The Arone–Goodwillie spectral sequence for $\Sigma^\infty\Omega^n$ and topological realization at odd primes

SEBASTIAN BÜSCHER

FABIAN HEBESTREIT

OLIVER RÖNDIGS

MANFRED STELZER

We employ the Goodwillie spectral sequence for the iterated loop space functor in order to provide realizability conditions on certain unstable modules over the Steenrod algebra at an odd prime.

1 Introduction

A basic task of algebraic topology is to construct meaningful algebraic invariants of topological spaces. Singular cohomology can be viewed as such an invariant, but in many different ways. In order to specify the algebraic natures of singular cohomology appearing in this article, fix a finite prime field $\mathbb{F}_p$ as coefficients. A rather elementary view is to consider $H^*(X)$, the direct sum of all cohomology groups of a topological space X, as a graded $\mathbb{F}_p$–vector space. Via cup product, it in addition becomes an $\mathbb{F}_p$–algebra. Furthermore, $H^*(X)$ is a module over the Steenrod algebra $\mathcal{A}_p$, the algebra of all stable operations in singular cohomology with $\mathbb{F}_p$–coefficients. The realization question under consideration is whether a given algebraic structure, for example an $\mathcal{A}_p$–module M or an $\mathcal{A}_p$–algebra A, is isomorphic to $H^*(X)$ for some topological space. There are some obvious restrictions (the module should be unstable), but also more delicate ones. Nick Kuhn formulated the following realization conjecture in [16].

Conjecture 1.1 *For any topological space X such that $H^*(X)$ is a finitely generated $\mathcal{A}_p$–module, $H^*(X)$ is already a finite-dimensional $\mathbb{F}_p$–vector space.*

Kuhn proved Conjecture 1.1 under an additional assumption on the behaviour of the Bockstein. He employed Lannes' T–functor to construct from a hypothetical counterexample to Conjecture 1.1 an even more hypothetical topological space Y

whose cohomology $H^*(Y)$ has a specific finite composition series. Further ingredients in his proof that such a space Y cannot exist are the mod p solution of the Hopf invariant one problem, and results of Ravenel and Mahowald on the mod p Kervaire invariant problem.

Lionel Schwartz proved the general case of Conjecture 1.1 for $p = 2$ (without assumptions on the Bockstein) in [23]. Continuing with Kuhn's dubious space Y, his key idea was to analyze the induced unstable $\mathcal{A}_p$–algebra structure of $H^*(\Omega^n Y)$ for an appropriate n via an iterated use of the Eilenberg–Moore spectral sequence. The article [23] contains a brief discussion of the odd primary case as well; an erratum by Schwartz [24] pointing to a complete proof by Gaudens and Schwartz in [9] appeared recently.

As explained by Kuhn in [18], the iterated Eilenberg–Moore spectral sequences may be replaced with a single spectral sequence, defined by the Goodwillie tower of the functor $X \mapsto \Sigma^\infty \Omega^n X$. This approach lead him to a streamlined proof of Conjecture 1.1 at the prime 2 via nonrealization theorems for certain finite unstable modules over the Steenrod algebra $\mathcal{A}_2$.

The present paper is rooted in our attempt to generalize Kuhn's approach in [18] to odd primes. Our study of the Arone–Goodwillie spectral sequence at an odd prime is parallel to Smith's classical investigation of the Eilenberg–Moore spectral sequence in [25]. In order to state our main theorem, let $\Phi(k, k+2)$ be the quotient of the sub-$\mathcal{A}_p$–module of $H^*(K(\mathbb{Z}/p, 1))$ generated by an element $t \neq 0$ with $|t| = 2$, having $\mathbb{F}_p$–basis $\{t^{p^k}, t^{p^{k+1}}, t^{p^{k+2}}\}$. The module $\Phi(k, k+2)$ is closely related to the free unstable module $F(1)$ on a one-dimensional class described in [16]. The notion of a desuspension class of even origin is given in Definition 5.2.

Theorem 5.3 *Let M be an unstable $\mathcal{A}_p$–module of finite type, concentrated in degrees $[\ell, m]$. Suppose M contains a desuspension class of even origin. If X is a topological space with*

$$\widetilde{H}^*(X) \cong M \otimes \Phi(k, k+2)$$

as $\mathcal{A}_p$–modules, then $2p^k \leq (p^2 - 1)m + p(m - \ell)$.

Theorem 5.3 generalizes Kuhn's nonrealization theorems mentioned above only in roughly half of all the cases. The reason for this may be traced back to the fact that, at odd primes, the parity of certain elements determines the type of operation detecting the desuspension behaviour of a given unstable $\mathcal{A}_p$–module M. Only in the case the Bockstein is not involved, a direct relation to an unstable algebra structure on M can be made. The arguments leading to the nonrealization theorems use this relation in the

unstable algebra $H^*(\Omega^n X)$ in an essential way. Theorem 5.3 implies a new proof of Kuhn's Conjecture 1.1 in the case that the Bockstein acts trivially in high degrees.

Gaudens and Schwartz gave a proof of a generalized form of Kuhn's Conjecture 1.1 for all primes [9]. However, contrary to Theorem 5.3, their argument gives no information about topologically realizing finite modules by its very nature.

The paper is organized as follows. Section 2 recalls some facts about the spectral sequence from work of Goodwillie, Arone, Ahearn and Kuhn. Besides the right $\mathcal{A}_p-$action, the homology spectral sequence comes with an action of Dyer–Lashof and Browder operations. The interplay between these structures is essential to our proofs. In order to employ algebra structures, we switch to cohomology. This forces us to define and study operations dual to the corresponding homology operations, which is presented in Section 3. The first nontrivial differentials are investigated and related to the operational structure in Section 4. We prove our nonrealization theorems in Section 5. They improve on the existing results in the literature.

Finally there are two appendices. In Appendix A the cohomology of a free algebra over a suitable E_n operad in spectra is computed by reduction to the unstable calculation by Cohen, Lada and May [8]. This result is needed in Section 5. Appendix B mainly consists of a proof for cohomological Nishida relations on the dual operations defined in Section 3, again needed in Section 5.
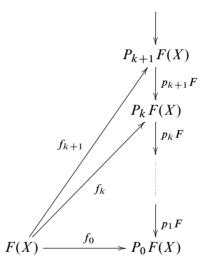
1.1 Notation and conventions

All (co)homology groups have coefficients in the prime field $\mathbb{F}_p$, where p is an odd prime. The corresponding Eilenberg–Mac Lane spectrum is denoted $H\mathbb{Z}/p$. Unless stated otherwise, topological spaces have the homotopy type of cell complexes. From Section 3 on all spectra or topological spaces X are bounded below with $H^*(X)$ of finite type, ie, finite-dimensional in each degree. The n–fold shift of a graded module M is denoted $\mathrm{sh}^n M$ and satisfies $(\mathrm{sh}^n M)^k = M^{k-n}$. Tensor products are over $\mathbb{F}_p$, unless stated otherwise.

2 The Arone–Goodwillie spectral sequence

Goodwillie's calculus of homotopy functors, as developed in [10; 11; 12], provides an interpolation between unstable and stable phenomena in homotopy theory. The layers of this interpolation carry a surprisingly rich structure. The purpose of this section is to describe the interpolation in the examples relevant to us, following work of Goodwillie, Johnson and Arone.

Let **Top**$_\bullet$ be a convenient category of pointed topological spaces (for example, compactly generated spaces), and let **Spt** be a convenient category of S^1–spectra in **Top**$_\bullet$ (for example, orthogonal spectra). Suppose $F \colon \mathcal{C} \to$ **Spt** is a functor preserving weak equivalences, where $\mathcal{C}$ is either the category **Top**$_\bullet$ or **Spt**. Goodwillie's calculus of functors provides a sequence of functors $P_n F \colon \mathcal{C} \to$ **Spt** and a natural tower of fibrations for any $X \in \mathcal{C}$:

The transformation $F \to P_k F$ is universal among k–excisive approximations of F. The fiber $D_k F$ of $p_k F \colon P_k F \to P_{k-1} F$ is called the k–th differential of F and may be thought of as the degree-k homogeneous part of the polynomial functor $P_k F$. Applying any cohomology functor h^* to the Goodwillie tower produces an exact couple and hence a spectral sequence (E, d) with

$$(1) \qquad E_1^{-j,k} = \begin{cases} h^{k-j}(D_j F(X)) & j \geq 0, \\ 0 & j < 0. \end{cases}$$

This Goodwillie spectral sequence will be exploited for the functors

$$\Sigma^\infty \Omega^n \colon \mathbf{Top}_\bullet \to \mathbf{Spt}, \quad \Sigma^\infty \Omega^\infty \colon \mathbf{Spt} \to \mathbf{Spt},$$

composed with $h^* = H^*(-; \mathbb{F}_p)$. Arone gave an explicit model of the Goodwillie tower for the functor

$$\Sigma^\infty \operatorname{Map}_\bullet(K, -) \colon \mathbf{Top}_\bullet \to \mathbf{Spt}$$

with K a finite (pointed) cell complex in [2]. This model was used by Ahearn and Kuhn in [1] to describe the k–th differential

$$D_{n,k} := D_k \Sigma^\infty \Omega^n$$

of the Goodwillie tower of $\Sigma^\infty \Omega^n = \Sigma^\infty \operatorname{Map}_\bullet(S^n, -)$ via the operad of little cubes introduced by Boardman and Vogt in [4]. In order to present this description, let $\mathcal{C}(n, k)$ be the space of k disjoint little n–cubes in a bigger n–cube (cf May [21]), with the obvious action of the symmetric group Σ_k.

Theorem 2.1 *Let $1 \leq n \in \mathbb{N}$.*

(1) *For any pointed topological space X, there is a natural weak equivalence of spectra*

$$D_{n,k}(X) \simeq \mathcal{C}(n, k)_+ \wedge_{h\Sigma_k} (\Sigma^{-n} \Sigma^\infty X)^{\wedge k}.$$

(2) *For any spectrum X, there is a natural weak equivalence of spectra*

$$D_{\infty,k}(X) \simeq \mathcal{C}(\infty, k)_+ \wedge_{h\Sigma_k} X^{\wedge k}.$$

Proof The first weak equivalence is stated as [1, Equations (1,2)] and is obtained as a consequence of [1, Theorem 7.1]. The second weak equivalence is [1, Corollary 1.3]. □

The special cases $D_{1,k}(X) \simeq X^{\wedge k}$, $D_{\infty,k}(X) \simeq X^{\wedge k}_{h\Sigma_k}$, $D_{n,1}(X) \simeq \Sigma^{-n} \Sigma^\infty X$ follow from various properties of the spaces $\mathcal{C}(n, k)$. Furthermore, by convention $D_{n,0}(X) = \mathbb{S}$ is the sphere spectrum. The case $n = \infty$ is used for universal examples in upcoming proofs. In order to simplify the exposition, it will sometimes be included without explicit notational changes.

Note that convergence of the Goodwillie tower requires further assumptions on the variable X. For example, in the first case of Theorem 2.1, convergence holds by [2] on n–connected pointed topological spaces, whereas in the second case, convergence holds on 0–connected spectra [1, Corollary 1.3]. The Arone–Goodwillie spectral sequence arising from $\Sigma^\infty \Omega^n$ in singular cohomology with $\mathbb{F}_p$ coefficients will simply be referred to as 'the spectral sequence for n', infinity included. Connectivity assumptions on X alone do not necessarily imply strong convergence of the spectral sequence for n. However, in case of finiteness assumptions on the cohomology of X, results of Bousfield on an analogous homology spectral sequence [6] imply the following statement.

Theorem 2.2 *The spectral sequence for n converges strongly to $H^*(\Omega^n X)$ if X is an n–connected pointed cell complex ($n < \infty$) respectively a connected spectrum ($n = \infty$) such that $H^*(X)$ is dimensionwise finite. Furthermore, it is a spectral sequence of $\mathcal{A}_p$–modules, where $\mathcal{A}_p$ acts columnwise on each page and the $\mathcal{A}_p$–module structure on the E_1–page is the obvious one. The spectral sequence converges to the usual $\mathcal{A}_p$–module structure on $H^*(\Omega^n X)$.*

These spectral sequences are tied together as n varies. More precisely, there are natural maps

$$(2) \qquad \epsilon_{r,n} \colon \Sigma^r \Sigma^\infty \Omega^{n+r} X \to \Sigma^\infty \Omega^n X$$

induced by the evaluation $\Sigma \Omega X \to X$ for any pointed topological space X, as well as natural equivalences

$$(3) \qquad \varphi_n \colon \Sigma^\infty \Omega^n \Omega^\infty \Sigma^n X \to \Sigma^\infty \Omega^\infty X$$

for any spectrum X. These maps induce maps on the Goodwillie towers, hence, of the associated spectral sequences. In particular, they induce maps on the differentials, which will be of use later on.

3 Operations

In this section, following Kuhn's treatment for the case $p = 2$, we define and study operations in the cohomology of $\mathcal{C}_n$–extended powers $D_{n,j} X$ in the category of spectra which are closely related to the Browder operations, Dyer–Lashof operations and Pontryagin products in homology. The standard references for these homology operations are [8] and Bruner, May, McClure and Steinberger [7], with some corrections to be found by Wellington in [28]. We refer to these sources for the properties of and relations between the homology operations. To be more precise, the reindexed operations Q^r are studied in loc.cit., but the operations which appear as Q_r or P_r in the literature are more convenient in our setup. We employ the Dyer–Lashof notation $Q_{(n-1)(p-1)}, Q_{(n-1)(p-1)-1}$ for the two top operations denoted by ξ and ζ in [8]. Of course one should keep in mind that ξ is the single nonlinear operation among these. Furthermore, the Bockstein of ξ is ζ only up to a term consisting of a sum of p–fold Browder operations if $n > 1$.

3.1 Definitions

We have to fix notation for the structure maps which show up in the definitions of the operations.

Definition 3.1 In the following, $D_{n,k}(X)$ is equipped with the $\mathcal{C}_n$–operad action via the weak equivalence given in Theorem 2.1.

(1) Let
$$\epsilon \colon \Sigma^r D_{n+r,j} X \to D_{n,j} \Sigma^r X$$

denote the map induced by the evaluation map $\Sigma^r \Sigma^\infty \Omega^{n+r} X \to \Sigma^\infty \Omega^n X$ from (2).

(2) Let
$$\mu \colon D_{n,i} X \wedge D_{n,j} X \to D_{n,i+j} X$$

be the map induced by the operad action θ of some fixed element $c_2 \in \mathcal{C}_{n,2}$
$$\theta(c_2, -, -) \colon \mathcal{C}_{n,i} \times \mathcal{C}_{n,j} \longrightarrow \mathcal{C}_{n,i+j}$$

(see Lewis, May, Steinberger and McClure [19, Chapter VII] for details).

