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The $D(2)$–problem for dihedral groups of order $4n$

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We give a full solution in terms of k–invariants of the $D(2)$–problem for D_{4n}, assuming that $\boldsymbol{Z}[D_{4n}]$ satisfies torsion-free cancellation.

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1 Introduction

The following question was first posed by Wall in [12]:

$D(2)$–problem. Let X be a finite connected 3–dimensional CW–complex, with universal cover $\widetilde{X}$, such that

$$H_3(\widetilde{X};\boldsymbol{Z}) = H^3(X;\mathcal{B}) = 0$$

for all coefficient systems $\mathcal{B}$ on X. Is it true that X is homotopy equivalent to a finite 2–dimensional CW–complex?

The $D(2)$–problem is parametrized by the fundamental group of X; we say that the $D(2)$–property holds for a finitely presented group G if the above question is answered in the affirmative for every X with $\pi_1(X) \cong G$.

We shall be concerned with the $D(2)$–problem for D_{4n}, the dihedral group of order $4n$. Johnson [7] has shown that the $D(2)$–property holds for the groups D_{4n+2} for any $n \geq 1$; however his result relies on the fact that D_{4n+2} has periodic cohomology, a property not shared by D_{4n}. Mannan [9] has shown that the $D(2)$–property holds for D_8. We say that *torsion-free cancellation* holds for a group ring $\boldsymbol{Z}[G]$ if

$$X \oplus M \cong X \oplus N \Rightarrow M \cong N$$

for any $\boldsymbol{Z}[G]$–lattices X, M and N. We shall show:

Theorem 1.1 *Suppose that $\boldsymbol{Z}[D_{4n}]$ satisfies torsion-free cancellation. Then the $D(2)$–property holds for D_{4n}.*

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The calculations of Swan [11] and Endo and Miyata [3] show that torsion-free cancellation holds for $\boldsymbol{Z}[D_{4p}]$ when p is prime and $3 \leq p \leq 31$, $p = 47, 179$ or 19379. To date the only finite nonabelian, nonperiodic groups for which the $D(2)$–property is known to hold are those of the form D_{4p}, where p is prime.

Let G be a group and set $\Lambda = \boldsymbol{Z}[G]$. Any finite 2–dimensional CW–complex K with $\pi_1(K) = G$ gives rise to an exact sequence of Λ–modules

$$0 \to \pi_2(K) \to C_2(K) \xrightarrow{\partial_2} C_1(K) \xrightarrow{\partial_1} C_0(K) \to \boldsymbol{Z} \to 0, \tag{1}$$

where $C_r(K) = H_r(\widetilde{K}_r, \widetilde{K}_{r-1}; \boldsymbol{Z})$ is the free Λ–module with basis the r–cells of K. By an *algebraic 2–complex* over a group G, we mean an exact sequence of right Λ–modules of the form

$$0 \to J \to F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\varepsilon} \boldsymbol{Z} \to 0, \tag{2}$$

where each F_i is finitely generated free. An algebraic 2–complex is said to be geometrically realizable if it is homotopy equivalent to a 2–complex of type (1). If every algebraic 2–complex over a group G is geometrically realizable we say that the realization property holds for G. The following result is due to Johnson [7] and Mannan [10]:

Theorem 1.2 *Let G be a finitely presented group. Then the $D(2)$–property holds for G if and only if the realization property holds for G.*

We are grateful to the referee for pointing out a paper of Latiolais [8], in which it is proved that the homotopy type of a CW–complex with fundamental group D_{4n} is determined by the Euler characteristic. This result was extended by Hambleton and Kreck [6] to include those complexes whose fundamental groups are finite subgroups of $\mathrm{SO}(3)$. Latiolais achieves this by realizing all values of the Browning obstruction group (see Browning [1], Gruenberg [4], Gutierrez and Latiolais [5]); combining this realization with Theorem 1.2, it seems possible to give a proof of Theorem 1.1 without assuming torsion-free cancellation.

