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Derivators, pointed derivators and stable derivators

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We develop some aspects of the theory of derivators, pointed derivators and stable derivators. Stable derivators are shown to canonically take values in triangulated categories. Similarly, the functors belonging to a stable derivator are canonically exact so that stable derivators are an enhancement of triangulated categories. We also establish a similar result for additive derivators in the context of pretriangulated categories. Along the way, we simplify the notion of a pointed derivator, reformulate the base change axiom and give a new proof that a combinatorial model category has an underlying derivator.

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0 Introduction and plan

The theory of stable derivators was initiated by Heller [16; 17] and Grothendieck [14]. It was studied further, at least in similar settings and among others, by Franke [12], Keller [24] and Maltsiniotis [32]. One way to motivate it is by saying that it provides an enhancement of triangulated categories. Triangulated categories suffer the well-known defect that the cone construction is not functorial. A consequence of this nonfunctoriality of the cone construction is the fact that there is no good theory of homotopy (co)limits for triangulated categories. One can still define these notions, at least in some situations where the functors are defined on categories which are freely generated by certain graphs. This is for example the case for the cone construction itself, the homotopy pushout and the homotopy colimit of a sequence of morphisms. But in all these situations, the "universal objects" are only unique up to *non*canonical isomorphism. The slogan used to describe this situation is the following one: diagrams in a triangulated category do not carry sufficient information to define their homotopy (co)limits in a *canonical* way.

But in the typical situations, as in the case of the derived category of an abelian category or in the case of the homotopy category of a stable model or $(\infty,1)$–category, the "model in the background" allows for such constructions in a functorial manner. So, the passage from the model to the derived or homotopy category truncates the available

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information too strongly. To be more specific, let $\mathcal{A}$ be an abelian category such that the derived categories which occur in the following discussion exist. Moreover, let us denote by $C(\mathcal{A})$ the category of chain complexes in $\mathcal{A}$. As usual, let [1] be the ordinal $0 \leq 1$ considered as a category $(0 \to 1)$. Hence, for an arbitrary category $\mathcal{C}$, the functor category $\mathcal{C}^{[1]}$ of functors from [1] to $\mathcal{C}$ is the arrow category of $\mathcal{C}$. With this notation, the cone functor at the level of abelian categories is a functor $\mathsf{C}\colon C(\mathcal{A}^{[1]}) \cong C(\mathcal{A})^{[1]} \to C(\mathcal{A})$. But to give a *construction* of the cone functor in terms of *homotopical* algebra only, one has to consider more general diagrams. For this purpose, let $f\colon X \to Y$ be a morphism of chain complexes in $\mathcal{A}$. Then the cone $\mathsf{C}f$ of f is the *homotopy* pushout of the following diagram:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \\ 0 & & \end{array}$$

At the level of derived categories, the cone construction is again functorial *when considered as a functor* $D(\mathcal{A}^{[1]}) \to D(\mathcal{A})$. The important point is that one forms the arrow categories *before* passage to the derived categories. Said differently, at the level of derived categories, we have, in general, $D(\mathcal{A}^{[1]}) \ncong D(\mathcal{A})^{[1]}$. Moreover, as we have mentioned, to actually give a construction of this functor one needs apparently also the derived category of diagrams in $\mathcal{A}$ of the above shape and a homotopy pushout functor. More systematically, one should not only consider the derived category of an abelian category but also the derived categories of diagram categories and restriction and homotopy Kan extension functors between them. This is the basic idea behind the notion of a derivator.

In this paper we give a complete and self-contained proof that the values of a stable derivator can be *canonically* endowed with the structure of a triangulated category (Theorem 4.16). Similarly, we show that the functors which are part of the derivator can be *canonically* turned into exact functors with respect to these structures (Proposition 4.18). This is in a sense the main work and will occupy the bulk of this paper. We build on ideas of Franke [12] from his theory of systems of triangulated diagram categories and adapt them to this alternative set of axioms.

These two results reveal certain advantages of the language of stable derivators over the language of triangulated categories. A triangulated category $\mathcal{T}$ is, by the very definition, a triple consisting of a category $\mathcal{T}$ together with a functor $\Sigma\colon \mathcal{T} \to \mathcal{T}$ and a class of distinguished triangle as additionally *specified structure*. These are then subject to a list of axioms. One advantage of the stable derivators is that this structure does not have to be specified but instead is canonically available. Once the derivator is stable, ie has some easily motivated *properties*, triangulated structures can be canonically constructed. In particular, the octahedron axiom does not have to be made explicit.

Similarly, the fact that a morphism F of triangulated categories is exact means, by the very definition, that the functor is endowed with an *additional structure* given by a natural isomorphism $\sigma\colon F \circ \Sigma \to \Sigma \circ F$ which behaves nicely with respect to the two chosen classes of distinguished triangles. But, in fact, the exactness of such a morphism should only be a property and not a structure. In most applications, the exact functors under consideration are "derived functors" of functors defined "on certain models in the background". And in this situation, the exactness then reflects the fact that this functor preserves (certain) finite homotopy (co)limits. In the setting of stable derivators this is precisely the notion of an exact morphism. In particular, the exactness of a morphism is again a property and not the specification of an additional structure.

But the theory of derivators is more than "only an enhancement of triangulated categories". In fact, it gives us an alternative axiomatic approach to an abstract homotopy theory. As in the theory of model categories and $(\infty, 1)$–categories, there is a certain hierarchy of such structures: the unpointed situation, the pointed situation and the stable situation. In the classical situation of topology, this hierarchy corresponds to the passage from spaces to pointed spaces and then to spectra. In classical homological algebra, the passage from the derived category of nonnegatively graded chain complexes to the unbounded derived category can be seen as a second example for passing to the stable situation. In the theory of derivators this threefold hierarchy of structures is also present and the corresponding notions are then derivators, pointed derivators and stable derivators. Franke [12] has introduced a theory of systems of triangulated diagram categories which is similar to the notion of a stable derivator. The fact that the theory of derivators admits the mentioned threefold hierarchy of structures is one main advantage over the approach of Franke.

Along the way we give a simplification of the axioms of a pointed derivator. The usual definition of a pointed derivator (see Cisinski and Neeman [9]), here called a strongly pointed derivator, is formulated using the notion of cosieves and sieves. One usually demands that the homotopy left Kan extension functor $i_!$ along a cosieve i has itself a left adjoint $i^?$, and similarly that the homotopy right Kan extension functor j_* along a sieve j has a right adjoint $j^!$. Motivated by algebraic geometry, these additional adjoints are then called exceptional and coexceptional inverse image functors respectively. We show that this definition can be simplified. It suffices to ask that the underlying category of the derivator is pointed, ie has a zero object. This definition is more easily motivated, more intuitive for topologists and, of course, simpler to check in examples. We give a proof of the equivalence of these two notions in Section 3.

The theory described in this paper is not completely new. In particular, it owes a lot to Maltsiniotis who exposed and expanded the foundations of the theory originating with Grothendieck. The first two sections can be considered as a review of these foundations, although our exposition deviates somewhat from existing ones. In particular, we make systematic use of the calculus of mates from the very beginning, resulting in a streamlined development of the theory. The more original part of the paper lies in the remaining three sections in which we use a simplified notion of pointed derivators as we discussed above.