(3) Let
$$\tau \colon D_{n,i+j} X \to D_{n,i} X \wedge D_{n,j} X$$

denote the composition of the maps
$$(\mathcal{C}(n, i+j)_+ \wedge X^{\wedge i+j})_{h\Sigma_{i+j}} \to (\mathcal{C}(n, i+j)_+ \wedge X^{\wedge i+j})_{h\Sigma_i \times \Sigma_j},$$
$$(\mathcal{C}(n, i+j)_+ \wedge X^{\wedge i+j})_{h\Sigma_i \times \Sigma_j} \to (\mathcal{C}(n, i)_+ \wedge \mathcal{C}(n, j)_+ \wedge X^{\wedge i+j})_{h\Sigma_i \times \Sigma_j},$$

of homotopy orbits, where the first map is the transfer associated to $\Sigma_i \times \Sigma_j \subset \Sigma_{i+j}$ and the second map is induced by the $\Sigma_i \times \Sigma_j$–equivariant inclusion of spaces $\mathcal{C}(n, i+j) \subset \mathcal{C}(n, i) \times \mathcal{C}(n, j)$.

(4) Let
$$\omega \colon D_{n,pj} X \to D_{\infty,p} D_{n,j} X$$

denote the composition of maps
$$(\mathcal{C}(n, pj)_+ \wedge X^{\wedge pj})_{h\Sigma_{pj}} \to (\mathcal{C}(n, pj)_+ \wedge X^{\wedge pj})_{h\Sigma_p \wr \Sigma_j},$$
$$(\mathcal{C}(n, pj)_+ \wedge X^{\wedge pj})_{h(\Sigma_p \wr \Sigma_j)} \to (\mathcal{C}(n, j)_+^p \wedge X^{\wedge pj})_{h(\Sigma_p \wr \Sigma_j)},$$

of homotopy orbits, where the first map is the transfer associated to $\Sigma_p \wr \Sigma_j \subset \Sigma_{pj}$ and the second map is induced by the $\Sigma_p \wr \Sigma_j$–equivariant inclusion of spaces $\mathcal{C}(n, pj) \subset \mathcal{C}(n, j)^p$.

In the following, the notion of naturality has to be understood in the context of E_n–spectra, or more generally of H_n–spectra. Note that the corresponding definition in the case $p = 2$ is less involved, as then $0 = p - 2$.

Definition 3.2 Let $\infty \geq n \geq 1$ and $n \geq r \geq 0$ be given. Let $\iota \in H_d(\Sigma^d H\mathbb{Z}/p)$ be the fundamental class given by the inclusion of the bottom cell. The natural operation

$$\mathcal{Q}^r \colon H^d(D_{n,j}X) \to H^{pd+r}(D_{n,pj}X)$$

is defined as follows.

$\mathcal{Q}^0$ For d even and $x \in H^d(X)$, let $\mathcal{Q}^0(x) \in H^{pd}(D_{\infty,p}X)$ be the composition

$$D_{\infty,p}X \xrightarrow{D_{\infty,p}(x)} D_{\infty,p}\Sigma^d H\mathbb{Z}/p \xrightarrow{\widehat{\iota}_0} \Sigma^{pd}H\mathbb{Z}/p,$$

where $\widehat{\iota}_0$ represents the dual of $Q_0(\iota) \in H_{pd}(D_{\infty,p}\Sigma^d H\mathbb{Z}/p)$, ie the Dyer–Lashof operation evaluated at the fundamental class. For d even and $x \in H^d(D_{n,j}X)$, let $\mathcal{Q}^0(x) \in H^{pd}(D_{n,pj}X)$ be the composition

$$D_{n,pj}X \xrightarrow{\omega} D_{\infty,p}D_{n,j}X \xrightarrow{\mathcal{Q}^0(x)} \Sigma^{pd}H\mathbb{Z}/p,$$

with ω introduced in [Definition 3.1(4)](#). For $x \in H^d(D_{n,j}X)$ with d odd, $\mathcal{Q}^0(x)$ is defined to be zero.

$\mathcal{Q}^{p-2}$ For $x \in H^d(X)$ with d odd, let $\mathcal{Q}^{p-2}(x) \in H^{pd+(p-2)}(D_{\infty,p}X)$ be the composition

$$D_{\infty,p}X \xrightarrow{D_{\infty,p}(x)} D_{\infty,p}\Sigma^d H\mathbb{Z}/p \xrightarrow{\widehat{\iota}_{p-2}} \Sigma^{pd+(p-2)}H\mathbb{Z}/p,$$

where $\widehat{\iota}_{p-2}$ represents the dual of the evaluation of the Dyer–Lashof operation $Q_{p-2}(\iota) \in H_{pd+(p-2)}(D_{\infty,p}\Sigma^d H\mathbb{Z}/p)$ at the fundamental class. For $x \in H^d(D_{n,j}X)$ with d odd, let $\mathcal{Q}^{p-2}(x) \in H^{pd+(p-2)}(D_{n,pj}X)$ be the composition

$$D_{n,pj}X \xrightarrow{\omega} D_{\infty,p}D_{n,j}X \xrightarrow{\mathcal{Q}^{p-2}(x)} \Sigma^{pd+(p-2)}H\mathbb{Z}/p,$$

with ω introduced in [Definition 3.1(4)](#). For $x \in H^d(D_{n,j}X)$ with d even, $\mathcal{Q}^{p-2}(x)$ is defined to be zero.

$\mathcal{Q}^{s(p-1)}$ Let $0 < s \leq n-1$ and $x \in H^d(D_{n,j}X)$, with $d+s$ even. Define $\mathcal{Q}^{s(p-1)}(x) \in H^{pd+s(p-1)}(D_{n,pj}X)$ to be the composition

$$D_{n,pj}X \xrightarrow{\epsilon} \Sigma^{-s}D_{n-s,pj}\Sigma^s X \xrightarrow{\Sigma^{-s}\mathcal{Q}^0(\Sigma^s x)} \Sigma^{pd+s(p-1)}H\mathbb{Z}/p,$$

with ϵ introduced in [Definition 3.1(1)](#). For $x \in H^d(D_{n,j}X)$ with $d+s$ odd, $\mathcal{Q}^{s(p-1)}(x)$ is defined to be zero.

$\mathcal{Q}^{s(p-1)-1}$ Let $0 < s \leq n - 1$ and $x \in H^d(D_{n,j}X)$, with $d + s$ even. Define $\mathcal{Q}^{s(p-1)-1}(x) \in H^{pd+r(p-1)-1}(D_{n,pj}X)$ to be the composition

$$D_{n,pj}X \xrightarrow{\epsilon} \Sigma^{1-s}D_{n+1-s,pj}\Sigma^{(s-1)}X \xrightarrow{\Sigma^{1-s}\mathcal{Q}^{p-2}(\Sigma^{(s-1)}x)} \Sigma^{pd+s(p-1)-1}H\mathbb{Z}/p,$$

with ϵ introduced in [Definition 3.1(1)](). For $x \in H^d(D_{n,j}X)$ with $d + s$ odd, we define $\mathcal{Q}^{s(p-1)-1}(x)$ to be zero.

$\mathcal{Q}^r$ All other $\mathcal{Q}^r$ are defined to be zero.

Thus the only (possibly) nonzero dual Dyer–Lashof operations are

$$\mathcal{Q}^0, \mathcal{Q}^{p-2}, \mathcal{Q}^{p-1}, \mathcal{Q}^{2p-3}, \mathcal{Q}^{2p-2}, \ldots, \mathcal{Q}^{(n-1)(p-1)}.$$

The dual Browder operations are obtained as follows.

Definition 3.3   Let $d = \sum_{i=1}^{j} d_i$. The dual Browder operation is the homomorphism

$$\mathcal{L}^{n-1} \colon H^{d_1}(X) \otimes \cdots \otimes H^{d_j}(X) \to H^{d+(j-1)(n-1)}(D_{n,j}X)$$

induced by the map

$$\epsilon^{n-1} \colon D_{n,j}X \to \Sigma^{1-n}D_{1,j}\Sigma^{n-1}X = \Sigma^{(j-1)(n-1)}X^{\wedge j},$$

with ϵ introduced in [Definition 3.1(1)]().

Note that $\mathcal{L}^0$ coincides with the Künneth isomorphism. Our final definition in this section describes a "product" resp. a "coproduct" structure on $\bigoplus_k H^*(D_{n,k}X)$.

Definition 3.4   The map t introduced in [Definition 3.1(3)]() induces a pairing

$$\star \colon H^*(D_{n,i}X) \otimes H^*(D_{n,j}X) \to H^*(D_{n,i+j}X).$$

The map μ introduced in [Definition 3.1(2)]() induces a copairing

$$\Psi \colon H^*(D_{n,i+j}X) \to H^*(D_{n,i}X) \otimes H^*(D_{n,j}X).$$

3.2   Properties

This section summarizes some properties of the operations and pairings defined in [Section 3.1]().

Proposition 3.5 *The map $\epsilon^*\colon H^*(D_{n,j}\Sigma X) \to \mathrm{sh}^1 H^*(D_{n+1,j}X)$ commutes with the dual Dyer–Lashof and dual Browder operations. More precisely, the following equalities hold:*

$$\epsilon^*(Q^r(\sigma x)) = Q^{r+p-1}(x),$$

$$\epsilon^*(\mathcal{L}^{n-1}(\otimes\sigma(x_i))) = \mathcal{L}^n(\otimes x_i).$$

Proof This is a direct consequence of Definitions 3.2 and 3.3.   □

Fix $1 \le n \in \mathbb{N}$. In order to recall some properties of the Dyer–Lashof operations

$$Q_r\colon H_*(D_{n,j}X) \to H_*(D_{n,pj}X),$$

the Browder operations

$$L_{n-1}\colon H_*(D_{n,i}X) \otimes H_*(D_{n,j}X) \to H_*(D_{n,i+j}X)$$

and the Pontryagin product

$$*\colon H_*(D_{n,i}X) \otimes H_*(D_{n,j}X) \to H_*(D_{n,i+j}X)$$

from the literature, let $q \in \mathbb{N}$ and $r = t(p-1) + k$ with $0 \le k < p-1$. Set

(4a)
$$\gamma(q,r) = \prod_{i=0}^{t-1}(-1)^{\frac{(p-1)(q+i)}{2}}\left(\left(\tfrac{p-1}{2}\right)!\right)^t,$$

(4b)
$$\nu(q) = (-1)^{\frac{(p-1)q}{2}}\left(\tfrac{p-1}{2}\right)!,$$

(4c)
$$\lambda(q) = \prod_{i=0}^{n-2}\nu(q+i).$$

The (co)homological degree of a homogeneous (co)homology class y is denoted $|y|$.

Proposition 3.6 *The map $\epsilon_*\colon \mathrm{sh}^1 H_*(D_{n+1,j}X) \to H_*(D_{n,j}\Sigma X)$ satisfies the following:*

(1) *The equation $\epsilon_*(\sigma Q_r(y)) = \nu(|y|)Q_{r-(p-1)}(\sigma(y))$ holds, where Q_i is understood to be trivial for negative i.*

(2) *For all $x, y \in H_*(D_{n,i}X)$ one has $\epsilon_*(\sigma L_n(x,y)) = L_{n-1}(\sigma x, \sigma y)$.*

(3) *All Pontryagin products of elements in positive degrees are in the kernel of ϵ_*.*

Proof The sources quoted above consider the operations in the homology of loop spaces, not C_n–spectra. As explained in Appendix A., one can reduce to this situation via a suitable Thom isomorphism. For parts (1) and (2) see [8, III.1.4.], which discusses all cases except $Q_{(n-1)(p-1)} = \zeta$. The remaining case is treated via the equality $\zeta(x) = \beta\xi(x) - ad^{p-1}(x)(\beta(x))$, where $\beta \in \mathcal{A}_p$ is the Bockstein. The assertion in part (3) is classical and can be found for example by Whitehead in [29]. $\qquad\square$

Proposition 3.7 *The following hold for* $\tau_*\colon H_*(D_{n,k}X) \to H_*(X)^{\otimes k}$:

(5a)
$$\tau_*(Q_r(y)) = 0,$$

(5b)
$$\tau_*(*_{i=1}^k y_i) = \sum_{\sigma \in \Sigma_k} \bigotimes_{i=1}^k y_{\sigma(i)},$$

(5c)
$$\tau_*(L_{n-1}(y, z)) = 0.$$

Proof This follows immediately from the definitions. $\qquad\square$

In the following $L_{n-1}(y_1, y_2, \ldots, y_k)$ denotes any k–fold iteration of Browder operations on elements $y_1, \ldots, y_n \in H_*(X)$.

Proposition 3.8 *Let* $k \in \mathbb{N}$, $x, x_1, \ldots, x_k \in H^*(X)$, $y, y_1, \ldots, y_k \in H_*(X)$, $w \in H^*(D_{n,k}X)$, $z \in H_*(D_{n,k}X)$ *and* $z_1, \ldots, z_k \in \bigoplus_j H_*(D_{n,j}X)$, *satisfying* $*_{j=1}^k z_j \in H_*(D_{n,k}X)$. *Under the Kronecker pairing the cohomology and the homology operations pair as follows:*

(1) $\langle \mathcal{Q}^r(w), Q_s(z)\rangle = \begin{cases} \gamma(|w|, r)\langle w, z\rangle & r = s = t(p-1) - \epsilon, \ \epsilon \in \{0, 1\}, \\ 0 & \text{otherwise}; \end{cases}$

(2) $\langle \mathcal{Q}^r(x), L_{n-1}(y_1, y_2, \ldots, y_k)\rangle = 0$;

(3) $\langle \mathcal{Q}^r(x), *_{i=1}^k y_i\rangle = \begin{cases} \prod_{i=1}^k \langle x, y_i\rangle & r = 0 \text{ and } k = p, \\ 0 & \text{otherwise}; \end{cases}$

(4) $\langle \mathcal{L}^{n-1}(\otimes_{i=1}^k x_i), Q_s(y)\rangle = \begin{cases} \lambda(|y|) \prod_{i=1}^p \langle x_i, y\rangle & k = p, s = (n-1)(p-1), \\ 0 & \text{otherwise}; \end{cases}$

(5) $\langle \mathcal{L}^{n-1}(\otimes_{i=1}^k x_i), L_{n-1}(y_1, y_2, \ldots, y_k)\rangle = \langle \otimes_{i=1}^k x_i, L_0(y_1, y_2, \ldots, y_k)\rangle$;

(6) $\langle \mathcal{L}^{n-1}(\otimes_{i=1}^k x_i), *_{j=1}^k z_j\rangle = 0$;

(7) $\langle \star_{i=1}^p x_i, Q_s(y)\rangle = 0$;

(8) $\langle \star_{i=1}^k x_i, L_{n-1}(y_1, y_2, \ldots, y_k)\rangle = 0$;

(9) $\langle \star_{i=1}^k x_i, *_{i=1}^k y_i\rangle = \sum_{\sigma \in \Sigma_k} \prod_{i=1}^k \langle x_i, y_{\sigma(i)}\rangle.$

Proof (1) This follows from the definitions and Proposition 3.6(1), together with the congruence $(m!)^2 = (-1)^{m+1} \mod p$, which is a consequence of Wilson's theorem.

