lift of v^+ with stabilizer gAg^{-1}, then this map identifies the edges coming from x and projecting to e, by the action of gC_1g^{-1}. Locally at x it looks like "folding". In [1], Bestvina and Feighn explain this from the viewpoint of graphs of groups.

If e is an edge of Γ, let Γ_e be the graph of groups obtained by collapsing the components of $\Gamma - e$.

Definition 2.6 A splitting Γ is *unfolded* when one of the following holds:

(1) Γ has only one edge, and there is no folding onto it. That is, there is no Γ_0 such that Γ is obtained as a folding of Γ_0.

(2) Γ has several edges, and Γ_e is unfolded for all of them.

2.3 $\mathbb{Z}$–Splittings, quadratically hanging subgroups

A $\mathbb{Z}$–*splitting* of the group G is a splitting whose edge groups are infinite cyclic. That is, a graph of groups Γ, with $\pi_1(\Gamma) \cong G$ and $G_e \cong \mathbb{Z}$ for all edges of Γ.

Definition 2.7 Let Γ be a graph of groups. A vertex group G_v is *quadratically hanging (QH)* if:

(1) $G_v \cong \pi_1(S)$ where S is a 2–orbifold, ie it has one of the following presentations:

$$\langle a_1, \ldots, a_g, b_1, \ldots, b_g, p_1, \ldots, p_m, s_1, \ldots, s_n \mid s_i^{k_i} = 1, \Pi_k p_k \Pi_i s_i \Pi_j [a_j, b_j] = 1 \rangle,$$

$$\langle a_1, \ldots, a_g, p_1, \ldots, p_m, s_1, \ldots, s_n \mid s_i^{k_i} = 1, \Pi_k p_k \Pi_i s_i \Pi_j a_j^2 = 1 \rangle.$$

We require S to be different from the disk, the cylinder, and a disk with one cone-point.

(2) The edges from v are in correspondence with the components of ∂S. Moreover, if these edges are $e_1, \ldots, e_m$, then we have $\partial_{e_i} : G_{e_i} \to \langle p_i \rangle$ (where p_i is the boundary loop corresponding to e_i), and G_{e_i} is nontrivial.

Definition 2.8 Let G be a group. Then $P \subset G$ is a *QH subgroup* if there is a $\mathbb{Z}$–splitting Γ_P of G with P occurring as a QH vertex group.

Our definition of QH vertex differs slightly from the one originally used by Rips and Sela in [14], in which they require the maps $\partial_{e_i} : G_{e_i} \to \langle p_i \rangle$ to be onto. This does not change the QH subgroups, since the additional condition on the QH vertex can be met by performing elementary expansions on Γ_P.

Definition 2.9　We say that a $\mathbb{Z}$–splitting is *reduced* if it does not admit elementary collapses, except possibly on the edges connecting to QH vertices.

Let Γ_1, Γ_2 be one-edged $\mathbb{Z}$–splittings of G, with edge groups C_1, C_2 respectively. That is, G is written as an amalgamation or HNN extension over C_i. We say that Γ_1 is *elliptic* in Γ_2 if the subgroup C_1 acts elliptically in T_{Γ_2}, the Bass–Serre tree of Γ_2. Otherwise, we say that Γ_1 is *hyperbolic* in Γ_2.

Proposition 2.10　[14, Theorem 2.1]　*Let G be freely indecomposable, and Γ_1, Γ_2 be one-edged $\mathbb{Z}$–splittings of G. Then Γ_1 is elliptic in Γ_2 if and only if Γ_2 is elliptic in Γ_1.*

2.4　The Rips–Sela JSJ decomposition

We will now state the fundamental theorem of Rips and Sela, which proves the existence of certain $\mathbb{Z}$–splittings that will be called JSJ decompositions. It applies to *one-ended* groups, that are defined as follows.

Definition 2.11　A space X is *one-ended* if there is an increasing sequence of compact sets K_n, such that $X = \bigcup_n K_n$ and $X - K_n$ is connected for all n.

Definition 2.12　A group G is *one-ended* if one/all of its Cayley graphs is/are one-ended. Equivalently, for G finitely generated, if it acts freely and cocompactly on a one-ended space.

Consider a class of groups $\mathcal{A}$, such as trivial, finite or cyclic groups. We say that a group G *splits over* $\mathcal{A}$ if it admits a nontrivial graph of groups decomposition with edge groups in $\mathcal{A}$. For example, G splits over infinite cyclic groups if it admits a nontrivial $\mathbb{Z}$–splitting.

According to a theorem of Stallings [18], a finitely generated infinite group is one-ended if and only if it does not split over finite groups. Thus it makes sense to study the splittings over infinite cyclic groups, $\mathbb{Z}$–splittings, of such a group as a next step.

Definition 2.13　A simple closed curve in a 2–orbifold S is *weakly essential* if it is not nullhomotopic, nor boundary parallel, nor the core of a Möbius band embedded in S, and does not circle around a branching point.