The author is aware of the fact that there will be a written up version of a proof of the existence of these canonical triangulated structures in a future paper by Maltsiniotis. In fact, Maltsiniotis presented an alternative, unpublished variant of Franke's theorem in a seminar in Paris in 2001. He showed that this notion of stable derivators is equivalent to a variant thereof (as used in the thesis of Ayoub [1; 2]) where the triangulations are part of the notion. Nevertheless, we give this independent account. The construction of the suspension functor by Cisinski and Neeman [9] and the axioms of Maltsiniotis [32] indicate that that proof will use the (co)exceptional inverse image functors. But one point here is to show that these functors are not needed for these purposes.

We now turn to a short description of the content of the paper. In Section 1, we give the central definitions and deduce some immediate consequences of the axioms. The existence of certain very special (co)limits can be explained using the so-called (partial) underlying diagram functors. We recall some aspects of the "calculus of mates" (Section 1.2) which is the main tool in most of the proofs in this paper. Using that calculus we are able to characterize derivators by saying that they satisfy base change for Grothendieck (op)fibrations. This in turn is the key ingredient to establish the theoretically important class of examples, that for a derivator $\mathbb{D}$ the prederivator $\mathbb{D}^M$ (see Example 1.3) is also a derivator (Theorem 1.25). As a further class of examples, we give a simple, ie completely formal, proof that combinatorial model categories have underlying derivators.

In Section 2, we introduce morphisms and natural transformations in the context of derivators which leads to the 2–category Der of derivators. We then turn to homotopy-colimit preserving morphisms and establish some basic facts about them. In particular, again using the fact that derivators satisfy base change for Grothendieck (op)fibrations, we show that homotopy Kan extensions in a derivator of the form $\mathbb{D}^M$ are calculated pointwise (Proposition 2.5) which will be of some importance in Section 4. Moreover, we study in some detail the notion of an adjunction between derivators.

In Section 3, we consider pointed derivators and give the typical examples. We prove that our "weaker" definition of a pointed derivator is equivalent to the "stronger" one using the (co)exceptional inverse image functors (Corollary 3.8). Moreover, in the pointed context homotopy right Kan extensions along sieves give "extension by zero functors" and dually for cosieves (Proposition 3.6). We briefly talk about (co)cartesian squares in a derivator and deduce some properties about them. An important example of this kind of results is the composition and cancellation property of (co)cartesian squares (Proposition 3.13). Another one is a "detection result" for (co)cartesian squares in larger diagrams (Proposition 3.10) which is due to Franke [12]. We close the section by a discussion of the important suspension, loop, cone and fiber functors.

In Section 4, we stick to stable derivators for which, by definition, the classes of cocartesian and cartesian squares coincide. Some nice consequences of this are that the suspension and the loop morphisms define inverse equivalences, that bicartesian squares satisfy the 2-out-of-3 property and that we are working in the additive context (Proposition 4.7 and Corollary 4.14). The main aim of the section is to establish the canonical triangulated structures on the values of a stable derivator (Theorem 4.16). These are preserved by exact morphisms of stable derivators (Proposition 4.18) and, in particular, by the functors belonging to the stable derivator itself (Corollary 4.19).

In the last section, we introduce additive derivators and show that the values of an additive derivator can be endowed with the structure of a pretriangulated category (Theorem 5.6) in the sense of Beligiannis [3]. Moreover, these pretriangulations are preserved by an adapted class of morphisms (Proposition 5.7 and Corollary 5.8). The results of this sections follow to a large extent from an adaptation of the techniques of Section 4.

There are two remarks in order before we begin with the paper. The first remark concerns duality. Many of the statements in this paper have dual statements which also hold true by the dual proof (the reason for this is Example 1.11). In most cases, we will not make these statements explicit and we will hardly ever give a proof of both statements. Nevertheless, we allow ourselves to refer to a statement also in cases where, strictly speaking, the dual statement is needed.

The second remark concerns the terminology employed here. In the existing literature on derivators, the term "triangulated derivator" is used instead of "stable derivator". We preferred to use this different terminology for two reasons: First, the terminology "triangulated derivator" (introduced by Maltsiniotis in [32]) is a bit misleading in that no triangulations are part of the initial data. One main point of this paper is to give a proof that these triangulations can be canonically constructed. Thus, from the perspective of the typical distinction between *structures* and *properties* the author does not like the former terminology too much. Second, in the related theories of model categories and $(\infty, 1)$–categories, corresponding notions exist and are called *stable* model categories and *stable* $(\infty, 1)$–categories respectively. So, the terminology stable derivator reminds us of the related theories.

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1 Derivators

1.1 Basic definitions

As we mentioned in the introduction, the basic idea behind a derivator is to consider simultaneously derived or homotopy categories of diagram categories of different shapes. So, the most basic notion in this business is the following one.

Definition 1.1 A *prederivator* $\mathbb{D}$ is a strict 2–functor $\mathbb{D}\colon \mathsf{Cat}^{\mathrm{op}} \to \mathsf{CAT}$.

Here, Cat denotes the 2–category of small categories, $\mathsf{Cat}^{\mathrm{op}}$ is obtained from Cat by reversing the direction of the functors, while CAT denotes the "2–category" of not necessarily small categories. There are the usual set-theoretical problems with the notion of the "2–category" CAT in that this will not be a category enriched over Cat. Since we will never need this nonfact in this paper, we use slogans as the "2–category CAT" as a convenient parlance and think instead of a prederivator as a function $\mathbb{D}$ as we describe it now. Given a prederivator $\mathbb{D}$ and a functor $u\colon J \to K$, an application of $\mathbb{D}$ to u gives us two categories $\mathbb{D}(J)$, $\mathbb{D}(K)$ and a functor

$$\mathbb{D}(u) = u^*\colon \mathbb{D}(K) \longrightarrow \mathbb{D}(J).$$

Similarly, given two functors $u, v\colon J \to K$ and a natural transformation $\alpha\colon u \to v$, we obtain an induced natural transformation α^* as depicted in the next diagram:

$$\begin{tikzcd} J \arrow[r, bend left, "u"] \arrow[r, bend right, "v"'] \arrow[r, phantom, "\Downarrow\alpha"] & K \end{tikzcd} \qquad\qquad \begin{tikzcd} \mathbb{D}(K) \arrow[r, bend left, "u^*"] \arrow[r, bend right, "v^*"'] \arrow[r, phantom, "\Downarrow\alpha^*"] & \mathbb{D}(J) \end{tikzcd}$$

This datum is compatible with compositions and identities in a strict sense, ie we have equalities of the respective expressions and not only coherent natural isomorphisms between them. For the relevant basic 2–categorical notions, which were introduced by

Ehresmann in [11], we refer to Kelly and Street [28] or Borceux [6, Chapter 7], but nothing deep from that theory is needed here.