We begin by briefly recalling the classification of algebraic complexes in terms of k–invariants — for a full treatment, see Johnson [7, Chapter 6]. Fix a finite group G and put $\Lambda = \boldsymbol{Z}[G]$. Let $\mathcal{P} = (0 \to J \to F_2 \to F_1 \to F_0 \to \boldsymbol{Z} \to 0)$ be an algebraic 2–complex over G and let $\mathcal{E} = (0 \to J \to E_2 \to E_1 \to E_0 \to \boldsymbol{Z} \to 0) \in \mathrm{Ext}^3_\Lambda(\boldsymbol{Z}, J)$ be an arbitrary extension of $\boldsymbol{Z}$ by J. Then by the universal property of projective modules, there exists a commutative diagram:

$$\begin{array}{ccccccccccccc}
\mathcal{P} & = & (0 & \longrightarrow & J & \longrightarrow & F_2 & \longrightarrow & F_1 & \longrightarrow & F_0 & \longrightarrow & \boldsymbol{Z} \longrightarrow 0) \\
\big\downarrow \alpha & & & & \big\downarrow \alpha_+ & & \big\downarrow \alpha_2 & & \big\downarrow \alpha_1 & & \big\downarrow \alpha_0 & & \big\downarrow \mathrm{Id} \\
\mathcal{E} & = & (0 & \longrightarrow & J & \longrightarrow & E_2 & \longrightarrow & E_1 & \longrightarrow & E_0 & \longrightarrow & \boldsymbol{Z} \longrightarrow 0)
\end{array}$$

We may extend α_+ thus:

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & J & \longrightarrow & F_2 & \longrightarrow & F_1 & \longrightarrow & F_0 & \longrightarrow & \mathbf{Z} & \longrightarrow & 0 \\
& & \big\downarrow \alpha_+ & & \big\downarrow \alpha_2' & & \big\downarrow \alpha_1' & & \big\downarrow \alpha_0' & & \big\downarrow \widetilde{\alpha} & & \\
0 & \longrightarrow & J & \longrightarrow & F_2 & \longrightarrow & F_1 & \longrightarrow & F_0 & \longrightarrow & \boldsymbol{Z} & \longrightarrow & 0
\end{array}$$

Then $\widetilde{\alpha}$ is unique up to congruence modulo $|G|$ and we have a well-defined map $\kappa\colon \mathrm{End}_\Lambda J \to \boldsymbol{Z}/|G|$ given by $\kappa(\alpha_+) = \widetilde{\alpha}$. The k–invariant of the transition $\alpha\colon \mathcal{P} \to \mathcal{E}$ is defined to be $k(\mathcal{P} \to \mathcal{E}) = \kappa(\alpha_+)$. Given $\alpha \in \mathrm{End}_\Lambda J$ we have a k–invariant $k(\mathcal{P} \to \alpha_*(\mathcal{P})) = \kappa(\alpha) k(\mathcal{P} \to \mathcal{P}) = \kappa(\alpha)$, where $\alpha_*(\mathcal{P})$ is the pushout extension. Since $\kappa(\alpha)$ is a unit if α is an isomorphism, this induces a mapping

$$\mathrm{Aut}_\Lambda J \to (\boldsymbol{Z}/|G|)^*$$

called the Swan map, which is independent of the choice of algebraic complex in which J appears. We have (see Johnson [7, Theorems 54.6 and 54.7]):

Theorem 1.3 *Suppose that the Swan map* $\mathrm{Aut}\, J \to (\boldsymbol{Z}/|G|)^*$ *is surjective. Then for each $n \geq 0$ there is, up to chain homotopy equivalence, a unique algebraic 2–complex of the form*

$$0 \to J \oplus \Lambda^n \to F_2 \to F_1 \to F_0 \to \boldsymbol{Z} \to 0.$$

2 The Swan map for D_{2n}

For any n the group D_{2n} may be described by the presentation

$$\langle x, y \mid x^n, y^2, y^{-1}xyx \rangle.$$

Write $\Lambda = \boldsymbol{Z}[D_{2n}]$ and $\Sigma = 1 + x + x^2 + \cdots + x^{n-1}$. Applying the Cayley complex construction to this presentation gives the following 2–complex:

(3) $$0 \to J \to \Lambda^3 \xrightarrow{\partial_2} \Lambda^2 \xrightarrow{\partial_1} \Lambda \xrightarrow{\varepsilon} \boldsymbol{Z} \to 0,$$

where ε is the augmentation map, $\partial_1 = (x-1, y-1)$ and $\partial_2 = \begin{pmatrix} \Sigma & 0 & 1+yx \\ 0 & 1+y & x-1 \end{pmatrix}$. The following proposition is easily verified:

Proposition 2.1 *Fix n and let k be any odd integer with $3 \le k \le n-1$. If we write $m = (k-1)/2$ then the following diagram commutes:*

