Complete intersections and mod p cochains

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We give homotopy invariant definitions corresponding to three well known properties of complete intersections, for the ring, the module theory and the endomorphisms of the residue field, and we investigate them for the mod p cochains on a space, showing that suitable versions of the second and third are equivalent and that the first is stronger. We are particularly interested in classifying spaces of groups, and we give a number of examples. The case of rational homotopy theory is treated in [27], and there are some interesting contrasts.

1 Introduction

1A The context

In algebraic geometry, the best behaved varieties are subvarieties of an affine space that are specified by the right number of equations: if they are of codimension c then only c equations are required. Considering this locally, we may work in commutative algebra. A commutative local ring R is a *complete intersection (ci)* if its completion is the quotient of a regular local ring Q by a regular sequence, $f_1, f_2, \ldots, f_c$. We will suppose that R is complete, so that

$$R = Q/(f_1, \ldots, f_c).$$

The smallest possible value of c (as Q and the regular sequence vary) is called the *codimension* of R.

If R is a commutative Noetherian local ring with residue field k that is ci of codimension c, one may construct a resolution of any finitely generated module growing like a polynomial of degree $c - 1$. In particular the ring $\mathrm{Ext}_R^*(k, k)$ has polynomial growth (we say that R is *gci*). Perhaps the most striking result about ci rings is the theorem of Gulliksen [29], which states that this characterises ci rings so that the ci and gci conditions are equivalent for local rings.

In fact one may go further and show that the resolutions are constructed in an eventually multiperiodic fashion. In particular, for a ci local ring, $\mathrm{Ext}^*_R(k, k)$ is finite as a module over a commutative Noetherian ring; see Gulliksen [28]. This in turn opens the way to the theory of support varieties for modules over a ci ring; see Avramov and Buchweitz [7].

1B   The aspiration

We are interested in extending the notion of ci rings to commutative differential graded algebras (DGAs) and commutative ring spectra. Indeed we have a particular interest in studying the cochains $C^*(BG; k)$ on the classifying space of a finite group G, where k is a field of characteristic p, partly because of the consequences for the cohomology ring $H^*(BG; k)$. We would like to follow the model of Dwyer, Greenlees and Iyengar [15], who consider the Gorenstein condition. In fact, they show $C^*(BG; k)$ is Gorenstein in a homotopical sense for all finite groups G. This structural result then establishes the existence of a local cohomology theorem for $H^*(BG; k)$. An immediate corollary is the result of Benson and Carlson [9] that if $H^*(BG; k)$ is Cohen–Macaulay it is also Gorenstein, and the fact that in any case $H^*(BG; k)$ is Gorenstein in codimension 0 in the sense that its localization at any minimal prime is Gorenstein.

By contrast, we only expect the complete intersection condition on $C^*(BG)$ to hold for a small subclass of groups G, and we expect the structural implications for $H^*(BG)$ to be at a more subtle level. To explain this, we note that by the Eilenberg–Moore theorem, the counterpart of the Ext algebra $\mathrm{Ext}^*_R(k, k)$ for $R = C^*(BG)$ is the loop space homology of the p–completed classifying space, $H_*(\Omega(BG^\wedge_p))$. Of course if G is a p–group, this is simply the group ring kG in degree 0. More generally, it is known to be of polynomial growth in certain cases (for instance A_4 or M_{11} in characteristic 2) and R Levi [33; 34; 35; 36] has proved there is a dichotomy between small growth and large growth, and given examples where the growth is exponential. Evidently groups whose p–completed classifying spaces have loop space homology that has exponential growth cannot be spherically resolvable, so Levi's groups disproved a conjecture of F Cohen.

We give homotopy invariant versions of all three characterizations of the ci condition:

(sci):   the "regular ring modulo regular sequence" condition,

(mci):   the "modules have eventually multiperiodic resolutions" and

(gci):   polynomial growth of the Ext algebra.

Before we do so, we need to give a counterpart of the Noetherian condition, which in effect corresponds to Noether normalisation. In crude terms, ignoring the variety of

variants, we show that under the Noetherian condition the mci and gci conditions are equivalent, and both are implied by the strictly stronger sci condition. These definitions acquire their interest because of the variety of examples available, and the insights that our results give.

We note that the Avramov–Quillen characterization of ci rings in terms of André–Quillen homology does not work for cochains on a space in the mod p context since Mandell [38] has shown that the topological André–Quillen cohomology vanishes rather generally in this case.

1C   Relation to other papers in the series

In [10] two of us observed that the work of Eisenbud [19] and Avramov and Buchweitz [7] allowed us to formulate a homotopy invariant version of the mci condition (the zci condition). The work of Gulliksen [29] shows that it is equivalent to the sci and gci conditions for classical commutative local rings. The lectures of Avramov [5] give a very accessible account of the relevant material making it clear that the zci condition is equivalent to the sci and gci conditions for classical commutative local rings.

In [27] two of us together with K Hess formulated a homotopy invariant version of the sci condition and applied it in the rational homotopy theory of simply connected spaces, by taking $R = C^*(X; \mathbb{Q})$ for some strictly commutative model of the rational cochains. The Noetherian condition is then simply that $H^*(X; \mathbb{Q})$ is Noetherian. The sci condition implies the gci condition, and under a strong Noetherian condition sci is equivalent to gci. However, even when they are equivalent, the zci condition is strictly stronger. Accordingly, we formulated a weakened homotopy invariant condition (eci) which is another counterpart of the mci condition and coincides with the zci condition for ungraded commutative rings. We then showed that for strongly Noetherian rational DGAs the sci, eci and gci conditions are equivalent. For finite complexes, the three conditions are closely related to the classical notion of an elliptic space.

In the present paper we consider the same conditions as in the rational case, and study the relationship between them when applied to $R = C^*(X; k)$ for a field k of characteristic p, and a connected p–complete space X (with particular interest in the case $X = BG_p^\wedge$). At the superficial level there is a technical difficulty in that there is no commutative DGA model for $C^*(X; k)$ in general, but this difficulty is easily circumvented by the use of commutative ring spectra, at the cost of having to work in a more sophisticated technical setting. There is also the difficulty that in mod p homotopy theory one cannot expect to make complete algebraic calculations. Neither of these are essential differences. On the other hand, a really significant difference comes from the fact that in the rational case the centre of $H_*(\Omega X)$ can have a smaller

growth rate than $H_*(\Omega X)$ itself. By contrast, in the mod p context work of Félix–Halperin–Thomas [23] shows that this does not happen. We are therefore able to prove results which are in some ways a little stronger, and it seems possible that the eci and zci conditions are again equivalent.

1D Organization of the paper

In Section 2 we introduce our philosophy in studying the cochain algebras: the idea is to formulate conditions in commutative algebra so that they are homotopy invariant and then to say that a space X has property P if the commutative ring spectrum $C^*(X; k)$ has property P. This picks out interesting classes of spaces.

In Section 3 we record results about regular spaces, introducing some terminology that becomes essential in discussions of complete intersections. There are two further sections of prerequisites. In Section 4 we introduce two finiteness conditions that we need, analogous to the Noetherian condition on rings and finite generation for modules. In Section 5 we explain how bimodules give rise to endomorphisms of the module category and how they are related to natural constructions on the module category.

This equips us to define in Section 6 a number of variants of the ci condition for spaces: the sci condition is a structural condition like the basic definition of a ci local ring, and it is the strongest. The eci condition is analogous to having a multiperiodic module theory and the gci condition is a growth condition on the loop space homology.

In Sections 7 to 12 we explain how these definitions are related. Very roughly speaking Section 7 explains how sci implies eci, Section 8 shows that eci implies gci, and Section 9 shows that gci implies Gorenstein. Finally, the most surprising fact is that the finiteness condition gci does have strong structural implications for the module categories. Section 10 explains some basic properties of maps between Hochschild cohomology groups, in preparation for Sections 11 and 12 which show that gci implies eci (and also zci if the space is finite).

Finally, we finish with a section of examples coming from representation theory, and pose some questions.

1E Grading conventions

We will have cause to discuss homological and cohomological gradings. Our experience is that this is a frequent source of confusion, so we adopt the following conventions. First, we refer to lower gradings as *degrees* and upper gradings as *codegrees*. As usual, one may convert gradings to cogradings via the rule $M_n = M^{-n}$. Thus both chain complexes and cochain complexes have differentials of degree -1 (which is to say, of codegree $+1$). This much is standard. However, since we need to deal with both homology and cohomology it is essential to have separate notation for homological suspensions (Σ^i) and cohomological suspensions (Σ_i): these are defined by

$$(\Sigma^i M)_n = M_{n-i} \quad \text{and} \quad (\Sigma_i M)^n = M^{n-i}.$$

Thus, for example, with reduced chains and cochains of a based space X, we have

$$C_*(\Sigma^i X) = \Sigma^i C_*(X) \quad \text{and} \quad C^*(\Sigma^i X) = \Sigma_i C^*(X).$$

2 Commutative algebra for spaces

2A Philosophy

For a space X we want to work with a model of the cochains $R = C^*(X; k)$ which behaves like a commutative ring in the sense that its category of modules has a symmetric monoidal tensor product $\otimes_R$, and so that we can form the derived category of $R-$modules. We then use algebraic behaviour of this commutative ring to pick out interesting classes of spaces. In accordance with the principle that $C^*(X; k)$ is a sort of ring of functions on X, we simplify terminology and say that X has a property P if the commutative ring $C^*(X; k)$ has the property P.

2B Building

We will be working in various triangulated categories. We say that *A finitely builds B* (or that *B is finitely built by A*) if B may be formed from A by finitely many cofibre sequences and retracts (in other words, B is in the thick subcategory generated by A). We then write $A \models B$.

Similarly if arbitrary coproducts are also permitted we say that *A builds B* (in other words, B is in the localizing subcategory generated by A). In this case we write $A \vdash B$

When we work in a category of $R-$modules, we say M is *small* if it is finitely built by R ($R \models M$). We say that M is *virtually small* [16] if it finitely builds a nontrivial small object W, called the *witness*: $M \models W$, $R \models W \not\simeq 0$. We say that M is *proxy-small* [15] if it is virtually small and some witness W builds M.

2C Convenient models

If k is a field of characteristic zero, the simplicial de Rham complex gives a strictly commutative model for the cochains, and the commutative algebra of this DGA has been extensively investigated by rational homotopy theorists, and from the present point of view in [27]. On the other hand, it is well-known that in general Steenrod operations give an obstruction to a natural commutative DGA model. Fortunately, there is a commutative ring spectrum model of functions from X to the Eilenberg–MacLane spectrum Hk [20; 30], so we use the notation

$$C^*(X;k) := \mathrm{map}(X, Hk).$$

The category of module spectra over this ring spectrum has a model structure with weak equivalences given by homotopy isomorphisms, and we write $Ho(C^*(X;k))$ for the homotopy category: this is a tensor triangulated category that provides a suitable setting for investigating the commutative algebra for spaces.

This terminology is consistent in the sense that if we take X to be a point, we have an equivalence

$$Ho(Hk) \simeq D(k)$$

of tensor triangulated categories, between the topological and algebraic derived categories [47]. If k is replaced by a noncommutative ring there is a similar equivalence of triangulated categories.

We will not be using special properties of the models, so it is not necessary to give further details, but [26] gives an introduction and guide to the available sets of foundations.

2D Some analogies

At the most basic level, cofibre sequences

$$X \longrightarrow Y \longrightarrow Z$$

of pointed spaces induce (additive) exact sequences

$$C^*(X;k) \longleftarrow C^*(Y;k) \longleftarrow C^*(Z;k)$$

of reduced cochains. On the other hand, fibrations

$$F \longrightarrow E \longrightarrow B$$

of spaces induce (multiplicative) exact sequences

$$C^*(F;k) \overset{EM}{\simeq} C^*(E;k) \otimes_{C^*(B;k)} k \longleftarrow C^*(E;k) \longleftarrow C^*(B;k)$$

provided an Eilenberg–Moore theorem (EM) holds.

More generally, a homotopy pullback square

$$\begin{CD} Z \times_X Y @>>> Z \\ @VVV @VVV \\ Y @>>> X \end{CD}$$

induces a homotopy pushout square

$$\begin{CD} C^*(Z \times_X Y; k) @<<< C^*(Z; k) \\ @AAA @AAA \\ C^*(Y; k) @<<< C^*(X; k) \end{CD}$$

in the sense that

$$C^*(Z \times_X Y; k) \simeq C^*(Z; k) \otimes_{C^*(X;k)} C^*(Y; k)$$

if the conditions of the Eilenberg–Moore theorem are satisfied (for example if X is 1–connected or connected and p–complete with $\pi_1(X)$ a p–group and $p^N = 0$ on k for some N [13]).

In view of the importance of the Ext algebra, one particular case will be especially significant for us.

Proposition 2.1 *We have an equivalence*

$$C_*(\Omega X; k) \simeq \mathrm{Hom}_{C^*(X;k)}(k, k)$$

provided either (i) X *is simply connected or* (ii) X *is connected and p–complete,* $\pi_1(X)$ *is a p–group and $p^N = 0$ on k for some N.*

Because of the importance of this condition we will always assume that our space satisfies either condition (i) or (ii). In particular, note that if X is the p–completion of a space with a finite fundamental group (such as $B\Gamma$ for a compact Lie group) it satisfies the second condition with $k = \mathbb{F}_p$.

2E Conventions

Throughout we will be working over a field k of characteristic p, and we take $k = \mathbb{F}_p$ for definiteness. We repeat that the space X is assumed to be p–complete, connected and to have fundamental group a finite p–group.

We will adapt our language accordingly, so that a space X is k–finite if $H_*(X; k)$ is finite-dimensional, and a sequence of spaces $F \to E \to B$ is a k–fibration if its k–completion is a fibration in the usual sense.

We will often omit notation for the coefficients k, and say "finite" when "k–finite" is intended. This amounts to a global assumption of working in the category of k–complete spaces.

The cochains, homology and cohomology of a space will always be taken with coefficients in k, which will be omitted from the notation. Taking the cochains of a space X results in a commutative Hk–algebra $C^*(X)$ in the terminology of [20], which we will sometimes refer to simply as a k–algebra. We will use k also to denote Hk whenever there is no cause for confusion.

In this model, the homology is given by the homotopy groups according to the formula $\pi_*(C^*(X)) = H^*(X)$. We say that a $C^*(X)$–module M is *bounded above* if its homotopy groups $\pi_*(M)$ are bounded above.

3   Regular spaces

Regular rings and spaces are not the main subject of this paper, but it is essential to deal with regular rings first, since they provide the basis for subsequent study: not only do they provide the basis for an essential finiteness condition, but also complete intersections are defined as regular quotients of regular rings.

3A   Commutative algebra

In commutative algebra there are three styles for a definition of a regular local ring: one ideal theoretic, one in terms of the growth of the Ext algebra and a derived version. Although these are equivalent for classical commutative rings, we distinguish the definitions for comparison with other contexts.

Definition 3.1   (i)   A local Noetherian ring R is *s–regular* if the maximal ideal is generated by a regular sequence.

  (ii)   A local Noetherian ring R is *g–regular* if $\mathrm{Ext}_R^*(k, k)$ is finite-dimensional.

  (iii)   A local Noetherian ring R is *m–regular* if every finitely generated module is small in the derived category $D(R)$.

It is not hard to see that g–regularity is equivalent to m–regularity or that s–regularity implies g–regularity. Serre [44] proved that g–regularity implies s–regularity, so the three conditions are equivalent.

3B *g*–regularity for spaces

Of the commutative algebra definitions, the only one with a straightforward counterpart for $C^*(X)$ is g–regularity. In view of Proposition 2.1 it takes the following form.