The following examples give an idea of how such prederivators arise. The third example assumes some knowledge about model categories (see Quillen [36]). See Dwyer and Spaliński [10] for a well written, leisurely introduction while much more material is treated in the monographs Hovey [19] and Hirschhorn [18]. Similarly, the last example uses the theory of $(\infty,1)$–categories (also known as ∞–categories, quasi-categories, weak Kan complexes), ie of simplicial sets satisfying the inner horn extension property. These were originally introduced by Boardman and Vogt in their work [5] on homotopy invariant algebraic structures. Detailed accounts of this theory are given in the tomes due to Joyal [20; 21; 22; 23] and Lurie [29; 30]. A short exposition of many of the central ideas and also of the philosophy of this theory can be found in the author's [13].

Example 1.2 (1) Every category $\mathcal{C}$ gives rise to the *prederivator represented by* $\mathcal{C}$:

$$y(\mathcal{C})=\mathcal{C}\colon\ J\mapsto\mathcal{C}^J$$

(2) Let $\mathcal{A}$ be a sufficiently nice abelian category, ie such that we can form the derived categories occurring in this example without running into set-theoretical problems. Recall that, by definition, the derived category $D(\mathcal{A})$ is the localization of the category of chain complexes at the class of quasi-isomorphisms. Since weak equivalences in diagram categories are defined pointwise, we have the *prederivator* $\mathbb{D}_{\mathcal{A}}$ *associated to an abelian category* $\mathcal{A}$:

$$\mathbb{D}_{\mathcal{A}}\colon\ J\mapsto\mathbb{D}_{\mathcal{A}}(J)=D(\mathcal{A}^J)$$

(3) Let $\mathcal{M}$ be a cofibrantly generated model category. Recall that this assumption implies that diagram categories $\mathcal{M}^J$ can be endowed with the so-called *projective* model structure (in which fibrations and weak equivalences are defined levelwise). The universal property of the localization functors guarantees that we obtain the *prederivator* $\mathbb{D}_{\mathcal{M}}$ *underlying a model category* $\mathcal{M}$ by setting

$$\mathbb{D}_{\mathcal{M}}\colon\ J\mapsto\mathbb{D}_{\mathcal{M}}(J)=\mathsf{Ho}(\mathcal{M}^J).$$

(4) Let $\mathcal{C}\in\mathsf{Set}$ be an $(\infty,1)$–category and let $K\in\mathsf{Set}$ be a simplicial set. Since the Joyal model structure [22] on the category of simplicial sets is cartesian it follows that the simplicial mapping space $\mathcal{C}_{\bullet}^{K}=\mathsf{hom}_{\mathsf{Set}}(\Delta^{\bullet}\times K,\mathcal{C})$ is again an $(\infty,1)$–category (as opposed to a more general simplicial set). Using the nerve functor $N\colon\mathsf{Cat}\to\mathsf{Set}$, we thus obtain the *prederivator* $\mathbb{D}_{\mathcal{C}}$ *underlying an* $(\infty,1)$*-category* $\mathcal{C}$:

$$\mathbb{D}_{\mathcal{C}}\colon\ J\mapsto\mathbb{D}_{\mathcal{C}}(J)=\mathsf{Ho}\left(\mathcal{C}^{N(J)}\right)$$

The functoriality of this construction follows from [21, Theorem 5.14].

Anticipating the fact that we have a 2–category PDer of prederivators (see Section 2 and in particular Example 2.1) we want to mention that Example 1.2(1) extends to a (2–categorical) Yoneda embedding $y\colon \mathsf{CAT} \to \mathsf{PDer}$. In this paper we introduce many notions for derivators which are analogs of well-known notions from category theory. Then it will be important to see that these notions are extensions of the classical ones in that both notions coincide on represented (pre)derivators.

The last example which we are about to mention now does not seem to be too interesting in its own right. But as we will see later it largely reduces the amount of work in many proofs (see Theorem 1.25).

Example 1.3 Let $\mathbb{D}$ be a prederivator and let M be a fixed category. Then the assignment

$$\mathbb{D}^M\colon \mathsf{Cat}^{\mathrm{op}} \to \mathsf{CAT}, \quad J \mapsto \mathbb{D}^M(J) = \mathbb{D}(M \times J)$$

is again a prederivator. This example extends to an action on PDer (see Example 2.1).

Let now $\mathbb{D}$ be a prederivator and let $u\colon J \to K$ be a functor. Motivated by the above examples we call the induced functor $\mathbb{D}(u) = u^*\colon \mathbb{D}(K) \to \mathbb{D}(J)$ a *restriction of diagram functor* or *precomposition functor*. As a special case of this, let $J = e$ be the terminal category, ie the category with one object and identity morphism only. For an object k of K, we denote by $k\colon e \to K$ the unique functor sending the unique object of e to k. Given a prederivator $\mathbb{D}$, we obtain, in particular, for each object $k \in K$ an associated functor $k^*\colon \mathbb{D}(K) \to \mathbb{D}(e)$ which takes values in the *underlying category* $\mathbb{D}(e)$. Let us call such a functor an *evaluation functor*. For a morphism $f\colon X \to Y$ in $\mathbb{D}(K)$ let us write $f_k\colon X_k \to Y_k$ for its image under k^*.

Definition 1.4 Let $\mathbb{D}$ be a prederivator and let $u\colon J \to K$ be a functor.

(1) The prederivator $\mathbb{D}$ admits *homotopy left Kan extensions along* u if the induced functor u^* has a left adjoint:

$$(u_! = \mathrm{HoLan}_u, u^*)\colon \mathbb{D}(J) \rightleftarrows \mathbb{D}(K)$$

(2) The prederivator $\mathbb{D}$ admits *homotopy right Kan extensions along* u if the induced functor u^* has a right adjoint:

$$(u^*, u_* = \mathrm{HoRan}_u)\colon \mathbb{D}(K) \rightleftarrows \mathbb{D}(J)$$

In the case of homotopy left Kan extensions along $p_J\colon J \to e$ we also speak of *homotopy colimits of shape* J and write $p_{J!} = \mathrm{Hocolim}_J$.

Let us recall from classical category theory that right Kan extensions in complete categories can be calculated pointwise by certain limits and dually. More precisely, consider $u\colon J \to K$ and $F\colon J \to \mathcal{C}$, where $\mathcal{C}$ is a complete category. Then the right Kan extension $\mathrm{Ran}_u(F)\colon K \to \mathcal{C}$ of F along u exists and can be described as follows. The slice category $J_{k/}$ has objects pairs (j, f) consisting of an object $j \in J$ together with a morphism $f\colon k \to u(j)$ in K. Morphisms are morphisms in J making the obvious triangles commute. The slice category comes with an obvious forgetful functor $\mathrm{pr}\colon J_{k/} \to J$. In this notation there is the following natural isomorphism:

$$\mathrm{Ran}_u(F)_k \cong \lim_{J_{k/}} \mathrm{pr}^*(F) = \lim_{J_{k/}} F \circ \mathrm{pr}, \quad k \in K$$

The corresponding property for *homotopy* Kan extensions holds in the case of model categories (see Section 1.3) and will be demanded axiomatically for a derivator. For this purpose, let $\mathbb{D}$ be a prederivator and consider a natural transformation of functors $\alpha\colon w \circ u \to u' \circ v$.