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & J & \longrightarrow & \Lambda^3 & \xrightarrow{\partial_2} & \Lambda^2 & \xrightarrow{\partial_1} & \Lambda & \xrightarrow{\varepsilon} & \mathbf{Z} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle k} & & \\
0 & \longrightarrow & J & \longrightarrow & \Lambda^3 & \xrightarrow{\partial_2} & \Lambda^2 & \xrightarrow{\partial_1} & \Lambda & \xrightarrow{\varepsilon} & \mathbf{Z} & \longrightarrow & 0
\end{array}$$

where $\partial_1 = (x-1, y-1)$, $\partial_2 = \left(\begin{smallmatrix} \Sigma & 0 & 1+yx \\ 0 & 1+y & x-1 \end{smallmatrix}\right)$,

$$\alpha_0 = (1 + x^{-1} + \cdots + x^{-m} + x^{-1}y + \cdots + x^{-m}y),$$

$$\alpha_1 = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$a = 1 + x^{-1} + \cdots + x^{-m} - x^{-2}y - \cdots - x^{-m-1}y$ *and* $\theta = \alpha_2|_J$.

Consider the commutative diagram above as a diagram of (free) $\mathbf{Z}$–modules and $\boldsymbol{Z}$–linear maps; taking determinants we have:

Proposition 2.2 $\quad k \det\theta \det\alpha_1 = \det\alpha_2 \det\alpha_0$.

Proof Let v denote the restriction of α_0 to $\ker\varepsilon$ and let u denote the restriction of α_1 to $\ker\partial_1$. Then $v(\ker\varepsilon) \subset \ker\varepsilon$, $u(\ker\partial_1) \subset \ker\partial_1$ and we have an commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker\partial_1 & \longrightarrow & \Lambda^2 & \xrightarrow{\partial_1} & \ker\varepsilon & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle v} & & \\
0 & \longrightarrow & \ker\partial_1 & \longrightarrow & \Lambda^2 & \xrightarrow{\partial_1} & \ker\varepsilon & \longrightarrow & 0
\end{array}$$

Considered as a diagram of (free) $\boldsymbol{Z}$–modules, both exact sequences split, and so there exists α_1' such that

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker\partial_1 & \longrightarrow & \ker\partial_1 \oplus \ker\varepsilon & \longrightarrow & \ker\varepsilon & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle \alpha_1'} & & \downarrow{\scriptstyle v} & & \\
0 & \longrightarrow & \ker\partial_1 & \longrightarrow & \ker\partial_1 \oplus \ker\varepsilon & \longrightarrow & \ker\varepsilon & \longrightarrow & 0
\end{array}$$

commutes with the obvious maps, and where $\det\alpha_1' = \det\alpha_1$. Therefore we have $\det\alpha_1' = \det\left(\begin{smallmatrix} u & w \\ 0 & v \end{smallmatrix}\right) = \det u \det v$. Similarly

$$\det\alpha_2 = \det\theta \det u \quad \text{and} \quad \det\alpha_0 = \det v \det k = k \det v.$$

Thus

$$\det \alpha_2 \det \alpha_0 = \det \theta \det u \det(v)\, k = k \det \theta \det \alpha_1$$

as required. □

Now, any Λ–homomorphism is a Λ–isomorphism if and only if it is an isomorphism as a $\boldsymbol{Z}$–linear map. Thus, in order to show that $[k]$ is in the image of the Swan map, it suffices to show that $\det \theta = \pm 1$.

Proposition 2.3 *Suppose that k is coprime to $2n$. Then $\det \alpha_0 = \pm k$.*

Proof Let $M(\alpha_0)$ be the matrix of the Λ–linear map given by $x \mapsto \alpha_0 x$ with respect to the $\boldsymbol{Z}$–basis $\{1, x, \dots, x^{n-1}, y, \dots, x^{n-1}y\}$, with the elements of Λ being interpreted as columns. Notice that $M(\alpha_0) = \left(\begin{smallmatrix} A & B \\ B & A \end{smallmatrix}\right)$ where $A = (a_{i,j})$ and $B = (b_{i,j})$ are $n \times n$ matrices. We know that $a_{i,1} = 1$ if α_0 contains an x^{i-1} term and $\alpha_{i,1} = 0$ otherwise. Thus

$$a_{i,1} = \begin{cases} 1 & \text{if } i \in \{1, n-m+1, n-m+2, \dots, n\}, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly

$$b_{i,1} = \begin{cases} 1 & \text{if } i \in \{n-m+1, n-m+2, \dots, n\}, \\ 0 & \text{otherwise.} \end{cases}$$