Theorem 2.14 (Rips–Sela) *Let G be a finitely presented one-ended group. Then there is a reduced, unfolded $\mathbb{Z}$–splitting Γ of G satisfying the following conditions:*

(1) (a) *A vertex group of Γ can either be a QH vertex group, or be elliptic in every $\mathbb{Z}$–splitting of G.*
   (b) *Edge groups are elliptic in every $\mathbb{Z}$–splitting of G.*
   (c) *Every maximal QH subgroup of G is conjugate to a QH vertex group of Γ.*

(2) *Let Γ_1 be a one-edged $\mathbb{Z}$–splitting of G, with edge group C. Suppose that Γ_1 is hyperbolic in some other one-edged $\mathbb{Z}$–splitting. Then there is a QH vertex group $G_v = \pi_1(S)$ of Γ, and a weakly essential simple closed curve $\gamma \subset S$ such that C is conjugate to the group generated by $[\gamma] \in G_v \subset G$.*

(3) *If Γ_1 is a one-edged $\mathbb{Z}$–splitting of G that is elliptic in every other one-edged $\mathbb{Z}$–splitting, then there is a morphism $T_\Gamma \to T_{\Gamma_1}$.*

(4) *Let Γ_1 be any $\mathbb{Z}$–splitting of G. Then there is a $\mathbb{Z}$–splitting $\widehat{\Gamma}$, which is a refinement of Γ obtained by splitting some QH vertex groups along weakly essential simple closed curves, and a morphism $T_{\widehat{\Gamma}} \to T_{\Gamma_1}$.*

A splitting Γ as in the theorem is called a *cyclic JSJ decomposition*, or *Rips–Sela JSJ decomposition* of G. Here we will consider only this version of JSJ decomposition. Due to our definition of QH vertices (Definition 2.7), our JSJ decompositions may differ a bit from the ones in [14], but they agree after elementary expansions at the QH vertices.

Condition (4) in the theorem is called *universality*. It says how every $\mathbb{Z}$–splitting of a group G can be obtained from a JSJ decomposition. Also, it is because of universality that the splitting in the theorem verifies the general definition of a JSJ decomposition (over infinite cyclic groups), given by Guirardel and Levitt in [10; 8]. Although we will not need that definition here.

There is some redundancy in the conditions for a Rips–Sela JSJ decomposition, as the following proposition shows.

Proposition 2.15 *Let G be a one-ended group. Suppose Γ is a reduced $\mathbb{Z}$–splitting of G satisfying universality, as in (4) of Theorem 2.14. Then it also satisfies conditions (1), (2) and (3) of Theorem 2.14.*

Proof For (a) and (b) of condition (1), let Γ_1 be any $\mathbb{Z}$–splitting of G. Let $\widehat{\Gamma}$ and $f\colon T_{\widehat{\Gamma}} \to T_{\Gamma_1}$ be the refinement and the morphism given by universality. If G_v is a vertex group of Γ that is not QH, then it is still elliptic in $\widehat{\Gamma}$, and so it is elliptic in Γ_1.

This proves (a). The edge groups of Γ are also elliptic in $\widehat{\Gamma}$, and so they are elliptic in Γ_1. This gives (b).

Now we prove (2). Let Γ_1 be a one-edged $\mathbb{Z}$–splitting of G that is hyperbolic in some other $\mathbb{Z}$–splitting. Let $\widehat{\Gamma}$ be the refinement of Γ given by condition (4), and $f\colon T_{\widehat{\Gamma}} \to T_{\Gamma_1}$ the corresponding morphism. Take e an edge in T_{Γ_1}, let $C = \operatorname{Stab}_G(e)$ be its stabilizer subgroup and $K = f^{-1}(e)$ be its preimage under f. There are two kinds of edges in $\widehat{\Gamma}$: those that were already present in Γ, and those that were obtained by cutting the surfaces of QH vertices along simple closed curves. Since $f(K) = e$, K is not a single point and it meets the interior of an edge e_1. Then $\operatorname{Stab}_G(e_1) \subset C$. Moreover, since C is cyclic, the generator of $\operatorname{Stab}_G(e_1)$ is a power of the one of C. If e_1 was of the first kind, then C would be elliptic in every $\mathbb{Z}$–splitting of G, which is a contradiction against our assumption on Γ_1. Thus e_1 is of the second kind, and K does not meet the interior of any edges of the first kind. Let K^+ be the union of the edges e' of $T_{\widehat{\Gamma}}$ so that $\operatorname{Stab}_G(e')$ intersects C in a nontrivial subgroup. Then K^+ is connected and contains K. (If $C = \langle c \rangle$, then $K^+ = \bigcup_{n \geq 1} \operatorname{Fix}(c^n)$, which is an increasing union of connected sets). The same reasoning used for e_1 shows that K^+ does not contain edges of the first kind. (Recall that an element g is elliptic if and only if g^n is elliptic for any $n \neq 0$).

Now let v be the QH vertex of Γ that corresponds to e_1. Let Γ_0 be the splitting of $G_v = \pi_1(S)$ obtained by cutting S along the same simple closed curves as in $\widehat{\Gamma}$. Then there is a copy of T_{Γ_0} embedded in $T_{\widehat{\Gamma}}$ containing e_1. Notice if $g\colon T_{\widehat{\Gamma}} \to T_{\Gamma}$ is the map that collapses all edges of the second kind, then g collapses T_{Γ_0} to a vertex w in the orbit of v. So $\operatorname{Stab}_G(T_{\Gamma_0}) = \operatorname{Stab}_G(w)$ and it is conjugate to $G_v = \pi_1(S)$. Observe K^+ must be contained in T_{Γ_0}, since it cannot cross edges of the first kind. In particular, any fixed point of C lies in T_{Γ_0}, and so it is mapped to w by g. Thus $C \subset \operatorname{Stab}_G(w)$ and is conjugate to G_v. And $C = \operatorname{Stab}_G(e_1)$, since a simple closed curve represents a primitive element of $\pi_1(S)$. This proves (2).