(2) The morphism of operads

$$\eta_n \colon C_n \to C_{n+1}$$

from [21, page 31] induces an H_n–map

$$s(n) \colon D_{n,i} X \to D_{n+1,i} X$$

by [7]. Any $x \in H^*(X) \cong H^*(D_{n,1}X)$ can be written as $x = s(n)^* x'$ for $x' \in H^*(D_{n+1,1}X) \cong H^*(X)$. Then

$$\begin{aligned}
\langle \mathcal{Q}^r(x), L_{n-1}(y_1, y_2, \ldots, y_k) \rangle &= \langle \mathcal{Q}^r(s(n)^* x'), L_{n-1}(y_1, \ldots, y_k) \rangle \\
&= \langle s(n)^* \mathcal{Q}^r(x'), L_{n-1}(y_1, \ldots, y_k) \rangle \\
&= \langle \mathcal{Q}^r(x'), s(n)_* L_{n-1}(y_1, \ldots, y_k) \rangle \\
&= \langle \mathcal{Q}^r(x'), L_{n-1}(s(n)_* y_1, \ldots, s(n)_* y_k) \rangle \\
&= \langle \mathcal{Q}^r(x'), 0 \rangle = 0,
\end{aligned}$$

because the Browder operations L_{n-1} vanish on C_{n+1}–spectra by their very nature.

(3) For r different from 0 or $p-2$ the assertion follows from the definition of $\mathcal{Q}^r$ and Proposition 3.6(3). Since $\mathcal{Q}^0$ is essentially the dual of the p–fold Pontryagin product operation, the assertion holds for $r = 0$. For $r = p - 2$ it suffices to consider classes of odd degree. Since $\iota^2 = 0$ for any generator $\iota \in H_d(D_{\infty,p} \Sigma^d H\mathbb{Z}/p)$ if d is odd, the group $H_{pd+p-2}(D_{\infty,p} \Sigma^d H\mathbb{Z}/p)$ is generated by the indecomposable element $Q_{p-2}(\iota)$. The assertion for $r = p - 2$ follows.

(4) This follows from Proposition 3.6(1) and the fact that Q_0 is the Frobenius.

(5) This follows from Proposition 3.5, Proposition 3.6(2) and the definition of a graded commutator, which describes L_0.

The assertions in (6), (7), (8) and (9) follow from Proposition 3.6(3), (5a), (5c) and (5b) respectively.   □

Proposition 3.9 *Let $x, y \in H^*(D_{n,j}X)$ and $r \in \mathbb{N}$. Then*

$$\mathcal{Q}^r(x + y) = \begin{cases} \mathcal{Q}^r(x) + \mathcal{Q}^r(y) & r > 0, \\ \mathcal{Q}^0(x) + \mathcal{Q}^0(y) + \sum_{k=1}^{p-1} \frac{1}{(p-1)!k!} x^{\star k} \star y^{\star(p-k)} & r = 0. \end{cases}$$

In particular, $\mathcal{Q}^r(u \cdot x) = u \cdot \mathcal{Q}^r(x)$ for every $u \in \mathbb{F}_p$.

Proof The proof is similar to the proof of Proposition 3.11 and will be given in Appendix B. □

In order to interpret the following proposition for all n including ∞, note that $\mathcal{L}^{n-1}(x) = x$ for any x, and that $\mathcal{L}^{\infty}(\bigotimes_{i=1}^{k} x_i) = 0$ whenever $k > 1$.

Proposition 3.10 Let $0 \le j < p^2$. The cohomology $H^*(D_{n,j} X)$ is generated as an algebra by elements $\mathcal{Q}^r \mathcal{L}^{n-1}(\bigotimes_{i=1}^{k} x_i)$ and $\mathcal{L}^{n-1}(\bigotimes_{i=1}^{k} x_i)$ where $r, k \in \mathbb{N}$ and $x_1, \ldots, x_k \in H^*(X)$. The degrees of these generators are

$$\left| \mathcal{L}^{n-1}\left(\bigotimes_{i=1}^{k} x_i \right) \right| = (k-1)(n-1) + \sum_{i=1}^{k} |x_i|,$$

$$\left| \mathcal{Q}^r \mathcal{L}^{n-1}\left(\bigotimes_{i=1}^{k} x_i \right) \right| = p\left((k-1)(n-1) + \sum_{i=1}^{k} |x_i| \right) + r.$$

Proof This follows from Corollary A.7 which describes $H^*(D_{n,*} X)$ as a free algebra with divided powers, generated by the duals of a generating set of $H_*(D_{n,*} X)$. This generating set is given by Dyer–Lashof operations applied to iterated Browder products of generators of $H_*(X)$. The dualities described in Proposition 3.8 guarantee that all generators in this generating set are detected by the elements given above. The statement about the degrees follows from the definition. □

The discussion of the dual operations concludes with Nishida-type relations. Due to its length, the proof is deferred to Appendix B.

Proposition 3.11 Let $s \in \mathbb{N}$ and $0 < r < (p-1)(n-1)$. The equation

$$\mathrm{P}^s \mathcal{Q}^r(x) = \sum_{i=0}^{\lfloor \frac{s}{p} \rfloor} a^{r,s,|x|}(i) c^{r,s,|x|}(i) \mathcal{Q}^{r+2(s-pi)(p-1)} \mathrm{P}^i(x)$$

(6)

$$+ \delta^r \sum_{i=0}^{\lfloor \frac{s-1}{p} \rfloor} b^{r,s,|x|}(i) d^{r,s,|x|}(i) \mathcal{Q}^{r+2(s-pi)(p-1)-p} \mathrm{P}^i \beta(x)$$

$$+ \xi^r \sum_{n \in N} \frac{1}{e_n} \left(\star_{i=1}^{p} \mathrm{P}^{n_i}(x) \right),$$

holds, where

$$\delta^r = \begin{cases} 0 & r \equiv 1 \mod 2, \\ 1 & r \equiv 0 \mod 2, \end{cases} \qquad \xi^r = \begin{cases} 0 & r > 0, \\ 1 & r = 0, \end{cases}$$

$$a^{r,s,|x|}(i) = \binom{\lfloor \frac{r}{2} \rfloor + \frac{(|x|-2i)(p-1)}{2}}{s - pi},$$

$$b^{r,s,|x|}(i) = (-1)^{\frac{p-1}{2}(|x|+1)+1} \left(\frac{p-1}{2}\right)! \binom{\lfloor \frac{r}{2} \rfloor + \frac{(|x|-2i)(p-1)}{2} - 1}{s - pi - 1},$$

$$c^{r,s,|x|}(i) = \frac{\gamma(|x|,r)}{\gamma(|x|+2(p-1)i, r-2(pi-s)(p-1))},$$

$$d^{r,s,|x|}(i) = \frac{\gamma(|x|,r)}{\gamma(|x|+1+2(p-1)i, r-p-2(pi-s)(p-1))},$$

$$N = \{n \in \mathbb{N}^p : n_i \leq n_{i+1}, \sum n_i = s, \exists j : n_j < n_{j+1}\},$$

and e_n is the residue class in $\mathbb{F}_p$ of the order of the isotropy group of $n \in N$, where Σ_p acts on $\mathbb{N}^p$ by permuting coordinates.

The point of the following lemma is that—contrary to a naive guess—every summand appearing in the right hand side contains less than p factors P^{p^k}. This fact is essential to the proof of Theorem 5.6.

Lemma 3.12 *For $k \in \mathbb{N}$, let $\mathcal{A}_p(k) \subset \mathcal{A}_p$ be the subalgebra generated by the elements $1, P^1, P^p, \ldots, P^{p^k}$. There is a decomposition*

$$P^{p^k} P^{(p-1)p^k} = \sum_{j=1}^{r} A_{j1} P^{p^k} A_{j2} P^{p^k} \cdots A_{je_j}$$

with $A_{ji} \in \mathcal{A}_p(k-1)$ and $0 \leq e_j \leq p$ for all $1 \leq j \leq r$.

Proof To provide the decomposition, note that

$$(7) \qquad P^{p^k} P^{(p-1)p^k} = \sum_{i=0}^{p^{k-1}} (-1)^{p^k+i} \binom{(p-1)((p-1)p^k-i)-1}{p^k-pi} P^{p^{k+1}-i} P^i$$

by the Adem relations. The binomial coefficient for $i = 0$ vanishes, because

$$\binom{(p-1)^2 p^k - 1}{p^k} \equiv \prod_{j=0}^{k+1} \binom{a_j}{b_j} \mod p,$$

where $(p-1)^2 p^k - 1 = \sum_{j=0}^{k+1} a_j p^j$ and $p^k = \sum_{j=0}^{k} b_j p^j$ are the p–adic expansions. However,

$$a_j = \begin{cases} p-1 & j < k, \\ 0 & j = k, \\ p-2 & j = k+1, \end{cases}$$

whereas $b_k \neq 0$. Thus the first summand in (7) vanishes. If $\mathrm{P}^i \in \mathcal{A}_p$ is indecomposable, then i is a power of p. In particular, $\mathrm{P}^{p^{k+1}-i} \in \mathcal{A}_p(k)$ for $i > 0$. For degree reasons, there is a decomposition of $\mathrm{P}^{p^{k+1}-i} \in \mathcal{A}_p(k)$ which does not contain a summand $A_1 \mathrm{P}^{p^k} A_2 \mathrm{P}^{p^k} \cdots A_j$ with $A_i \in \mathcal{A}_p(k-1)$ and $j > p$. Since $\mathrm{P}^i \in \mathcal{A}_p(k-1)$ for $i \leq p^{k-1}$, the assertion follows. □

4 Some differentials

Recall the Goodwillie spectral sequence

$$E_1^{-j,k} = \begin{cases} H^{k-j}(D_{n,j}X) & j \geq 0, \\ 0 & j < 0, \end{cases}$$

from (1) for the n–fold loop space functor. In this chapter we will compute its first nontrivial differentials on certain classes, namely dual Dyer–Lashof operations with input from the -1–st column. Furthermore we show that dual Browder operations define permanent cycles in the cohomology of a suspension.

Proposition 4.1 *Let $1 \leq n < \infty$, $r \leq n$ and $i < p-1$ be given. Then*

$$d_i \mathcal{Q}^r(x) = 0$$

for all $x \in E_1^{-1,k+1} \cong H^{k+n}(X)$. Furthermore, for all $k, s \in \mathbb{N}$ with $k + s \equiv 0$ mod 2 there are elements $u^{k,s}, v^{k,s} \in (\mathbb{F}_p)^\times$ such that the equations

(8a) $$d_{p-1} \mathcal{Q}^{s(p-1)}(x) = u^{k,s} \cdot \beta \mathrm{P}^{\frac{k+s}{2}}(x),$$

(8b) $$d_{p-1} \mathcal{Q}^{s(p-1)-1}(x) = v^{k,s} \cdot \mathrm{P}^{\frac{k+s}{2}}(x),$$

hold for every $x \in E_1^{-1,k+1} \cong H^{k+n}(X)$.

Proposition 4.1 completely describes all differentials on classes of the form $\mathcal{Q}^r(x)$ with $x \in E^{-1,*}$, since the mentioned ones are the only nontrivial dual Dyer–Lashof operations and higher differentials leave the second quadrant. In the proof given below, the formulae are reduced to $n = \infty$ and $s \in \{0, 1\}$, and to the universal case described in the following lemma.

Lemma 4.2 Let $\iota \in H^0(\mathrm{H}\mathbb{Z}/p)$ be a generator. For every $k \in \mathbb{N}$ there exists an element $u^k \in (\mathbb{F}_p)^\times$ such that in the Goodwillie spectral sequence for $\Sigma^k \mathrm{H}\mathbb{Z}/p$ and $n = \infty$ one has

$$d_{p-1}\mathcal{Q}^0(\sigma^k\iota) = u_k \cdot \beta \mathrm{P}^{\frac{k}{2}}(\sigma^k\iota), \quad \text{if } k \text{ is even,}$$

$$d_{p-1}\mathcal{Q}^{p-2}(\sigma^k\iota) = u_k \cdot \mathrm{P}^{\frac{k+1}{2}}(\sigma^k\iota), \quad \text{if } k \text{ is odd.}$$

All other differentials on these classes are zero.

Proof The spectral sequence for $\Sigma^k \mathrm{H}\mathbb{Z}/p$ converges to $H^*(K(\mathbb{Z}/p, k))$. Suppose that $k = 2i$ is even. Then $\beta \mathrm{P}^i(\iota_{2i}) = 0$, where ι_{2i} is the image of ι in $H^{2i}(K(\mathbb{Z}/p, 2i))$ under the canonical map

$$\Sigma^\infty \Sigma^{2i} K(\mathbb{Z}/p, 2i) \to \mathrm{H}\mathbb{Z}/p.$$

Hence the element $0 \neq \beta \mathrm{P}^i(\sigma^{2i}\iota) \in H^{2pi+1}(\Sigma^{2i}\mathrm{H}\mathbb{Z}/p) = E^{-1, 2pi+2}$ lies in the image of some differential by [Theorem 2.2](#). The possibilities for this differential are

$$d_r \colon E_r^{-(r+1), 2pi+r+1} \longrightarrow E_r^{-1, 2pi+2},$$

for varying r. Since the -1–column (a shift of the Steenrod algebra $\mathcal{A}_p$) consists entirely of permanent cycles and the differentials are derivations, all products of elements in the -1–column are permanent cycles. For degree reasons, the first $p - 1$ columns solely consist of such products, whence the differentials are zero in this range. Consider the column $E_{p-1}^{-p, *}$. Again for degree reasons, only the dual Dyer–Lashof operations $\mathcal{Q}^j(x)$ with $x \in E^{-1, *}$ can be mapped nontrivially by any differential d_r. If $r > p - 1$, then

$$E_1^{-(r+1), 2pi+r+1} = H^{2pi}(D_{\infty, r+1}) = 0,$$

because $2pi < 2(r + 1)i$ and $D_{\infty, r+1}(\Sigma^{2i}\mathrm{H}\mathbb{Z}/p)$ is $(2i(r + 1) - 1)$–connected. Therefore $\beta \mathrm{P}^i(\sigma^{2i}\iota)$ lies in the image of

$$d_{p-1} \colon E_{p-1}^{-p, 2pi+p} \longrightarrow E_{p-1}^{-1, 2pi+2}$$

and a killer of $\beta \mathrm{P}^i(\sigma^{2i}\iota)$ is a sum of elements of the form $\mathcal{Q}^j(x)$ for degree reasons. In order for all degrees to match, such a nonzero x has to be an element in $E^{-1, k} = H^{k-1}(\Sigma^{2i}\mathrm{H}\mathbb{Z}/p)$ with $pk + j = 2pi + p$, which is only possible if $k = 2i + 1$ and $j = 0$ by connectivity. Now $H^{2i}(\Sigma^{2i}\mathrm{H}\mathbb{Z}/p)$ is generated by $\sigma^{2i}\iota$, so for some $u \in (\mathbb{F}_p)^\times$ the equality

$$\beta \mathrm{P}^i(\sigma^{2i}\iota) = d_{p-1}(u\mathcal{Q}^0(\sigma^{2i}\iota))$$

holds by Proposition 3.9. With $u_{2i} = \frac{1}{u}$ one obtains

$$d_{p-1}\big(\mathcal{Q}_0(\sigma^{2i}\iota)\big) = u_{2i}\beta\mathrm{P}^i(\sigma^{2i}\iota),$$

which finishes the universal case for an even dimensional class. The other case is similar and left to the reader. □