Definition 3.2 A space X is *g–regular* if $H_*(\Omega X; k)$ is finite-dimensional.

Remark 3.3 Breaking our conventions for a moment, we might say that a not necessarily complete space X' is globally regular if $\Gamma' = \Omega X'$ is a finite complex, so that $X = B\Gamma$ is the classifying space of a finite loop space. For instance, if Γ is any compact Lie group we obtain such a space X.

Reverting to the standard situation that $k = \mathbb{F}_p$ and X is p–complete and connected, the g–regularity condition that $H_*(\Omega X; \mathbb{F}_p)$ is finite is precisely the condition $\Gamma = \Omega X$ is a p–compact group in the sense of Dwyer and Wilkerson [17] with classifying space $X = B\Gamma$. For example if Γ' is a compact Lie group with component group a p–nilpotent group, the p–completion of $B\Gamma'$ is an example, although there are many examples not of this form.

3C Some small objects

To discuss possible definitions of m–regularity we need to identify some small objects.

Lemma 3.4 *For a map $f\colon Y \to X$ between p–complete spaces with fundamental groups finite p–groups, $H_*(F(f))$ is finite-dimensional if and only if $C^*(Y)$ is small as a $C^*(X)$–module.*

Proof Suppose first that $H_*(F(f))$ is finite-dimensional. Note first that since $\pi_1(X)$ is a finite p–group, the only simple module over $H_0(\Omega X) = k\pi_1 X$ is the trivial module k. Accordingly, the hypothesis implies that k finitely builds $C_*(F(f))$ as a $C_*(\Omega X)$–module. Applying $\mathrm{Hom}_{C_*(\Omega X)}(\cdot, k)$, we deduce from the Rothenberg–Steenrod equivalence that $C^*(X)$ finitely builds $C^*(Y)$. In symbols,

$$k \models_{C_*(\Omega X)} C_*(F(f))$$

and hence

$$C^*(X) \simeq \mathrm{Hom}_{C_*(\Omega X)}(k, k) \models \mathrm{Hom}_{C_*(\Omega X)}(C_*(F(f)), k) \simeq C^*(Y).$$

The reverse implication is similar, but using the Eilenberg–Moore equivalence. □

3D What should finitely generated mean?

The m–regularity condition in commutative algebra states that finitely generated modules are small. It would be nice to have a counterpart of this for spaces, but it is not clear what should play the role of finitely generated modules.

Our solution to this problem is to treat smallness as a *definition* of "finitely generated" for modules over g–regular rings. We then use g–regular spaces to define the notion in general. We pause to show that over g–regular spaces, at least for modules M arising from maps of spaces, this notion of finitely generated has the familiar form that $H^*(M)$ is finitely generated over $H^*(X)$. This also holds in the rational context [27].

Lemma 3.5 (Dwyer) *If Y is a space with a map $f\colon Y \to X$ to a regular space X so that $H^*(Y)$ is finitely generated over $H^*(X)$ then in fact the fibre $F(f)$ is small.*

Proof First, by the fundamental work of Dwyer–Wilkerson on p–compact groups [17], $H^*(X)$ is Noetherian. Accordingly, since $H^*(Y)$ is a finitely generated algebra over $H^*(X)$ it is also Noetherian.

Finally, we argue that $H^*(F)$ is finitely generated over $H^*(Y)$. For this we consider the Serre spectral sequence of

$$\Omega X \longrightarrow F \longrightarrow Y.$$

The E_2 page has a finite number of rows, and each is a finitely generated module over $H^*(Y)$. It follows inductively that E_r has a finite number of rows, each finitely generated over $H^*(Y)$. Since the spectral sequence collapses at a finite stage, this also applies to $r = \infty$. Piecing the rows together, we see that $H^*(F)$ is finitely generated over $H^*(Y)$ as claimed.

Finally, we see that $H^*(F)$ is finite. Indeed $H^*(Y)$ is Noetherian and the action of $H^*(Y)$ on $H^*(F)$ factors through the finite quotient ring $k \otimes_{H^*X} H^*(Y)$. $\square$

4 A Noetherian condition

Before we turn to complete intersections there are two matters we need to discuss. The first, which we deal with here, is the notion of a normalisable space, which is a counterpart of the Noetherian condition. This lets us use the ideas of the previous section to give a notion of "finitely generated" module for all normalisable spaces. Thanks to some major theorems in the theory of p–compact groups we can show that the simplest possible definition has good properties.

4A Normalisable spaces

In commutative algebra, it is natural to impose the Noetherian finiteness assumption. One of the most useful consequences for k–algebras is Noether normalisation, stating that a Noetherian k–algebra is a finitely generated module over a polynomial subring.

Definition 4.1 A space X is *normalisable* if there is a connected g–regular space $B\Gamma$ and a map $\nu\colon X \to B\Gamma$ so that the homotopy fibre $F(\nu)$ is a k–finite complex. We say that ν is a *normalisation* and $F(\nu)$ is a *Noether fibre*. The normalisation is called *polynomial* if $B\Gamma$ is simply connected and has polynomial cohomology.

Remark 4.2 (i) By Lemma 3.4, the condition that $F(\nu)$ is k–finite is equivalent to asking that $C^*(X)$ is small over $C^*(B\Gamma)$.

  (ii) In the rational context, whenever $H^*(X)$ is Noetherian the space X is normalisable [27].

  (iii) If X is normalisable then $H^*(X)$ is Noetherian: since $H^*(B\Gamma)$ is Noetherian [17] the first remark implies $H^*(X)$ is a finitely generated $H^*(B\Gamma)$–module.

  (iv) Since our spaces X are assumed to be connected, the Noether fibre of a polynomial normalisation is connected.

Example 4.3 If G is a compact Lie group (for example a finite group), then BG is polynomially normalisable by choosing a faithful representation $G \to U(n)$, giving a fibration $U(n)/G \to BG \to BU(n)$.

It is clear that requiring the existence of a map $X \to B\Gamma$ with $C^*(X)$ small over $C^*(B\Gamma)$ is a finiteness condition, but we need to give examples to see how stringent it is. For example, it is natural to assume $H^*(X)$ is Noetherian, but this does not guarantee normalisability.

Example 4.4 (Castellana) A space X for which $H^*(X; \mathbb{Z}_p^\wedge)$ has unbounded p–torsion is not normalisable. Indeed, if $X \to B\Gamma$ is a normalisation, it is shown by Dwyer–Wilkerson [17] that $H^*(X; \mathbb{Z}_p^\wedge)$ is finitely generated over the Noetherian ring $H^*(B\Gamma; \mathbb{Z}_p^\wedge)$ and hence Noetherian.

Various classes of such spaces with $H^*(X; \mathbb{F}_p)$ Noetherian are known. A simple example is the 3–connected cover of S^3. Others come up naturally when considering homotopical generalizations of classifying spaces of groups. For instance, some of the classifying spaces of rank 2 Kac–Moody groups described by Kitchloo [31] have this property, as do the Aguadé–Broto–Notbohm [1, 5.5 and 5.6] spaces $X_k(r)$.

4B   Finitely generated modules

Next we need to define a notion of finitely generated modules, and for this we restrict to normalisable spaces X. We choose a normalisation $\nu\colon X \to B\Gamma$ and then take

$$\mathcal{FG}_\nu := \{C^*(Y) \mid Y \longrightarrow X \xrightarrow{\nu} B\Gamma \text{ makes } C^*(Y) \text{ small over } C^*(B\Gamma)\},$$

and

$$\mathcal{FG}(X) := \bigcup_{\substack{\text{normali-}\\\text{sations } \nu}} \mathcal{FG}_\nu.$$

In any case, it is convenient to be able to compare different normalisations. Thus if ν_1, ν_2 are two normalisations we can form

$$\{\nu_1, \nu_2\}\colon X \longrightarrow B\Gamma_1 \times B\Gamma_2.$$

We may compare $\{\nu_1, \nu_2\}$ to ν_1 in the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\ \{\nu_1,\nu_2\}\ } & B\Gamma_1 \times B\Gamma_2 \\
{\scriptstyle =}\big\downarrow & & \big\downarrow \\
X & \xrightarrow{\ \ \nu_1\ \ } & B\Gamma_1.
\end{array}
$$

Lemma 4.5   *Given two normalisations, ν_1, ν_2, the map $\{\nu_1, \nu_2\}$ is also a normalisation, and we have*

$$\mathcal{FG}_{\nu_1} \subseteq \mathcal{FG}_{\{\nu_1,\nu_2\}}.$$

Proof   For the first statement, we need only take iterated fibres to obtain a fibration

$$\Gamma_2 \longrightarrow F(\{\nu_1, \nu_2\}) \longrightarrow F(\nu_1)$$

from which the statement follows. For the second statement we repeat the proof with X replaced by Y and the normalisations replaced by their composites with $Y \to X$. $\square$

We next show that the class $\mathcal{FG}$ is independent of the chosen normalisation when the following conjecture holds.

Conjecture 4.6   (Linear representation)   The conjecture is that every p–compact group has a faithful linear representation. More precisely that if $B\Gamma$ is regular there is a map $B\Gamma \to B\mathrm{SU}(n)$, for some n, whose homotopy fibre is $\mathbb{F}_p$–finite. We understand that the conjecture has in fact been proved, but we state it as a conjecture so that the dependence of our work on it is made clear.

For the prime 2 the classification of 2–compact groups (see Andersen and Grodal [3] and Møller [39; 40]) shows that any 2–compact group is the product of classical groups and copies of DI(4). Ziemianski [48] has constructed a faithful linear representation of $B\mathrm{DI}(4)$.

For odd primes p this is a consequence of the classification of p–compact groups (see Andersen, Grodal, Møller, and Viruel [4, 1.6]) using verifications in some exotic cases by Castellana [11; 12].

Note the Linear Representation Conjecture (LRC) implies that if $\nu\colon X \to B\Gamma$ is a normalisation, then we may compose with a faithful linear representation to obtain another normalisation $X \to B\mathrm{SU}(n)$.

Thus for every normalisable space X we can always assume without loss of generality that the normalisation is $\nu\colon X \to B\mathrm{SU}(n)$. In particular we may assume that our normalisation is polynomial.

Lemma 4.7 *Assuming the LRC 4.6, if X is a normalisable space then $Y \in \mathcal{FG}(X)$ if and only if $H^*(Y)$ is finitely generated as a module over $H^*(X)$.*

Proof Suppose $Y \in \mathcal{FG}_\nu$, so that there are maps $Y \to X \xrightarrow{\nu} B\Gamma$ making $C^*(Y)$ into a small $C^*(B\Gamma)$–module. Since $H^*(B\Gamma)$ is Noetherian, this implies $H^*(Y)$ is finitely generated over $H^*(B\Gamma)$ and hence also over $H^*(X)$.

Conversely, suppose given a map $Y \to X$ such that $H^*(Y)$ is finitely generated as an $H^*(X)$–module. Since X is normalisable, the LRC 4.6 shows we can choose a normalisation $\nu\colon X \to B\mathrm{SU}(n)$ for some n. Now $H^*(Y)$ is a finitely generated module over $H^*(B\mathrm{SU}(n))$, and the Eilenberg–Moore spectral sequence shows that the homotopy fibre of $Y \to B\mathrm{SU}(n)$ is k–finite. Hence $Y \in \mathcal{FG}_\nu$. □

Even more is true.

Lemma 4.8 *Assuming the LRC 4.6, if X has a polynomial normalisation $\nu\colon X \to B\Gamma$ then $\mathcal{FG}_\nu = \mathcal{FG}$.*

The proof is left as an exercise to the reader.

In the following sections we will give several definitions where we require a polynomial normalisation, so that we can rely on this lemma.

5 Bimodules and natural endomorphisms of R–modules

The second topic we need to treat before coming to complete intersections is that of bimodules.

5A The centre of the derived category of R–modules

If R is a commutative Noetherian ring and M is a finitely generated R–module with an eventually n–periodic resolution, comparing the resolution with its n–fold shift gives a map $M \to \Sigma^n M$ in the derived category whose mapping cone is a small R–module. If R is a hypersurface ring, any finitely generated module has an eventually 2–periodic resolution. In fact the construction of these resolutions can be made in a very uniform way. The lesson learnt from commutative algebra is that if we want to use this property to characterize hypersurfaces we need to use this uniformity. In fact the uniform construction can be formulated as a natural transformation $1 \to \Sigma^n 1$ of the identity functor with a mapping cone which is small on finitely generated modules, and it turns out that the existence of such a transformation does characterize hypersurface rings.

By definition the centre $ZD(R)$ is the graded ring of all such natural transformations of the identity functor. There are various ways of constructing elements of the centre, and various natural ways to restrict the elements we consider. Some of these work better than others, and it is the purpose of this section is to introduce these ideas.

5B Bimodules

We consider a map $Q \to R$, where Q is regular and R is small over Q. We may then consider $R^e = R \otimes_Q R$, and R^e–modules are $(R \mid Q)$–bimodules. The Hochschild cohomology ring is defined by

$$HH^*(R \mid Q) = \text{Ext}^*_{R^e}(R, R).$$

If $f \colon X \to Y$ is a map of $(R \mid Q)$–bimodules, for any R–module M we obtain a map $f \otimes 1 \colon X \otimes_R M \to Y \otimes_R M$ of (left) R–modules.

The simplest way for us to use this is that if we have isomorphisms $X \cong R$ and $Y \cong \Sigma^n R$ as R–bimodules, the map $f \otimes 1 \colon M \to \Sigma^n M$ is natural in M and therefore gives an element of codegree n in $ZD(R)$: we obtain a map of rings

$$HH^n(R) = \text{Hom}_{R^e}(R, \Sigma^n R) \longrightarrow ZD(R)^n.$$

Continuing, if $X \models_{R^e} Y$ then $X \otimes_R M \models_R Y \otimes_R M$. In particular, if $X = R$ builds a small R^e–module Y then

$$M = R \otimes_R M \models_R Y \otimes_R M \dashv R^e \otimes_R M = R \otimes_Q M.$$

Thus if M is finitely generated (ie small over Q), this shows M finitely builds a small R–module.

We could then restrict the maps permitted in showing that $X \models_{R^e} Y$. We could restrict ourselves to using maps of positive codegree coming from Hochschild cohomology, and say $X \models_{hh} Y$, more generally we could permit any maps of positive codegree from the centre $ZD(R^e)$ and say $X \models_z Y$, or we could relax further and require only that all the maps involved in building are endomorphisms of nonzero degree for some object and say $X \models_e Y$.

We shall have occasion to use Koszul constructions of modules. Given $z_1, \ldots, z_c \in ZD(R)$ and an R–module M we define M/z_1 to be the homotopy cofibre of $z_1 \colon \Sigma^n M \to M$ and $M/z_1/\cdots/z_c$ is defined inductively. Given elements $x_1, \ldots, x_n \in HH^*(R \mid Q)$, we have already observed that these define elements in $ZD(R)$ (because R is a unit for $\otimes_R$), and we note here that they also define elements in $ZD(R^e)$ for the same reason. Accordingly, for a bimodule M we may construct $M/x_1/\cdots/x_c$ as a bimodule.

Example 5.1 In the topological context we have a map $X \to B\Gamma$ and take $R = C^*(X)$, $Q = C^*(B\Gamma)$ and $R^e = C^*(X \times_{B\Gamma} X)$. The associated Hochschild cohomology can be abbreviated

$$HH^*(X \mid B\Gamma) = \pi_*(\mathrm{Hom}_{C^*(X \times_{B\Gamma} X)}(C^*(X), C^*(X))).$$

6 Complete intersection spaces

6A The definition in commutative algebra

In commutative algebra there are three styles for a definition of a complete intersection ring: structural, in terms of the growth of the Ext algebra and module theoretic. See Avramov [5] and Benson and Greenlees [10] for a more complete discussion.