Let us assume that $\mathbb{D}$ admits homotopy right Kan extensions along u and u'. The calculus of mates applied to α^* gives us a natural transformation $\alpha_*\colon w^* \circ u'_* \to u_* \circ v^*$.

Dually, if we have a natural transformation $\alpha\colon u' \circ v \to w \circ u$ and if the prederivator admits homotopy left Kan extensions along u and u' the calculus of mates gives rise to a natural transformation $\alpha_!\colon u_! \circ v^* \to w^* \circ u'_!$ (a precise definition in the relevant case will be given in the next paragraph).

This calculus of mates [28] will be studied a bit more systematically in Section 1.2. At the very moment, we are interested in the following situation. Let $u\colon J \to K$ be a functor and $k \in K$ an object. Identifying k again with the corresponding functor $k\colon e \to K$, we have the following natural transformation α in the context of the slice constructions:

$$\begin{tikzcd} J_{/k} \arrow[r, "\mathrm{pr}"] \arrow[d, "p=p_{J_{/k}}"'] \arrow[dr, phantom, "\swArrow"] & J \arrow[d, "u"] \\ e \arrow[r, "k"'] & K \end{tikzcd} \qquad \begin{tikzcd} \mathbb{D}(e) \arrow[dr, phantom, "\swArrow"] & \mathbb{D}(J_{/k}) \arrow[l, "p_!"'] \arrow[dr, phantom, "\swArrow"] & \mathbb{D}(J) \arrow[l, "\mathrm{pr}^*"'] \arrow[dr, phantom, "\swArrow"] & \\ & \mathbb{D}(e) \arrow[ul, bend left, "="] \arrow[u, "p^*"'] & \mathbb{D}(K) \arrow[l, "k^*"] \arrow[u, "u^*"'] & \mathbb{D}(J) \arrow[l, "u_!"] \arrow[ul, bend right, "="'] \end{tikzcd}$$

The component of α at $(j, f\colon u(j) \to k)$ is f. Assuming $\mathbb{D}$ to be a prederivator admitting the necessary homotopy Kan extensions, the calculus of mates gives a natural transformation:

$$\mathrm{Hocolim}_{J_{/k}} \circ \mathrm{pr}^*(X) \to \mathrm{HoLan}_u(X)_k, \quad X \in \mathbb{D}(J).$$

It is obtained from the diagram on the right by pasting of α^* with the undecorated adjunction morphisms. There is a dual such construction in the context of the slice category $J_{k/}$.

Definition 1.5 A prederivator $\mathbb{D}$ is called a *derivator* if it satisfies the following axioms:

(Der1) $\mathbb{D}$ sends coproducts to products. In particular, $\mathbb{D}(\varnothing)$ is trivial.

(Der2) A morphism $f\colon X \to Y$ in $\mathbb{D}(J)$ is an isomorphism if and only if $f_j\colon X_j \to Y_j$ is an isomorphism in $\mathbb{D}(e)$ for every object $j \in J$.

(Der3) For every functor $u\colon J \to K$, there are homotopy left and right Kan extensions along u:

$$(u_!, u^*)\colon \mathbb{D}(J) \rightleftarrows \mathbb{D}(K) \quad \text{and} \quad (u^*, u_*)\colon \mathbb{D}(K) \rightleftarrows \mathbb{D}(J)$$

(Der4) For every functor $u\colon J \to K$ and every $k \in K$, the canonical morphisms

$$\mathrm{Hocolim}_{J_{/k}} \mathrm{pr}^*(X) \to u_!(X)_k \quad \text{and} \quad u_*(X)_k \to \mathrm{Holim}_{J_{k/}} \mathrm{pr}^*(X)$$

are isomorphisms for all $X \in \mathbb{D}(J)$.

The last two axioms of course encode a "homotopical bicompleteness property" together with the pointwise formulas. One could easily develop a more general theory of prederivators which are only homotopy (co)complete or even only have a certain class of homotopy (co)limits.

Example 1.6 Let $\mathcal{C}$ be a category. The represented prederivator $y(\mathcal{C})\colon J \mapsto \mathcal{C}^J$ is a derivator if and only if $\mathcal{C}$ is bicomplete. Thus, the 2–category of bicomplete categories is embedded into the 2–category of derivators.

The idea is of course that the derivator encodes additional structure on its values and in particular on its underlying category $\mathbb{D}(e)$. One nice feature of this approach is that this structure does not have to be chosen but its existence can be deduced from the axioms. Note that all axioms are of the form that they demand a *property*; the only actual *structure* is the given prederivator. As a first example of this "higher structure" we give the following result. We will pursue this more systematically from Section 1.2 on.

Proposition 1.7 *Let $\mathbb{D}$ be a derivator and let J be a category.*

(1) *The category $\mathbb{D}(J)$ admits an initial object $\varnothing$ and a terminal object $*$.*

(2) *The category $\mathbb{D}(J)$ admits coproducts and products.*

We want to emphasize that, in general, the values of a derivator only have very few *categorical* (co)limits. In order to relate this to the homotopical variants, let us introduce the underlying diagram functors and their partial variants. This discussion will also give a proof of the proposition. We saw already that an object $m \in M$ induces an evaluation functor $m^*\colon \mathbb{D}(M) \to \mathbb{D}(e)$. Similarly, a morphism $\alpha\colon m_1 \to m_2$ in M can be considered as a natural transformation of the corresponding classifying functors and thus gives rise to $\alpha^*\colon m_1^* \to m_2^*$. Under the categorical exponential law we hence obtain an *underlying diagram functor*

$$\mathsf{dia}_M\colon \mathbb{D}(M) \to \mathbb{D}(e)^M.$$

Similarly, given a product $M \times J$ we obtain a *partial underlying diagram functor*

$$\mathsf{dia}_{M,J}\colon \mathbb{D}(M \times J) \to \mathbb{D}(J)^M.$$

The natural isomorphism $M \cong M \times e$ induces an identification of dia_M and $\mathsf{dia}_{M,e}$. Now, the functor $p_M\colon M \to e$ gives rise to the following diagram

$$\begin{tikzcd}
\mathbb{D}(M) \arrow[r, "\mathsf{dia}_M"] \arrow[d, bend right=60, "\mathrm{Hocolim}_M"'] \arrow[d, bend left=60, "\mathrm{Holim}_M"] & \mathbb{D}(e)^M \arrow[d, bend right=60, dotted] \arrow[d, bend left=60, dotted] \\
\mathbb{D}(e) \arrow[u, "p_M^*"] \arrow[r, "\mathrm{id}"'] & \mathbb{D}(e), \arrow[u, "\Delta_M"']
\end{tikzcd}$$

which commutes in the sense that we have

$$\mathsf{dia}_M \circ p_M^* = \Delta_M\colon \mathbb{D}(e) \to \mathbb{D}(e)^M.$$

If the underlying diagram functor dia_M happens to be an equivalence for a certain category M, then also Δ_M has adjoints on both sides, ie the category $\mathbb{D}(e)$ has then (co)limits of shape M. Similar remarks apply to the case of the partial underlying diagram functor $\mathsf{dia}_{M,J}$, where we would then deduce a conclusion about the category $\mathbb{D}(J)$. Now, axiom (Der1) implies that the partial underlying diagram functors

$$\mathsf{dia}_{\varnothing,J}\colon \mathbb{D}(\varnothing) \to \mathbb{D}(J)^{\varnothing} = e \quad \text{and} \quad \mathsf{dia}_{S,J}\colon \mathbb{D}(S \times J) \to \mathbb{D}(J)^S$$

are equivalences. In the second case, S denotes a set considered as a discrete category, ie with identity morphisms only. This explains why we are able to deduce Proposition 1.7 from the axioms but, in general, do not have other categorical (co)limits.