The other columns of A and B are obtained by cyclically permuting the first column; let $\sigma_+, \sigma_- \colon \{1, \dots, n\} \to \{1, \dots, n\}$ be the permutations given by $\sigma_+(i) = i + 1 \mod n$ and $\sigma_-(i) = i - 1 \mod n$. We now have

$$a_{i,j} = a_{\sigma_-^{j-1}(i),1} \quad \text{and} \quad b_{i,j} = b_{\sigma_+^{j-1}(i),1}.$$

Now label the columns of $M(\alpha_0)$ by $v_1, \dots, v_{2n}$. Let N be the matrix with columns $v'_1, \dots, v'_{2n}$ where $v'_i = v_i$ for $1 \le i \le n$ and $v'_{n+i} = v_{n+i} - v_{n+1-i}$ for $1 \le i \le n$. For example, if $n = 4$ and $k = 3$ (so that $m = 1$), we would have:

$$M(\alpha_0) = \begin{pmatrix} 1&1&0&0&0&0&0&1 \\ 0&1&1&0&0&0&1&0 \\ 0&0&1&1&0&1&0&0 \\ 1&0&0&1&1&0&0&0 \\ 0&0&0&1&1&1&0&0 \\ 0&0&1&0&0&1&1&0 \\ 0&1&0&0&0&0&1&1 \\ 1&0&0&0&1&0&0&1 \end{pmatrix}; \quad N = \begin{pmatrix} 1&1&0&0&0&0&-1&0 \\ 0&1&1&0&0&-1&0&0 \\ 0&0&1&1&-1&0&0&0 \\ 1&0&0&1&0&0&0&-1 \\ 0&0&0&1&0&1&0&0 \\ 0&0&1&0&0&0&1&0 \\ 0&1&0&0&0&0&0&1 \\ 1&0&0&0&1&0&0&0 \end{pmatrix}$$

If $N = \left(\begin{smallmatrix} A & C \\ B & D \end{smallmatrix}\right)$ for matrices $C = (c_{i,j})$ and $D = (d_{i,j})$ then $c_{i,j} = b_{i,j} - a_{i,n+1-j}$ and $d_{i,j} = a_{i,j} - b_{i,n+1-j}$. Now,

$$a_{i,n} = a_{\sigma_-^{n-1}(i),1} = \begin{cases} 1 & \text{if } \in \{n-m, n-m+1, \dots, n\}, \\ 0 & \text{otherwise,} \end{cases}$$

and so

$$c_{i,1} = \begin{cases} -1 & \text{if } i = n-m, \\ 0 & \text{otherwise,} \end{cases}$$

Similarly,

$$d_{i,1} = \begin{cases} 1 & \text{if } i = n-m+1, \\ 0 & \text{otherwise.} \end{cases}$$

We also have

$$c_{i,j} = c_{\sigma_+^{j-1}(i),1} \quad \text{and} \quad d_{i,j} = d_{\sigma_-^{j-1}(i),1}.$$

There is precisely one -1 appearing in the i–th row of C; fix i, j such that $c_{i,j} = -1$. Then $c_{\sigma_+^{j-1}(i),1} = -1 \Rightarrow \sigma_+^{j-1}(i) = n-m \Rightarrow j = \sigma_-^{i-1}(n-m)$. The row of D containing $+1$ in the j–th position is the k–th, where

$$\begin{aligned} d_{k,\sigma_-^{i-1}(n-m)} = 1 &\Rightarrow d_{\sigma_-^{\sigma_-^{i-1}(n-m)-1}(k),1} = 1 \\ &\Rightarrow \sigma_-^{\sigma_-^{i-1}(n-m)-1}(k) = n-m+1 \\ &\Rightarrow k - \sigma_-^{i-1}(n-m) + 1 = n-m+1 \mod n \\ &\Rightarrow k - n + m + i = n - m + 1 \mod n \\ &\Rightarrow k = n - 2m - i + 1 \mod n \\ &\Rightarrow k = \sigma_-^{i-1}(n-2m). \end{aligned}$$