Definition 6.1 (i) A local Noetherian ring R is an *s–complete intersection (sci)* ring if $R = Q/(f_1, f_2, \ldots, f_c)$ for some regular ring Q and some regular sequence $f_1, f_2, \ldots, f_c$. The minimum such c (over all Q and regular sequences) is called the *codimension* of R.

(ii)   A local Noetherian ring R is *gci* if $\operatorname{Ext}_R^*(k,k)$ has polynomial growth. The *g–codimension* of R is one more than the degree of the growth.

(iii)   A local Noetherian ring R is *zci* [10] if there are elements $z_1, z_2, \ldots z_c \in ZD(R)$ of nonzero degree so that $M/z_1/z_2/\cdots/z_c$ is small for all finitely generated modules M. The minimum such c is called the *z–codimension* of R. Similarly R is *hhci* if the elements z_i can be chosen to come from Hochschild cohomology.

Remark 6.2   The following condition has proved less useful. If R is a commutative ring or CDGA, it is said to be a *quasicomplete intersection (qci)* [16] if every finitely generated object is virtually small.

Remark 6.3   If Q is a regular local ring with a map $Q \to R$ making R into a small Q–module we may consider a number of bimodule conditions.

We say R is *bci, eci, zbci* or *hhbci* if R finitely builds, e–builds, z–builds or hh–builds a nontrivial small R^e–module. Evidently these are increasingly strong conditions. Similarly, it is clear that zbci implies zci and hhbci implies hhci.

Theorem 6.4   [10]   *For a local Noetherian ring the conditions sci, gci, hhbci, hhci, zbci, zci and eci are all equivalent, and the corresponding codimensions are equal. These conditions imply the bci and qci conditions.*

It is a result of Shamash [45] that if R is ci of codimension c, one may construct a resolution of any finitely generated module growing like a polynomial of degree $c - 1$. Furthermore, the resolution is constructed by Eisenbud [19] as a module over a polynomial ring on c generators of degree -2, which shows directly that R is hhbci. (the form of the resolution due to Avramov–Buchweitz [7] and described by Avramov [5, Section 9] makes this very clear). Considering the module k shows that the ring $\operatorname{Ext}_R^*(k,k)$ has polynomial growth. Perhaps the most striking result about ci rings is the theorem of Gulliksen [29], which states that gci implies sci for commutative local rings so that the ci and gci conditions are equivalent.

Remark 6.5   In commutative algebra, Avramov [6] proved Quillen's conjectured characterization of complete intersections by the fact that the André–Quillen cohomology is bounded. The natural counterpart of this is false in homotopy theory since, by results of Dwyer and Mandell [38], the topological André–Quillen cohomology vanishes much too generally in characteristic p.

When k is of characteristic 0, the André–Quillen cohomology of $C^*(X)$ gives the dual homotopy groups of X, so Avramov's characterization corresponds to the gci condition.

6B   Definitions for spaces

In view of the fact that regular elements correspond to spherical fibrations, adapting the above definitions for spaces is straightforward.

Definition 6.6 (i)   A space X is *spherically ci (sci)* if it is formed from a connected g–regular space $B\Gamma$ using a finite number of spherical fibrations. More precisely, we require that there is a g–regular space $X_0 = B\Gamma$ and fibrations

$$S^{n_1} \longrightarrow X_1 \longrightarrow X_0 = B\Gamma, \quad S^{n_2} \longrightarrow X_2 \longrightarrow X_1, \quad \ldots, \quad S^{n_c} \longrightarrow X_c \longrightarrow X_{c-1}$$

with $X = X_c$. The least such c is called the *s–codimension* of X.

(i)$'$ A space X is *weakly spherically ci (wsci)* if there is a g–regular space $B\Gamma$ and a fibration

$$F \longrightarrow X \longrightarrow B\Gamma$$

and $F = F_1$ is *spherically resolvable* in the sense that there are fibrations

$$F_2 \longrightarrow F_1 \longrightarrow S^{n_1}, \quad F_3 \longrightarrow F_2 \longrightarrow S^{n_2}, \quad \ldots, \quad * \longrightarrow F_c \longrightarrow S^{n_c}.$$

The least such c is called the *ws–codimension* of X.

(ii)   A space X is a *gci* space if $H^*(X)$ is Noetherian and $H_*(\Omega X)$ has polynomial growth. The *g–codimension* of X is one more than the degree of growth.

(iii)   A space X is a *zci* space if X is normalisable and there are elements $z_1, z_2, \ldots, z_c$ in $ZD(C^*(X))$ of nonzero degree so that $C^*(Y)/z_1/z_2/\cdots/z_c$ is small for all $C^*(Y) \in \mathcal{FG}(X)$.

(iii)$'$ A space X is a *hhci* space if there is a polynomial normalisation $\nu \colon X \to B\Gamma$ and $R = C^*(X)$ finitely hh–builds a small R^e–module, where $R^e = C^*(X \times_{B\Gamma} X)$. In particular, assuming the LRC 4.6, there are elements $z_1, z_2, \ldots, z_c \in HH^*(X \mid B\Gamma)$ of positive codegree so that $C^*(Y)/z_1/z_2/\cdots/z_c$ is small for all $C^*(Y) \in \mathcal{FG}(X)$.

(iii)$''$ A space X is a *eci* space if X has a polynomial normalisation as above and R finitely e–builds a small R^e–module. In particular, there are homotopy cofibration sequences of R^e–modules

$$M_0 \longrightarrow \Sigma^{n_0} M_0 \longrightarrow M_1,$$
$$M_1 \longrightarrow \Sigma^{n_1} M_1 \longrightarrow M_2, \quad \ldots,$$
$$M_{c-1} \longrightarrow \Sigma^{n_{c-1}} M_{c-1} \longrightarrow M_c$$

with $n_i \neq 0$ and M_c small over R^e.

Remark 6.7 Two further variants have proved to be less useful.

We say X is *bci* space if X is normalisable and $C^*(X)$ is virtually small as a $C^*(X \times_{B\Gamma} X)$–module for some g–regular space $B\Gamma$ and map $X \to B\Gamma$ with $C^*(X)$ small over $C^*(B\Gamma)$.

We say X is *qci* space if X is normalisable and each $C^*(Y) \in \mathcal{FG}(X)$ is virtually small.

There are also further variants, ωsci and wωsci where we are permitted to use loop spaces on spheres rather than spheres. These conditions arose in Levi's work. They are evidently weakenings of sci and wsci, but they still imply gci.

It is easiest to verify the gci condition. The simplest example gives a good supply.

Example 6.8 If $H^*(X)$ is a complete intersection then $\mathrm{Ext}^{*,*}_{H^*(X)}(k,k)$ has polynomial growth, so that if the Eilenberg–Moore spectral sequence converges, X is gci.

For normalisable spaces X, working over $k = \mathbb{F}_p$ we will establish the implications

$$\text{sci} \Rightarrow \text{wsci} \Rightarrow \text{eci} \Longleftrightarrow \text{gci}.$$

We will also show that for k–finite spaces hhci $\Longleftrightarrow$ gci. It is shown in [27] that in the rational case sci $\Rightarrow$ eci $\Rightarrow$ gci, and that under an additional finiteness hypothesis the three conditions sci, eci and gci are equivalent.

The implication, that sci implies wsci is straightforward. Indeed, if X is sci, the fibre of the composite

$$X = X_c \longrightarrow X_{c-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow B\Gamma$$

is clearly an iterated spherical fibration.

In Section 8 we explain that a rather straightforward calculation with Hilbert series shows that eci implies gci. In Section 7 we give a direct construction to show that an s–hypersurface is a z–hypersurface. This argument can be iterated if all the spherical fibrations have odd-dimensional spheres, but we give a less direct general argument.

The main result is that (assuming normalisability) the gci finiteness condition implies the structural condition eci: this occupies Sections 9 to 12.

Problem 6.9 Give an example of a gci space that is not wsci or show that no such space exists.

6C   Hypersurface rings

A hypersurface is a complete intersection of codimension 1. The first three definitions adapt to define s–hypersurfaces, g–hypersurfaces and z–hypersurfaces. The notion of g–hypersurface (ie the dimension of the groups $\mathrm{Ext}^i_R(k, k)$ is bounded) may be strengthened to the notion of p–hypersurface where we require that they are eventually periodic, given by multiplication with an element of the ring. All four of these conditions are equivalent by results of Avramov.

One possible formulation of b–hypersurface would be to require that R builds a small R^e–module in one step (or equivalently, that R is a z–hypersurface but z arises from $HH^*(R \mid Q)$). This is also equivalent to the above definitions.

Finally, we may say that R is a q–hypersurface if every finitely generated module M has a self map with nontrivial small mapping cone.

6D   Hypersurface spaces

All six of these conditions have obvious formulations for spaces. A space X is an s–*hypersurface* if there is a fibration

$$S^n \longrightarrow X \longrightarrow B\Gamma,$$

where $X \to B\Gamma$ is a polynomial normalisation. It is a g–*hypersurface* if the dimensions of $H_i(\Omega X)$ are bounded, and a p–*hypersurface* if they are eventually periodic given by multiplication by an element of the ring.

The space X is an hh–*hypersurface* if $C^*(X)$ builds a small $C^*(X \times_{B\Gamma} X)$–module in one step using an hh–map. An e–*hypersurface* is the same as a hh–hypersurface. It is a z–*hypersurface* if there is an element z of $ZD(C^*(X))$ of nonzero degree so that $C^*(Y)/z$ is small for every module $C^*(Y)$ in $\mathcal{FG}(X)$. Finally, X is a q–*hypersurface* if every module $C^*(Y)$ in $\mathcal{FG}(X)$ has a self map with nontrivial small mapping cone.

The point about this condition is that there are a number of interesting examples (see Section 13 below). We will show in Section 7 that an s–hypersurface is a z–hypersurface.

7 s–hypersurface spaces and z–hypersurface spaces

In the algebraic setting the remarkable fact is that modules over hypersurfaces have eventually periodic resolutions, and hence that they are hhci of codimension 1. The purpose of this section is to prove a similar result for spaces. The result will be proved

in a form that provides the inductive step for the general result on ci spaces, but the iteration will only show it is an eci space (rather than zci or hhci).

Theorem 7.1 *If X is an s–hypersurface space with fibre sphere of dimension ≥ 2 then X is a z–hypersurface space.*

Remark 7.2 The proof will show a similar result holds for the total space of a circle fibration over a connected regular space, but the definition of a z–hypersurface would need to be adapted along the Jacobson radical lines of [10] to give a uniform statement.

7A   Split spherical fibrations

The key in algebra was to consider bimodules, for which we consider the (multiplicative) exact sequence

$$R \longrightarrow R^e \longrightarrow R^e \otimes_R k,$$

where the first map is a monomorphism split by the map μ along which R acquires its structure as an R^e–module structure. This corresponds to the pullback fibration

$$X \longleftarrow X \times_{B\Gamma} X \longleftarrow S^n,$$

split by the diagonal

$$\Delta\colon X \longrightarrow X \times_{B\Gamma} X$$

along which the cochains on X becomes a bimodule. To simplify notation, we consider a more general situation: a fibration

$$B \longleftarrow E \longleftarrow S^n$$

with section $s\colon B \longrightarrow E$. The case of immediate interest is $B = X$, $E = X \times_{B\Gamma} X$, where a $C^*(E)$–module is a $C^*(X)$–bimodule.

Since s is a section of p there is a fibration

$$\Omega S^n \longrightarrow B \xrightarrow{s} E.$$

This gives the required input for the following theorem. The strength of the result is that the cofibre sequences are of $C^*(E)$–modules.

Theorem 7.3 *Suppose given a fibration $\Omega S^n \to B \xrightarrow{s} E$ with $n \geq 2$.*

(i) *If n is odd, then there is a cofibre sequence of $C^*(E)$–modules*

$$\Sigma_{n-1} C^*(B) \longleftarrow C^*(B) \longleftarrow C^*(E).$$

(ii) *If n is even, then there are cofibre sequences of $C^*(E)$–modules*

$$C \longleftarrow C^*(B) \longleftarrow C^*(E)$$

and

$$\Sigma_{2n-2}C^*(B) \longleftarrow C \longleftarrow \Sigma_{n-1}C^*(E).$$

In particular the fibre of the composite

$$C^*(B) \longrightarrow C \longrightarrow \Sigma_{2n-2}C^*(B)$$

is a small $C^*(E)$–module constructed with one cell in codegree 0 and one in codegree $n-1$.

Remark 7.4 The case $n = 1$ is slightly different since $\Omega S^1 \simeq \mathbb{Z}$. In this case, the map $B \to E$ is an infinite cyclic covering up to homotopy. Taking appropriate models, B is a free $\mathbb{Z}$–space with $E = B/\mathbb{Z}$ and there is a cofibre sequence

$$C^*(E) \longrightarrow C^*(B) \xrightarrow{1-z} C^*(B),$$

where z is a generator of $\mathbb{Z}$.

7B Strategy

We will first prove the counterparts in cohomology by looking at the Serre spectral sequence of the fibration from part (i) and then lift the conclusion to the level of cochains.

Note that in either case we obtain a cofibre sequence

$$K \longleftarrow C^*(B) \longleftarrow \Sigma_a C^*(B)$$

of $C^*(E)$–modules with K small.

In the motivating example we see that if X is an s–hypersurface, it is an e–hypersurface with $n \geq 2$ and so it is a z–hypersurface as required. If $n = 1$, the condition on nontriviality for a z–hypersurface needs to be adapted along the lines of [10]. Thus Theorem 7.1 follows from Theorem 7.3.

To describe the strategy in more detail, we consider the Serre spectral sequence of the fibration in part (i).

If n is odd, $H^*(\Omega S^n)$ is a free divided power algebra on one generator Φ of codegree $n-1$. If n is even, $H^*(\Omega S^n)$ is the tensor product of an exterior algebra on one generator Φ of codegree $n-1$ and a free divided power algebra on one generator Ψ of codegree $2n-2$.

In either case, $H^*(\Omega S^n)$ corresponds precisely to the natural algebraic resolution of $R = C^*(B)$ over $C^*(E)$. Indeed, the Serre spectral sequence of this fibration gives precisely such a resolution, with the generators of $H^*(\Omega S^n)$ giving an $H^*(E)$–basis for the E_2 term. One might imagine realizing the associated filtration by codegree in $H^*(\Omega S^n)$. Using the equivalence

$$E \simeq E\Omega B \times_{\Omega B} \Omega S^n$$

the filtration would correspond to the skeletal filtration of ΩS^n provided this was a filtration of ΩB–spaces. We could then take

$$E^k = E\Omega B \times_{\Omega B} (\Omega S^n)^{((n-1)k)}$$

to realize the filtration, but the terms would have to be ΩB–spaces so it seems unlikely that this can be realized. In any case we can make do with a little less.

To start with we consider the map

$$C^*(B) \xleftarrow{s^*} C^*(E)$$

induced by the section, using it to give $C^*(B)$ the structure of a $C^*(E)$–module. Thus s^* is a map of $C^*(E)$–modules, and we can take its mapping cone C in the category of $C^*(E)$–modules.

By looking at the Serre spectral sequence of the original spherical fibration, we see that $\pi_*(C) \cong \Sigma_{n-1} H^*(B)$ as $H^*(B)$–modules, and hence C is equivalent to $\Sigma_{n-1} C^*(B) \simeq C^*(\Sigma^{n-1} B)$ as a $C^*(B)$–module. However, we need to consider $\pi_*(C)$ not just as an $H^*(B)$–module, but as an $H^*(E)$–module.