Proof of Proposition 4.1 Let $k \in \mathbb{N}$ and $1 \le n < \infty$. The natural equivalence

$$\Sigma^\infty \Omega^n K(\mathbb{Z}/p, k+n) \longrightarrow \Sigma^\infty \Omega^\infty \Sigma^k H\mathbb{Z}/p$$

induces maps

$$D_{n,j}(\Sigma^{-n}\Sigma^\infty K(\mathbb{Z}/p, k+n)) \longrightarrow D_{\infty,j}(\Sigma^k H\mathbb{Z}/p)$$

which for $j = 1$ coincide with the canonical map $\Sigma^{-n}\Sigma^\infty K(\mathbb{Z}/p, k+n) \longrightarrow \Sigma^k H\mathbb{Z}/p$ sending $\sigma^k \iota$ to $\sigma^{-n}\iota_{k+n}$ in cohomology. Any class in $E_1^{-1,k+1} = H^k(\Sigma^{-n}\Sigma^\infty X)$ may be represented as $\sigma^{-n}f^*(\iota_{k+n})$ for some map $f\colon X \longrightarrow K(\mathbb{Z}/p, k+n)$. The naturality of the spectral sequences, the Dyer–Lashof and the Steenrod operations with respect to these maps then supply (8a) and (8b) in Proposition 4.1 for $s = 0$. Proposition 3.5 relates $\mathcal{Q}^{(s+1)(p-1)}(x)$ in the spectral sequence for n to $\mathcal{Q}^{s(p-1)}(\sigma x)$ in the spectral sequence for $n-1$. By hypothesis, the latter is just $\beta\mathrm{P}^{k+1+s/2}(\sigma x)$. This proves (8a) by induction, since the evaluation induces the identity map on the first column. Equation (8b) follows by the same argument. □

The next statement shows that all differentials vanish on dual Browder operations originating in the -1–st column, provided the topological space in question is a suspension.

Proposition 4.3 Let $1 \le n < \infty$ and $y_i \in E_1^{-1,*}$ for all $1 \le i \le m$ in the spectral sequence for ΣX. Then

$$d_s \mathcal{L}^{n-1}(y_1 \otimes \cdots \otimes y_m) = 0$$

for all s.

Proof The spectral sequence for $n = 1$ converges to $H^*(\Omega\Sigma X)$. By a theorem of Bott and Samelson [5], the homology of $\Omega\Sigma X$ is—as a graded algebra with Pontryagin product—isomorphic to the tensor algebra over the homology of X:

$$H_*(\Omega\Sigma X) \cong TH_*(X).$$

Thus $H^*(\Omega\Sigma X) \cong TH^*(X)$ at least as vector spaces (and coalgebras). However, already

$$E_1^{-j,*} = \mathrm{sh}^j H^*(D_{1,j}\Sigma^\infty X) \cong \mathrm{sh}^j H^*(X^{\wedge j}) \cong \mathrm{sh}^j H^*(X)^{\otimes j},$$

and since these isomorphisms are compatible with the filtration, E_∞ has to be equal to E_1. In particular, all differentials are zero. This can be used as the start of an induction (along n this time) using Proposition 3.5. ☐

The following consequence of Proposition 4.3 uses the notion of desuspension index, which is introduced in Definition 5.2.

Corollary 4.4 Let $1 \le n < \infty$. Suppose that $H^*(X)$ has desuspension index at least $n-1$. Then all elements in the spectral sequence for ΣX and n contained in the first $2p-1$ columns are permanent cycles.

Proof Proposition 3.10 implies that all indecomposable base elements in the range in question are those appearing in Propositions 4.1 and 4.3, together with a copy of $\mathbb{F}_p$ in the 0–column. The assumption on the desuspension index of $H^*(X)$ implies that d_{p-1} vanishes on Dyer–Lashof operations and higher differentials do also (see the comment below Proposition 4.1). ☐

5   Nonrealization results

In [23] Schwartz reduced Kuhn's Conjecture to the nonrealizability of certain modules over the Steenrod algebra. For $p = 2$, a very readable account of this reduction is given at the end of [18]. In order to describe these modules, consider the tensor product $\mathbb{F}_p[t] \otimes \Lambda_{\mathbb{F}_p}[s]$ of a polynomial $\mathbb{F}_p$–algebra on a generator of degree $|t| = 2$, and an exterior $\mathbb{F}_p$–algebra on a generator $|s| = 1$, with the following $\mathcal{A}_p$–action:

$$\beta \cdot (t^n \otimes 1) = 0,$$
$$\beta \cdot (t^n \otimes s) = t^{n+1} \otimes 1,$$
$$\mathrm{P}^i \cdot (t^n \otimes 1) = \binom{n}{i} t^{n+i(p-1)} \otimes 1,$$
$$\mathrm{P}^i \cdot (t^n \otimes s) = \binom{n}{i} t^{n+i(p-1)} \otimes s.$$

With this structure $\mathbb{F}_p[t] \otimes \Lambda_{\mathbb{F}_p}[s] \cong H^*(K(\mathbb{Z}/p, 1))$ as an $\mathcal{A}_p$–module. We identify the submodule $\{a \otimes 1 : a \in \mathbb{F}_p[t]\}$ with $\mathbb{F}_p[t]$. Let

$$\Phi(k) := \langle t^{p^k} \rangle_{\mathcal{A}_p} \subseteq \mathbb{F}_p[t]$$

be the submodule generated by t^{p^k}. The module structure of $\Phi(k)$ simplifies to

$$\beta \cdot t^{p^j} = 0,$$

$$\mathrm{P}^i \cdot t^{p^j} = \begin{cases} t^{p^j} & i = 0, \\ t^{p^{j+1}} & i = p^j, \\ 0 & \text{otherwise}, \end{cases}$$

whence the set $\{t^{p^i} : i \geq k\}$ is a $\mathbb{F}_p$–basis for $\Phi(k)$. Because $\mathrm{P}^{p^k} \cdot t^{p^k} = t^{p^{k+1}}$, $\Phi(\ell)$ is a submodule of $\Phi(k)$ for $k < \ell$.

Definition 5.1 Let $k < \ell$. Then $\Phi(k, \ell) := \Phi(k)/\Phi(\ell + 1)$ is the quotient in the category of graded $\mathcal{A}_p$–modules.

The module $\Phi(k, \ell)$ admits the residue classes of $\{t^{p^i} : k \leq i \leq \ell\}$ as an $\mathbb{F}_p$–basis. Only the module $\Phi(k, k+2)$ will occur in the sequel. It consists of three vector spaces $\mathbb{F}_p \cdot t^{p^k}$, $\mathbb{F}_p \cdot t^{p^{k+1}}$ and $\mathbb{F}_p \cdot t^{p^{k+2}}$. The operation P^{p^k} maps the first vector space isomorphically onto the second, and $\mathrm{P}^{p^{k+1}}$ maps the second vector space isomorphically onto the third.

As mentioned in the introduction, our methods only suffice for roughly half the cases needed to provide a full proof of Kuhn's Conjecture 1.1. A precise formulation requires the notion of a *desuspension class*.

Definition 5.2 Let M be an unstable $\mathcal{A}_p$–module. The *desuspension index* of M is the largest natural number n such that the shift $\mathrm{sh}^n(M)$ is unstable. If M has desuspension index n, an element $x \in M$ with

$$\begin{cases} \mathrm{P}^{\frac{|x|-n}{2}} \cdot x \neq 0 & |x| - n \text{ even}, \\ \beta \mathrm{P}^{\frac{|x|-n-1}{2}} \cdot x \neq 0 & |x| - n \text{ odd}, \end{cases}$$

is called *desuspension class of even* resp. *odd origin*.

Every nonzero unstable $\mathcal{A}_p$–module contains a desuspension class. Note that in the odd primary cohomology of a topological space

$$\mathrm{P}^{\frac{|y|}{2}}(y) = y^p$$

holds for cohomology classes of even degree only, whereas the equation

$$\mathrm{Sq}^{|y|}(y) = y^2$$

holds for every cohomology class in $H^*(X, \mathbb{F}_2)$. Recall Theorem 5.3 from the introduction.

Theorem 5.3 Let M be an unstable $\mathcal{A}_p$–module concentrated in degrees $[\ell, m]$. Suppose M contains a desuspension class of even origin. If X is a pointed topological space with

$$H^*(X) \cong M \otimes \Phi(k, k+2)$$

as $\mathcal{A}_p$–modules, then $2p^k \le (p^2 - 1)m + p(m - \ell)$.

The idea of the proof of Theorem 5.3 is to consider the cohomology of $\Omega^n X$, where n is the desuspension index of M, by means of the spectral sequence developed above. In the easiest case ($n = 0$), no loop space (hence no spectral sequence and—as it turns out—not even a space) is necessary. The following result proves the strong realization conjecture from [18] in the special case $n = 0$.

Proposition 5.4 Let M be an unstable $\mathcal{A}_p$–module with $|M| \subseteq [\ell, m]$, and let $x \in M$ be an element of degree $2i$ such that $\mathrm{P}^i \cdot x \ne 0$. Then $M \otimes \Phi(k, k+1)$ admits an unstable algebra structure only if $2(p-2)p^{k+1} \le pm + m - p\ell$.

The conditions on x precisely say that M has desuspension index 0 and that x is a desuspension class of even origin.

Proof Since $|\mathrm{P}^i \cdot x| = 2pi$ and $|x| = 2i$, and both these elements are nonzero, one deduces $0 \le \ell \le 2i \le 2pi \le m$. Let $M_1 := M \otimes (\mathbb{F}_p \cdot t^{p^k})$ and $M_2 := M \otimes (\mathbb{F}_p \cdot t^{p^{k+1}})$ as sub-$\mathbb{F}_p$–modules in $M \otimes \Phi(k, k+1) = M_1 \oplus M_2$. Then $|M_1| \subseteq [\ell + 2p^k, m + 2p^k]$ and $|M_2| \subseteq [\ell + 2p^{k+1}, m + 2p^{k+1}]$. Assume $M \otimes \Phi(k, k+1)$ is equipped with an unstable algebra structure, and that $2(p-2)p^{k+1} > pm + m - p\ell$ holds. This inequality implies in particular

$$m + 2p^k < \ell + 2p^{k+1},$$

whence there is a gap between the sub-$\mathbb{F}_p$–modules M_1 and M_2. Let $b = x \otimes t^{p^k}$ with $|b| = 2i + 2p^k$. On the one hand, $\mathrm{P}^{p^k+i} \cdot b = b^p \ne 0$. In fact,

$$\mathrm{P}^{p^k+i} \cdot b = \mathrm{P}^{p^k+i} \cdot \left(x \otimes t^{p^k}\right) = \sum_{j=0}^{p^k+i} \left(\mathrm{P}^j \cdot x\right) \otimes \left(\mathrm{P}^{p^k+i-j} \cdot t^{p^k}\right)$$

$$= \left(\mathrm{P}^i \cdot x\right) \otimes t^{p^{k+1}} + \left(\mathrm{P}^{p^k+i} \cdot x\right) \otimes t^{p^k} = \left(\mathrm{P}^i \cdot x\right) \otimes t^{p^{k+1}}$$

is nonzero because neither $\mathrm{P}^i \cdot x$ nor $t^{p^{k+1}}$ is zero. On the other hand, the sequence $b, b^2, \ldots, b^{p-1}, b^p$ starts in M_1 and ends in M_2 and so has to pass the gap. Let j

denote the largest number such that $b^j \in M_1$. The inequality $2(p-2)p^{k+1} > pm + m - p\ell$ implies

$$|b^{j+1}| = 2(j+1)i + 2(j+1)p^k \leq m + 2p^k + 2i + 2p^k \leq m + 4p^k + \frac{m}{p} < \ell + 2p^{k+1},$$

whence b^{j+1} lies in the gap. Therefore also $b^p = 0$. This contradiction shows that there can be no unstable algebra structure on $M \otimes \Phi(k, k+1)$. □

In the case $p = 2$, the first half of the proof of Proposition 5.4 leads to $0 \neq b^2$. However, the sequence $b, \ldots, b^p$ from the second half of the proof consists only of b and b^2 which lie in M_1 and M_2 respectively, and therefore b^2 has no reason to be trivial. The remedy for this is to include a third direct summand, ie study $M \otimes \Phi(k-1, k+1) = M_0 \oplus M_1 \oplus M_2$. The class $a = x \otimes t^{k-1} \in M_0$ then satisfies $b^2 = Sq^{2^{k-1}}(ab)$ by the Cartan formula and the observation that all summands but one vanish due to either instability or gaps. However, the element ab lying halfway between b and b^2 may be forced to fall into the gap between M_1 and M_2. This is exactly Kuhn's argument in [18]. In the case p odd, the corresponding statement is as follows.

Proposition 5.5 *Let M be an unstable $\mathcal{A}_p$–module with $|M| \subseteq [\ell, m]$ and let $x \in M$ be an element of (even) degree $2i$, such that $P^i \cdot x \neq 0$. Then $M \otimes \Phi(k, k+2)$ admits an unstable algebra structure only if $2p^k \leq m$.*

Proof Let $a = x \otimes t^{p^k}$, $b = x \otimes t^{p^{k+1}}$ and $c = x \otimes t^{p^{k+2}}$. The argument just described for $p = 2$ works, provided ab is replaced with ab^{p-1} as follows. The proof of Proposition 5.4 applies to give $0 \neq P^{p^k+i} \cdot b = b^p$. This element may also be expressed as

$$b^p = P^{p^k+i} \cdot b = \sum_{j=0}^{i} (P^j \cdot x) \otimes (P^{i-j} \cdot t^{p^{k+1}}) = (P^i \cdot x) \otimes t^{p^{k+1}} = P^i \cdot c,$$

because $i < p^{k+1}$. Furthermore $P^{p^k} \cdot (ab^{p-1}) = b^p$ by the following calculation:

$$P^{p^k} \cdot (ab^{p-1}) = \sum_{j=0}^{p^k} (P^{p^k-j} \cdot a)(P^j \cdot b^{p-1})$$

$$= (P^{p^k} \cdot a)b^{p-1} + \sum_{j=1}^{p^k} (P^{p^k-j} \cdot a)(P^j \cdot b^{p-1}).$$

The second summand in this expression is zero, because either $P^{p^k-j} \cdot a$ is zero by instability, or $P^j \cdot b^{p-1}$ is in the gap between M_1 and M_2. The equality $P^{p^k} \cdot a = b$ follows by essentially the same computation as in the proof of Proposition 5.4. However, assuming $2p^k > m$, the element ab^{p-1} lies in the gap between M_1 and M_2, giving a contradiction. $\qquad\square$

Note that Proposition 5.5 is also a corollary of Proposition 5.4 (except for $p = 3$, where the conclusion is a little bit stronger). In fact, an unstable algebra structure on $M \otimes \Phi(k, k+2)$ induces one on the quotient $M \otimes \Phi(k, k+1)$. The reason for giving a proof of Proposition 5.5 is that it may be carried through the Goodwillie spectral sequence for higher desuspension indices.