Although, in general, we do not want to assume that also other partial underlying diagram functors are equivalences, the following definition is very important. This definition again emphasizes the importance of the distinction between the categories $\mathbb{D}(K)$ and $\mathbb{D}(e)^K$.

Definition 1.8 A derivator $\mathbb{D}$ is called *strong* if it satisfies the following axiom:

(Der5) The partial underlying diagram functor $\mathsf{dia}_{[1],J}\colon \mathbb{D}([1]\times J)\to\mathbb{D}(J)^{[1]}$ is full and essentially surjective for each category J.

This axiom is a bit harder to motivate. Derivators associated to bicomplete categories or model categories are strong (see the next lemma). In this paper, (Der5) will play a key role in the construction of the triangulated structures on the values of a stable derivator. The point is that it allows one to lift morphisms in the underlying category $\mathbb{D}(e)$ to objects in the category $\mathbb{D}([1])$ where we can apply certain constructions to it. Similarly, in the context of pointed derivators it allows us to construct fiber and cofiber sequences associated to a morphism in the underlying category. One might expect that in later developments of the theory this property will also be useful in the unpointed context. Nevertheless, we follow Maltsiniotis in not including this property as an axiom of the basic notion of a derivator.

Let us already give the argument needed later to show that the underlying derivators of model categories are strong. Given an arbitrary model category $\mathcal{M}$ the arrow category $\mathcal{M}^{[1]}$ can be endowed with projective model structure. In fact, this the Reedy model structure in the special case of the direct index category $[1]=(0<1)$ [19]. The fibrations and weak equivalences are defined levelwise. Given two objects $X=(X_0\to X_1)$ and $Y=(Y_0\to Y_1)$ a morphism $f=(f_0,f_1)\colon X\to Y$ is a cofibration (respectively acyclic cofibration) if and only f_0 and the pushout product map $Y_0\sqcup_{X_0}X_1\to Y_1$ are cofibrations (respectively acyclic cofibrations). In particular, the bifibrant objects are cofibrations $X_0\to X_1$ between bifibrant objects.

Lemma 1.9 *Let $\mathcal{M}$ be a model category. The underlying diagram functor*

$$\mathsf{Ho}(\mathcal{M}^{[1]})\to\mathsf{Ho}(\mathcal{M})^{[1]}$$

is full and essentially surjective.

Proof As model for the homotopy categories we use the category given by the bifibrant objects and homotopy classes of morphisms. The essential surjectivity is immediate. The fullness follows from the following mapping cylinder argument. Let $u_X\colon X_0\to X_1$ and $u_Y\colon Y_0\to Y_1$ be bifibrant objects and let us consider a morphism $[f]\colon X\to Y$ in $\mathsf{Ho}(\mathcal{M})^{[1]}$. Thus the following diagram

$$\begin{array}{ccc} X_0 & \xrightarrow{u_X} & X_1 \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f_1} \\ Y_0 & \xrightarrow[u_Y]{} & Y_1 \end{array}$$

commutes up to a homotopy H. Choose a cylinder object $\mathsf{Cyl}(X)$ of X in $\mathcal{M}^{[1]}$. Evaluation yields a cylinder object $\mathsf{Cyl}(X_i) = \mathsf{Cyl}(X)_i$ for X_i for $i = 0, 1$ and we choose our homotopy H to be with respect to $\mathsf{Cyl}(X_0)$. The acyclic cofibration $X \to \mathsf{Cyl}(X)$ yields an acyclic cofibration $\mathsf{Cyl}(X_0) \sqcup_{X_0} X_1 \to \mathsf{Cyl}(X_1)$. A solution to the lifting problem

$$\begin{tikzcd}
\mathsf{Cyl}(X_0) \sqcup_{X_0} X_1 \arrow[r, "H \sqcup f_1"] \arrow[d] & Y_1 \\
\mathsf{Cyl}(X_1) \arrow[ur, dashed, bend right] &
\end{tikzcd}$$

allows us to rigidify the above homotopy commutative diagram concluding the proof. □

Let us quickly recall the dualization process for derivators. The point is that given a 2–category $\mathcal{C}$ we obtain a new 2–category $\mathcal{C}^{\mathrm{op}}$ by inverting the direction of the 1–morphisms and we get a further 2–category $\mathcal{C}^{\mathrm{co}}$ by inverting the direction of the 2–morphisms. Moreover, these operations can be combined so that given a 2–category using the various dualizations we obtain 4 different 2–categories (more generally, an n–category has 2^n different dualizations).

Definition 1.10 Let $\mathbb{D}$ be a prederivator. Then we define the *dual prederivator* $\mathbb{D}^{\mathrm{op}}$ by the following diagram:

$$\begin{tikzcd}
\mathsf{Cat}^{\mathrm{op}} \arrow[r, "\mathbb{D}^{\mathrm{op}}"] \arrow[d, "(-)^{\mathrm{op}}"'] & \mathsf{CAT} \\
\mathsf{Cat}^{\mathrm{op,co}} \arrow[r, "\mathbb{D}"'] & \mathsf{CAT}^{\mathrm{co}} \arrow[u, "(-)^{\mathrm{op}}"']
\end{tikzcd}$$

Example 1.11 A prederivator $\mathbb{D}$ is a derivator if and only if its dual $\mathbb{D}^{\mathrm{op}}$ is a derivator.

This result implies that in many general statements about derivators and morphisms between derivators we only have to prove claims about, say, homotopy left Kan extensions while the corresponding claim for homotopy right Kan extensions follows by duality.

We close this subsection by a brief discussion of admissible shapes of diagrams. Depending on the context it might be useful to consider "derivators defined on certain sub-2–categories of Cat". The theory as developed in this paper and the sequels works equally well as soon as the subcategory satisfies the following closure properties.

Definition 1.12 A full 2–subcategory $\mathsf{Dia} \subseteq \mathsf{Cat}$ is called a *category of diagrams* if it satisfies the following axioms:

(1) All finite posets considered as categories belong to Dia.

(2) Dia is closed under finite coproducts and under pullbacks.

(3) For every $J \in \mathsf{Dia}$ and every $j \in J$, the slice constructions $J_{j/}$ and $J_{/j}$ belong to Dia.