If n is odd, we will show $\pi_*(C)$ is $\Sigma_{n-1} H^*(B)$. If n is even this need not be true, but we may repeat the construction, once more to obtain a $C^*(E)$–module C_1 so that $\pi_*(C_1) \cong \Sigma_{2n-2} H^*(B)$ as an $H^*(E)$–module.

In either case we have a $C^*(E)$–module M so that $\pi_*(M) \cong H^*(B)$ as $H^*(E)$–modules. We will show in Section 7D that the algebraic simplicity of the $H^*(E)$–module $H^*(B)$ is such that we can lift the equivalence to the level of cochains.

If n is odd, we therefore have a triangle

$$\Sigma_{n-1} C^*(B) \longleftarrow C^*(B) \longleftarrow C^*(E)$$

of $C^*(E)$–modules. If n is even we have cofibre sequence

$$C \longleftarrow C^*(B) \longleftarrow C^*(E)$$

and

$$\Sigma_{2n-2} C^*(B) \longleftarrow C \longleftarrow \Sigma_{n-1} C^*(E)$$

of $C^*(E)$–modules, and by the octahedral axiom, the fibre of the composite

$$C^*(B) \longrightarrow C \longrightarrow \Sigma_{2n-2} C^*(B)$$

is a $C^*(E)$–complex with one cell in codegree 0 and one cell in codegree $n-1$. In particular, it is small.

7C The situation in homology

We start with the case that n is odd, since it is a little simpler.

Proposition 7.5 *If n is odd, there is a short exact sequence:*

$$0 \longrightarrow \Sigma_n H^*(B) \longrightarrow H^*(E) \longrightarrow H^*(B) \longrightarrow 0$$

Proof Since $p\colon E \to B$ is split, $H^*(E)$ is a free module over $H^*(B)$ on two generators. We have a split surjection $s^*\colon H^*(E) \to H^*(B)$, so the kernel is a cyclic $H^*(B)$–module, and therefore a principal ideal in $H^*(E)$, generated by an element τ of codegree n. We therefore have a short exact sequence

$$0 \longrightarrow (\tau) \longrightarrow H^*(E) \longrightarrow H^*(B) \longrightarrow 0,$$

and $(\tau) \cong H^*(E)/\operatorname{ann}(\tau)$. Since n is odd, $\tau^2 = 0$, and hence $(\tau) \subseteq \operatorname{ann}(\tau)$. Since $H^*(E)$ is a free module on 1 and τ it follows that we have equality: $(\tau) = \operatorname{ann}(\tau)$, giving the isomorphism

$$(\tau) \cong H^*(E)/\operatorname{ann}(\tau) \cong H^*(E)/(\tau) \cong H^*(B)$$

of $H^*(E)$–modules as required. $\square$

If n is even, it is easy to find examples where the conclusion fails.

Example 7.6 Take $B = S^4$, $E = S^4 \times S^4$, with p being the projection onto the first factor and s the diagonal. Then $H^*(E) = \Lambda(x_1, x_2)$, $H^*(B) = H^*(E)/(x_1 - x_2)$ and

$$\ker(s^*) = \Sigma_n H^*(E)/(x_1 + x_2)$$

so we do not have the conclusion of Proposition 7.5 unless the characteristic is 2.

Proposition 7.7 *If n is even, there are short exact sequences*

$$0 \longrightarrow \Sigma_n H^*(B)' \longrightarrow H^*(E) \longrightarrow H^*(B) \longrightarrow 0$$

and

$$0 \longrightarrow \Sigma_n H^*(B) \longrightarrow H^*(E) \longrightarrow H^*(B)' \longrightarrow 0,$$

where $H^*(B)'$ is an $H^*(E)$–module isomorphic to $H^*(B)$ as an $H^*(B)$–module.

Proof The proof begins as for Proposition 7.5, but we need no longer have $\tau^2 = 0$. To control the situation we use the fact that the map $s^*\colon H^*E \to H^*B$ is the edge homomorphism of the Serre spectral sequence of the fibration

$$\Omega S^n \longrightarrow B \xrightarrow{s} E,$$

and that $H^*(\Omega S^n)$ is the tensor product of an exterior algebra on the generator Φ of codegree $n-1$ and the divided power algebra on Ψ of codegree $2n-2$. We have $d_n(\Phi) = \tau$ and we define τ_1 by $d_n(\Psi) = \tau_1 \Phi$. It will be convenient to write

$$d_n^i \colon E_n^{*,i(n-1)} \longrightarrow E_n^{*,(i-1)(n-1)}$$

for the $H^*(E)$–module map giving the part of the differential from the i–th nonzero row to the $(i-1)$–st.

By comparison with the path-loop fibration for S^n we see that τ restricts to a generator of $H^n(S^n)$, and therefore, from the Serre spectral sequence of $S^n \to E \to B$, that 1 and τ give an $H^*(B)$–basis for $H^*(E)$. This means that $E_{n+1}^{*,0} = H^*(E)/(\tau) \cong H^*(B)$, and the edge homomorphism shows that we have an exact sequence

$$0 \longrightarrow (\tau) \longrightarrow H^*(E) \xrightarrow{s^*} H^*(B) \longrightarrow 0.$$

Since $H^*(E)$ is free as an $H^*(B)$–module on two generators, we conclude (τ) is free as an $H^*(B)$–module on one generator of degree n, and we obtain the first exact sequence.

In particular (τ) has a single copy of k in the bottom codegree. Next, we have the exact sequence

$$0 \longrightarrow \mathrm{ann}(\tau) \longrightarrow H^*(E) \longrightarrow (\tau) \longrightarrow 0,$$

and, since $H^*(E)$ is $H^*(B)$–free on two generators, $\mathrm{ann}(\tau)$ is free as an $H^*(B)$–module on one generator of degree n, and in particular it has a single copy of k in the bottom codegree. Of course $\ker(d_n^1) = \mathrm{ann}(\tau)$, whilst on the other hand, by comparison

with the path-loop fibration, $d_n^2(\Psi)$ generates a copy of k in degree n, and since it lies in $\ker(d_n^1)$ we see τ_1 generates $\ker(d_n^1)$, and

$$(\tau_1) = \mathrm{ann}(\tau).$$

Finally, we have the exact sequence

$$0 \longrightarrow \mathrm{ann}(\tau_1) \longrightarrow H^*(E) \longrightarrow (\tau_1) \longrightarrow 0.$$

Since $H^*(E)$ is $H^*(B)$–free on two generators, $\mathrm{ann}(\tau_1)$ is free as an $H^*(B)$–module on one generator of degree n, and in particular it has a single copy of k in the bottom codegree. Of course $\ker(d_n^2) = \mathrm{ann}(\tau_1)$, whilst on the other hand $d_n(\Phi\Psi) = \tau\Psi + \Phi^2\tau_1 = \tau\Psi$ lies in this kernel and generates a copy of k in degree n. Since $d_n(\Phi\Psi)$ lies in $\ker(d_n^2)$, we see τ generates $\ker(d_n^2)$ so that

$$(\tau) = \mathrm{ann}(\tau_1).$$

This identifies (τ_1) with $H^*(B) = H^*(E)/(\tau)$ as an $H^*(E)$–module, giving the second exact sequence as required. $\qquad\qquad\square$

7D   Lifting to cochains

Whether n is even or odd, we have constructed a $C^*(E)$–module M for which $\pi_*(M) = H^*(B)$ as $H^*(E)$–modules. We now show that we may lift this conclusion to the cochain level.

Proposition 7.8 *There is a unique $C^*(E)$–module M with $\pi_*(M) \cong H^*(B)$ as $H^*(E)$–modules.*

Proof   We will first give the proof assuming there is a short exact sequence

$$0 \longrightarrow \Sigma_n H^*(B) \longrightarrow H^*(E) \longrightarrow H^*(B) \longrightarrow 0$$

of $H^*(E)$–modules.

The first step is to show that M is equivalent to a module constructed by adding free $C^*(E)$–cells in codegrees $0, n-1, 2n-2, 3n-3, \ldots$ as one might hope. We take $M_0 = M$, and then construct a diagram

$$M = M_0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow \cdots,$$

where

$$\pi_*(M_i) = \Sigma_{(n-1)i} H^*(B).$$

To construct M_{i+1} from M_i we construct a cofibre sequence

$$\Sigma_{ni-i}C^*(E) \xrightarrow{a_i} M_i \longrightarrow M_{i+1},$$

where a_i is a chosen generator of $\pi_*(M_i)$. It follows from the hypothesis that $\pi_*(M_{i+1}) = \Sigma_{(n-1)(i+1)}H^*(B)$ as required to proceed. Since each map $M_i \to M_{i+1}$ is zero in homotopy, $M_\infty = \underrightarrow{\text{holim}}_i M_i$ has zero homotopy and is thus contractible.

Now construct the dual tower by the cofibre sequences:

$$
\begin{array}{ccccc}
\downarrow & & \downarrow & & \downarrow \\
M^i & \longrightarrow & M & \longrightarrow & M_i \\
\downarrow & & \downarrow & & \downarrow \\
M^{i+1} & \longrightarrow & M & \longrightarrow & M_{i+1} \\
\downarrow & & \downarrow & & \downarrow
\end{array}
$$

Passing to direct limits we see that

$$M^\infty := \underrightarrow{\text{holim}}_s M^s \xrightarrow{\simeq} M$$

is an equivalence. On the other hand, we have $M^0 \simeq 0$ and by the octahedral axiom we have cofibre sequences

$$M^i \longrightarrow M^{i+1} \longrightarrow \Sigma_{(n-1)i}C^*(E),$$

so the module M^∞ is a version of M built with periodic cells $C^*(E)$ as required.

To see that any two such modules $M(0)$ and $M(1)$ are equivalent, we perform the above construction to replace the two modules $M(a)$ ($a = 0, 1$) with filtered modules $\{M(a)^i\}_i$ with $M(a)^{i+1}/M(a)^i = \Sigma_{(n-1)i}C^*(E)$. We suppose given an isomorphism $f_*: \pi_*(M(0)) \to \pi_*(M(1))$, and we show it can be realized with a map $f: M(0) \to M(1)$. Indeed, we recursively construct maps

$$f_i: M(0)^i \longrightarrow M(1)$$

agreeing with the map $M(0)^i \to M(0)$ in homotopy, where we identify $\pi_*(M(0))$ and $\pi_*(M(1))$ using f_*. Since $M(0)^0 = 0$, there is nothing to prove for $i = 0$. After

that we are faced with the extension problem:

$$\Sigma_{(n-1)i-1}C^*(E)$$

$$\downarrow b_i$$

$$M(0)^i \xrightarrow{\ f_i\ } M(1)$$

$$\downarrow \qquad \nearrow f_{i+1}$$

$$M(0)^{i+1}$$

Since the map

$$\Sigma_{(n-1)i-1}C^*(E) \xrightarrow{\ b_i\ } M(0)^i \longrightarrow M(0)$$

is zero in homotopy by construction, the same is true for the map

$$\Sigma_{(n-1)i-1}C^*(E) \xrightarrow{\ b_i\ } M(0)^i \xrightarrow{\ f_i\ } M(1).$$

Since the domain is free, this shows $f_i b_i \simeq 0$, and we can solve the extension problem. Passing to limits, the Milnor exact sequence gives a map $M(0) \to M(1)$ inducing a homotopy isomorphism. This completes the proof if n is odd.

If n is even, the proof is precisely similar, except that the odd image modules are suspensions of $H^*(B)'$ rather than of the standard $H^*(E)$–module $H^*(B)$. □

8 Growth conditions

In this section we prove perhaps the simplest implication between the ci conditions: for spaces of finite type, eci implies gci.

8A Polynomial growth

Throughout algebra and topology it is common to use the rate of growth of homology groups as a measurement of complexity. We will be working over $H^*(X)$, so it is natural to assume that our modules M are *locally finite* in the sense that $H^*(M)$ is cohomologically bounded below and $\dim_k(H^i(M))$ is finite for all i.

Definition 8.1 We say that a locally finite module M has polynomial growth of degree $\leq d$ and write growth$(M) \leq d$ if there is a polynomial $p(x)$ of degree d with

$$\dim_k(H^n(M)) \leq p(n)$$

for all $n \gg 0$.

Remark 8.2 Note that a complex with bounded homology has growth ≤ -1. For complexes with growth $\leq d$ with $d \geq 0$, by adding a constant to the polynomial, we may insist that the bound applies for all $n \geq 0$.

8B Mapping cones reduce degree by one

We give the following estimate on growth.

Lemma 8.3 *Given locally finite modules M, N in a triangle*

$$\Sigma_n M \xrightarrow{\chi} M \longrightarrow N$$

with $n \neq 0$ then

$$\mathrm{growth}(M) \leq \mathrm{growth}(N) + 1.$$

Proof The homology long exact sequence of the triangle includes

$$\cdots \longrightarrow H^{i-n}(M) \xrightarrow{\chi} H^i(M) \longrightarrow H^i(N) \longrightarrow \cdots.$$

This shows

$$\dim_k(H^i(M)) \leq \dim_k(H^i(N)) + \dim_k(\chi H^{i-n}(M)).$$

Iterating s times, we find

$$\dim_k(H^i(M)) \leq \dim_k(H^i(N)) + \dim_k(H^{i-n}(N))$$
$$+ \cdots + \dim_k(H^{i-(s-1)n}(N)) + \dim_k(\chi^s H^{i-sn}(M)).$$

To obtain growth estimates, it is convenient to collect the dimensions of the homogeneous parts into the Hilbert series $h_M(t) = \sum_n \dim_k(H^i(M)) t^i$. An inequality between such formal series means that it holds between all coefficients.

First suppose that $n > 0$. Since $H^*(M)$ is bounded below, if $h_M(t)$ is the Hilbert series of $H^*(M)$ then we have

$$h_M(t) \leq h_N(t)(1 + t^n + t^{2n} + \cdots) = \frac{h_N(t)}{1 - t^n},$$

giving the required growth estimate.

If $n = -n' < 0$ we rearrange to obtain

$$N' \longrightarrow M \longrightarrow \Sigma_{n'} M,$$

where $N' = \Sigma_{n'-1} N$ and argue precisely similarly. $\qquad \square$

8C Growth of eci spaces

The implication we require is now straightforward.

Theorem 8.4 *If X is eci then it is also gci, and if X has e–codimension c it has g–codimension $\leq c$.*

Proof It is sufficient to show $C^*(\Omega X) \simeq k \otimes_{C^*(X)} k$ has polynomial growth.

By hypothesis there are elements $\chi_1, \chi_2, \ldots, \chi_c$ of nonzero degree in $ZD(C^*(X))$ so that $k/\chi_c/\cdots/\chi_1$ is small, and thus, applying $k \otimes_{C^*(X)} (\cdot)$ we obtain a complex with growth ≤ -1. By [Lemma 8.3](#) if we apply $k \otimes_{C^*(X)} (\cdot)$ to $k/\chi_c/\cdots/\chi_2$ we obtain a complex of growth ≤ 0. Doing this repeatedly, we deduce that when we apply $k \otimes_{C^*(X)} (\cdot)$ to k itself we obtain a complex with growth $\leq c - 1$ as required. □

9 Properties of gci spaces

The main results here are a smallness condition for modules over the cochains of a normalisable gci space X and a structural result for the homology of ΩX which generalizes results of Félix, Halperin and Thomas. Throughout this section we assume the LRC [4.6](#), alternatively the reader may forego assuming the LRC and instead replace the normalisability condition with polynomial normalisability.

9A Cellular approximation, completion and Gorenstein condition

We have need of recalling concepts from [14] and [15].