Theorem 5.6 Let M be an unstable $\mathcal{A}_p$–module with $|M| \subseteq [\ell, m]$, and let $x \in M$ be an element of (even) degree $2i$, such that $P^i \cdot x \neq 0$. Suppose that X is a locally finite pointed cell complex with

$$H^*(X) \cong (\mathrm{sh}^n M) \otimes \Phi(k, k+2)$$

as graded $\mathcal{A}_p$–modules. Then $2p^k \leq (p^2 - 1)m + p(m - \ell) + (p^2 - 2)n + 1$.

Proof In order to trivialize several differentials which would otherwise become a nuisance later in the proof, let $X' = X/X_{2i+n+2p^k-1}$ be the quotient collapsing the $2i + n + 2p^k - 1$–skeleton. The proof employs the spectral sequence for $Y = \Sigma^{2i+1} X'$ and $n' := n + 2i + 1$, which converges to the cohomology of $\Omega^{n+2i+1} \Sigma^{2i+1} X'$ instead of $\Omega^n X'$. Note that the passage from X to X' appears already in [18]. The projection $X \to X'$ induces a surjection $\varphi \colon H^j(X') \longrightarrow H^j(X)$ for $j = 2i + n + 2p^k$ and an isomorphism for $j > 2i + n + 2p^k$. Let $N_j = \varphi^{-1}(M \otimes (\mathbb{F}_p \cdot t^{p^{k+j}}))$ for $j \in \{0, 1, 2\}$ and $N = \bigoplus_{j=0}^2 N_j$. Then $\mathrm{sh}^n N^* = H^*(Y)$ and $|N_0| \subseteq [2i + 2p^k, m + 2p^k]$, $|N_j| \subseteq [\ell + 2p^{k+j}, m + 2p^{k+j}]$ for $j \in \{1, 2\}$. Let $a, b, c \in N$ be elements which satisfy $\varphi(a) = x \otimes t^{p^k}$, $\varphi(b) = x \otimes t^{p^{k+1}}$ and $\varphi(c) = x \otimes t^{p^{k+2}}$. Note that b and c are uniquely determined, because φ is an isomorphism in high degrees. Suppose that

$$(9) \qquad\qquad 2p^k > (p^2 - 1)m + p(m - \ell) + (p^2 - 2)n + 1.$$

Then in the spectral sequence for Y and n'

$$E_1^{-r,*} = \mathrm{sh}^r H^*(D_{n',r} \Sigma^{-n'} \Sigma^\infty Y) \cong \mathrm{sh}^r H^*(D_{n',r} \Sigma^{-n} \Sigma^\infty X'),$$

hence every element in the -1–st column

$$E_1^{-1,*} = \mathrm{sh}^{1-n} H^*(X') = \mathrm{sh}^1 N$$

is a permanent cycle, since the 0–column consists of a $\mathbb{F}_p$ in degree 0 only. Thus there are (unique) elements

$$(10) \quad \alpha \in H^{2i+2p^k}(\Omega^{n'}Y), \quad \beta \in H^{2i+2p^{k+1}}(\Omega^{n'}Y), \quad \gamma \in H^{2i+2p^{k+2}}(\Omega^{n'}Y)$$

representing $a \in E_\infty^{-1,2i+2p^k+1}, b \in E_\infty^{-1,2i+2p^{k+1}+1}$ and $c \in E_\infty^{-1,2i+2p^{k+2}+1}$ respectively. Before proceeding to the actual argument, observe that by Proposition 3.10,

$$E_1^{-r,*} = \sum_{\substack{(u,v,w)\in\mathbb{N}^3 \\ u+v+w=r}} \mathrm{sh}^r(N_0^u \cdot N_1^v \cdot N_2^w),$$

where $N_0^u \cdot N_1^v \cdot N_2^w \subseteq H^*(D_{n',u+v+w}\Sigma^{-n'}\Sigma^\infty Y)$ is the $\mathbb{F}_p$–vector space generated by 'products' with u factors coming from N_0, v from N_1 and w from N_2. Here 'products' refers to Pontryagin products, Dyer–Lashof operations (which count as p–fold 'products') and Browder operations with s inputs (which count as s–fold 'products'). Then

$$|N_0^u \cdot N_1^v \cdot N_2^w| \subseteq [2ui + (v+w)\ell + 2(u+vp+wp^2)p^k,$$
$$(u+v+w)m + 2(u+vp+wp^2)p^k + (u+v+w-1)(n'-1)]$$

with the lower bound coming from true products and the upper bound stemming from pure Browder operations. This concludes the preliminaries of the proof. The aim is to show that the element $\mathrm{P}^{p^k}(\alpha \cup \beta^{p-1})$ is both zero and nonzero under the assumption (9). This requires a few auxiliary computations, listed in (11), (12) and (13):

$$(11) \qquad \mathrm{P}^{p^k}(\alpha) = \beta, \quad \mathrm{P}^{p^{k+1}}(\beta) = \gamma.$$

The equations in (11) hold for $a,b,c \in E_1^{-1,*}$ by the same computation as in the proof of Proposition 5.4. They carry over to cohomology, because the lower filtrations (ie the 0–column) are trivial away from degree 0. Furthermore,

$$(12) \qquad \mathrm{P}^{p^k}(\alpha \cup \beta^{p-1}) = \beta^p.$$

In fact, the equation

$$\mathrm{P}^{p^k}(\alpha \cup \beta^{p-1}) = \sum_{j_1+\cdots+j_p=p^k} \mathrm{P}^{j_1}(\alpha) \cup \mathrm{P}^{j_2}(\beta) \cup \cdots \cup \mathrm{P}^{j_p}(\beta)$$

simplifies to the expression $\mathrm{P}^{p^k}(\alpha \cup \beta^{p-1}) = \mathrm{P}^{p^k}(\alpha) \cup \beta^{p-1}$ which implies (12) by (11). One may simplify, because $\mathrm{P}^{j_1} \cdot a = 0$ once $p^k > j_1 > i$, and because in the case $j_1 \leq i$ there exists $2 \leq r \leq p$ with $j_r \geq (p^k-i)/(p-1)$, which in turn implies

that $P^{jr} \cdot b$ is in the gap between N_1 and N_2 by the numerical assumption (9). The conclusion for α and β follows as in the case of (11). Finally,

$$P^i(\gamma) = \beta^p. \tag{13}$$

Here the calculation from the proof of Proposition 5.5 translates immediately to $P^{p^{k+1}+i} \cdot b = P^i \cdot c$, which extends to $\beta^p = P^{p^{k+1}+i}(\beta) = P^i(\gamma)$ as before. The next calculation, formulated as a lemma, is the first step significantly different from the easy case. A priori $P^i \cdot c \neq 0$ does not imply $P^i(\gamma) \neq 0$, because $P^i \cdot c$ might lie in the image of some differential. In fact, passing from X to Y is needed precisely to ensure this is not the case.

Lemma 5.7 *The element $P^i(\gamma)$ is nonzero.*

Proof The computation from the proof of Proposition 5.4 and (13) shows $P^i(c) \neq 0$ as before. Thus the claim holds if $P^i \cdot c \in E^{-1,2pi+2p^{k+2}+1}$ is not hit by any differential

$$d_r \colon E_r^{-(r+1),2pi+2p^{k+2}+r} \longrightarrow E_r^{-1,2pi+2p^{k+2}+1}.$$

As $E_r^{-(r+1),2pi+2p^{k+2}+r}$ is a subquotient of

$$E_1^{-(r+1),2pi+2p^{k+2}+r} = H^{2pi+2p^{k+2}-1}(D_{n',r+1}\Sigma^{-n}\Sigma^{\infty}X')$$

and $D_{n',r+1}\Sigma^{-n}\Sigma^{\infty}X'$ is $(r+1)(2i+2p^k)-1$–connected, this group is trivial once $r \geq p^2 - 1$. If $r < p^2 - 1$,

$$H^*(D_{n',r+1}\Sigma^{-n}\Sigma^{\infty}X') = \sum_{u+v+w=r+1} N_0^u \cdot N_1^v \cdot N_2^w,$$

with no iterated Dyer–Lashof operations appearing. If furthermore the inequality $u + vp + wp^2 < p^2$ holds, the top degree of $N_0^u \cdot N_1^v \cdot N_2^w$ is too low:

$$(u+v+w)m + 2(u+vp+wp^2)p^k + (u+v+w-1)(n'-1)$$
$$\leq (p^2-1)m + 2(p^2-1)p^k + (p^2-2)(n+2i)$$
$$\leq 2p^{k+2} - 2p^k + (p^2+p-1)m + (p^2-2)n$$
$$< 2pi + 2p^{k+2} - 1.$$

Here the last step is the numerical assumption (9). The case $u+vp+wp^2 \geq p^2$ is a little more complicated, since different algebra generators do not hit $P^i \cdot c$ for different reasons. Elements in $E_r^{-(r+1),2pi+2p^{k+2}+r}$ represented by (sums of) products of Browder operations and elements from the -1–column are permanent cycles by Proposition 4.3 and the fact that products of cycles are again cycles. A potential killer of $P^i \cdot c$

therefore has to be represented by a sum of products with some Dyer–Lashof operation appearing. If $y \in E_1^{-s,s'}$ is the input for this operation, then $1 \le s < p$ by the above discussion. Furthermore, Proposition 4.1 rules out $s = 1$. By Proposition 3.10 it remains to consider classes represented by sums of products with at least one Dyer–Lashof operation applied to a Browder operation appearing. For such an element the lower bound may be sharpened slightly: its degree in $N_0^u \cdot N_1^v \cdot N_2^w$ is greater than or equal to

$$2ui + (v + w)l + (u + vp + wp^2)p^k + p(n' - 1),$$

because a Browder operation with s inputs raises degree by $(s-1)(n'-1)$ in comparison to a simple product, and then applying a Dyer–Lashof operation adds at least $p(n'-1)$. This sharpened bound in the case $u + vp + wp^2 \ge p^2$ implies

$$2ui + (v + w)l + (u + vp + wp^2)p^k + p(n' - 1) \ge l + 2p^{k+2} + p(n + 2i)$$
$$> 2pi + 2p^{k+2} - 1,$$

which was the reason for suspending the space $2i + 1$ times. With all the differentials covered, $\mathrm{P}^i(\gamma)$ is nonzero as claimed. $\qquad\square$

Equations (12), (13) and Lemma 5.7 imply that

$$(14) \qquad\qquad \mathrm{P}^{p^k}\left(\alpha \cup \beta^{p-1}\right) \ne 0.$$

However, in the current situation there is no obvious reason for $\alpha \cup \beta^{p-1}$ to be zero. The gap into which one might hope a product of this type would fall is filled by its corresponding Pontryagin product, at least on the E_1–page. However, by Corollary 4.4, the element $Q^0(a) \in E_1^{-p,2pi+2p^{k+1}+p}$ is a permanent cycle. Hence there is a class $\delta \in H^{2pi+2p^{k+1}}(\Omega^{n'} Y)$ representing $Q^0(a) \in E_\infty^{-p,2pi+2p^{k+1}+p}$ with the following property.

Lemma 5.8 *The equation $\mathrm{P}^{(p-1)p^k}(\delta) = -\alpha \cup \beta^{p-1}$ holds.*

Proof The first step is a calculation in the spectral sequence. The equality

$$\mathrm{P}^{(p-1)p^k} Q^0(a) = \sum_{j=0}^{(p-1)p^{k-1}} a^{0,(p-1)p^k,|a|}(j) c^{0,(p-1)p^k,|a|}(j) Q^{2((p-1)p^k - pj)(p-1)} \mathrm{P}^j(a)$$

$$+ \sum_{j=0}^{(p-1)p^{k-1}-1} b^{0,(p-1)p^k,|a|}(j) d^{0,(p-1)p^k,|a|}(j) Q^{2((p-1)p^k - pj)(p-1)-p} \mathrm{P}^j \beta(a)$$

$$+ \sum_{t \in N} \frac{1}{e_t} \mathrm{P}^{t_1}(a) \star \cdots \star \mathrm{P}^{t_p}(a)$$

holds by the Nishida relations, Proposition 3.11 (we apologize for the two-fold use of the letters 'a' and 'β'). In the first and second summands, $P^j \cdot a$ and $P^j \beta \cdot a$ are zero if $j > i$ by instability. However, if $j \leq i$, then the index for the Dyer–Lashof operation exceeds $(n-1)(p-1)$, so both summands vanish entirely. The third sum reduces to the term indexed by $t := (0, p^k, \dots, p^k)$. In fact, if $t_2 \leq i$, then $t_j > i + p^k$ for some j with $2 < j \leq p$, which in turn yields zero in the product by instability. If $i < t_2 < p^k$, then $P^{t_2}(a) = 0$, whence the only element in N that contributes to the sum nontrivially is $(0, p^k, \dots, p^k)$. Thus

$$P^{(p-1)p^k} Q^0(a) = -a * b^{p-1},$$

and both $P^{(p-1)p^k}(\delta)$ and $-\alpha \cup \beta^{p-1}$ represent the same element in $E_\infty^{-p, q+p} = F_q^{-p} / F_q^{-(p-1)}$, where $q := 2pi + 2p^k - 2p^{k+1} + 2p^{k+2}$. Proving $F_q^{-(p-1)} = 0$ is equivalent to proving $E_\infty^{-r, q+r} = 0$ for all $0 \leq r < p$. However, for $0 \leq r < p$, the group $E_1^{-r, q+r}$ lies in a gap by the following argument. If $w = 0$, then the top degree of $N_0^u \cdot N_1^v \cdot N_2^w$ is too low, because

$$(u + v + w)m + 2(u + vp + wp^2)p^k + (u + v + w - 1)(n' - 1)$$
$$\leq (p-1)m + 2(p-1)p^{k+1} + (p-2)(n + 2i)$$
$$\leq 2(p-1)p^{k+1} + pm - \ell + (p-2)n$$
$$< p\ell + 2p^k + 2(p-1)p^{k+1} \leq q.$$

If $w \neq 0$, its bottom degree is too high, because $2(p-1)p^k > m - \ell$ and

$$2ui + (v + w)\ell + 2(u + pv + p^2 w)p^k \geq \ell + 2p^{k+2}$$
$$> m - 2(p-1)p^k + 2p^{k+2} \geq q.$$

It follows that $F_q^{-(p-1)} = 0$. Therefore $P^{(p-1)p^k}(\delta)$ and $-\alpha \cup \beta^{p-1}$ coincide in $F_q^{-p} \subseteq H^q(\Omega^{n'} Y)$. $\qquad\square$

Introducing this element δ goes back to Schwartz' paper [23]. The inequality (14) and Lemma 5.8 combine to give $P^{p^k} P^{(p-1)p^k}(\delta) \neq 0$. However, the operation $P^{p^k} P^{(p-1)p^k}$ can be decomposed using the Adem relations in the form of Lemma 3.12, which then implies the following computation.