Let the rows of N be labelled by $w_1, \dots, w_{2n}$. Put $w'_i = w_i$ for $n+1 \le i \le 2n$ and $w'_i = w_i + w_{n+\sigma_-^{i-1}(n-2m)}$ for $1 \le i \le n$. If we let P be the matrix with rows $w'_1, \dots, w'_{2n}$ then by the preceding argument P is of the form $P = \left(\begin{smallmatrix} E & 0 \\ B & D \end{smallmatrix}\right)$. Here D is a permutation matrix, and so we have $\det M(\alpha_0) = \pm \det E$. In the case $n = 4$, $k = 3$ we have:

$$E = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

If $E = (e_{i,j})$, then

$$e_{i,j} = a_{i,j} + b_{\sigma_-^{i-1}(n-2m),j}.$$

Consider

$$\begin{aligned} e_{i,j} - e_{\sigma_-^{j-1}(i),1} &= a_{i,j} + b_{\sigma_-^{i-1}(n-2m),j} - a_{\sigma_-^{j-1}(i),1} - b_{\sigma_-^{\sigma_-^{j-1}(i)-1}(n-2m),1} \\ &= b_{\sigma_+^{j-1}(\sigma_-^{i-1}(n-2m)),1} - b_{\sigma_-^{\sigma_-^{j-1}(i)-1}(n-2m),1}, \end{aligned}$$

where we have cancelled the a terms. Now,

$$\begin{aligned} \sigma_-^{\sigma_-^{j-1}(i)-1}(n-2m) = n-2m-\sigma_-^{j-1}(i)+1 &= n-2m-(i-j+1)+1 \mod n \\ &= n-2m+j-i \mod n. \end{aligned}$$

However,

$$\begin{aligned} \sigma_+^{j-1}(\sigma_-^{i-1}(n-2m)) = \sigma_-^{i-1}(n-2m)+j-1 &= n-2m-i+1+j-1 \mod n \\ &= n-2m+j-i \mod n, \end{aligned}$$

so the b terms also cancel, and we can conclude that $e_{i,j} = e_{\sigma_-^{j-1}(i),1}$.

Consider the first column of E: we know that

$$b_{\sigma_-^{i-1}(n-2m),1} = \begin{cases} 1 & \text{if } \sigma_-^{i-1}(n-2m) \in \{n-m+1, n-m+2, \dots, n\}, \\ 0 & \text{otherwise.} \end{cases}$$

However,

$$\sigma_-^{i-1}(n-2m) \in \{n-m+1, \dots, n\} \iff [n-2m-i+1] \in \{[n-m+1], \dots, [n]\},$$

where [ ] represents class modulo n. This is equivalent to

$$[-i] \in \{[2m-1], [2m-2], \dots, [m]\},$$

or $i \in \{n-2m+1, n-2m+2, \dots, n-m\}$. Comparing this with the $a_{i,1}$ s, we see that

$$e_{i,1} = \begin{cases} 1 & \text{if } i \in \{1, n-2m+1, \dots, n\}, \\ 0 & \text{otherwise,} \end{cases}$$

so that E has $2m+1 = k$ 1s in each column. We may cyclically permute the rows of E to form a new matrix $F = (f_{i,j})$ with $f_{i,j} = f_{\sigma_-^{j-1}(i),1}$ and

$$f_{i,1} = \begin{cases} 1 & \text{if } 1 \le i \le k, \\ 0 & \text{otherwise.} \end{cases}$$

The matrix F is the circulant matrix associated to the row vector $(v_0, v_1, \dots, v_{n-1})$ with $v_i = 1$ for $0 \le i \le k-1$ and $v_i = 0$ for $k-1 \le i \le n-1$. The determinant of

F is given by the well-known formula (see for example [2]):

$$\det F = \prod_{i=0}^{n-1} \sum_{j=0}^{n-1} \zeta^{ij} v_j,$$

where ζ is a primitive n–th root of unity. Write $\lambda_i = \sum_{j=0}^{n-1} \zeta^{ij} v_j$; clearly $\lambda_0 = k$. However, for each $i \geq 1$, we have

$$\lambda_i = \sum_{j=0}^{k-1} (\zeta^i)^j = \frac{\zeta^{ik} - 1}{\zeta^i - 1},$$

and hence

$$\det F = k \prod_{i=1}^{n-1} \frac{\zeta^{ik} - 1}{\zeta^i - 1}.$$

We note that since k is coprime to n, the sets $\{\zeta^{ik} \mid i \in \{1, 2, \ldots, n-1\}\}$ and $\{\zeta^i \mid i \in \{1, 2, \ldots, n-1\}\}$ coincide, and hence $\det \alpha_0 = \pm \det F = \pm k$. □

Proposition 2.4 $\det \alpha_1 = \det \alpha_2 \neq 0$.