Let $R = C^*(X)$. A map $U \to V$ of R–modules is a k–*equivalence* if the induced map $\operatorname{Hom}_R(k, U) \to \operatorname{Hom}_R(k, V)$ is an equivalence. A module N is k–*null* if $\operatorname{Hom}_R(k, N) \simeq 0$. A module C is k–*cellular* if $\operatorname{Hom}_R(C, N) \simeq 0$ for every k–null module N. A module C is k–*complete* if $\operatorname{Hom}_R(N, C) \simeq 0$ for every k–null module N. A map $f\colon C \to X$ of R–modules is a k–*cellular approximation (of X)* if C is k–cellular and f is a k–equivalence. Finally, a map $f\colon X \to C$ is a k–*completion (of X)* if C is k–complete and f is a k–equivalence.

We shall not recall the general definition of the Gorenstein condition from [15]; instead we give a definition which is equivalent under the mild assumption of proxy-smallness.

Recall that an object is *proxy-small* if the thick category it generates includes a nontrivial small object building the original object. Suppose that k is proxy-small as an R–module. Then R is *Gorenstein* if $\operatorname{Hom}_R(k, R) \simeq \Sigma^a k$ as left k–modules for some a [15, Proposition 8.4].

In the situation we consider, where X is normalisable, then k is always proxy-small by the following lemma.

Lemma 9.1 *Let X be a connected space. If $H^*(X)$ is Noetherian then k is proxy-small over $C^*(X)$.*

Proof Since $H^*(X)$ is Noetherian, it is finitely generated as a module over a polynomial subring. We may use the polynomial generators to form a Koszul complex K on $C^*(X)$. Since its homology is finite-dimensional over k, it is finitely built by k. Finally, it is easy to see that $K \otimes_{C^*(X)} k$ is nonzero. Since k is a retract of $K \otimes_{C^*(X)} k$, we see that K builds k. $\qquad\square$

9B The Gorenstein property of gci spaces

It is well-known that for commutative rings a complete intersection is Gorenstein. We prove an analogue, using the weakest of the analogues of ci.

Proposition 9.2 *If X is a normalisable gci space then both $C_*(\Omega X)$ and $C^*(X)$ are Gorenstein.*

In effect we rely on [23] for the simply connected finite case, and then apply the fibration lemma for Gorenstein spaces [15, 10.2] twice to deduce it in general. We will present the proof in stages. Recall that $k = \mathbb{F}_p$.

Lemma 9.3 *Let F be a k–finite simply connected gci space, then $C^*(F)$ is Gorenstein.*

Proof We begin by observing that F is equivalent to the p–completion of a finite simply connected CW–complex F', which is thus a space of finite LS category. Since $C_*(\Omega F') \simeq C_*(\Omega F)$, F' is also gci and therefore we see that $H_*(\Omega F')$ is an elliptic Hopf algebra in the sense of [23]. By [23, Proposition 3.1], it is Gorenstein in the sense that $\mathrm{Ext}^*_{H_*(\Omega F')}(k, H_*(\Omega F'))$ is one-dimensional (nonzero in degree a, say), and the same is true with F' replaced by F. Accordingly $\mathrm{Hom}_{C_*(\Omega F)}(k, C_*(\Omega F)) \simeq \Sigma^a k$ and $C_*(\Omega F)$ is Gorenstein in the sense of [15].

By Proposition 2.1, the Eilenberg–Moore spectral sequence for $C^*(F)$ converges, so it follows from [15, Proposition 8.5] that $C^*(F)$ is also Gorenstein, as required. $\quad\square$

Next we deal with the case of a finite complex which need not be simply connected.

Lemma 9.4 *Let F be a connected k–finite gci space with finite fundamental group such that its universal cover $\widetilde{F}$ is Gorenstein. Then $C^*(F)$ is Gorenstein.*

Proof Consider the fibration $\widetilde{F} \longrightarrow F \longrightarrow B\pi_1 F$. Since $\pi_1(F)$ is a p–group, $C^*(B\pi_1(F))$ is dc–complete in the sense of [15, 4.16] and so $C^*(B\pi_1(F))$ is Gorenstein, because $C_*(\Omega B\pi_1(F))$ is. Since $\widetilde{F}$ has finite homology, $C^*(F)$ is a small $C^*(B\pi_1(F))$–module. Now we can apply [15, Proposition 8.10], which shows that $C^*(F)$ is Gorenstein. □

Finally we are ready to deal with the general case.

Proof of Proposition 9.2 First, suppose $F \to X \to B$ is a polynomial normalisation for X. By [15, 10.2] $C^*(B)$ is Gorenstein.

Lemma 9.5 *The Noether fibre F is gci.*

Proof First note that both F and ΩB are connected since the normalisation is polynomial. Since ΩB is a finite H–space it is gci.

Next, we may assume that X is simply connected. Indeed, we may replace X by its universal cover and F by its corresponding cover and since $\pi_1(X)$ is finite, this will not affect the conclusion. Now consider the fibration $\Omega^2 B \to \Omega F \to \Omega X$. It is principal and the $\pi_1(\Omega X)$ action on $H_*(\Omega^2 B)$ is trivial. The fact that ΩF has polynomial growth follows from the Serre spectral sequence. □

It now follows from Lemma 9.4 that the Noether fibre F is Gorenstein. Since F is k–finite, $C^*(X)$ is small as a $C^*(B)$–module. As before, we apply [15, Proposition 8.10], which shows that $C^*(X)$ is Gorenstein. Finally, since $C^*(X)$ is dc–complete [15, 4.22], then $C_*(\Omega X)$ is also Gorenstein, by [15, Proposition 8.5].

This completes the proof of Proposition 9.2 □

9C Completion and smallness criteria

We shall need criteria for both smallness and k–completeness of $C^*(X)$–modules.

Lemma 9.6 *Suppose X is a normalisable gci space. Then the k–completion of any $C^*(X)$–module M is given by*

$$\mathrm{Hom}_{C_*(\Omega X)}(\Sigma^a k, \mathrm{Hom}_{C^*(X)}(k, M)),$$

where a is the Gorenstein shift of $C^*(X)$.

Proof Set $R = C^*(X)$ and $\mathcal{E} = C_*(\Omega X)$. Since R is dc–complete, $\mathcal{E} \simeq \operatorname{Hom}_R(k, k)$. As we saw above, k is proxy-small as an R–module, so by [15]

$$\operatorname{Cell}_k R \simeq \operatorname{Hom}_R(k, R) \otimes_{\mathcal{E}} k.$$

Using the proxy-smallness of k once again, it is an easy exercise to show that the k–completion of any R–module M is given by

$$M_k^{\wedge} \simeq \operatorname{Hom}_R(\operatorname{Cell}_k R, M).$$

Bearing in mind that k is an $R \otimes \mathcal{E}$–module, we have the following equivalences given by standard adjunctions:

$$\operatorname{Hom}_R(\operatorname{Cell}_k R, M) \simeq \operatorname{Hom}_R(\operatorname{Hom}_R(k, R) \otimes_{\mathcal{E}} k, M)$$
$$\simeq \operatorname{Hom}_{\mathcal{E}^{\mathrm{op}}}(\operatorname{Hom}_R(k, R), \operatorname{Hom}_R(k, M))$$

Since X is Gorenstein, we see that $\operatorname{Hom}_R(k, R) \simeq \Sigma^a k$ as left k–modules. Because $\pi_1(X)$ is a p–group and $k = \mathbb{F}_p$ there is only one simple $k[\pi_1(X)]$–module, which is the trivial module k. Hence the $\pi_0(\mathcal{E})$–module $\pi_a(\operatorname{Hom}_R(k, R))$ is that simple module k. Now we can employ [15, Proposition 3.9], which shows that $\operatorname{Hom}_R(k, R) \simeq \Sigma^a k$ as right $\mathcal{E}$–modules $\square$

It is useful to have a criterion for k–completeness.

Lemma 9.7 If X is a Gorenstein space with $H^*(X)$ Noetherian then every bounded-above $C^*(X)$–module of finite type is k–complete.

Remark 9.8 (i) By Proposition 9.2 this applies to any normalisable gci space.

(ii) The exact generality of the lemma is not clear, but we note that k need not be k–complete in general. For example if $R = S^0$ and $k = H\mathbb{F}_p$ then R is itself k–null by Lin's theorem [37], and hence k is not k–complete.

Proof Denote by R the k–algebra $C^*(X)$. We first note that it is sufficient to show that k is k–complete. Indeed, in that case every R–module finitely built from k is k–complete. Since there are dual Postnikov sections in the category of R–modules (see [15, Proposition 3.3]) then every bounded-above R–module of finite type is the homotopy limit of k–complete modules and hence is itself k–complete.

Since k is proxy-small, there is a small object K so that

$$k \models K \vdash k.$$

Because R is Gorenstein (Proposition 9.2) and coconnective with $\pi_0(R)$ a field we have that $\operatorname{Hom}_R(k, R) \simeq \Sigma^a k$ as a left R–module (see [15, Proposition 3.9]).

Now suppose N is k–null, so that $\operatorname{Hom}_R(k, N) \simeq 0$. For an R–module M we denote by DM the R–module $\operatorname{Hom}_R(M, R)$. Since $k \vdash K$ we find $Dk \vdash DK$, so that by the Gorenstein condition $k \vdash DK$. Hence

$$0 \simeq \operatorname{Hom}_R(DK, N) \simeq DDK \otimes_R N \simeq K \otimes_R N.$$

Since $K \vdash k$ it follows that $k \otimes_R N \simeq 0$. Finally,

$$\operatorname{Hom}_R(N, k) \simeq \operatorname{Hom}_R(N, \Sigma^n \operatorname{Hom}_R(k, R))$$
$$\simeq \operatorname{Hom}_R(k \otimes_R N, \Sigma^n R) \simeq 0$$

and k is k–complete as required. $\qquad\square$

We get the following criteria for smallness.

Corollary 9.9 *Suppose X is a normalisable gci space and let M be a bounded-above $\mathrm{C}^*(X)$–module of finite type. The following are equivalent:*

(1) *M is small.*

(2) *$\pi_*(M \otimes_{\mathrm{C}^*(X)} k)$ is finite.*

Proof Set $R = \mathrm{C}^*(X)$. Clearly if M is small then $\pi_*(M \otimes_R k)$ is finite.

So suppose that $M \otimes_R k$ has finite homotopy. Let $\mathcal{E} = \mathrm{C}_*(\Omega X)$. Since k is proxy-small and M is k–complete we see that $M = \operatorname{Hom}_{\mathcal{E}}(\Sigma^a k, \overline{M})$, where $\overline{M}$ is the $\mathcal{E}$–module $\operatorname{Hom}_R(k, M)$.

By [15, Proposition 4.17] k is proxy-small also as an $\mathcal{E}$–module. It follows that $\Sigma^a M \otimes_R k$ is a k–cellular approximation of $\overline{M}$. Hence there is a k–equivalence of $\mathcal{E}$–module $\Sigma^a M \otimes_R k \to \overline{M}$. In particular

$$M \simeq \operatorname{Hom}_{\mathcal{E}}(k, M \otimes_R k).$$

Since $\pi_1(X)$ is a finite p–group it follows that $\pi_n(M \otimes_R k)$ is finitely built by k as a $k[\pi_1 X]$–module for every n. Using [15, Proposition 3.2] and the fact that $M \otimes_R k$ has finite homotopy we see that $k \models_{\mathcal{E}} M \otimes_R k$ and therefore

$$R \sim \operatorname{Hom}_{\mathcal{E}}(k, k) \models_R \operatorname{Hom}_{\mathcal{E}}(k, M \otimes_R k) \sim M. \qquad\square$$

9D   Loop space homology of gci spaces

The following rests heavily on results of Félix, Halperin and Thomas.

Lemma 9.10   *Let X be a Gorenstein gci space. Then*

(1) $H_*(\Omega X)$ *is left and right Noetherian, and*

(2) $H_*(\Omega X)$ *is a finitely generated module over a central polynomial subalgebra.*

Proof   First suppose that X is simply connected. The depth of the Hopf algebra $H_*(\Omega X)$ is the least integer m such that $\operatorname{Ext}^m_{H_*(\Omega X)}(k, H_*(\Omega X))$ is nonzero. We claim that $H_*(\Omega X)$ has finite depth. Observe that there is conditionally convergent spectral sequence

$$E_2^{p,q} = \operatorname{Ext}^{p,q}_{H_*(\Omega X)}(k, H_*(\Omega X)) \;\Rightarrow\; \operatorname{Ext}^{p+q}_{C_*(\Omega X)}(k, C_*(\Omega X)).$$

Since $C_*(\Omega X)$ is Gorenstein, the E_2 term of this spectral sequence cannot be zero. Hence $H_*(\Omega X)$ has finite depth. Now, by [22, Theorems B and C], $H_*(\Omega X)$ has the desired properties.

Now suppose that X is not simply connected. Let $\widetilde{X}$ be the universal cover of X. Clearly $\widetilde{X}$ is gci. Using [15, Proposition 8.10] it follows from the fibration $\pi_1(X) \to \widetilde{X} \to X$ that $\widetilde{X}$ is also Gorenstein. Hence $H_*(\Omega \widetilde{X})$ has the desired properties, ie it is left and right Noetherian and is a finite module over a central polynomial subalgebra.

One can identify $H_*(\Omega X)$ with the semidirect product $H_*(\Omega \widetilde{X}) \rtimes \pi_1(X)$ in a natural way. Since $\pi_1(X)$ is a finite group, we see that $H_*(\Omega X)$ is a left and right Noetherian $H_*(\Omega \widetilde{X})$–module. In particular $H_*(\Omega X)$ is left and right Noetherian (over itself).

Note that $\pi_1(X)$ acts on $H_*(\Omega \widetilde{X})$ via algebra maps and therefore the center of $H_*(\Omega \widetilde{X})$, which we shall denote by $\widetilde{Z}$, is invariant under this action. Because there is a polynomial subalgebra $P \subset \widetilde{Z}$ such that $H_*(\Omega \widetilde{X})$ is a finitely generated P–module, $\widetilde{Z}$ is Noetherian. The Hilbert–Noether theorem shows that the ring $\widetilde{Z}^{\pi_1(X)}$ of invariants is Noetherian and $\widetilde{Z}$ is a finitely generated over it. We conclude that $H_*(\Omega X)$ is finitely generated over $\widetilde{Z}^{\pi_1(X)}$.

From the identification $H_*(\Omega X) \cong H_*(\widetilde{X}) \rtimes \pi_1(X)$ we see that $\widetilde{Z}^{\pi_1(X)}$ is contained in the center of $H_*(\Omega X)$. Therefore $H_*(\Omega X)$ is finitely generated over its center Z, and Z is Noetherian. From the Noether normalisation theorem we conclude that $H_*(\Omega X)$ is a finitely generated module over a central polynomial subalgebra. $\square$

10 The Hochschild cohomology shearing map

We need to discuss certain ring homomorphisms

$$\psi \colon HH^*(X) \longrightarrow H_*(\Omega X)$$

in the absolute case and

$$\chi \colon HH^*(X \mid B) \longrightarrow H_*(\Omega F)$$

when we have a fibration $F \to X \to B$. We refer to these as *shearing maps*, and the purpose of the present section is to show that several different possible definitions agree. Note that ψ is well-known in various contexts; see [24].

10A General context

In this section we give two equivalent definitions for the shearing map. Throughout we assume Q is a commutative ring spectrum and that there are maps of ring spectra

$$Q \xrightarrow{\rho} R \xrightarrow{\sigma} S.$$

Thus, R and S are Q–algebras, note that we do not assume R and S are commutative Q–algebras. This is a precaution, as it is often all too easy to assume the wrong bimodule structure when working in a commutative setting.