Lemma 5.9 *The equation* $P^{p^k} P^{(p-1)p^k}(\delta) = 0$ *holds.*

Proof The idea is to find p gaps in the filtration F^{-p} which are large enough. Unfortunately, there are only two obvious such gaps, but $P^{p^k} P^{(p-1)p^k}(\delta)$ is shown to

have nontrivial contribution outside $F^{-(p-1)}$ before performing its final 'jump'. In $F^{-p}/F^{-(p-1)}$ we then find the required $p-2$ additional gaps. To this end define

$$V_0 = \sum_{\substack{(u,v)\in\mathbb{N}^2,\, u+v\leq p \\ (0,p)\neq(u,v)\neq(1,p-1)}} N_0^u \cdot N_1^v,$$

$$V_1 = N_0 \cdot N_1^{p-1},$$

$$V_2 = N_1^p + \sum_{\substack{(u,v,w)\in\mathbb{N}^3 \\ u+v+w\leq p,\, w\neq 0}} N_0^u \cdot N_1^v \cdot N_2^w,$$
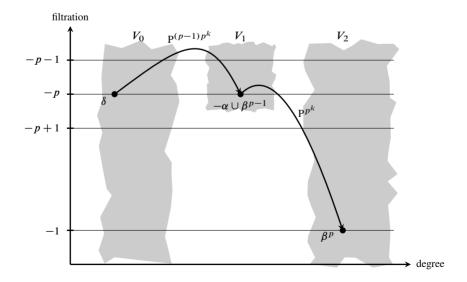
and recall that shifted versions of these modules assemble to the first p columns of E_1 in such a way that

$$F_*^p = V_0^\infty + V_1^\infty + V_2^\infty,$$

where V_j^∞ denote the subquotients of V_j in E_∞. By definition, the degrees of elements are distributed as follows:

$$|V_0| \subseteq [2i + 2p^k, (p-1)m + 2(p-1)p^{k+1} + (p-2)(n'-1)],$$

$$|V_1| \subseteq [2i + (p-1)\ell + (p^2 - p + 1)2p^k, pm + (p^2 - p + 1)2p^k + (p-1)(n'-1)],$$

$$|V_2| \subseteq [\ell + 2p^{k+2}, pm + 2p^{k+3} + (p-1)(n'-1)].$$

Hence both between V_0 and V_1 and between V_1 and V_2 there is a gap spanning at least $2(p-1)p^{k-1}$. Moreover, the distance between V_0 and V_2 is strictly larger than $2(p-1)p^k$.

Lemma 3.12 provides a decomposition of the form

$$\mathrm{P}^{p^k}\mathrm{P}^{(p-1)p^k} = \sum_{j=1}^{r} A_{j1}\mathrm{P}^{p^k}A_{j2}\mathrm{P}^{p^k}\cdots A_{je_j},$$

with $A_{jj'} \in \mathcal{A}_p(k-1)$ and $0 \le e_j \le p$ for all $1 \le j \le r$. If $\mathrm{P}^{p^k}\mathrm{P}^{(p-1)p^k}(\delta) \ne 0$, there exists $1 \le j \le r$ such that $\big(A_{j1}\mathrm{P}^{p^k}A_{j2}\mathrm{P}^{p^k}\cdots A_{je_j}\big)(\delta) \ne 0$. Expressing the $A_{jj'}$ via indecomposables yields (as a summand) a product

(15) $$\mathrm{P}^{p^{\nu_1}}\cdots\mathrm{P}^{p^{\nu_t}}, \text{ with } 1 \le \nu_j \le k, \quad \big(\mathrm{P}^{p^{\nu_1}}\cdots\mathrm{P}^{p^{\nu_t}}\big)(\delta) \ne 0,$$
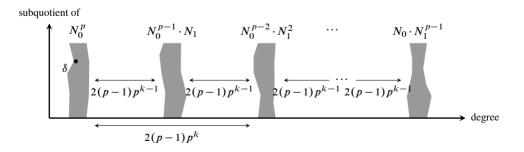
where $\nu_j = k$ for at most $p-1$ indices. The gap between V_0 and V_2 is too large to be passed by any single $\mathrm{P}^{p^{\nu_j}}$. Since $\delta \in (N_0^p)^\infty \subseteq V_0^\infty$ (or rather a subquotient of N_0) and $\mathrm{P}^{p^k}\mathrm{P}^{(p-1)p^k}(\delta) = \beta^p \in (N_1^p)^\infty \subseteq V_2^\infty$, there exists a number $1 \le s \le t$ such that $(\mathrm{P}^{p^{\nu_s}}\cdots\mathrm{P}^{p^{\nu_t}})(\delta) \in V_1$. The only indecomposable operation appearing in (15) which can pass the gap between V_1 and V_2 is P^{p^k}. Hence there exists $1 \le r < s$ with $\nu_r = k$. In particular, $\nu_j = k$ for at most $p-2$ indices $j \ge s$. The same degree calculation appearing in the proof of Lemma 5.8 yields $V_1^\infty \cap F^{-(p-1)} = \{0\}$ on the infinity page, so

$$\big(\mathrm{P}^{p^{\nu_s}}\cdots\mathrm{P}^{p^{\nu_t}}\big)(\delta) \ne 0 \text{ in } F^{-p}/F^{-(p-1)}.$$

However, the image of $V_0 \oplus V_1$ in $F^{-p}/F^{-(p-1)}$ is a subquotient of the module

$$N_0^p \oplus \big(N_0^{p-1}\cdot N_1\big) \oplus \big(N_0^{p-2}\cdot N_1^2\big) \oplus \cdots \oplus \big(N_0 \cdot N_1^{p-1}\big),$$

which contains at least $(p-1)$ gaps. More precisely, between any $N_0^u \cdot N_1^v$ and $N_0^{u-1} \cdot N_1^{v+1}$ there is a gap spanning at least $2(p-1)p^{k-1}$. Similarly, the distance between $N_0^u \cdot N_1^v$ and $N_0^{u-2} \cdot N_1^{v+2}$ spans more than $2(p-1)p^k$.

Since no indecomposable operation in (15) can pass across two gaps at once, the product $(\mathrm{P}^{p^{\nu_s}}\cdots\mathrm{P}^{p^{\nu_t}})(\delta)$ has to factor through a trivial module in $F^{-p}/F^{-(p-1)}$. It follows that $\mathrm{P}^{p^k}\mathrm{P}^{(p-1)p^k}(\delta) = 0$. $\qquad\square$

The element $P^{p^k}P^{(p-1)p^k}(\delta) = -P^k(\alpha \cup \beta^{p-1})$ is both zero and nonzero, as shown by inequality (14) and Lemmas 5.8 and 5.9. Hence the assumption (9) is wrong, which proves Theorem 5.6. $\square$

The theorem as stated in the beginning can now be obtained by a simple reindexing using some obvious estimates. Note that M has a desuspension class of even origin if the Bockstein acts trivially on M. Following Kuhn's observations in [16], we recover a theorem of his.

Theorem 5.10 If $H^*(X)$ is finitely generated over the Steenrod algebra $\mathcal{A}_p$ and the Bockstein acts trivially on it in high degrees, then $H^*(X)$ is finite.

Proof The proof proceeds as in [18]. Let $\overline{T}$ be the reduced Lannes functor and $\Delta(X) = \mathrm{Map}(B\mathbb{Z}/p, X)/X$ where X is embedded as the space of constant maps. Iterated applications of Δ produce a space Y which still has finitely generated, but infinite dimensional, cohomology and whose cohomology satisfies $\overline{T}^2(H^*(Y)) = 0$. Structure theorems of Kuhn and Schwartz imply that in large degrees, $H^*(Y)$ is isomorphic to the module $M \otimes \Phi(k, \infty)$ with $M \cong \overline{T}H^*(Y)$. By a result of Winstead recorded in [16, Theorem 1.3], the Bockstein still acts trivially on M in large degrees, forcing all desuspension classes to have even origin. For large k one can produce a subquotient space Z with $H^*(Z) \cong M \otimes \Phi(k, k+2)$, which is a contradiction of Theorem 5.6. $\square$

The case of a desuspension class of odd origin remains unsolved. The operation βP^i has an interpretation in terms of p–fold symmetric Massey products

$$\beta P^i(x) = \langle x, \ldots, x \rangle_s$$

by work of Kraines [15]. A similar formula holds for Q_{p-2} by a result of Kochman [14]. Consequently, one may try to replace p–fold cup products in the arguments for the even case by p–fold Massey products. This attempt works if the module M does not desuspend at all (the case $n = 0$), via a Cartan formula for Massey products. If $n > 0$, one may try to replace $\mathcal{Q}^0$ by $\mathcal{Q}^{p-2}$ and use the interpretation in terms of Massey products mentioned above. Currently we do not know how to make this work. Another strategy is to introduce an extra looping to reduce to the even case. This strategy already appears in the K–theoretical Adams–Atiyah proof of the Hopf invariant one theorem at odd primes, and it was also used in [23]. However, in that situation the element $d_{p-1}\mathcal{Q}^0(\alpha)$ is possibly nonzero. There are several ways to deal with this. One approach is to bring the projective plane into play, as was done in [23]. As noted in [24], it is unclear whether $\mathcal{Q}^0(\alpha)$ still survives in the spectral sequence for the projective

plane. Another approach is to switch to the homotopy fibre F of $\beta P^i(\alpha)$. Here the class corresponding to $Q^0(\alpha)$ surely survives, but an extension problem related to the H–deviation of the splitting map for ΩF as a product of the loop space on X and the appropriate Eilenberg–Mac Lane space prevents the argument to go through. In both approaches, one uses the fact that βP^i loops down to zero. Harper shows that the problems in both approaches are closely related in [13].

Appendix A Homology of free algebras over E_n–operads

Our aim in this section is to compute the (co)homology of the free algebra on a spectrum E over a suitable E_n–algebra. The case of an E_∞–operad was already treated in [7]. The homology of the free C_n–algebra $C_n X$ on a space X has been computed in [8]. More precisely, he defined a category of so called allowable $AR_{n-1}\Lambda_{n-1}$–Hopf algebras and constructed (among other things) a free functor W_{n-1} from a category of coalgebras to this category of allowable $AR_{n-1}\Lambda_{n-1}$–Hopf algebras. Cohen then proved that the homology $H_*C_n X$ of the free algebra over the operad of little n–cubes on a space X is isomorphic to $W_{n-1}H_*X$ as an $AR_{n-1}\Lambda_{n-1}$–Hopf algebra. Besides the Hopf algebra structure, the action of the Steenrod reduced powers, the Dyer–Lashof and the Browder operations are encoded in the $AR_{n-1}\Lambda_{n-1}$–Hopf algebra structure. The precise definition is given in the first few pages of [8].

Building on Cohen's calculation, most of the axioms where shown to hold for the homology of a suitable E_n–spectrum (or even an H_n–spectrum) in [7]. The part of the structure which was not studied there is the comultiplication and its interaction with the operational structure. In the papers [1; 17], Kuhn studied a natural coproduct on the homology of a free operad algebra on a spectrum, and we rely very much on these results. The unstable case of an E_n–operad was treated already in [8], and Strickland and Turner considered the E_∞ case in [26; 27].

Let C be a topological operad augmented over the operad $\mathcal{L}$ of linear isometries, let E be a spectrum and $E_+ = E \vee \mathbb{S}$, where $\mathbb{S}$ is the sphere spectrum. Then the free C–algebra on E_+ is defined in [19]. Let $\mathbf{D}_k$ denote the k–th extended powers with respect to C. Write $\Delta\colon E_+ \to E_+ \vee E_+$ for the stable pinch map and consider the map η given by the composition

$$C E_+ \xrightarrow{C\Delta} C(E_+ \vee E_+) \simeq \bigvee_{k=0}^{\infty} \mathbf{D}_k(E_+ \vee E_+) \xrightarrow{\rho} \bigvee_{k=0}^{\infty} \bigvee_{i+j=k} \mathbf{D}_i E_+ \wedge \mathbf{D}_j E_+ \simeq C E_+ \wedge C E_+,$$

where ρ is induced by the degeneracies of the operad (see [21, page 12]). Roughly, it associates to a labeled "configuration" all partitions of the configuration. Following

Kuhn, we choose

$$\psi_*\colon H_*\mathcal{C}\Sigma^\infty X \to H_*\mathcal{C}\Sigma^\infty X \otimes H_*\mathcal{C}\Sigma^\infty X$$

as coproduct. The homomorphism induced by ρ on homology can be described by suitable transfer homomorphisms. It is part of the main result in [17] that, for a co-H–space X, ψ coincides with the usual coproduct. Here the isomorphism

$$\mathcal{C}\Sigma^\infty X \cong \Sigma^\infty \mathcal{C} X$$

enters. Since $\mathcal{C}_n$ is not augmented over the operad $\mathcal{L}$ of linear isometries, the free $\mathcal{C}_n$–algebra $\mathcal{C}_n E$ on a spectrum E in the sense of [19, Chapter VII] cannot be formed. However, a cofibrant model $\bar{\mathcal{C}} \to \mathcal{C}_n$ in the model category of topological operads is augmented over $\mathcal{L}$; see Berger and Moerdijk [3]. Alternatively, one can choose the operad $\mathcal{C}_n \times \mathcal{L}$ instead. Moreover, functorial cellular approximation leads to a cellular operad, still denoted $\bar{\mathcal{C}}$, because it commutes with products.

Remark A.1 There is a notion of *naive* (MIT) $\mathcal{C}$–ring spectra, for which one does not need to assume that the operad $\mathcal{C}$ is augmented over $\mathcal{L}$. No operadic internalization of external smash products is needed, instead one uses the symmetric monoidal smash product in S–modules and the fact that the model category of S–modules is tensored over topological spaces. However, we have to rely on certain results in [19] and there is no comparison between these two notions of operad ring spectra in the literature as far as we know (but see May [22]).