(4) If $J \in \mathsf{Dia}$ then also $J^{\mathrm{op}} \in \mathsf{Dia}$.

(5) For every Grothendieck fibration $u \colon J \to K$, if all fibers $J_k, k \in K$, and the base K belong to Dia then also J lies in Dia.

Given such a category of diagrams there is the corresponding notion of a *(pre)derivators of type* Dia. The reader is invited to replace "(pre)derivator" by "(pre)derivator of type Dia" throughout this paper and to check that all results we establish here also work for that notion. Depending on Dia, it might be the case that derivators of type Dia only admit *finite* (co)products (Proposition 1.7).

Example 1.13 The full 2–subcategory of finite posets is the smallest category of diagrams, Cat itself is the largest one. Further examples are given by the full 2–subcategories spanned by the finite categories or the finite-dimensional categories. Moreover, the intersection of a family of categories of diagrams is again a category of diagrams.

1.2 Homotopy exact squares and some properties of homotopy Kan extensions

Let us collect some basic facts about the calculus of mates [28] which will be of constant use in the remainder of this paper. Although this calculus is available in any 2–category we restrict attention to the case of CAT. The choice of notation is motivated by our later applications to derivators so that functors are decorated by $(-)^*$, while adjoint functors to, say, u^* will be denoted by $u_!$ and u_* respectively.

In this discussion we assume all necessary adjoint functors to exist. Given a natural transformation $\alpha \colon v^*u'^* \to u^*w^*$ there is a canonical natural transformation $\alpha_! \colon u_!v^* \to w^*u'_!$. Similarly, given a natural transformation $\alpha \colon u^*w^* \to v^*u'^*$ we obtain canonically $\alpha_* \colon w^*u'_* \to u_*w^*$. Both natural transformations $\alpha_!$ and α_* are referred to as mates of α.

Lemma 1.14 (1) *The calculus of mates is compatible with pasting.*

(2) *The two different formations of mates $\alpha \mapsto \alpha_!$ and $\alpha \mapsto \alpha_*$ are inverse to each other, ie we have $\alpha = (\alpha_!)_* = (\alpha_*)_!$.*

(3) *Given a natural transformation $\alpha\colon v^*u'^* \to u^*w^*$ then the mates $\alpha_!\colon u_!v^* \to w^*u'_!$ and $\alpha_*\colon u'^*w_* \to v_*u^*$ are conjugate. In particular, $\alpha_!$ is an isomorphism if and only if α_* is an isomorphism.*

For the convenience of the reader and because of its central importance later on we give a proof of the first fact. For this purpose, let us consider the pasting $\alpha_1 \odot \alpha_2$ of α_1 and α_2 in CAT:

$$\begin{tikzcd}
\mathcal{C}_1 & \mathcal{C}_2 \arrow[l, "v_1^*"'] & \mathcal{C}_3 \arrow[l, "v_2^*"'] \\
\mathcal{D}_1 \arrow[u, "u_1^*"] \arrow[ur, phantom, "\swarrow"] & \mathcal{D}_2 \arrow[l, "w_1^*"] \arrow[u, "u_2^*"'] \arrow[ur, phantom, "\swarrow"] & \mathcal{D}_3 \arrow[l, "w_2^*"] \arrow[u, "u_3^*"']
\end{tikzcd}$$

By definition the pasting $\alpha_{2!} \odot \alpha_{1!}$ is given by

$$\begin{tikzcd}
\mathcal{D}_1 & \mathcal{C}_1 \arrow[l, "u_{1!}"'] & \mathcal{C}_2 \arrow[l, "v_1^*"'] & & & \\
& \mathcal{D}_1 \arrow[ul, bend left, "="] \arrow[u, "u_1^*"] \arrow[ul, phantom, "\swarrow"] \arrow[ur, phantom, "\swarrow"] & \mathcal{D}_2 \arrow[l, "w_1^*"] \arrow[u, "u_2^*"'] & \mathcal{C}_2 \arrow[l, "u_{2!}"] \arrow[ul, bend right, "="'] \arrow[ul, phantom, "\swarrow"] & \mathcal{C}_3 \arrow[l] & \\
& & & \mathcal{D}_2 \arrow[ul, bend left, "="] \arrow[u, "u_2^*"'] \arrow[ul, phantom, "\swarrow"] \arrow[ur, phantom, "\swarrow"] & \mathcal{D}_3 \arrow[l, "w_2^*"] \arrow[u, "u_3^*"'] & \mathcal{C}_3, \arrow[l, "u_{3!}"] \arrow[ul, bend right, "="'] \arrow[ul, phantom, "\swarrow"]
\end{tikzcd}$$

where the additional undecorated 2–cells are given by adjunction morphisms. Now, by a triangular identity for adjunctions the two triangles in the middle cancel and we obtain $(\alpha_1 \odot \alpha_2)_!$ as intended. The proof of the compatibility of vertical pasting and also for the assignment $\alpha \mapsto \alpha_*$ is similar. Note, that we only have a compatibility with respect to pasting and not "a functoriality". In particular, it can (and will) be the case that we start with a commutative square but that the mates of the identity transformation are even not isomorphisms. Nevertheless, this compatibility with respect to pasting combined with the 2-out-of-3-property for isomorphisms will be a key ingredient in many proofs of this paper.

Let us now apply this formalism to derivators. Given a derivator $\mathbb{D}$ and a natural transformation $\alpha\colon u'v \to wu$ in Cat we will abuse notation by setting $\alpha_! = (\alpha^*)_!$ and $\alpha_* = (\alpha^*)_*$.

Definition 1.15 Let $\mathbb{D}$ be a derivator and let us consider a natural transformation α as indicated in the following square in Cat:

$$\begin{tikzcd} J_1 \arrow[r, "v"] \arrow[d, "u_1"'] \arrow[dr, phantom, "\swarrow"] & J_2 \arrow[d, "u_2"] \\ K_1 \arrow[r, "w"'] & K_2 \end{tikzcd}$$

The square is $\mathbb{D}$*-exact* if the natural transformation $\alpha_!\colon u_{1!} \circ v^* \to w^* \circ u_{2!}$ (or, by Lemma 1.14, equivalently $\alpha_*\colon u_2^* \circ w_* \to v_* \circ u_1^*$) is an isomorphism. The square is called *homotopy exact* if it is $\mathbb{D}$–exact for all derivators $\mathbb{D}$.

We will also apply the terminology of $\mathbb{D}$–exact squares in the context of a prederivator $\mathbb{D}$ admitting the necessary homotopy Kan extensions. For a derivator $\mathbb{D}$ it follows immediately from Lemma 1.14 that $\mathbb{D}$–exact squares are stable under horizontal and vertical pasting.

Warning 1.16 We want to include a warning on a certain risk of ambiguity if the natural transformation α under consideration happens to be an isomorphism. In that case it can (and will) happen that, say, $\alpha_!$ is an isomorphism without this being the case for $(\alpha^{-1})_!$ (see for example Section 2.2). In particular, this can happen for commutative squares. Thus, in case there is a risk of ambiguity we will always give a direction to natural isomorphisms and even to identity transformations (see for example Proposition 1.24).