Proof The following commutes:

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & J & \longrightarrow & \Lambda^3 & \xrightarrow{\partial_2} & \Lambda^2 & \xrightarrow{\partial_1} & \Lambda & \xrightarrow{\varepsilon} & \mathbf{Z} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \theta'} & & \downarrow{\scriptstyle \alpha_2'} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle k} & & \\
0 & \longrightarrow & J & \longrightarrow & \Lambda^3 & \xrightarrow{\partial_2} & \Lambda^2 & \xrightarrow{\partial_1} & \Lambda & \xrightarrow{\varepsilon} & \mathbf{Z} & \longrightarrow & 0
\end{array}$$

where

$$\alpha_2' = \begin{pmatrix} m+1-my & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and θ' is the restriction of α_2' to J. We proceed to calculate $\det \alpha_2' = \det(m+1-my)$. If we represent $(m+1-my)$ with respect to the basis $\{1, x, \ldots, x^{n-1}, y, \ldots, x^{n-1}y\}$, then we form the matrix:

$$M = \begin{pmatrix} A & B \\ B & A \end{pmatrix}$$

Here A is diagonal with each diagonal entry equal to $m+1$, and B is equal to $-m$ times the permutation matrix associated to $\left(\begin{smallmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & n & n-1 & \cdots & 2 \end{smallmatrix}\right)$. Label the rows of M by $v_1, \ldots, v_{2n}$ and let N be the matrix with rows $v_1', \ldots, v_{2n}'$, where $v_1' = v_1 + v_{n+1}$, $v_i' = v_i + v_{2n-i+2}$ for $2 \leq i \leq n$, and $v_i' = v_i$ for $n+1 \leq i \leq 2n$. Now label the columns of M by $w_1, \ldots, w_{2n}$ and let L be the matrix with columns $w_1', \ldots, w_{2n}'$

where $w_i' = w_i$ for $1 \le i \le n$, $w_{n+1}' = w_{n+1} - w_1$ and $w_{n+i}' = w_{n+1} - w_{n-i+2}$ for $2 \le i \le n$. For example, if $n = 4$ and $k = 3$ (so that $m = 1$) we have:

$$N = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 2 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}; \quad L = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 3 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 3 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

It is easy to see that L is lower triangular with n diagonal entries equal to 1 and n diagonal entries equal to $2m+1 = k$. Then $\det \alpha_2' = \det(m+1-my) = \det L = k^n$. Using $k \det \theta' \det \alpha_1 = \det \alpha_0 \det \alpha_2' = \pm k^{n+1}$ we see that $\det \alpha_1 = \det \alpha_2 \neq 0$. □

Therefore by Propositions 2.2, 2.3 and 2.4:

Proposition 2.5 *If $3 \le k \le n-1$ is coprime to $2n$ then $\det \theta = \pm 1$ and so θ is an isomorphism. Thus $[k]$ is in the image of the Swan map.*

Clearly $[-1]$ is in the image of the Swan map and so:

Corollary 2.6 *The Swan map $\operatorname{Aut} J \to (\mathbf{Z}/2n)^*$ is surjective for each D_{2n}.*

Mannan [9] has previously shown that the Swan map is surjective for D_{2^n}.

3 The $D(2)$–property for $\mathbf{Z}[D_{4n}]$

We now restrict to the case D_{4n}. An application of Schanuel's lemma shows that the module J appearing in (2) is determined up to stable equivalence; that is, if $0 \to J \to F_2 \to F_1 \to F_0 \to \mathbf{Z} \to 0$ and $0 \to J' \to F_2' \to F_1' \to F_0' \to \mathbf{Z} \to 0$ are two algebraic 2–complexes, we have $J \oplus \Lambda^n \cong J' \oplus \Lambda^m$ for some n, m. Write $\Omega_3(\mathbf{Z})$ for the class of modules J' appearing in an algebraic 2–complex over D_{4n}. Now take $J = \ker \partial_2$ in (3); the following proposition is due to Mannan [9]:

Proposition 3.1 *J has minimal $\mathbf{Z}$–rank in $\Omega_3(\mathbf{Z})$.*

Let Γ be an order over a Dedekind domain R. We say that *torsion-free cancellation* holds for Γ if $X \oplus M \cong X \oplus N \Longrightarrow M \cong N$ for lattices X, M and N over Γ (so that X, M and N are finitely generated as Γ–modules and torsion-free over R). There are very few finite groups G for which $\Gamma = \boldsymbol{Z}[G]$ has torsion-free cancellation; if G is nonabelian then the only possible candidates are A_4, A_5, S_4 and D_{2n} for certain values of n. Clearly we have:

Proposition 3.2 *Suppose that $\boldsymbol{Z}[D_{4n}]$ has torsion-free cancellation. Then every $J' \in \Omega_3(\boldsymbol{Z})$ is of the form $J' \cong J \oplus \Lambda^m$ for some $m \geq 0$.*

For a finite group G, the integral group ring $\boldsymbol{Z}[G]$ is a $\boldsymbol{Z}$–order in the semisimple algebra $\mathbf{Q}[G]$; we may choose a maximal $\boldsymbol{Z}$–order Γ in $\mathbf{Q}[G]$ containing $\boldsymbol{Z}[G]$, and define $D(\boldsymbol{Z}[G]) = \ker(\widetilde{K}_0(\boldsymbol{Z}[G]) \to \widetilde{K}_0(\Gamma))$. A necessary condition for $\boldsymbol{Z}[G]$ to possess torsion-free cancellation is $D(\boldsymbol{Z}[G]) = 0$. The following is due to Swan [11]:

Theorem 3.3 *Let p be a prime. Then D_{4p} satisfies torsion-free cancellation if and only if $D(\boldsymbol{Z}[D_{4p}]) = 0$.*

Endo and Miyata [3] calculate the order of $D(\boldsymbol{Z}[D_{2n}])$ for various values of n. In particular they show $D(\boldsymbol{Z}[D_{4p}]) = 0$ for prime p when $3 \leq p \leq 31$, $p = 47, 179$ or 19379. However, there do exist values of n for which $D(\boldsymbol{Z}[D_{4n}]) \neq 0$, for example $n = 37$. Moreover, results of Swan show that $D(\boldsymbol{Z}[D_{4n}]) = 0$ is not a sufficient condition for torsion-free cancellation to hold. For example, $D(\boldsymbol{Z}[D_{2^n}]) = 0$ for all n, yet torsion-free cancellation fails when $n \geq 7$ (see [11, Theorem 8.1]). Of course, although values of n exist for which $\boldsymbol{Z}[D_{4n}]$ does not have torsion-free cancellation, it may still be the case that cancellation of finitely generated free modules holds within $\Omega_3(\boldsymbol{Z})$ for such n.

If torsion-free cancellation holds for D_{4n} then, by Theorem 1.3, Corollary 2.6 and Proposition 3.2, up to congruence, the only algebraic 2–complexes over D_{4n} are of the form

$$\mathcal{E}_m = (0 \to J \oplus \Lambda^m \to \Lambda^3 \oplus \Lambda^m \xrightarrow{\partial_2 \pi_1} \Lambda^2 \xrightarrow{\partial_1} \Lambda \to \boldsymbol{Z} \to 0),$$

where $\pi_1\colon \Lambda^3 \oplus \Lambda^m \to \Lambda^3$ denotes projection onto the first factor. If a pair of algebraic 2–complexes are congruent then they are homotopy equivalent (see Johnson [7, page 182]), and so the $\mathcal{E}_m$ represent all homotopy classes of algebraic 2–complexes over D_{4n}. However, $\mathcal{E}_m$ is geometrically realized by the Cayley complex arising from the presentation

$$\mathcal{G}_m = \langle x, y \mid x^{2n}, y^2, y^{-1}xyx, 1, \ldots, 1 \rangle,$$

where there are m trivial relators added to the standard presentation for D_{4n}. Therefore every homotopy class of algebraic 2–complex over D_{4n} is geometrically realized and hence by Theorem 1.2 we have proved Theorem 1.1. By Theorems 1.1 and 3.3 we have:

Corollary 3.4 *Let p be a prime and suppose that $D(\mathbf{Z}[D_{4p}]) = 0$. Then the $D(2)$–property holds for D_{4p}.*

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