Now we will prove (3). The setup is the same as in the previous case, but this time Γ_1 is elliptic in every $\mathbb{Z}$–splitting of G. This time, K cannot meet the interior of any edge of the second kind. To see that, suppose K intersects the interior of an edge e_1 of the second kind. Let $G_v = \pi_1(S)$ be the QH vertex group of Γ corresponding to e_1, and let α be the simple closed curve in S such that $\operatorname{Stab}_G(e_1)$ conjugates to $\langle [\alpha] \rangle$. Since K meets the interior of e_1, we have $\operatorname{Stab}_G(e_1) \subset C$. Thus, if we write $C = \langle c \rangle$, we get c^n is conjugate to $[\alpha]$ for some n. Let β be a simple closed curve in S that intersects α nontrivially and minimally. Then consider the one-edged splitting Γ_2 of G obtained from $[\beta]$. Since $[\alpha]$ acts hyperbolically on T_{Γ_2}, so does c. Thus Γ_1 is hyperbolic in Γ_2 (and viceversa, by Proposition 2.10), which goes against our assumption. So K does not intersect any edges of the second kind. This was shown for $K = f^{-1}(e)$

where e was any edge in T_{Γ_1}, so all the edges of the second kind are collapsed to points under f. Let $g\colon T_{\widehat{\Gamma}} \to T_\Gamma$ be the map obtained by collapsing the edges of the second kind. Then f factors through g, and so we obtain the morphism in (3).

Finally, for (a), let H be a QH subgroup of G. Let Γ_1 be a $\mathbb{Z}$–splitting realizing it as a QH vertex. Write $H = \pi_1(S)$ as given by Γ_1. Again, condition (4) gives a morphism $f\colon T_{\widehat{\Gamma}} \to T_{\Gamma_1}$ for some refinement $\widehat{\Gamma}$ of Γ as before.

If c is the class of a boundary component of S, then c acts elliptically on $T_{\widehat{\Gamma}}$. To see that, note that some power of c fixes an edge e of T_{Γ_1} (the incident edge at v corresponding to this boundary curve), and $f^{-1}(e)$ meets the interior of some edge e_1. We obtain that $\mathrm{Stab}_G(e_1) \subset \mathrm{Stab}_G(e) \subset \langle c \rangle$, thus e_1 is fixed by a power of c.

Consider the action of H on $T_{\widehat{\Gamma}}$ by restriction, and let $\widehat{T}$ be a minimal subtree for this action. Then we have the boundary classes of S are elliptic in $\widehat{T}$, since they are elliptic in $T_{\widehat{\Gamma}}$.

Consider the decomposition Γ_H of H induced by $\widehat{T}$. If e is an edge in $\widehat{T}$, then we have $\mathrm{Stab}_H(e) \subset \mathrm{Stab}_G(e)$, so the edge groups of Γ_H are either trivial or infinite cyclic. Since the boundary classes of S are elliptic in $\widehat{T}$, then Γ_H can be extended to Γ_2, a splitting of G obtained by refining Γ_1. And since G is one-ended, all edge groups of Γ_2 are infinite cyclic. Hence all edge groups of Γ_H are infinite cyclic.

Using Corollary 5.3, Γ_H is obtained by splitting S along some disjoint, weakly essential simple closed curves. Now, if e is an edge in $\widehat{T}$, then $\mathrm{Stab}_H(e)$ is generated by a conjugate of one of these curves. So $\mathrm{Stab}_H(e) = \mathrm{Stab}_G(e)$ since the generator of $\mathrm{Stab}_H(e)$ is primitive. And it is also hyperbolic in some $\mathbb{Z}$–splitting of G, so e is of the second kind.

We conclude as in the proof of (2), obtaining that H is conjugate into G_v, for v a QH vertex of Γ. $\square$

Corollary 2.16 *Let G be a one-ended group. If Γ is a reduced, unfolded $\mathbb{Z}$–splitting of G that verifies universality, as in Theorem 2.14(4), then it is a Rips–Sela JSJ decomposition for G.*

3 Generalized Baumslag–Solitar groups

Here we discuss the results in Forester's paper [6] that are relevant to this paper.

Definition 3.1 A *Generalized Baumslag–Solitar (GBS) graph* is a graph of groups in which all vertex and edge groups are infinite cyclic.

Note this is a special case of $\mathbb{Z}$–splitting. A *GBS group* is a group obtained as a fundamental group of a GBS graph, and a *GBS tree* is the associated Bass–Serre tree.

Lemma 3.2 [6, Lemma 2.6] *Let Γ be a GBS graph, $G = \pi_1(\Gamma)$. Assume $G \ncong \mathbb{Z}$, and let $T = T_\Gamma$ be the Bass–Serre tree of Γ. Then:*

1. *G is not free;*
2. *G acts freely on $T \times \mathbb{R}$;*
3. *G is torsion-free, one-ended and has cohomological dimension 2;*
4. *T contains an invariant line if and only if $G \cong \mathbb{Z} \rtimes \mathbb{Z}$ (ie either $\mathbb{Z}^2$ or the Klein bottle group).*

The following is the most general statement about JSJ decompositions of GBS groups.