In this section we shall denote by R_Q^e the Q–algebra $R \otimes_Q R^{\mathrm{op}}$. We will denote by S^σ the Q–algebra $R \otimes_Q S^{\mathrm{op}}$. Note there are maps of Q–algebras

$$R_Q^e \xrightarrow{1 \otimes \sigma} S^\sigma \xrightarrow{\sigma \otimes 1} S_Q^e.$$

Since S is an S_Q^e–module, this structure is pulled back to make S both an S^σ–module and an R_Q^e–module. The shearing map is a map of graded algebras

$$HH^*(R \mid Q) \longrightarrow \mathrm{Ext}_{S^\sigma}^*(S, S).$$

This is probably the place to mention we have two main settings in mind. In the first there is a fibration $F \to X \to B$ and we set $Q = C^*(B)$, $R = C^*(X)$ and $S = C^*(F)$ with ρ and σ being the obvious maps induced by the fibration. In this first setting S^σ is equivalent to the k–algebra $S \otimes_k S$ and so $\mathrm{Ext}_{S^\sigma}^*(S, S)$ is the Hochschild cohomology $HH^*(F)$. In the second setting $R = C^*(X)$ for some space X and both S and Q are k. Here S^σ is equivalent to R and $\mathrm{Ext}_{S^\sigma}^*(S, S)$ turns out to be $H_*(\Omega X)$. There is also a third setting, which is an amalgamation of the first two. We shall say more on all of these settings towards the end of this section.

10B Two descriptions of the shearing map

Before describing the shearing map we need a pair of lemmas. In what follows we refer to the right copy of R in R_Q^e as R_r, thus there is a k–algebra map $R_r \to R_Q^e$.

Lemma 10.1 *There is a functor $F\colon Ho(R_Q^e) \to Ho(S^\sigma)$ given by $M \mapsto M \otimes_{R_r} S$. This functor is naturally equivalent to the functor $M \mapsto S^\sigma \otimes_{R_Q^e} M$ and therefore F is left adjoint to the forgetful functor $Ho(S^\sigma) \to Ho(R_Q^e)$.*

Lemma 10.2 *There is an equivalence of S^σ–modules: $R \otimes_{R_r} S \simeq S$.*

At first glance it is perhaps not clear what is the claim in Lemma 10.2 above. Recall that S is an S_Q^e–module, and it is this module structure that we pulled back to make S into an S^σ–module. So it is not immediately apparent that the S^σ–module structure on $R \otimes_{R_r} S$ agrees with the one on S.

The proofs of Lemmas 10.1 and 10.2 are both based on chasing the same diagram. There is a commuting square of maps of Q–algebras:

$$
\begin{array}{ccc}
R_r^{\mathrm{op}} & \longrightarrow & S^{\mathrm{op}} \\
\downarrow & & \downarrow \\
R_Q^e & \longrightarrow & S^\sigma
\end{array}
$$

This induces a diagram of homotopy categories and adjoint functor between them:

$$
\begin{array}{ccc}
Ho(R_r^{\mathrm{op}}) & \rightleftarrows & Ho(S^{\mathrm{op}}) \\
\updownarrow & & \updownarrow \\
Ho(R_Q^e) & \rightleftarrows & Ho(S^\sigma)
\end{array}
$$

Chasing the diagram above will easily yield proofs for Lemmas 10.1 and 10.2, and we leave it to the reader to complete the details.

Definition 10.3 Define the *shearing map for the transitivity triple $Q \to R \to S$* to be the map $\chi\colon \mathrm{Ext}^*_{R_Q^e}(R, R) \to \mathrm{Ext}^*_{S^\sigma}(S, S)$ of graded rings given by $\chi(f) = f \otimes_{R_r} S$.

By Lemmas 10.1 and 10.2 there is an isomorphism $\tau_R\colon \mathrm{Ext}^*_{R_Q^e}(R, S) \to \mathrm{Ext}^*_{S^\sigma}(S, S)$. This allows us to construct another map

$$
\alpha\colon \mathrm{Ext}^*_{R_Q^e}(R, R) \longrightarrow \mathrm{Ext}^*_{S^\sigma}(S, S),
$$

where α is the composition

$$\mathrm{Ext}^*_{R^e_Q}(R, R) \xrightarrow{\sigma} \mathrm{Ext}^*_{R^e_Q}(R, S) \xrightarrow{\tau R} \mathrm{Ext}^*_{S^\sigma}(S, S).$$

We will show that α and χ are equal, but for that we must first have an explicit description of τ.

Since τ is the isomorphism given by an adjunction, it only depends on a choice of counit map $m\colon S \otimes_{R_r} S \to S$. This then becomes a bootstrapping problem, namely describing the counit map. For that we state the following lemma.

Lemma 10.4 *There is the following commutative diagram of S^e_Q–modules*

$$
\begin{array}{ccc}
S \otimes_Q S & \xrightarrow{m'} & S \\
\downarrow & & \big\Vert \\
S \otimes_R S & \xrightarrow{m} & S,
\end{array}
$$

where m' is the multiplication map of S as a Q–algebra. The map m satisfies the following identities:

$$m(\sigma \otimes_R S) = 1_S,$$
$$m(S \otimes_R \sigma) = 1_S,$$

where the first is an identity of morphisms in $Ho(S^\sigma)$ and the second is an identity of morphisms in $Ho(S \otimes_Q R^{\mathrm{op}})$. $\qquad\square$

Remark 10.5 Suppose $F \to X$ is a fibration and set $Q = k$, $R = C^*(X)$ and $S = C^*(F)$. Then the map m' is induced by the diagonal $\Delta'\colon F \to F \times F$ while m is induced by the diagonal $\Delta\colon F \to F \times_X F$. Clearly $F \times_X F \subseteq F \times F$ and the composition

$$F \xrightarrow{\Delta} F \times_X F \longrightarrow F \times F$$

is Δ'. This gives a topological explanation to the commuting diagram above.

Denote by F the functor $- \otimes_{R_r} S\colon Ho(R^e_Q) \to Ho(S^\sigma)$ and by G the forgetful functor $Ho(S^\sigma) \to Ho(R^e_Q)$. Let $\epsilon\colon FG \to 1$ be given by $\epsilon_Y = Y \otimes_S m$ and let $\eta\colon 1 \to GF$ be $\eta_X = X \otimes_{R_r} \sigma$. From Lemma 10.4 we get the following identities:

$$1_{FX} = \epsilon_{FX} \circ F(\eta X),$$
$$1_{GY} = G(\epsilon Y) \circ \eta_{GY}.$$

Accordingly, ϵ and η are the counit and the unit for the adjunction of F and G. In particular we get the following corollary.

Corollary 10.6 *Define natural transformations*

$$\tau\colon \operatorname{Ext}^*_{R^e_Q}(-,S) \longrightarrow \operatorname{Ext}^*_{S^\sigma}(-\otimes_{R_r} S, S),$$

$$\delta\colon \operatorname{Ext}^*_{S^\sigma}(-\otimes_{R_r} S, S) \longrightarrow \operatorname{Ext}^*_{R^e_Q}(-,S),$$

in the following manner. Given a map $f\colon M \to \Sigma^n S$ of R^e_Q–modules, let $\tau(f)$ be the composition $m(f \otimes_{R_r} S)$, and given a map $g\colon M \otimes_R S \to \Sigma^n S$ of S^σ–modules, let $\delta(g) = g(1_M \otimes_{R_r} \sigma)$. Then $\delta\tau = 1$ and $\tau\delta = 1$.

Proof The adjoint to a morphism $f\colon M \to GS$ is the composition

$$FM \xrightarrow{\ Ff\ } FGS \xrightarrow{\ \epsilon_S\ } S.$$

The adjoint to a morphism $g\colon FM \to S$ is the composition

$$M \xrightarrow{\ \eta_M\ } GFM \xrightarrow{\ Gg\ } GS.$$

Since $\epsilon_S = m$ and $\eta_M = 1 \otimes_{R_r} \sigma$, we are done. □

Lemma 10.7 *The map α defined above is equal to the shearing map χ. In particular α is a map of graded algebras. Moreover, the right $\operatorname{Ext}^*_{R^e_Q}(R, R)$–module structure on $\operatorname{Ext}^*_{R^e_Q}(R, S)$ is the same as the module structure induced by the map of algebras χ.*

Proof Let f be an element of $\operatorname{Ext}^*_{R^e_Q}(R, R)$, then

$$\tau\sigma(f) = \tau(\sigma f) = m(\sigma f \otimes S) = m(\sigma \otimes S)(f \otimes S) = m(\sigma \otimes S)\chi(f) = \chi(f).$$

The last identity comes from $m(\sigma \otimes S) = 1_S$.

The second statement requires us to show that for f in $\operatorname{Ext}^*_{R^e_Q}(R, R)$ and g in $\operatorname{Ext}^*_{R^e_Q}(S, S)$,

$$m(g\sigma f \otimes S) = g(f \otimes S).$$

Thus, it is enough to show that $m(g\sigma \otimes S) = g$, but this is precisely the statement $\tau\delta(g) = g$ which is proved above. □

10C The shearing map for a fibration

Let $F \xrightarrow{q} X \xrightarrow{p} B$ be a fibration and set $Q = C^*(B)$, $R = C^*(X)$ and $S = C^*(F)$. We shall show that S^σ is equivalent to the k–algebra $S \otimes_k S$ and so $\operatorname{Ext}^*_{S^\sigma}(S, S)$ is the Hochschild cohomology $HH^*(F)$.

Lemma 10.8 *There is an equivalence of k–algebras $S^\sigma \simeq S \otimes_k S^{\mathrm{op}} = S^e$.*

Proof Recall that $S \simeq R \otimes_Q k$. Thus,

$$R_Q^e \otimes_R S^{\mathrm{op}} \simeq (R \otimes_Q R) \otimes_R S^{\mathrm{op}} \simeq R \otimes_Q S^{\mathrm{op}}$$
$$\simeq R \otimes_Q (k \otimes_Q R) \simeq (R \otimes_Q k) \otimes_k (k \otimes_Q R)$$
$$\simeq S \otimes_k S^{\mathrm{op}}. \qquad \square$$

Remark 10.9 There is also topological argument for the equivalence above. It is easy to show that the homotopy pullback of the diagram

$$X \times_B X \xrightarrow{p_2} X \xleftarrow{q} F$$

is $F \times F$, where p_2 is the projection onto the second coordinate.

Remark 10.10 The equivalence $S^\sigma \simeq S^e$ allows us to translate the S^σ–module $R \otimes_R S$ into an S^e–module, which we shall provisionally denote by T. Clearly T is equivalent to S as a k–module, but one might worry that T would not have the correct S^e–module structure we want. The module structure we need is the diagonal one, induced by the diagonal map $\Delta \colon F \to F \times F$ (whereas, for example, the $S \otimes_k S$– module $k \otimes_k S$ has a "wrong" module structure). We sketch a topological argument as to why T has the correct module structure. Consider the map of homotopy pullback squares below, where the left and right walls are homotopy pullbacks and the map $X \times_B X \to X$ is projection to the right coordinate:
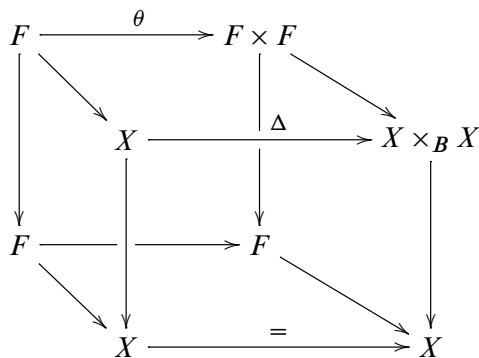
One easily sees that the induced morphism θ of homotopy pullbacks is indeed the diagonal map, as required. Hence T has the correct module structure.

Thus we have shown:

Corollary 10.11 *In the setting above the shearing map*

$$\chi \colon HH^*(X \mid B) \longrightarrow HH^*(F)$$

is a map of graded algebras.

10D   The shearing map to the loop space homology

In this setting $R = C^*(X)$ for some space X and both S and Q are k. In this case $S^\sigma = R$ and $\mathrm{Ext}^*_{S^\sigma}(S, S) \cong H_*(\Omega X)$. One immediately sees that the shearing map has the form

$$\psi\colon HH^*(X) \longrightarrow H_*(\Omega X).$$

Note that [24] describes a map $I\colon HH^*(X) \to H_*(\Omega X)$ which is the same as the map α introduced earlier and so, by Lemma 10.7, it is the shearing map. It is shown in [24] that the image of I is central in $H_*(\Omega X)$, we give here a different argument for that fact.

In the current setting the shearing map is clearly equal to the composition

$$HH^*(R) \longrightarrow ZD(R) \longrightarrow \mathrm{Ext}^*_R(k, k),$$

where the map $ZD(R) \to \mathrm{Ext}^*_R(k, k)$ is given by $\zeta \mapsto \zeta_k\colon k \to \Sigma^n k$. The following is now obvious.

Lemma 10.12   *In this setting the image of the shearing map ψ is contained in the graded centre of $H_*(\Omega X)$.*

10E   The relative shearing map to the loop space homology

The setting here is as follows. Let $F \to X \to B$ be a fibration and set $Q = C^*(B)$, $R = C^*(X)$ and $S = k$. Then $S^\sigma = R \otimes_Q S$ turns out to be $C^*(F)$. Thus the shearing map is:

$$\chi\colon HH^*(X \mid B) \longrightarrow H_*(\Omega F)$$

Lemma 10.13   *The composition*

$$HH^*(X \mid B) \xrightarrow{\chi} H_*(\Omega F) \longrightarrow H_*(\Omega X)$$

is equal to the composition

$$HH^*(X \mid B) \longrightarrow HH^*(X) \xrightarrow{\psi} H_*(\Omega X),$$

where ψ is the shearing map from Section 10D above.

Proof Let $a: R \to k$ be the augmentation map and let $\varphi: R^e \to R_Q^e$ be the obvious map. Note that φ induces the forgetful functor $\varphi^*: Ho(R_Q^e) \to Ho(R^e)$. In light of Lemma 10.7 we need to show that the two compositions

$$(1) \qquad \operatorname{Ext}^*_{R_Q^e}(R, R) \xrightarrow{a} \operatorname{Ext}^*_{R_Q^e}(R, k) \xrightarrow{\varphi^*} \operatorname{Ext}^*_{R^e}(R, k)$$

$$(2) \qquad \operatorname{Ext}^*_{R_Q^e}(R, R) \xrightarrow{\varphi^*} \operatorname{Ext}^*_{R^e}(R, R) \xrightarrow{a} \operatorname{Ext}^*_{R^e}(R, k)$$

are equal.

Let $f: R \to \Sigma^n R$ be an element of $\operatorname{Ext}^*_{R_Q^e}(R, R)$. The first composition yields the morphism $\varphi^*(af) \in \operatorname{Ext}^*_{R^e}(R, k)$ while the second composition yields $\varphi^*(a)\varphi^*(f) \in \operatorname{Ext}^*_{R^e}(R, k)$. Both compositions are equal, because φ^* is a functor. $\qquad\square$

Corollary 10.14 *The shearing map $\chi: HH^*(X \mid B) \to H_*(\Omega F)$ is equal to the composition*

$$HH^*(X \mid B) \xrightarrow{\alpha} HH^*(F) \xrightarrow{\beta} H_*(\Omega F),$$

where α and β are the appropriate shearing maps.

Proof Given $f \in HH^*(X \mid B)$ we have

$$\chi(f) = f \otimes_{C^*(X)} k = f \otimes_{C^*(X)} C^*(F) \otimes_{C^*(F)} k = \beta(\alpha(\chi)). \qquad\square$$

11 Finite gci spaces are hhci

In this section we assume the LRC 4.6.

11A A sufficient condition for being hhci

This condition (Theorem 11.2 below) will also be used in Section 12, and for that reason we phrase it in greater generality than is strictly needed in this section.