We remind the reader how $\mathbf{D}_k\Sigma^d X$ can be viewed as a relative Thom complex in the case where X is a pointed space. Consider the bundle

$$p\colon \bar{\mathcal{C}}_k \times_{\Sigma_k} (\mathbb{R}^d)^k \to \bar{\mathcal{C}}_k/\Sigma_k = \bar{\mathcal{B}}_k$$

and let p_X the pullback along the map

$$q\colon \bar{\mathcal{C}}_k \times_{\Sigma_k} X^k \to \bar{\mathcal{B}}_k.$$

Then $\mathbf{D}_k\Sigma^d X$ is the quotient of Thom complexes $T(p_X)/Tp_\bullet$, where $\bullet$ is the base point in X. Extended powers of spectra do not commute with suspension. However, there is a Thom isomorphism Φ^{-1} for a spectrum E and d even whose inverse

$$\Phi\colon H_*\mathbf{D}_k\Sigma^d E \to H_*\mathbf{D}_k E$$

is more convenient for us. In the E_∞–case, such a Thom isomorphism is provided in [7, Theorem VII.3.3] (see also [8, Chapter VII]), and the proof generalizes. The homomorphism Φ is induced by the composite

$$H_*\mathbf{D}_k\Sigma^d E \xrightarrow{\delta_*} H_*\mathbf{D}_k S^d \wedge \mathbf{D}_k E \cong H_*\mathbf{D}_k S^d \otimes H_*\mathbf{D}_k E \xrightarrow{\epsilon \otimes \mathrm{id}} H_*\mathbf{D}_k E,$$

where $\delta\colon \mathbf{D}_k \Sigma^d E \to \mathbf{D}_k S^d \wedge \mathbf{D}_k E$ is induced by the diagonal in $\bar{\mathcal{C}}$ and ϵ is evaluation at the orientation class $e_0 \otimes \iota^k \in H^*\mathbf{D}_k S^d$. Here ι is the fundamental class of S^d and $e_0 = 1$ is the image of the bottom class of the standard free acyclic $\mathbb{Z}/p$–complex W in $H^0\bar{\mathcal{C}}_k$; see [19, page 403] and [7, pages 244-247] for details. The isomorphism Φ is the colimit of a compatible system of isomorphisms Φ_s of (inverse) Thom isomorphisms for a system of bundles defined by the augmentation to $\mathcal{L}$, followed by suitable desuspension isomorphisms. Snaith's splitting assembles these isomorphisms Φ for various k to an isomorphism

$$\Phi\colon H_*\mathcal{C}\Sigma^d E_+ \to H_*\mathcal{C}E_+.$$

Lemma A.2 *The isomorphism*

$$\Phi\colon H_*\bar{\mathcal{C}}\Sigma^d E_+ \to H_*\bar{\mathcal{C}}E_+$$

is compatible with the comultiplication ψ.

Proof This follows from naturality of Φ applied to $\rho\mathcal{C}(\Delta)$, just as in [26, 7.1]. □

The next lemma shows that Φ is also compatible with Browder operations, Dyer–Lashof operations and the Pontryagin product. A version of it is proved for cyclic powers in [19, Lemma VIII.3.3.]. The arguments given there generalise once one replaces the standard acyclic $\mathbb{Z}/p$ complex W [20] by the spaces in $\bar{\mathcal{C}}$. Before we state the lemma, recall that [19, VIII 2.9.] supplies an isomorphism

$$C_*(\mathbf{D}_k E) \cong C_*(\bar{\mathcal{C}}_k) \otimes_{\Sigma_k} C_*(E)^{\otimes k}$$

for a cellular operad $\mathcal{C}$ and a cellular spectrum E, where C_* denotes the cellular chain complex. The chain complex $C_*(E)^{\otimes k}$ is equivariantly chain homotopy equivalent to $H_*(E)^{\otimes k}$. Thus, any class in $H_*\mathbf{D}_k E$ may be represented as $[c \otimes x_1 \otimes \cdots \otimes x_k]$ with $c \in C_*(\bar{\mathcal{C}}_k)$ and $[x_i] \in H_*E$.

Lemma A.3 *The equality*

$$\Phi([c \otimes \sigma^d x_1 \otimes \cdots \otimes \sigma^d x_k]) = [c \otimes x_1 \otimes \ldots x_k]$$

holds.

Proof Consider a class $z \in H_*\Sigma^d \mathbf{D}_k E$ and represent it as

$$z = [c \otimes \sigma^d x_1 \otimes \cdots \otimes \sigma^d x_k].$$

Express $\widetilde{\Delta} c = \sum_{i=1} c_i' \otimes c_i''$, where $\widetilde{\Delta}$ is a diagonal approximation for $\bar{\mathcal{C}}_k$. By definition of $\widetilde{\Delta}$, the class z is mapped to

$$\sum_{i=1} (c_i' \otimes \sigma \otimes \cdots \otimes \sigma) \otimes (c_i'' \otimes x_1 \otimes \cdots \otimes x_k),$$

where σ is interpreted as the generator of $H_* S^d$. Evaluating the orientation class $1 \otimes i^k \otimes \mathrm{id}$ on that cycle yields $[c \otimes x_1 \otimes \cdots \otimes x_k]$, which finishes the proof. $\qquad\square$

Corollary A.4 *The Thom isomorphism Φ commutes with Browder operations, Dyer–Lashof operations Q_i and the Pontryagin product.*

Proof The operations are defined to be the image of a class $[c \otimes x_1 \otimes \cdots \otimes x_k]$ under the algebra structure map, where c is either one of Cohen's classes e_i, or the fundamental class ι for $\bar{\mathcal{C}}_2 \simeq S^{n-1}$, or a class in $H_0 \bar{\mathcal{C}}_2$ [21, page 66]. Moreover, it suffices to consider free $\bar{\mathcal{C}}$–algebras. The assertion then follows directly from Lemma A.3. $\qquad\square$

The coproduct ψ on the homology of a $\bar{\mathcal{C}}$–algebra on a space X is related to the other operations by diagonal Cartan formulas [8]. These read

$$(16) \qquad \psi Q^k(x) = \sum_{i+j=k} \sum_r Q^i(x_r') \otimes Q^j(x_r''),$$

$$\psi \lambda_{n-1}(x, y) = \sum_{r,s} (-1)^{(n-1)|x_r'| + |x_r''||y_s'|} \lambda_{n-1}(x_r' y_s') \otimes x_r'' y_s''$$

$$(17) \qquad\qquad + \sum_{r,s} (-1)^{(n-1)|y_s'| + |x_r''||y_s'|} x_r' y_s' \otimes \lambda_{n-1}(x_r'', y_s''),$$

$$(18) \qquad \text{if } \psi(x) = \sum_r x_r' \otimes x_r'', \quad \psi(y) = \sum_s y_s' \otimes y_s''.$$

The nonlinear top operation ξ satisfies a diagonal formula only up to an error term consisting of p–fold Browder operations in the elements x', x''. This fact was overlooked in [8] and corrected by Wellington [28]. The diagonal formula for the operation ζ follows from the formulae for ξ and λ_{n-1}, together with the relation $\zeta(x) = \beta \xi(x) - \mathrm{ad}^{p-1}(x)(\beta x)$.

Corollary A.5 *The diagonal Cartan formulas hold in $H_* \bar{\mathcal{C}} E$.*

Proof This assertion is proven in [8] for a suspension spectrum $\Sigma^\infty X$. By [19, I.4.7] there is a natural isomorphism $E \cong \operatorname{colim}_m \Sigma^{-m} \Sigma^\infty E_m$. Since extended powers commute with colimits by [7, Proposition I.1.2] it suffices to prove the assertion for a shifted suspension spectrum $\Sigma^{-m} \Sigma^\infty X$, where X is a pointed cell complex. Furthermore, one can restrict to the case where X is connected and m is even because of the natural isomorphism

$$\Sigma^{-m} \Sigma^\infty X \cong \Sigma^{-(m+1)} \Sigma^\infty \Sigma X$$

(see [19, I.4.2]). Then the assertion follows from Lemma A.3. □

The following is now immediate.

Proposition A.6 Let $E = \Sigma^{-m} \Sigma^\infty X$ for a connected cell complex X and m even. Then the composition

$$W_{n-1} H_* \Sigma^m E \xrightarrow{\eta} H_* \bar{C}_{n-1} \Sigma^m E \xrightarrow{\Phi^{\frac{m}{2}}} H_* \bar{C}_{n-1} E$$

is an isomorphism of $A R_{n-1} \Lambda_{n-1}$–Hopf algebras.

Note that the isomorphism in Proposition A.6 does not preserve the degree. As an algebra, $W_{n-1} H_* \Sigma^m E$ is isomorphic to a free commutative graded algebra ΛM_{n-1}. The vector space M_{n-1} has a basis which consists of classes $Q^I y$, where I runs through certain admissible sequences and y runs through basic products in the λ_{n-1}–algebra underlying $W_{n-1} H_* E$ (see [8, page 227] for precise definitions). This information and Proposition A.6 imply the next result, which is the raison d'être of this appendix.

Corollary A.7 Let $E = \Sigma^{-m} \Sigma^\infty X$ for a connected cell complex X and m even. As an algebra, $H^* \bar{C}_{n-1} E$ is the free algebra $\Gamma \operatorname{sh}^k M_{n-1}^*$ with divided powers on the dual of a shifted copy $\operatorname{sh}^k M_{n-1}$ of M_{n-1}.

Proof The assertion follows from Proposition A.6 and [28, 3.13]. □

Appendix B   Nishida relations

This final section presents the proofs of Proposition 3.9 and the Nishida relations in cohomology, Proposition 3.11. The standard Nishida relations in homology given by May ([20, page 209, Theorem 9.4]) are a major ingredient in this proof. The modified version of the two top operations from [8] is not required here.

Proposition B.1 (Nishida relations) *Let $s \in \mathbb{N}$ and $0 \leq r < (p-1)(n-1)$. Then the equality*

$$P_s(Q_r(x)) = \sum_{i=0}^{\lfloor \frac{s}{p} \rfloor} a_{r,s,|x|}(i) Q_{r+2(pi-s)(p-1)} P_i(x)$$

$$+ \delta_r \sum_{i=0}^{\lfloor \frac{s-1}{p} \rfloor} b_{r,s,|x|}(i) Q_{r+p+2(pi-s)(p-1)} \beta P_i(x)$$

holds, where

$$\delta_r = \begin{cases} 0 & r \equiv 0 \mod 2, \\ 1 & r \equiv 1 \mod 2, \end{cases}$$

$$a_{r,s,|x|}(i) = \binom{s-ps+\lfloor \frac{r}{2} \rfloor + \frac{p-1}{2}|x|}{s-pi},$$

$$b_{r,s,|x|}(i) = (-1)^{\frac{p-1}{2}|x|+1} \left(\frac{p-1}{2}\right)! \binom{s-ps-1+\lfloor \frac{r+1}{2} \rfloor + \frac{p-1}{2}|x|}{s-pi-1}.$$

We turn now to the proof of [Proposition 3.11](#).

Proof Let Σ_p act on $\mathbb{N}^p$ by permuting coordinates, set

$$N := \left\{ n \in \mathbb{N}^p \colon n_i \leq n_{i+1}, \sum n_i = s, \exists j \colon n_j < n_{j+1} \right\} \subset \mathbb{N}^p$$

and choose $n \in N$. To justify that the residue class e_n of the order of the isotropy group is invertible in $\mathbb{F}_p$, observe that the prime p divides e_n if and only if $n_i = n_j$ for all i, j for the following reason. In fact, if all coordinates of $n \in \mathbb{N}^p$ are equal, the isotropy group of n has order $p!$, being Σ_p. If p divides e_n, there exists an element τ of order p in the isotropy group of n by Cauchy's theorem from group theory. The only elements of order p in Σ_p are p–cycles. Thus $\tau \cdot n = n$ implies $n_i = n_j$ for all i, j. It follows that $\frac{1}{e_n}$ is well defined.

The proof is basically a computation of pairings with generating elements of the corresponding homology, together with references to [Proposition B.1](#). Pairing [(6)](#) with the Dyer–Lashof operations yields as left hand side

$$\langle P^s \mathcal{Q}^r(x), Q_j(y) \rangle = \langle \mathcal{Q}^r(x), P_s Q_j(y) \rangle,$$

whereas the right hand side will be considered separately for each of the three summands (with the third only appearing if $r = 0$). The first summand is

$$\sum_{i=0}^{\left\lfloor \frac{s}{p} \right\rfloor} a^{r,s,|x|}(i) c^{r,s,|x|}(i) Q^{r+2(s-pi)(p-1)} \mathrm{P}^i(x).$$

On the cohomological side, the pairing $\langle Q^{r+2(s-pi)(p-1)} \mathrm{P}^i(x), Q_j(y) \rangle$ is nonzero only if $j = r + 2(s - pi)(p-1)$ and either

$$2(p-1) \mid (p-1)|y| + j \quad \text{or} \quad 2(p-1) \mid (p-1)|y| + j + 1.$$

For this proof, we call such an index j *good*. Let $q := (r - j + 2s(p-1))/(2(p-1)p)$, then

$$\left\langle \sum_{i=0}^{\left\lfloor \frac{s}{p} \right\rfloor} a^{r,s,|x|}(i) c^{r,s,|x|}(i) Q^{r+2(s-pi)(p-1)} \mathrm{P}^i(x), Q_j(y) \right\rangle$$
$$= \begin{cases} a^{r,s,|x|}(q) \gamma(|x|, r) \langle \mathrm{P}^q(x), y \rangle & q \in \mathbb{N}, \ j \text{ good}, \\ 0 & \text{else.} \end{cases}$$

On the homological side, we have $\langle Q^r(x), Q_{j+2(pi-s)(p-1)} \mathrm{P}_i(x) \rangle$ is nonzero only if $j = r + 2(s - pi)(p-1)$ and either

$$2(p-1) \mid j + 2(pi-s)(p-1) + (p-1)(|y| - 2(p-1)i)$$

or

$$2(p-1) \mid j + 2(pi-s)(p-1) + (p-1)(|y| - 2(p-1)i) + 1$$

which reduces to j being good. As above

$$\left\langle Q^r(x), \sum_{i=0}^{\left\lfloor \frac{s}{p} \right\rfloor} a_{j,s,|y|}(i) Q_{j+2(pi-s)(p-1)} \mathrm{P}_i(y) \right\rangle$$
$$= \begin{cases} a_{j,s,|y|}(q) \gamma(|x|, r) \langle x, \mathrm{P}_q(y) \rangle & q \in \mathbb{N}, \ j \text{ good}, \\ 0 & \text{else.} \end{cases}$$