Theorem 3.3 [6, Theorem 2.15] *Let Γ be a GBS graph, $G = \pi_1(\Gamma)$. Suppose Γ is reduced, unfolded, and T_Γ is not a point or a line ($G \ncong \mathbb{Z}, \mathbb{Z} \rtimes \mathbb{Z}$). Then Γ is a JSJ decomposition of G.*

In general, it is hard to check wether a splitting is unfolded or not. The following result proves unfoldedness for most GBS graphs.

Proposition 3.4 [6, Proposition 2.17] *Let Γ be a GBS graph. If every edge group is a proper subgroup of its neighboring vertex groups, then Γ is unfolded.*

The combination the two last statements permits us to recognize most GBS graphs as JSJ decompositions of their fundamental groups.

4 Criterion for unfoldedness

Here we give a criterion for the unfoldedness of a general $\mathbb{Z}$–splitting. It is a generalization of Proposition 3.4, due to Forester, and the proof follows the same lines.

Lemma 4.1 *Let G be a freely indecomposable group. Suppose that Γ is a $\mathbb{Z}$–splitting of G, e is an edge of T_Γ with endpoints v_0, v_1 and $H \le Stab_G(v_1)$ contains $Stab_G(e)$ properly. If Γ_1 is a nontrivial unfolding of Γ_e at the endpoint v_0 of e, then H cannot be elliptic in Γ_1.*

In the statement of the lemma, we abused notation and still called e, v_0 and v_1 their respective projections in Γ and Γ_e. Recall that Γ_e is the graph obtained from Γ by collapsing all edges but the projection of e.

Proof Let X be the Bass–Serre tree corresponding to Γ_e and Y the one corresponding to Γ_1. Notice X can be obtained from T_Γ by collapsing the components of $T_\Gamma - Ge$.

Let $q\colon T_\Gamma \to X$ be the quotient map, and $f\colon Y \to X$ be the folding map. Let e' be an edge of Y, with endpoints v_0', v_1', such that $f(e') = q(e)$ and the fold occurs at v_0'.

Let g be the generator of $\operatorname{Stab}_G(e)$ and g^m the one of $\operatorname{Stab}_G(e')$. We know $m \neq 0$ since G is freely indecomposable, and so $|m| > 1$ since the fold is nontrivial ($\operatorname{Stab}_G(e')$ is strictly contained in $\operatorname{Stab}_G(e) = \operatorname{Stab}_G(q(e))$). We may assume $m > 1$, the case for $m < -1$ being analogous.

Define Y_0, $g^k Y_1$ for $k = 0, \ldots, m-1$ to be the components of Y minus the edges $g^k e'$, containing v_0', $g^k v_1'$ respectively. Also let X_0, X_1 be the components of $X - q(e)$ containing $q(v_0)$, $q(v_1)$, and T_0, T_1 the ones of $T_\Gamma - e$ containing v_0, v_1. Observe that $f(Y_0) = X_0$, $f(g^k Y_1) = X_1$, $q(T_0) = X_0$ and $q(T_1) = X_1$.

Seeking a proof by contradiction, suppose H is elliptic in Γ_1. Thus H fixes a point x' in Y. Since $g \in H$ and g fixes no point of $g^k Y_1$ for any k, we get x' must belong to Y_0. Then H fixes the point $x = f(x')$ in X_0 and stabilizes the subtree $q^{-1}(x)$ in T_0.

Now, e separates $q^{-1}(x)$ from v_1, and H stabilizes both. So H must also stabilize e, which is a contradiction, since H contained $\operatorname{Stab}_G(e)$ strictly. $\qquad \square$

We can obtain Proposition 3.4 from this lemma as follows.

Proof of Proposition 3.4 Suppose that Γ is a GBS graph in the conditions of Proposition 3.4. Notice that if Γ is not a single vertex, then $G = \pi_1(\Gamma) \not\cong \mathbb{Z}$ and so it is one-ended. If Γ is not unfolded, then there is an edge e of Γ and a nontrivial unfolding Γ_1 of Γ_e. In the Bass–Serre tree T_Γ, let v_0 be the endpoint of e at which the unfolding occurs, and v_1 be the other endpoint. Let e' be the edge of T_{Γ_1} with stabilizer contained in $\operatorname{Stab}_G(e)$. Put $H = \operatorname{Stab}_G(v_1)$. Then $\operatorname{Stab}_G(e') \leq \operatorname{Stab}_G(e) \leq H$, where both inclusions are strict (the first one because the unfolding is nontrivial, the second one by the hypothesis of Proposition 3.4). These three subgroups are infinite cyclic, and $\operatorname{Stab}_G(e')$ is elliptic in Γ_1, so H must also be elliptic in Γ_1 (if g^n acts elliptically on a tree, so acts g). This contradicts Lemma 4.1. $\qquad \square$

The following is the main result of this section. It gives an unfoldedness criterion for universal $\mathbb{Z}$–splittings.

Theorem 4.2 *Let G be a one-ended group. Suppose Γ is a reduced $\mathbb{Z}$–splitting of G satisfying universality. If every edge group is a proper subgroup of its neighboring vertex groups, then it is unfolded, and is therefore a cyclic JSJ decomposition for G.*

Proof Again, suppose that Γ is not unfolded. Let e be an edge of Γ and Γ_1 a nontrivial unfolding of Γ_e. Let v_0 and v_1 be the endpoints of e, when considered in T_Γ, and assume the unfolding occurs at v_0.