Throughout this section we denote by χ the shearing map $HH^*(X \mid B) \to H_*(\Omega F)$ where $X \to B$ is a polynomial normalisation and F is the Noether fiber. Recall (Lemma 10.12) that the image of χ is contained in the graded-commutative centre of $H_*(\Omega F)$, which we shall denote by Z. We also remind the reader that by Lemma 9.5 the space F is gci whenever X is gci.

Lemma 11.1 *Let X be a normalisable gci space. Denote by A the image of the shearing map χ. Suppose that Z is finitely generated as a module over the subalgebra A. Then A is a Noetherian algebra.*

Proof By Lemma 9.10 the algebra $H_*(\Omega F)$ is a free finitely generated module over a central polynomial subalgebra P. Let $Z \subseteq H_*(\Omega F)$ be the graded-commutative centre. Then $A \subseteq Z$ and Z is a Noetherian graded-commutative ring (Z is Noetherian because it is also finitely generated as a P–module). In [18] Eisenbud proves that if a central subring of a Noetherian ring finitely generates the whole ring as a module, then that subring is Noetherian. A simple generalization of this result to the graded-commutative setting completes the proof. □

Theorem 11.2 *Let X be a normalisable gci space and suppose that the shearing map χ makes Z into a finitely generated $HH^*(X \mid B)$–module. Then X is hhci.*

Proof Let A be the image of χ. Since A is a graded-commutative Noetherian algebra (by Lemma 11.1) with $A_0 = k$, then A has a Noether normalisation. There are elements $x_1, \ldots, x_c \in A$ which generate a polynomial subalgebra $P = k[x_1, \ldots, x_c]$ such that A is a finitely generated P–module. Since Z is finitely generated as an A–module, it is also finitely generated as a P–module.

We know from Lemma 9.10 that $H_*(\Omega F)$ is finitely generated as a Z–module and hence it is also finitely generated as a P–module.

Set $R = C^*(X)$ and $Q = C^*(B)$ and choose elements $z_1, \ldots, z_c \in HH^*(R \mid Q)$ such that $\chi(z_i) = x_i$. We will show that these elements satisfy the hhci definition. Let $B = R/z_1/\cdots/z_c$ be the relevant R_Q^e–module; we must show that B is small.

Clearly B is bounded-above and of finite type, thus we may use the smallness criterion of Corollary 9.9. Note we are applying the criterion to modules over the cochains of the space $X \times_B X$, which is also a normalisable gci space using the normalisation $X \times_B X \to B$. Thus we must show that $B \otimes_{R_Q^e} k$ has finite homology.

Recall from Section 10E that the transitivity triple used for defining the shearing map χ is $Q = C^*(B)$, $R = C^*(X)$ and $S = k$ and thus S^σ is $C^*(F)$. As noted in Lemma 10.1 $B \otimes_R k \simeq S^\sigma \otimes_{R_Q^e} B$ and so

$$B \otimes_{R_Q^e} k \simeq k \otimes_{S^\sigma} (B \otimes_R k).$$

Next we claim it is enough to show that

$$\mathrm{Ext}^*_{S^\sigma}(B \otimes_R k, k)$$

is finite-dimensional. Indeed

$$\mathrm{Hom}_{S^\sigma}(B \otimes_R k, k) \simeq \mathrm{Hom}_k(k \otimes_{S^\sigma} (B \otimes_R k), k).$$

Recall there is an isomorphism $\operatorname{Ext}^*_{S^\sigma}(k, k) \cong H_*(\Omega F)$. For every $i = 1, \dots, c$ the map $z_i \otimes_R k \in \operatorname{Ext}^*_{S^\sigma}(k, k)$ represents the element $x_i = \chi(z_i)$. From this it is easy to see that the map

$$\operatorname{Ext}^*_{S^\sigma}(z_i \otimes_R k, k) \colon H_*(\Omega F) \longrightarrow H_{*-|z_i|}(\Omega F)$$

is simply right multiplication by x_i.

The Koszul filtration on $B = R/z_1/\cdots/z_c$ induces a filtration on $B \otimes_R k$. This gives rise to a spectral sequence whose E^2 term is the Koszul homology

$$H_{s,t}(H_*(\Omega F)/x_1/\cdots/x_c) \cong \operatorname{Tor}^P_{s,t}(H_*(\Omega F), k),$$

which strongly converges to $\operatorname{Ext}^{-s-t+c}_{S^\sigma}(k, B \otimes_R k)$. Since $H_*(\Omega F)$ is a finitely generated P–module the E^2 term of this spectral sequence is finite-dimensional. Therefore $\operatorname{Ext}^*_{S^\sigma}(k, B \otimes_R k)$ is finite-dimensional and the proof is done. $\qquad\square$

11B   Using a Hochschild cohomology spectral sequence

The Hochschild cohomology spectral sequence of [46] identifies one case where $H_*(\Omega F)$ is finitely generated over the image of the shearing map.

Theorem 11.3   *A finite gci space is hhci.*

Proof   Let X be a finite gci space. The polynomial normalisation we consider is of course $X \to pt$ and the resulting shearing map is $\chi \colon HH^*(X) \to H_*(\Omega X)$.

Since X is a normalisable gci space, then by Lemma 9.10 the loop space homology $H_*(\Omega X)$ is finitely generated over a central polynomial subalgebra. We can now use the results of [46] which show that the centre of $H_*(\Omega X)$ is finitely generated over the image of the shearing map. Invoking Theorem 11.2 completes the proof. $\qquad\square$

12   Normalisable gci spaces are eci

Throughout X is a normalisable gci space and we assume the LRC 4.6. Our goal is to prove the following result.

Theorem 12.1   *If X is a normalisable gci space then X is eci.*

The missing ingredient for showing that a normalisable gci space X is hhci is described in the next result.

Proposition 12.2 Let X be a normalisable gci space, let $v\colon X \to B$ be a polynomial normalisation for X and let F be its Noether fibre. If the relative Hochschild cohomology shearing map $\chi\colon HH^*(X \mid B) \to HH^*(F)$ makes $HH^*(F)$ into a finitely generated $HH^*(X \mid B)$–module, then X is hhci. In particular, if $HH^*(X \mid B)$ is Noetherian then X is hhci.

Example 12.3 For two closely related examples for $X \to B$ to which this applies suppose p is an odd prime. The two examples are (i) $BC_p \to B\mathrm{SO}(2)$ and (ii) $BD_{2p} \to BO(2)$. In both cases $H^*(B) = k[x]$ is polynomial on one generator (of degree 2 or 4 respectively) and $H^*(X) = k[x] \otimes \Lambda(\tau)$ where $|\tau| = |x| - 1$. Since the spectral sequences relating algebra and topology collapse the shearing map is

$$HH^*(k[x] \otimes \Lambda(\tau) \mid k[x]) \longrightarrow HH^*(\Lambda(\tau) \mid k)$$

and easily seen to be surjective.

12A Normalisable gci spaces

Recall from Lemma 9.5 that because X is gci, then so is the Noether fibre F of any polynomial normalisation $X \to B$. Theorem 11.3 above shows that F is hhci; we use this together with the normalisability of X to deduce it is eci.

Proof of Theorem 12.1 Let $v\colon X \to B$ be a polynomial normalisation for X and let F be the fibre of this map, then F is k–finite and is gci.

Set $Q = C^*(B)$, $R = C^*(X)$ and $S = C^*(F)$ and let $x_1, \dots, x_n \in H^*(B)$ be the polynomial generators of the cohomology algebra. So $k \simeq Q/x_1/\cdots/x_n$ and

$$S \simeq R \otimes_Q k \simeq R/x_1/\cdots/x_m$$

as R–modules.

There is a commutative diagram of algebras

$$
\begin{array}{ccc}
R^e_Q & \longrightarrow & S^e \\
\downarrow & & \downarrow \\
R & \longrightarrow & S
\end{array}
$$

coming from the obvious diagram of spaces. Pulling back along the left-hand vertical, $R/x_1/\cdots/x_n \simeq S$ as R^e_Q–modules. Thus we have shown that $R \models_{hh} S$.

By Theorem 11.3 the Noether fibre F is hhci. In particular there are elements $z_1, \dots, z_c \in \mathrm{Ext}^*_{S^e}(S, S)$ such that $T = S/z_1/\cdots/z_c$ is a small S^e–module. Using

the diagram above we can pull this construction back to the category of R_Q^e–modules, thereby showing that $S \models_e T$ as R_Q^e–modules. Combined with the fact that $R \models_{hh} S$ this yields $R \models_e T$.

It remains to show that T is small as an R_Q^e–module. The homotopy fibre of the map $F \times F \to X \times_B X$ is easily seen to be ΩB. This implies that S^e is small as an R_Q^e–module and therefore T is small as an R_Q^e–module. $\qquad\square$

12B Proof of Proposition 12.2

Let $\nu\colon X \to B$ be a polynomial normalisation with Noether fibre F. As before F is connected and gci.

By assumption, the shearing map $\chi\colon HH^*(X \mid B) \to HH^*(F)$ makes $HH^*(F)$ into a finitely generated module over $HH^*(X \mid B)$. As we saw in the proof of Theorem 11.3 above, the shearing map $\psi\colon HH^*(F) \to H_*(\Omega F)$ makes Z (the centre of $H_*(\Omega F)$) into a finitely generated $HH^*(F)$–module. By Corollary 10.14, the relative shearing map $\chi'\colon HH^*(X \mid B) \to H_*(\Omega F)$ is the composition $\psi\chi$. Thus χ' makes Z into a finitely generated $HH^*(X \mid B)$–module. By Theorem 11.2, X is hhci.

To complete the proof of Proposition 12.2 we must show that if $HH^*(X \mid B)$ is Noetherian then χ makes $HH^*(F)$ into a finitely generated $HH^*(X \mid B)$–module. This easily follows from the next lemma.

Lemma 12.4 *Set $Q = C^*(B)$, $R = C^*(X)$ and $S = C^*(F)$. Then $\mathrm{Hom}_{S^e}(S, S)$ is a small $\mathrm{Hom}_{R_Q^e}(R, R)$–module.*

Proof Since $H^*(\Omega B)$ is finite, we see that S is a small R–module. As we saw earlier, the R_Q^e–module structure on S coming from the map of algebras $R_Q^e \to R$ is the same as the usual R_Q^e–module structure on S. Hence, S is finitely built by R over R_Q^e and therefore $\mathrm{Hom}_{R_Q^e}(R, S)$ is finitely built by $\mathrm{Hom}_{R_Q^e}(R, R)$ as a $\mathrm{Hom}_{R_Q^e}(R, R)$–module. $\qquad\square$

This completes the proof of Proposition 12.2. $\qquad\square$

13 Some examples from group theory

In this section we restrict attention to the special case $R = C^*(BG; k)$, and seek to understand both the eci condition and the gci condition.

13A Chevalley groups

Now consider Quillen's p–adic construction of the classifying space of a Chevalley group. As usual, all spaces are completed at p, and we omit notation for this. If $q \neq p$ and $\Psi^q \colon BG \to BG$ is an Adams map we follow Quillen in defining $F\Psi^q$ by the homotopy pullback square

$$
\begin{array}{ccc}
F\Psi^q & \longrightarrow & BG \\
\downarrow & & \downarrow{\scriptstyle B\Delta} \\
BG & \xrightarrow{\{1,\Psi^q\}} & BG \times BG
\end{array}
$$

and then find a p–adic equivalence

$$F\Psi^q \simeq BG(q).$$

This gives a p–adic fibration

$$G \longrightarrow BG(q) \longrightarrow BG.$$

If G is a sphere this shows that $BG(q)$ is an s–hypersurface space. For example, $G = U(1)$ this shows again that the classifying space $BGL_1(q)$ of the cyclic group $GL_1(q)$ is an s–hypersurface space. More interesting is the case $G = SU(2)$, which shows that $BPSL_2(q)$ is an s–hypersurface space. When G is an iterated sphere bundle (eg if G is one of the classical groups) this shows $BG(q)$ is wsci, and hence also gci and therefore eci.

Problem 13.1 Show that when $G = SU(3)$ and q is such that the cohomology ring is not periodic, then the space $BG(q)$ is not sci.

Quillen [43], Fiedorowicz–Priddy [25] and Kleinerman [32] show that $H^*(BG(q))$ is ci provided $H^*(BG)$ has no p–torsion. Similarly Quillen shows the extraspecial groups [42] have ci cohomology rings. As in Example 6.8 this shows they are gci from the Eilenberg–Moore spectral sequence and hence eci by Theorem 12.1.

13B Squeezed homology

Since we are working with groups, it is illuminating to recall the first author's purely representation theoretic calculation of the loop space homology $H_*(\Omega(BG_p^\wedge))$ [8]. In fact

$$H_*(\Omega(BG_p^\wedge)) \cong H_*^{\Omega}(G; k),$$

where $H_*^{\Omega}(G; k)$ is defined algebraically.

More precisely $H_*^\Omega(G;k)$ is the homology of

$$\cdots \longrightarrow P_3 \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0,$$

a so-called *squeezed resolution* of k. The sequence of projective kG–modules P_i is defined recursively as follows. To start with $P_0 = P(k)$ is the projective cover of k. Now if P_i has been constructed, take $N_i = \ker(P_i \to P_{i-1})$ (where we take $P_{-1} = k$), and M_i to be the smallest submodule of N_i so that N_i/M_i is an iterated extension of copies of k. Now take P_{i+1} to be the projective cover of M_i.

13C Some simple cases

Note that if G is a p–group, we have $\Omega(BG_p^\wedge) \simeq G$ so that the topology focuses on $H_*(\Omega BG) \cong kG$ and since k is the only simple module, $M_0 = 0$ and we again find $H_*^\Omega(G) = kG$.

We would expect the next best behaviour to be when $H^*(BG)$ is a hypersurface. Indeed, if $H^*(BG)$ is a polynomial ring modulo a relation of codegree d, the Eilenberg–Moore spectral sequence

$$\mathrm{Ext}_{H^*(BG)}^{*,*}(k,k) \Rightarrow H_*(\Omega(BG_p^\wedge))$$

shows that there is an ultimate periodicity of period $d-2$. The actual period therefore divides $d-2$. This same phenomenon can be seen in the algebraic construction, where complete information about products is also available.

Here is an example where we can understand both the loop space homology and the Hochschild cohomology explicitly.

Example 13.2 $(G = p:q$ with $q \mid p-1)$ We take G to be a nontrivial semidirect product of C_p with C_q, where q is a divisor of $p-1$. Then

$$H^*(BG, k) = \Lambda(y) \otimes k[x],$$

where $|y| = -(2q-1)$ and $|x| = -2q$ (we grade everything homologically). So

$$H_*(\Omega BG_p^\wedge; k) = \Lambda(\xi) \otimes k[\eta],$$

where $|\xi| = 2q-1$ and $|\eta| = 2q-2$. The spectral sequence [46] for the Hochschild cohomology collapses to show

$$HH^*(C^*(BG)) = H^*(BG, H_*(\Omega BG_p^\wedge; k)) = \Lambda(y, \xi) \otimes k[x, \eta].$$

Alternatively, the spectral sequence

$$HH^*(H^*(BG, k)) \Rightarrow HH^*(C^* BG)$$

collapses, giving

$$HH^*(C^*(BG)) = \Lambda(y, \xi) \otimes k[x, \eta].$$

Since the spectral sequence of [46] collapses on the E^2 term, Theorem 11.2 shows BG is hhci. Explicitly, we can take the element $1 \otimes \eta \in HH^*(BG)$ to show the hhci condition holds.

Example 13.3 ($G = A_4$ with $p = 2$) To start with we use a homotopy theoretic proof, showing that BA_4 is an s–hypersurface space and hence also hhci.