We will next illustrate the notion of homotopy exact squares by giving some examples which are central to the development of the theory of derivators (for a more systematic discussion we refer to [33]). Using the 2–functoriality of prederivators, the following is immediate.

Lemma 1.17 *Let $\mathbb{D}$ be a prederivator and let $(L,R)\colon J \rightleftarrows K$ be an adjunction. Then we obtain an adjunction*

$$(R^*, L^*)\colon \mathbb{D}(J) \rightleftarrows \mathbb{D}(K).$$

Moreover, if L (respectively R) is fully faithful, then so is R^* (respectively L^*).

A related result using the notion of homotopy exact squares can be formulated as follows. This result expresses the cofinality of right adjoints.

Proposition 1.18 *For a right adjoint functor $R\colon J \to K$ the following square is homotopy exact:*

$$\begin{tikzcd} J \arrow[r, "R"] \arrow[d, "p_J"'] \arrow[dr, phantom, "\swArrow"] & K \arrow[d, "p_K"] \\ e \arrow[r, "\mathrm{id}"'] & e \end{tikzcd}$$

Proof We have to show that the natural transformation $p_{J!}R^* \to p_{K!}$ is an isomorphism for an arbitrary derivator. But this transformation is conjugate to $\mathrm{id}\colon p_K^* \to L^*p_J^* = (p_J L)^* = p_K^*$ by Lemma 1.14. □

Thus, for a derivator $\mathbb{D}$, a right adjoint functor $R\colon J \to K$, and an object $X \in \mathbb{D}(K)$ we have a canonical isomorphism

$$\mathrm{Hocolim}_J\ R^*(X) \xrightarrow{\cong} \mathrm{Hocolim}_K\ X.$$

For later reference, let us spell out the important special case where the right adjoint $R = t\colon e \to K$ just specifies a terminal object in K. The second part of the lemma follows immediately by passing to the conjugate of the natural transformation showing up in the first part.

Lemma 1.19 *Let $\mathbb{D}$ be a derivator and let K be a category admitting a terminal object t.*

(1) *For $X \in \mathbb{D}(K)$ we have a natural isomorphism $X_t \xrightarrow{\cong} \mathrm{Hocolim}_K\ X$.*

(2) *We have a canonical isomorphism of functors $p_K^* \xrightarrow{\cong} t_*$. The essential image of t_* consists of precisely those objects for which all structure maps in the underlying diagram are isomorphisms.*

Here is another important result about homotopy Kan extensions.

Proposition 1.20 *Let $u\colon J \to K$ be a fully faithful functor. Then the following square is homotopy exact:*

$$\begin{tikzcd} J \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & J \arrow[d, "u"] \\ J \arrow[r, "u"'] & K \end{tikzcd}$$

Thus, the adjunction morphisms $\eta\colon \mathrm{id} \to u^*u_!$ and $\epsilon\colon u^*u_* \to \mathrm{id}$ are isomorphisms, ie homotopy Kan extension functors along fully faithful functors are fully faithful.

Proof Since isomorphisms can be detected pointwise we can reduce our task to showing that the following pasting is homotopy exact for all $j \in J$:

$$\begin{tikzcd} J_{/j} \arrow[r, "\mathrm{pr}"] \arrow[d, "p"'] \arrow[dr, phantom, "\Swarrow"] & J \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] \arrow[dr, phantom, "\Swarrow"] & J \arrow[d, "u"] \\ e \arrow[r, "j"'] & J \arrow[r, "u"'] & K \end{tikzcd}$$

But, the fully faithfulness of u implies that we have an isomorphism $J_{/u(j)} \to J_{/j}$ so that it suffices (by Proposition 1.18) to show that the next pasting is homotopy exact:

$$\begin{tikzcd} J_{/u(j)} \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\Swarrow"] & J_{/j} \arrow[r, "\mathrm{pr}"] \arrow[d, "p"'] \arrow[dr, phantom, "\Swarrow"] & J \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] \arrow[dr, phantom, "\Swarrow"] & J \arrow[d, "u"] \\ e \arrow[r] & e \arrow[r, "j"'] & J \arrow[r, "u"'] & K \end{tikzcd}$$

But this is guaranteed by axiom (Der4). □

Since we now know that, for fully faithful $u\colon J \to K$, the homotopy Kan extension functors $u_!, u_*\colon \mathbb{D}(J) \to \mathbb{D}(K)$ are fully faithful, we would like to obtain a characterization of the objects in the essential images. The point of the next lemma is that one only has to control the adjunction morphisms at arguments $k \in K - u(J)$.

Lemma 1.21 *Let $\mathbb{D}$ be a derivator, $u\colon J \to K$ a fully faithful functor and $X \in \mathbb{D}(K)$.*

(1) *X lies in the essential image of $u_!$ if and only if the adjunction counit $\epsilon\colon u_!u^* \to \mathrm{id}$ induces an isomorphism $\epsilon_k\colon u_!u^*(X)_k \to X_k$ for all $k \in K - u(J)$.*

(2) *X lies in the essential image of u_* if and only if the adjunction unit $\eta\colon \mathrm{id} \to u_*u^*$ induces an isomorphism $\eta_k\colon X_k \to u_*u^*(X)_k$ for all $k \in K - u(J)$.*

Proof We give a proof of (2), so let us consider the adjunction $(u^*, u_*)\colon \mathbb{D}(K) \rightleftarrows \mathbb{D}(J)$. By Proposition 1.20, u_* is fully faithful. Thus, $X \in \mathbb{D}(K)$ lies in the essential image of u_* if and only if the adjunction unit $\eta\colon X \to u_*u^*X$ is an isomorphism. Since isomorphisms can be tested pointwise, this is the case if and only if we have an isomorphism $\eta_k\colon X_k \to u_*u^*(X)_k$ for all $k \in K$. For the converse direction, one of the triangular identities for our adjunction reads as $\mathrm{id} = \epsilon u^* \cdot u^*\eta$. Thus like ϵ, $u^*\eta$ is an isomorphism so that it suffices to check at points which do not lie in the image. □

There are two important classes of fully faithful functors where the essential image of homotopy Kan extensions can be characterized more easily. So let us give their definition.

Definition 1.22 Let $u\colon J \to K$ be a fully faithful functor which is injective on objects.

(1) The functor u is called a *cosieve* if whenever we have a morphism $u(j) \to k$ in K then k lies in the image of u.

(2) The functor u is called a *sieve* if whenever we have a morphism $k \to u(j)$ in K then k lies in the image of u.

The next proposition and a variant for the case of pointed derivators (see Proposition 3.6) will be frequently used throughout this paper.