By the universality of Γ, it has a refinement $\hat{\Gamma}$, obtained as in Theorem 2.14(4), that admits a morphism $T_{\hat{\Gamma}} \to T_{\Gamma_1}$. Let w_0, w_1 be the vertices of e as an edge of $T_{\hat{\Gamma}}$, that correspond to v_0, v_1 respectively. Put $H = \mathrm{Stab}_G(w_1)$.

Since H is elliptic in $\hat{\Gamma}$ and there is a morphism $T_{\hat{\Gamma}} \to T_{\Gamma_1}$, then H must also be elliptic in Γ_1.

On the other hand, $H \leq \mathrm{Stab}_G(v_1)$ and it contains $\mathrm{Stab}_G(e)$. If v_1 is not a QH vertex, then it does not get split in the refinement $\hat{\Gamma}$. So $H = \mathrm{Stab}_G(v_1)$, which contains $\mathrm{Stab}_G(e)$ strictly by hypothesis. If v_1 is a QH vertex with $G_{v_1} = \pi_1(S)$, then H is conjugate to $\pi_1(S_0)$, where S_0 is a component of S cut by some weakly essential simple closed curves. Thus H is not cyclic, and so must contain $\mathrm{Stab}_G(e)$ strictly.

By the lemma, H cannot be elliptic in Γ_1, which is a contradiction. $\square$

5   Adding surface vertices to universal graphs

In this section we deduce the universality of a $\mathbb{Z}$–splitting, given the universality of certain subgraphs of it. We start with some preliminaries.

Definition 5.1 Let Γ be a $\mathbb{Z}$–splitting of a finitely generated group and e an edge in Γ. Let v^+ and v^- be the endpoints of e, and a be a generator of G_e. Define m_e^+ as the supremum of the m such that $\partial_e^+(a) = b^m$ for some $b \in G_{v^+}$. Define m_e^- in the same manner.

The number m_e^+ will be called the *label* of e at the endpoint v^+. (With some abuse of notation, for when e is a loop, it gets two labels, one for each boundary map). We remark that it is possible to have $m_e^+ = +\infty$, although this will not happen in the cases that concern us. If v^+ is a QH vertex with $G_{v^+} = \pi_1(S)$, then the element b in the definition is the class of the boundary component of S corresponding to ∂_e^+. In particular m_e^+ is finite. Also, in the case when G_{v^+} is cyclic, the element b is a generator of G_{v^+} and the label is also finite.

The following theorem, due to Zieschang, will be crucial in the proof of Theorem 5.5. The proof is referred, and the corollary results from iterated use of the theorem.

Theorem 5.2 [20, Theorem 4.12.1, page 140] *Let S be a 2–orbifold with boundary components $\gamma_1, \ldots, \gamma_n$. Let Δ be a one-edged $\mathbb{Z}$–splitting of $\pi_1(S)$ in which $[\gamma_1], \ldots, [\gamma_n]$ are elliptic. Then there is a weakly essential simple closed curve c in S such that Δ is obtained by cutting S along c (via Van-Kampen's theorem).*

Corollary 5.3 *Let S be a 2–orbifold with boundary components $\gamma_1, \ldots, \gamma_n$. If Δ is a general $\mathbb{Z}$–splitting in which $[\gamma_1], \ldots, [\gamma_n]$ are elliptic, then Δ is obtained by cutting S along $c_1, \ldots, c_m$, disjoint weakly essential simple closed curves.*

We also need the following simple lemma about coverings of surfaces and 2–orbifolds.

Lemma 5.4 *Let S be a connected 2–orbifold with boundary that is neither a disk nor a cylinder with cone-points. Then there is a 4–sheeted cover $\widehat{S}$ of S, such that every boundary component γ of S is covered by two boundary components $\widehat{\gamma}_0$, $\widehat{\gamma}_1$ of $\widehat{S}$, and each one is a double cover of γ.*

Proof Assume S is an orientable surface, the general case is analogous. Write

$$\pi_1(S) = \langle a_1, \ldots, a_g, b_1, \ldots, b_g, p_1, \ldots, p_m \mid \Pi_k p_k \Pi_j [a_j, b_j] = 1 \rangle.$$

Observe if the genus is positive, then the kernel of the map $\pi_1(S) \to \mathbb{Z}_2$ sending a_1 to 1 and all other generators to 0 defines a double cover S_0 of S where each boundary component of S is covered by two homeomorphic copies of itself. On the other hand, when m is even, the map $\pi_1(S) \to \mathbb{Z}_2$ sending all p_i to 1 and a_j, b_j to 0 is a well defined homomorphism, and its kernel gives a double cover S_1 of S in which each boundary component of S is covered twice by a single boundary curve of S_1. Notice S_1 always has positive genus, by the Euler characteristic computation for a finite cover.

Combining these covers produces the desired 4–sheeted cover in the cases when m is even, or m is odd but S has positive genus. There remains the case of a sphere with an odd number of punctures. In this case, we have

$$\pi_1(S) = \langle p_1, \ldots, p_m \mid p_1 \cdots p_m = 1 \rangle.$$

Consider the map $\pi_1(S) \to \mathbb{Z}_2 \times \mathbb{Z}_2$ sending $p_1, \ldots, p_{m-2}$ to $(1, 0)$, p_{m-1} to $(0, 1)$ and p_m to $(1, 1)$. This map is a well defined homomorphism and gives the desired covering. $\square$

The following result is the main point of this section. Under some conditions, it allows us to recognize the universality of a $\mathbb{Z}$–splitting built from the union of smaller universal graphs and some extra QH vertices.