Indeed, the natural 3–dimensional representation $A_4 \to \mathrm{SO}(3)$ gives a 2–adic fibration

$$S^3 \longrightarrow BA_4 \longrightarrow B\mathrm{SO}(3),$$

and BA_4 is an s–hypersurface space at 2 with $B\Gamma = B\mathrm{SO}(3)$, and $n = 3$ (ie $d = -4$).

The stable cofibre sequence establishing that BA_4 is hhci will then be

$$BA_4 \times_{B\mathrm{SO}(3)} BA_4 \longleftarrow BA_4 \longleftarrow \Sigma^2 BA_4,$$

and the periodicity element will be

$$\chi \in HH^{-2}(BA_4 | B\mathrm{SO}(3)).$$

Next we outline a purely algebraic proof. To start with, we would like to see algebraically that $H_*(\Omega(BA_4)_2^\wedge)$ is eventually periodic. Although this calculation is already in [8], we recall it briefly, since we refer to it below.

This case is small enough to be able to compute products in $H_*(\Omega BG_p^\wedge, k)$ using squeezed resolutions, and we get

$$H_*(\Omega BG_p^\wedge, k) = \Lambda(\alpha) \otimes k\langle\beta, \gamma\rangle/(\beta^2, \gamma^2)$$

with $|\alpha| = 1$ and $|\beta| = |\gamma| = 2$. Beware that β and γ do not commute, so that a k–basis for $H_*(\Omega BG_p^\wedge, k)$ is given by alternating words in β and γ (such as $\beta\gamma\beta$ or the empty word), and α times these alternating words.

First note that

$$H^*(BA_4) = H^*(BV_4)^{A_4/V_4} = k[x_2, y_3, z_3]/(r_6),$$

where $r_6 = x_2^3 + y_3^2 + y_3 z_3 + z_3^2$. From the Eilenberg–Moore spectral sequence we see that the loop space homology will eventually have period dividing 4, and by calculation with the squeezed resolution we find the eventual period is exactly 4.

There are three simple modules. Indeed, the quotient of A_4 by its normal Sylow 2–subgroup is of order 3; supposing for simplicity that k contains three cube roots of

unity $1, \omega, \overline{\omega}$, the simples correspond to how a chosen generator acts. The projective covers of the three simple modules are

$$P(k) = \omega \overset{\displaystyle k}{\underset{\displaystyle k}{\diamondsuit}} \overline{\omega} \,, \quad P(\omega) = \overline{\omega} \overset{\displaystyle \omega}{\underset{\displaystyle \omega}{\diamondsuit}} k \,, \quad P(\overline{\omega}) = k \overset{\displaystyle \overline{\omega}}{\underset{\displaystyle \overline{\omega}}{\diamondsuit}} \omega \,.$$

Turning to Hochschild cohomology, $HH^*(C^*(BG, k))$ is necessarily graded commutative, and we have

$$HH^*(H^*(BG, k)) = k[x, y, z]/(x^3 + y^2 + yz + z^2) \otimes \Lambda(\xi, \eta, \zeta) \otimes k[\rho]$$

with $|\rho| = 4$. We expect that both spectral sequences collapse.

Example 13.4 $(G = L_3(2)$ at $p = 2)$ The behaviour is essentially the same as that of $G = A_4$ with $p = 2$, as described in [8], and we give details here of a different group by way of variation.

First

$$H^*(BL_3(2)) = k[x_2, y_3, z_3]/(r_6),$$

where $r_6 = y_3 z_3$. From the Eilenberg–Moore spectral sequence we see that the loop space homology will eventually have period dividing 4.

There are three simple modules, the trivial module k, and two others, M and N. The projective covers of the three simple modules are

$$P(k) = M \overset{\displaystyle k}{\underset{\displaystyle k}{\diamondsuit}} N \,, \quad P(M) = \begin{matrix} & M & \\ N & & \\ | & & \\ M & & k \\ | & & \\ N & & \\ & M & \end{matrix} \,, \quad P(N) = \begin{matrix} & N & \\ M & & \\ | & & \\ N & & k \\ | & & \\ M & & \\ & N & \end{matrix} \,.$$

The squeezed resolution takes the following form, where the top row records the projective modules P_i, and the second row records the modules N_i; the modules M_i are obtained by deleting the copies of k marked with an asterisk.

Thus we see that

$$M_k \cong M_{k+2} \quad \text{for } k \geq 1.$$

$$\to P(M) \oplus P(N) \longrightarrow P(N) \oplus P(M) \longrightarrow P(N) \oplus P(M) \longrightarrow P(M) \oplus P(N) \longrightarrow P(k)$$

Example 13.5 (The cochains on $BL_3(3)$ at the prime 2) To start with, we note that this is also an example for M_{11}. Indeed, the principal blocks of $L_3(3)$ and of M_{11} are Morita equivalent at the prime 2 [2; 21], and there is a 2–adic equivalence $BL_3(3) \simeq BM_{11}$ [41].

In any case, we take $X = BL_3(3)$ because we want to use the Chevalley group properties. This example is of interest since $BL_3(3)$ is gci of codimension 1, whereas it is not wsci of codimension 1. Since $BL_3(3) \simeq BSU_3(3)$ it is wsci of codimension 2 as in Section 13A in view of the 2–adic fibration

$$SU(3) \longrightarrow BL_3(3) \longrightarrow BSU(3).$$

Since it has cohomology ring

$$H^*(BL_3(3); k) \cong k[x_3, y_4, z_5]/(r_{10}), \quad \text{with } r_{10} = x^2 y + z^2,$$

we see that to be sci of codimension 1, we would need a space with polynomial cohomology $k[x_3, y_4, z_5]$, which does not exist [3].

From the Eilenberg–Moore spectral sequence we see that the loop space homology of G will eventually have period dividing 8, and we may see this explicitly in terms of representation theory.

There are three simple modules: the trivial module k, a module M of dimension 10 and a module N of dimension 44. The associated projective covers $P(k)$, $P(M)$ and $P(N)$ are as follows

The squeezed resolution takes the following form, where the top row records the projective modules P_i, and the second row records the modules N_i; the modules M_i are obtained by deleting the copies of k marked with an asterisk. The somewhat delicate part of the following calculation is the fact that the kk^* in N_5 has the effect of creating a module M_5 with the same structure as M_1.
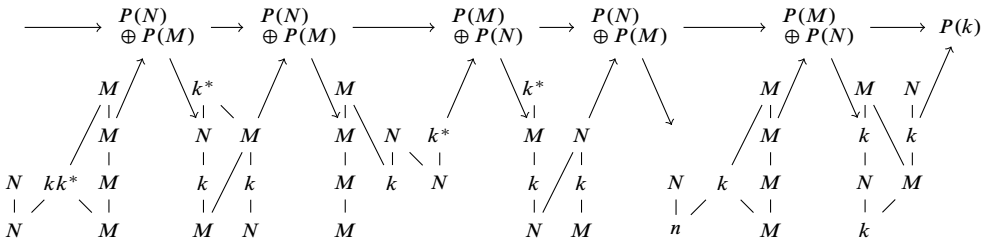
Thus we see that

$$M_k \cong M_{k+4} \quad \text{for } k \geq 1.$$

References

[1] **J Aguadé**, **C Broto**, **D Notbohm**, *Homotopy classification of spaces with interesting cohomology and a conjecture of Cooke, I*, Topology 33 (1994) 455–492 MR1286926

[2] **J L Alperin**, **R Brauer**, **D Gorenstein**, *Finite groups with quasi-dihedral and wreathed Sylow 2–subgroups*, Trans. Amer. Math. Soc. 151 (1970) 1–261 MR0284499

[3] **K K S Andersen**, **J Grodal**, *The classification of 2–compact groups*, J. Amer. Math. Soc. 22 (2009) 387–436 MR2476779

[4] **K K S Andersen**, **J Grodal**, **J M Møller**, **A Viruel**, *The classification of p–compact groups for p odd*, Ann. of Math. 167 (2008) 95–210 MR2373153

[5] **L L Avramov**, *Infinite free resolutions*, from: "Six lectures on commutative algebra", (J Elias, J M Giral, R M Miró-Roig, S Zarzuela, editors), Progr. Math. 166, Birkhäuser, Basel (1998) 1–118 MR1648664

[6] **L L Avramov**, *Locally complete intersection homomorphisms and a conjecture of Quillen on the vanishing of cotangent homology*, Ann. of Math. 150 (1999) 455–487 MR1726700

[7] **L L Avramov**, **R-O Buchweitz**, *Support varieties and cohomology over complete intersections*, Invent. Math. 142 (2000) 285–318 MR1794064

[8] **D Benson**, *An algebraic model for chains on $\Omega BG_p^\wedge$*, Trans. Amer. Math. Soc. 361 (2009) 2225–2242 MR2465835

[9]　**D J Benson**, **J F Carlson**, *Projective resolutions and Poincaré duality complexes*, Trans. Amer. Math. Soc. 342 (1994) 447–488　MR1142778

[10]　**D J Benson**, **J P C Greenlees**, *Complete intersections and derived categories* (2009) arXiv:0906.4025

[11]　**N Castellana**, *The homotopic adjoint representation for the exotic p–compact groups*, preprint (1999)　Available at http://mat.uab.es/~natalia/articles/adjointodd.pdf

[12]　**N Castellana**, *On the p–compact groups corresponding to the p–adic reflection groups $G(q, r, n)$*, Trans. Amer. Math. Soc. 358 (2006) 2799–2819　MR2216246

[13]　**W G Dwyer**, *Strong convergence of the Eilenberg–Moore spectral sequence*, Topology 13 (1974) 255–265　MR0394663

[14]　**W G Dwyer**, **J P C Greenlees**, *Complete modules and torsion modules*, Amer. J. Math. 124 (2002) 199–220　MR1879003

[15]　**W G Dwyer**, **J P C Greenlees**, **S Iyengar**, *Duality in algebra and topology*, Adv. Math. 200 (2006) 357–402　MR2200850

[16]　**W Dwyer**, **J P C Greenlees**, **S Iyengar**, *Finiteness in derived categories of local rings*, Comment. Math. Helv. 81 (2006) 383–432　MR2225632

[17]　**W G Dwyer**, **C W Wilkerson**, *Homotopy fixed-point methods for Lie groups and finite loop spaces*, Ann. of Math. 139 (1994) 395–442　MR1274096

[18]　**D Eisenbud**, *Subrings of Artinian and Noetherian rings*, Math. Ann. 185 (1970) 247–249　MR0262275

[19]　**D Eisenbud**, *Homological algebra on a complete intersection, with an application to group representations*, Trans. Amer. Math. Soc. 260 (1980) 35–64　MR570778

[20]　**A D Elmendorf**, **I Kriz**, **M A Mandell**, **J P May**, *Rings, modules, and algebras in stable homotopy theory*, Mathematical Surveys and Monographs 47, American Mathematical Society (1997)　MR1417719

[21]　**K Erdmann**, *Algebras and semidihedral defect groups, II*, Proc. London Math. Soc. 60 (1990) 123–165　MR1023807

[22]　**Y Félix**, **S Halperin**, **J-C Thomas**, *Elliptic Hopf algebras*, J. London Math. Soc. 43 (1991) 545–555　MR1113392

[23]　**Y Félix**, **S Halperin**, **J-C Thomas**, *Rational homotopy theory*, Graduate Texts in Mathematics 205, Springer, New York (2001)　MR1802847

[24]　**Y Felix**, **J-C Thomas**, **M Vigué-Poirrier**, *The Hochschild cohomology of a closed manifold*, Publ. Math. Inst. Hautes Études Sci. (2004) 235–252　MR2075886

[25]　**Z Fiedorowicz**, **S Priddy**, *Homology of classical groups over finite fields and their associated infinite loop spaces*, Lecture Notes in Mathematics 674, Springer, Berlin (1978)　MR513424

[26] **J P C Greenlees**, *Spectra for commutative algebraists*, from: "Interactions between homotopy theory and algebra", (L L Avramov, J D Christensen, W G Dwyer, M A Mandell, B E Shipley, editors), Contemp. Math. 436, Amer. Math. Soc., Providence, RI (2007) 149–173 MR2355773

[27] **J P C Greenlees**, **K Hess**, **S Shamir**, *Complete intersections in rational homotopy theory*, J. Pure Appl. Algebra 217 (2013) 636–663 MR2983840

[28] **T H Gulliksen**, *A change of ring theorem with applications to Poincaré series and intersection multiplicity*, Math. Scand. 34 (1974) 167–183 MR0364232

[29] **T H Gulliksen**, *On the deviations of a local ring*, Math. Scand. 47 (1980) 5–20 MR600076

[30] **M Hovey**, **B Shipley**, **J Smith**, *Symmetric spectra*, J. Amer. Math. Soc. 13 (2000) 149–208 MR1695653

[31] **N R Kitchloo**, *Topology of Kac–Moody groups*, Ph.D. thesis, Massachusetts Institute of Technology (1998) MR2716803 Available at http://search.proquest.com/docview/304473679

[32] **S N Kleinerman**, *The cohomology of Chevalley groups of exceptional Lie type*, Mem. Amer. Math. Soc. 39 (1982) viii+82 MR668808

[33] **R Levi**, *On finite groups and homotopy theory*, Mem. Amer. Math. Soc. 118 (1995) xiv+100 MR1308466

[34] **R Levi**, *A counter-example to a conjecture of Cohen*, from: "Algebraic topology: New trends in localization and periodicity", (C Broto, C Casacuberta, G Mislin, editors), Progr. Math. 136, Birkhäuser, Basel (1996) 261–269 MR1397737

[35] **R Levi**, *On homological rate of growth and the homotopy type of $\Omega BG_p^\wedge$*, Math. Z. 226 (1997) 429–444 MR1483541

[36] **R Levi**, *On p–completed classifying spaces of discrete groups and finite complexes*, J. London Math. Soc. 59 (1999) 1064–1080 MR1709097

[37] **T Y Lin**, *Duality and Eilenberg–Mac Lane spectra*, Proc. Amer. Math. Soc. 56 (1976) 291–299 MR0402738

[38] **M A Mandell**, *E_∞ algebras and p–adic homotopy theory*, Topology 40 (2001) 43–94 MR1791268

[39] **J M Møller**, *N–determined 2–compact groups, I*, Fund. Math. 195 (2007) 11–84 MR2314074

[40] **J M Møller**, *N–determined 2–compact groups, II*, Fund. Math. 196 (2007) 1–90 MR2338539

[41] **B Oliver**, *Equivalences of classifying spaces completed at the prime two*, Mem. Amer. Math. Soc. 180 (2006) vi+102 MR2203209

[42] **D Quillen**, *The* mod 2 *cohomology rings of extra-special* 2*–groups and the spinor groups*, Math. Ann. 194 (1971) 197–212   MR0290401

[43] **D Quillen**, *On the cohomology and K–theory of the general linear groups over a finite field*, Ann. of Math. 96 (1972) 552–586   MR0315016

[44] **J-P Serre**, *Sur la dimension homologique des anneaux et des modules noethériens*, from: "Proceedings of the international symposium on algebraic number theory", Science Council of Japan, Tokyo (1956) 175–189   MR0086071

[45] **J Shamash**, *The Poincaré series of a local ring*, J. Algebra 12 (1969) 453–470   MR0241411

[46] **S Shamir**, *Hochschild cohomology of cochain-algebras and local coefficients*, In preparation

[47] **B Shipley**, *H$\mathbb{Z}$–algebra spectra are differential graded algebras*, Amer. J. Math. 129 (2007) 351–379   MR2306038

[48] **K Ziemiański**, *A faithful unitary representation of the 2–compact group* DI(4), J. Pure Appl. Algebra 213 (2009) 1239–1253   MR2497572

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