Both these terms are nonzero only if $|\mathrm{P}^q(x)| = |y|$, and then a simple calculation shows $a_{j,s,|y|}(q) = a^{r,s,|x|}(q)$. The equality

$$
(19) \qquad \left\langle \sum_{i=0}^{\left\lfloor \frac{s}{p} \right\rfloor} a^{r,s,|x|}(i) c^{r,s,|x|}(i) Q^{r+2(s-pi)(p-1)} \mathrm{P}^i(x), Q_j(y) \right\rangle
$$

$$
= \left\langle Q^r(x), \sum_{i=0}^{\left\lfloor \frac{s}{p} \right\rfloor} a_{j,s,|y|}(i) Q_{j+2(pi-s)(p-1)} \mathrm{P}_i(y) \right\rangle
$$

follows. The second summand is

$$
\delta^r \sum_{i=0}^{\left\lfloor \frac{s-1}{p} \right\rfloor} b^{r,s,|x|}(i) d^{r,s,|x|}(i) Q^{r+2(s-pi)(p-1)-p} \mathrm{P}^i \beta(x).
$$

On the cohomological side, the term $\delta^r \langle Q^{r-p+2(s-pi)(p-1)} \mathrm{P}^i \beta(x), Q_j(y) \rangle$ is nonzero only if r is even, $j = r - p + 2(s - pi)(p - 1)$ and either

$$
2(p-1) \mid (p-1)|y| + j \quad \text{or} \quad 2(p-1) \mid (p-1)|y| + j + 1.
$$

Note that j is odd if r is even, so $2(p-1) \nmid (p-1)|y| + j$, eliminating the first of these two cases. Let $q := (r - j - p + 2(p-1)s)/2(p-1)p$ and call an index j *bad* if $j = r - p + 2(s - pi)(p - 1)$ and $2(p-1) \mid (p-1)|y| + j + 1$. Then the equality

$$
(20) \qquad \left\langle \delta^r \sum_{i=0}^{\left\lfloor \frac{s-1}{p} \right\rfloor} b^{r,s,|x|}(i) d^{r,s,|x|}(i) Q^{r-p+2(s-pi)(p-1)} \mathrm{P}^i \beta(x), Q_j(y) \right\rangle
$$

$$
= \begin{cases} \delta^r b^{r,s,|x|}(q) \gamma(|x|, r) \langle \mathrm{P}^q \beta(x), y \rangle & q \in \mathbb{N}, \ j \text{ bad}, \\ 0 & \text{else}, \end{cases}
$$

holds. On the homological side, $\delta_j \langle Q^r(x), Q_{j+p+2(pi-s)(p-1)} \beta \mathrm{P}_i(y) \rangle$ is nonzero only if j is odd, $r = j + p + 2(pi - s)(p - 1)$ (implying r even) and either

$$
2(p-1) \mid (p-1)(|y| - 2(p-1)i - 1) + j + p + 2(pi - s)(p - 1)
$$

or

$$
2(p-1) \mid (p-1)(|y| - 2(p-1)i - 1) + j + p + 2(pi - s)(p - 1) + 1.
$$

Note that the second divisibility cannot occur if j is odd, thereby reducing these conditions on j simply to being bad. The equality

$$\left\langle \mathcal{Q}^r(x), \delta_j \sum_{i=0}^{\lfloor \frac{s-1}{p} \rfloor} b_{j,s,|y|}(i) Q_{j+p+2(pi-s)(p-1)} \beta P_i(y) \right\rangle$$

$$= \begin{cases} \delta_j b_{j,s,|y|}(q)\gamma(|x|,r)\langle x, \beta P_q(y) \rangle & q \in \mathbb{N}, j \text{ bad,} \\ 0 & \text{else,} \end{cases}$$

follows. Since $|P^q\beta(x)| = |y|$ and $\delta^r = \delta_j$ imply $b^{r,s,|x|}(q) = b_{j,s,|y|}(q)$,

$$(21) \quad \left\langle \delta^r \sum_{i=0}^{\lfloor \frac{s-1}{p} \rfloor} b^{r,s,|x|}(i) d^{r,s,|x|}(i) \mathcal{Q}^{r-p+2(s-pi)(p-1)} P^i \beta(x), Q_j(y) \right\rangle$$

$$= \left\langle \mathcal{Q}^r(x), \delta_j \sum_{i=0}^{\lfloor \frac{s-1}{p} \rfloor} b_{j,s,|y|}(i) Q_{j+p+2(pi-s)(p-1)} \beta P_i(y) \right\rangle$$

results. Adding (19) and (21) yields the desired result for $r > 0$, since the sum of the right sides is just $P_s Q_j(y)$ by Proposition B.1. The extra term in cohomology for $r = 0$ does not contribute, since

$$\left\langle \sum_{n \in N} \frac{1}{e_n}(\star_{i=1}^p P^{n_i}(x)), Q_j(y) \right\rangle$$

is always zero by Proposition 3.8 (7). Consider now parings with Browder operations. The left hand side is

$$\langle P^s \mathcal{Q}^r(x), L_{n-1}(y,z) \rangle = \langle \mathcal{Q}^r(x), P_s L_{n-1}(y,z) \rangle$$

$$= \sum_{i=0}^s \langle \mathcal{Q}^r(x), L_{n-1}(P_i(y), P_{s-i}(z)) \rangle$$

$$= 0,$$

and thus coincides with the right hand side, because $\langle \mathcal{Q}^r(x), L_{n-1}(y,z) \rangle = 0 = \langle \star_{i=0}^p x_i, L_{n-1}(y,z) \rangle$.

Finally, pairings with products have to be treated. Proposition 3.8 (3) implies that only p–fold products may contribute. Let $N'' = \{n \in \mathbb{N}^p : n_1 + \cdots + n_p = s\}$. The left

hand side equals

$$\langle \mathrm{P}^s \mathcal{Q}^r(x), *_{i=1}^p y_i \rangle = \langle \mathcal{Q}^r(x), \mathrm{P}_s(*_{i=1}^p y_i) \rangle$$
$$= \sum_{n \in N''} \langle \mathcal{Q}^r(x), *_{i=1}^p \mathrm{P}_{n_i}(y_i) \rangle,$$

which is zero unless $r = 0$. In the case $r = 0$, the result is

$$\langle \mathrm{P}^s \mathcal{Q}^0(x), *_{i=1}^p y_i \rangle = \sum_{n \in \mathbb{N}''} \prod_{i=1}^p \langle x, \mathrm{P}_{n_i}(y_i) \rangle.$$

The right hand side $\langle \mathcal{Q}^{r+2(s-pi)(p-1)} \mathrm{P}^i(x), *_{i=0}^p y_i \rangle$ is zero unless we have that $r + 2(s - pi)(p - 1) = 0$. In that case

$$\frac{s}{p} \geq \left\lfloor \frac{s}{p} \right\rfloor \geq i = \frac{r + 2(p-1)s}{2(p-1)p},$$

which implies $r = 0$ (and then also $i = \frac{s}{p}$). It follows that

$$\left\langle \sum_{i=0}^{\lfloor \frac{s}{p} \rfloor} a^{r,s,|x|}(i) c^{r,s,|x|}(i) \mathcal{Q}^{r+2(s-pi)(p-1)} \mathrm{P}^i(x), *_{i=1}^p y_i \right\rangle$$
$$= \begin{cases} \prod_{i=1}^p \langle \mathrm{P}^{\frac{s}{p}}(x), y_i \rangle & r = 0, \, p \mid s, \\ 0 & \text{else,} \end{cases}$$

because $a^{0,s,|x|}(s/p) = c^{0,s,|x|}(s/p) = 1$. By the same reasoning, a nonzero pairing $\langle \mathcal{Q}^{r-p+2(s-pi)(p-1)} \mathrm{P}^i \beta(x), *_{i=1}^p y_i \rangle$ can only occur if $r - p + 2(s - pi)(p - 1) = 0$. But then $\delta^r = 0$, so the second term

$$\left\langle \delta^r \sum_{i=0}^{\lfloor \frac{s-1}{p} \rfloor} b^{r,s,|x|}(i) d^{r,s,|x|}(i) \mathcal{Q}^{r-p+2(s-pi)(p-1)} \mathrm{P}^i \beta(x), *_{i=1}^p y_i \right\rangle = 0$$

vanishes completely. Again, this proves the claim in the case $r > 0$, as both sides of the claimed equation pair to zero. In the case $r = 0$ one additionally calculates

$$\left\langle \sum_{n \in N} \frac{1}{e_n} (P^{n_1}(x) \star \cdots \star P^{n_p}(x)), *_{i=1}^p y_i \right\rangle = \sum_{n \in N} \frac{1}{e_n} \sum_{\tau \in \Sigma_p} \prod_{i=1}^p \langle P^{n_i}(x), y_{\tau(i)} \rangle$$

$$= \sum_{n \in N} \frac{1}{e_n} \sum_{\tau \in \Sigma_p} \prod_{i=1}^p \langle P^{(\tau \cdot n)_i}(x), y_i \rangle$$

$$= \sum_{\tau \in \Sigma_p} \sum_{n \in \tau \cdot N} e_{\tau^{-1} \cdot n}^{-1} \prod_{i=1}^p \langle P^{n_i}(x), y_i \rangle$$

$$= \sum_{\tau \in \Sigma_p} \sum_{n \in \tau \cdot N} e_n^{-1} \prod_{i=1}^p \langle P^{n_i}(x), y_i \rangle,$$

since $e_{\tau^{-1} \cdot n} = e_n$, where $(\tau \cdot n)_i = n_{\tau^{-1}(i)}$ for $\tau \in \Sigma_p$. Let $N' = \{ n \in N'' : \exists\, 1 \leq i, j \leq p : n_i \neq n_j \}$. For every $n \in \tau \cdot N$ there exist i, j with $n_i \neq n_j$ by the definition of N, thus the sums may be combined to give

$$\left\langle \sum_{n \in N} \frac{1}{e_n} (\star_{i=1}^p P^{n_i}(x)), *_{i=1}^p y_i \right\rangle = \sum_{n \in N'} \frac{|\{ \tau \in \Sigma_p : n \in t \cdot N \}|}{e_n} \prod_{i=1}^p \langle P^{n_i}(x), y_i \rangle,$$

because $e_n \neq 0$ for all such n, as mentioned at the beginning of the proof. If $\tau \cdot n = \sigma \cdot m$ for $n, m \in N$ and $\tau, \sigma \in \Sigma_p$, then $n = m$, since n and m are ascending. Hence for every $n \in \mathbb{N}^p$ there exists an $m \in N$ such that

$$(22) \qquad \{ \tau \in \Sigma_p : n \in t \cdot N \} = \{ \tau \in \Sigma_p : n = \tau \cdot m \},$$

and m is unique if this set is nonempty. In that case, it is a coset of the isotropy group of n. By the definition of N, the set (22) is empty exactly if $\sum n_i \neq s$, whereas its cardinality is just e_n if it is nonempty. Hence

$$\left\langle \sum_{n \in N} \frac{1}{e_n} (\star_{i=1}^p P^{n_i}(x)), *_{i=1}^p y_i \right\rangle = \sum_{n \in N'} \prod_{i=1}^p \langle P^{n_i}(x), y_i \rangle$$

$$= \begin{cases} \sum_{n \in N''} \prod_{i=1}^p \langle P^{n_i}(x), y_i \rangle - \prod_{i=1}^p \langle P^{\frac{s}{p}}(x), y_i \rangle & p \mid s, \\ \sum_{n \in N''} \prod_{i=1}^p \langle P^{n_i}(x), y_i \rangle & p \nmid s. \end{cases}$$

Summing these equations finishes the proof. □

It remains to supply the proof of the (non)linearity statement 3.9.

Proof The second equality follows from the first and Corollary A.7. The proof of the first equation proceeds as the proof of Proposition 3.11 given above. It suffices to prove the equations for homogeneous elements, so let $x, y \in H^d(D_{n,j}X)$ and $z \in H_d(D_{n,j}X)$. Pairing the equation

$$\mathcal{Q}^r(x+y) = \mathcal{Q}^r(x) + \mathcal{Q}^r(y)$$

with $Q_s(z)$ yields

$$\langle \mathcal{Q}^r(x+y), Q_s(z) \rangle = \begin{cases} \gamma(d,r)\langle x+y, z \rangle & r = s = t(p-1) - \epsilon, \; \epsilon \in \{0,1\}, \\ 0 & \text{otherwise,} \end{cases}$$

$$\langle \mathcal{Q}^r(x) + \mathcal{Q}^r(y), Q_s(z) \rangle = \langle \mathcal{Q}^r(x), Q_s(z) \rangle + \langle \mathcal{Q}^r(y), Q_s(z) \rangle$$

$$= \begin{cases} \gamma(d,r)\langle x+y, z \rangle & r = s = t(p-1) - \epsilon, \; \epsilon \in \{0,1\}, \\ 0 & \text{otherwise,} \end{cases}$$

by Proposition 3.8(1), since $|x| = |y| = d$. In the case $r = 0$, the summand $\sum_{k=1}^{p-1} \frac{1}{(p-1)!k!} x^{\star k} \star y^{\star(p-k)}$ pairs trivially with $Q_s(z)$ by Proposition 3.8(7). Pairing each of these summands with the value of a homological Browder operation also yields zero by Proposition 3.8 parts (2) and (8).

It remains to check pairings with products. Let $z_1, \ldots, z_k \in H_*(D_{n,*}X)$. If $r > 0$ or $k \neq p$,

$$\langle \mathcal{Q}^r(x+y), *_{i=1}^k z_i \rangle = \langle \mathcal{Q}^r(x) + \mathcal{Q}^r(y), *_{i=1}^k z_i \rangle = 0$$

by Proposition 3.8(3), and if $r = 0$ and $k = p$,

$$\langle \mathcal{Q}^0(x+y), *_{i=1}^p z_i \rangle = \prod_{i=1}^p \langle x+y, z_i \rangle$$

$$= \prod_{i=1}^p (\langle x, z_i \rangle + \langle y, z_i \rangle)$$

$$= \prod_{i=1}^p \langle x, z_i \rangle + \prod_{i=1}^p \langle y, z_i \rangle + \sum_{j=1}^{p-1} \frac{1}{(p-j)!j!} \langle x^{\star j} y^{\star(p-j)}, *_{i=1}^p z_i \rangle$$

holds by a calculation. Computing the pairing

$$\left\langle \mathcal{Q}^0(x) + \mathcal{Q}^0(y) + \sum_{j=1}^{p-1} \frac{1}{(p-j)!j!} x^{\star j} y^{\star(p-j)}, *_{i=1}^p z_i \right\rangle$$

yields the same result, which provides the desired equation. $\qquad\square$

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SB, OR, MS: *Institut für Mathematik, Universität Osnabrück*
D-D-49069, Osnabrück, Germany

Mathematisches Institut, Universität Münster
D-D-48149, Münster, Germany

sbuesche@uos.de, f.hebestreit@uni-muenster.de, oroendig@math.uos.de,
mstelzer@uos.de