Theorem 5.5 *Let Γ be a $\mathbb{Z}$–splitting of the one-ended group G. Let $V = \{v_1, \ldots, v_m\}$ be a subset of the QH vertices of Γ, such that their corresponding 2–orbifolds are not disks nor cylinders with cone-points. Let $\Gamma_1, \ldots, \Gamma_k$ be the components of the subgraph spanned by the vertices not in V, and put $G_i = \pi_1(\Gamma_i)$. Assume*

(1) *If e is an edge with endpoints in V, then $m_e^+, m_e^- > 1$.*

(2) *If the vertices $v_j \in V$ and $w \in \Gamma_i$ are connected by an edge, then w is not a QH vertex of Γ_i.*

(3) *Each G_i is one-ended, and each Γ_i satisfies universality as a $\mathbb{Z}$–splitting of G_i.*

Then Γ satisfies universality.

Proof First we observe that if w is a vertex of Γ_i, then it is QH in Γ if and only if it is QH in Γ_i. If w is QH in Γ_i, then it has no more incident edges in Γ by (2), and so it is also QH in Γ. And if w is QH in Γ, then it cannot be connected by an edge to $v_j \in V$, for that would cause G_i to be freely decomposable. (To see that, let $p_1, \ldots, p_n$ be the boundary classes in G_w, and suppose that the edge assigned to p_1 is not in Γ_i. Observe that $p_2, \ldots, p_n$ are part of a free basis for G_w. This induces a free splitting of G_w that allows us to refine Γ_i to a graph with some trivial edge groups).

Let Γ' be a $\mathbb{Z}$–splitting of G, and $T' = T_{\Gamma'}$ its Bass–Serre tree. Consider the action of G_i on T' by restriction of the action of G. Passing to a minimal invariant subtree, G_i acts cocompactly and with cyclic edge stabilizers (since G_i is one-ended). So this action gives rise to a $\mathbb{Z}$–splitting of G_i. By universality of Γ_i, there is a refinement $\hat{\Gamma}_i$ and a morphism $T_{\hat{\Gamma}_i} \to T'$ so that $\hat{\Gamma}_i$ is obtained from Γ_i by splitting QH vertex groups along weakly essential simple closed curves. Then all the non-QH vertex groups, and all the edge groups of Γ_i are elliptic in Γ'.

This proves that all the non-QH vertex groups of Γ are elliptic in Γ', since V consists only of QH vertices.

It also implies if an edge e has an endpoint in some Γ_i, then G_e is elliptic in Γ': If e is contained in Γ_i we have already shown it. If e has endpoints $v_j \in V$ and $w \in \Gamma_i$, then w is non-QH by (2), and so G_w is elliptic in Γ'. Since $G_e \subset G_w$, then G_e must also be elliptic in Γ'.

Claim All edge groups of Γ are elliptic in Γ'.

Proof of the claim If e has an endpoint in some Γ_i, we have already proved it.

Now let e be an edge with endpoints $v^{\pm} \in V$ (which can be the same vertex).

Let $\gamma^{\pm}$ be the boundary components of the orbifolds $S^{\pm}$ corresponding to $G_{v^{\pm}}$, so that $\partial_e^{\pm} \colon G_e \to \langle [\gamma^{\pm}] \rangle$. Let $H_e = \langle [\gamma^+], [\gamma^-] \rangle \subset G$ be the subgroup generated by the classes of $\gamma^{\pm}$. Note that H_e is a GBS group.

If either $m_e^+ > 2$ or $m_e^- > 2$, then the splitting of H_e with edge e satisfies the conditions in Theorem 3.3, that are direct consequences of those over $m_e^{\pm}$. So it is a JSJ decomposition of H_e, and so G_e is elliptic in T' (as we have done for the Γ_i).

If $m_e^+ = m_e^- = 2$, we proceed by contradiction. Suppose G_e is hyperbolic in Γ', and let c be the generator of G_e. Take an edge e' of Γ' that has a lift to T' lying on the axis of c. Then $\Gamma'' = \Gamma'_{e'}$ is a one-edged $\mathbb{Z}$–splitting of G in which c is hyperbolic. Let $T'' = T_{\Gamma''}$ be its Bass–Serre tree, and let a be the generator of the edge group of Γ''.

On one hand, we consider the subgroup H_e. Note that $H_e = \pi_1(K)$ where K is a Klein bottle. (K is obtained by gluing two Möbius bands by their boundaries. In this case γ^+ and γ^- are the core circles of the Möbius bands, and c is their common boundary circle). The action of H_e on T'' by restriction gives rise to a $\mathbb{Z}$–splitting of H_e (for H_e is freely indecomposable). Note that c is hyperbolic in it, since it is so in T''. So this $\mathbb{Z}$–splitting is nontrivial, and we can take $b \in H_e$ a generator of an edge group. Now observe that the edge groups of this decomposition of H_e are all conjugate in G into $\langle a \rangle$. This is so because the only elements that fix an edge of T'' are the conjugates of a power of a. So we obtain an element $b \in H_e$, $b \neq 1$, which is conjugate to a power of a.

On the other hand, we consider the subgroup M constructed as follows.

Take the graph formed by the vertices in V and the edges of Γ with endpoints in V and both labels equal to 2. Let Δ be the component of this graph that contains e. For each vertex $v_j \in \Delta$ write $G_{v_j} = \pi_1(S_j)$, where S_j is the orbifold that corresponds to v_j as a QH vertex of Γ. Let $\widehat{S}_j$ be the 4–sheeted cover of S_j given by Lemma 5.4. These covers can be extended to a 4–sheeted cover of the whole graph Δ, that can be constructed as follows. Define the graph $\widehat{\Delta}$ to have the same vertices as Δ, with $\pi_1(\widehat{S}_j) < \pi_1(S_j)$ as vertex group at v_j. And for each edge f of Δ, we put in four edges f_0, f_1, f_2 and f_3 in $\widehat{\Delta}$, with infinite cyclic edge groups. The boundary maps are described as follows: Suppose v_j is an endpoint of f and δ is the boundary component of S_j corresponding to f. Then let δ_0 and δ_1 be the boundary components of $\widehat{S}_j$ that cover δ and assign f_0, f_2 to δ_0 and f_1, f_3 to δ_1. So the generator of G_{f_0} maps to $[\delta_0]$ and similarly for the others. This is a 4–sheeted cover, in the sense that $\pi_1(\widehat{\Delta}) < \pi_1(\Delta)$ with index 4. (This is best seen by building a presentation 2–complex of $\pi_1(\Delta)$, using S_j for the vertex v_j, and tubes for the edges. Then extend the covers

$\widehat{S}_j$ of S_j to covers of the tubes.) Note that the labels of the edges of $\widehat{\Delta}$ are all 1. The local picture at each edge is as in the example on Figure 1.
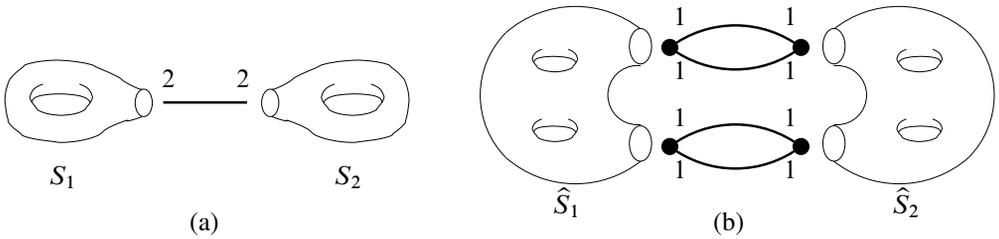

Figure 1: Example of the cover $\widehat{\Delta}$
  (a) Original graph Δ, with two QH vertices and an edge with $m_e^{\pm} = 2$
  (b) Its cover $\widehat{\Delta}$

Now let M be the subgroup of $\pi_1(\widehat{\Delta})$ generated only by the vertex groups and the stable letters of the f_0 and f_1 edges. This is equivalent to saying that M is the fundamental group of the graph resulting from $\widehat{\Delta}$ by erasing all f_2 and f_3 edges (and keeping the f_0 and f_1 edges). So $M = \pi_1(S)$, where S is the orbifold that results from gluing the $\widehat{S}_j$ along their boundary curves, so that two boundary curves are identified if they are connected by an edge of $\widehat{\Delta}$. Note that in this subgroup, c is the class of one of the common boundaries of $\widehat{S}^+$ and $\widehat{S}^-$ that corresponds to a lift of the edge e. (Say, to e_0). Lets call this curve β, so that $[\beta] = c$.

If p is a boundary curve of S, then some power of $[p]$ is in an edge group G_f of Γ, so that f is not in Δ. (All boundaries corresponding to edges in Δ were glued). Since f is not in Δ, but connects to a vertex in Δ, we know that f is one of the edges for which we have already proved that G_f is elliptic in Γ'. Thus, the classes of the boundary curves of S are elliptic in Γ' (and so in Γ'').

Again, restrict to M the action on T''. This gives a $\mathbb{Z}$–splitting of M, in which c is hyperbolic and all the boundary classes of S are elliptic. By Corollary 5.3, this decomposition of M is obtained by cutting S along disjoint, weakly essential simple closed curves. Let α be one of these curves, so that it intersects β essentially (ie the intersection cannot be removed by homotopy). There must be such α, since $c = [\beta]$ is hyperbolic in this decomposition.

Now, since $[\alpha]$ is a generator of an edge group in the $\mathbb{Z}$–splitting of M induced by T'', then $[\alpha]$ must be conjugate in G to a power of a. This is by the same argument we used for the element b.

Since both $[\alpha]$ and b are conjugate to a power of a, then they have the same dynamics in every action of G on a tree. That is to say, in a given G–tree, they are either both elliptic or both hyperbolic, depending on the behaviour of a.

For the contradiction, consider Γ^*, the one-edged splitting of G over $[\gamma^-]$. This splitting is obtained from Γ_e by folding at v^+. In the case of an amalgamation, Γ_e corresponds to $A *_{\langle c \rangle} B$ and Γ^* to $A *_{\langle [\gamma^-] \rangle} (H_e *_{\langle c \rangle} B)$. The case of an HNN extension is similar.

In both cases H_e is contained in a vertex group, so b must be elliptic in Γ^*. We will show $[\alpha]$ is hyperbolic in Γ^*. This will give the contradiction, thus proving the claim.

Consider the action of M on T_{Γ^*} by restriction. It gives a splitting of $M = \pi_1(S)$ in which the boundary classes are elliptic, so we may use Corollary 5.3 again. This time $c = [\beta]$ stabilizes an edge on T_{Γ^*}, thus β is one of the curves that cut S to form this decomposition. Since α intersects β essentially, then $[\alpha]$ must be hyperbolic in this splitting of M, and therefore in Γ^*. ◊

Thus far we know that all non-QH vertex groups and all edge groups of Γ are elliptic in Γ'. For each QH vertex v of Γ, write $G_v = \pi_1(S_v)$ where S_v is the corresponding orbifold. Then G_v acts on T' by restriction. Since edge groups of Γ are elliptic in Γ', it follows that the boundary classes of S_v act elliptically on T'. Applying Corollary 5.3, the $\mathbb{Z}$–splitting of G_v induced by its action on T' is obtained by cutting S_v along some disjoint, weakly essential simple closed curves. The vertex groups of this decomposition correspond to the pieces of S_v after the cutting, and are elliptic in Γ'. Also note that each boundary curve of S_v lies in exactly one of these pieces. So the splitting of G_v is compatible with Γ, giving rise to a refinement of Γ.

Let $\hat{\Gamma}$ be the refinement of Γ that results from splitting all the QH vertex groups G_v as above. Then all vertex and edge groups of $\hat{\Gamma}$ are elliptic on Γ'. Equivalently, there is a morphism $T_{\hat{\Gamma}} \to T'$. Since Γ' was an arbitrary $\mathbb{Z}$–splitting of G, this concludes the proof. □

6 Quadratic Baumslag–Solitar graphs

Now we consider graphs of groups Γ with edge groups infinite cyclic, and vertex groups either QH surface groups or infinite cyclic. We will call these graphs *quadratic Baumslag–Solitar (QBS) graphs*. For simplicity, we restrict the QH vertex groups to be surface groups instead of general 2–orbifold groups. Notice that in a GBS graph all labels are finite, and easily computed from the boundary maps as indicated in the remarks after Definition 5.1.

A group G will be called a *QBS group* if it can be written as $\pi_1(\Gamma)$, where Γ is a QBS graph.

If Γ is a QBS graph, let $\Gamma_1, \ldots, \Gamma_k$ be the components of the subgraph spanned by the non-QH vertices. That is, the components that are left after removing all QH vertices and the edges connecting to them. Note that each Γ_i is then a GBS graph. The Γ_i will be called the *GBS components* of Γ.

A GBS component of Γ will be called a *leaf* if it is reduced to a single vertex w, and is attached to only one edge e with $G_w = G_e$.

Proposition 6.1 *Let Γ be a reduced QBS graph with no leaves, and $G = \pi_1(\Gamma)$. Assume $G \ncong \mathbb{Z}$. Then G is one ended.*

This is a corollary of [19, Theorem 18].

We now turn to the main theorem of the paper. It allows us to recognize the defining graph of a QBS group as a Rips–Sela JSJ decomposition, in most cases.

Theorem 1.1 *Let Γ be a QBS graph, $G = \pi_1(\Gamma)$. Suppose that Γ is reduced, has no leaves, and satisfies the following conditions:*

(1) *Each edge e of Γ has labels $m_e^+, m_e^- > 1$.*

(2) *Each GBS component Γ_i of Γ is reduced, and T_{Γ_i} is not a point or a line.*

Then Γ is a Rips–Sela JSJ decomposition for G.

Proof Let V be the set of QH vertices of Γ. The components of Γ minus V are the GBS components Γ_i of Γ. By condition (1) and Proposition 3.4, each Γ_i is unfolded. This, together with condition (2), allows us to apply Theorem 3.3 (Forester's result). We conclude each Γ_i is a JSJ decomposition of $G_i = \pi_1(\Gamma_i)$. By Lemma 3.2, the G_i are one-ended. By these facts and condition (1), we have verified the hypotheses of Theorem 5.5 for Γ and V. So Γ satisfies universality. Now we can use Theorem 4.2 to conclude that Γ is unfolded. Therefore Γ is a JSJ decomposition of G, by Corollary 2.16. $\square$

When some edge label equals 1, then Γ may fail to be a JSJ decomposition. This was already true for GBS graphs. In Figure 2 there is an example, in which the edge e with a label equal to 1 is not in a GBS component. However, if in the same figure we change the label 1 for some $m_e^- > 1$, and make $k = 1$ instead, we do get a JSJ decomposition (by Theorem 5.5 and then Theorem 4.2), which is not covered by Theorem 1.1.
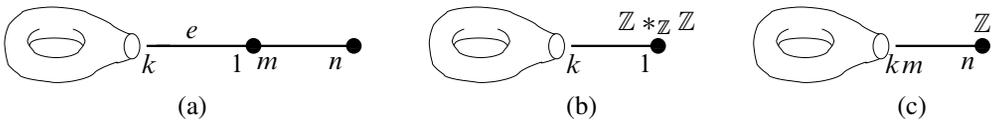
Figure 2: (a)   A QBS graph that satisfies universality (for $m, n > 1$), but with $m_e^- = 1$. It admits an unfolding at the surface vertex, as shown in (b) and (c).

(b)   The one-edged splitting corresponding to the edge e of the graph in (a)

(c)   An unfolding of the splitting in (b)

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