Proposition 1.23 *Let $\mathbb{D}$ be a derivator.*

(1) *Let $u\colon J \to K$ be a cosieve. Then the homotopy left Kan extension $u_!$ is fully faithful and $X \in \mathbb{D}(K)$ lies in the essential image of $u_!$ if and only if $X_k \cong \varnothing$ for all $k \in K - u(J)$.*

(2) *Let $u\colon J \to K$ be a sieve. Then the homotopy right Kan extension u_* is fully faithful and $X \in \mathbb{D}(K)$ lies in the essential image of u_* if and only if $X_k \cong *$ for all $k \in K - u(J)$.*

Proof We give a proof of the first statement. The functor $u_!$ is fully faithful by Proposition 1.20. To describe the essential image we use the criterion of Lemma 1.21. But for $k \in K - u(J)$ we have

$$u_!u^*(X)_k \cong \mathrm{Hocolim}_{J_{/k}} \mathrm{pr}^* u^*(X) = \mathrm{Hocolim}_{\varnothing} \mathrm{pr}^* u^*(X) = \varnothing.$$

Thus for $k \in K - u(J)$ the map $\epsilon_k\colon u_!u^*(X)_k \to X_k$ is an isomorphism if and only if $X_k \cong \varnothing$. □

1.3 Examples

The first aim is to show that given a derivator $\mathbb{D}$ and a small category M then $\mathbb{D}^M$ is again a derivator. The harder part is to establish (Der4) for $\mathbb{D}^M$. In order to achieve this we include a short detour and establish some reformulations of this axiom which are of independent interest and will be used further below. For this purpose let us consider a pullback diagram in Cat:

$$\begin{tikzcd} J_1 \arrow[r, "v"] \arrow[d, "u_1"'] \arrow[dr, phantom, "\nearrow"] & J_2 \arrow[d, "u_2"] \\ K_1 \arrow[r, "w"'] & K_2 \end{tikzcd}$$

For the notion of Grothendieck (op)fibrations we refer to [7, Section 8.1] or [41].

Proposition 1.24 *Using the above notation, a pullback diagram is homotopy exact, if u_2 is a Grothendieck fibration or if w is a Grothendieck opfibration.*

Proof We give the proof in the case where u_2 is a Grothendieck fibration. For a derivator $\mathbb{D}$ we thus have to show that the canonical map $\mathrm{id}_*\colon w^*u_{2*} \to u_{1*}v^*$ is a natural isomorphism. Since isomorphisms can be tested pointwise, (Der4) implies that it suffices to show that the following pasting is a homotopy exact square for all $k_1 \in K_1$:

$$\begin{tikzcd}
(J_1)_{k_1/} \arrow[r, "\mathrm{pr}"] \arrow[d, "p"'] \arrow[dr, phantom, "\nearrow"] & J_1 \arrow[r, "v"] \arrow[d, "u_1"'] \arrow[dr, phantom, "\nearrow"] & J_2 \arrow[d, "u_2"] \\
e \arrow[r, "k_1"'] & K_1 \arrow[r, "w"'] & K_2
\end{tikzcd}$$

Since our diagram in Cat is a pullback diagram, we deduce that like u_2 also u_1 is a Grothendieck fibration. Let us denote by $(J_1)_{k_1}$ the fiber of u_1 over k_1, ie the subcategory of J_1 consisting of all objects sent to k_1 and all morphisms sent to id_{k_1}. It then follows that the canonical functor

$$c\colon (J_1)_{k_1} \to (J_1)_{k_1/}, \quad j_1 \mapsto (j_1, k_1 \xrightarrow{\mathrm{id}} u_1(j_1))$$

is a left adjoint functor [37]. Now, Lemma 1.14 and Proposition 1.18 imply that the above pasting is homotopy exact if and only if this is the case for the following top pasting:

$$\begin{tikzcd}
(J_1)_{k_1} \arrow[r, "c"] \arrow[d, "p"'] \arrow[dr, phantom, "\nearrow"] & (J_1)_{k_1/} \arrow[r, "\mathrm{pr}"] \arrow[d, "p"'] \arrow[dr, phantom, "\nearrow"] & J_1 \arrow[r, "v"] \arrow[d, "u_1"'] \arrow[dr, phantom, "\nearrow"] & J_2 \arrow[d, "u_2"] \\
e \arrow[r] & e \arrow[r, "k_1"'] & K_1 \arrow[r, "w"'] & K_2
\end{tikzcd}$$

$$\begin{tikzcd}
(J_1)_{k_1} \arrow[r, "w"] \arrow[d, "p"'] \arrow[dr, phantom, "\nearrow"] & (J_2)_{w(k_1)} \arrow[r, "c"] \arrow[d, "p"'] \arrow[dr, phantom, "\nearrow"] & (J_2)_{w(k_1)/} \arrow[r, "\mathrm{pr}"] \arrow[d, "p"'] \arrow[dr, phantom, "\nearrow"] & J_2 \arrow[d, "u_2"] \\
e \arrow[r] & e \arrow[r] & e \arrow[r, "w(k_1)"'] & K_2
\end{tikzcd}$$

It is easy to check that the above two pastings define the same natural transformation. Thus, by exactly the same arguments again it suffices to show that the square

$$\begin{tikzcd}
(J_1)_{k_1} \arrow[r, "w"] \arrow[d, "p"'] \arrow[dr, phantom, "\nearrow"] & (J_2)_{w(k_1)} \arrow[d, "p"] \\
e \arrow[r] & e
\end{tikzcd}$$

is homotopy exact. But, since we started with a pullback diagram, w restricted in this way is an isomorphism of categories so that our claim follows (again by Proposition 1.18). $\square$

We will refer to this proposition by saying that a "derivator satisfies base change for Grothendieck (op)fibrations". This proposition allows us to establish the next theorem.

Theorem 1.25 *Let $\mathbb{D}$ be a derivator and let M be a small category. Then the prederivator $\mathbb{D}^M\colon \mathsf{Cat}^{\mathrm{op}} \to \mathsf{CAT}\colon K \mapsto \mathbb{D}(M \times K)$ is a derivator.*

Proof The axioms (Der1)–(Der3) are immediate so we only have to establish axiom (Der4) for $\mathbb{D}^M$. By duality, it suffices to give the proof for the case of homotopy right Kan extensions. In other words, we have to show that the following square on the left is $\mathbb{D}$–exact:

$$\begin{tikzcd}
M \times J_{k/} \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow"] & M \times e \arrow[d] \\
M \times J \arrow[r] & M \times K
\end{tikzcd}
\qquad
\begin{tikzcd}
M \times J_{k/} \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow"] & M \times K_{k/} \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow"] & M \times e \arrow[d] \\
M \times J \arrow[r] & M \times K \arrow[r] & M \times K
\end{tikzcd}$$

But this 2–cell can be obtained as the pasting of the diagram on the right in which the square on the left is a pullback diagram such that the right vertical arrow is a Grothendieck opfibration. Thus, by Proposition 1.24 it suffices to show that the square on the right is $\mathbb{D}$–exact. Using the isomorphism $M_{m/} \times K_{k/} \cong (M \times K_{k/})_{m/}$ and again that isomorphisms are detected pointwise it suffices to show that the pasting obtained by the following left diagram is $\mathbb{D}$–exact for every $m \in M$:

$$\begin{tikzcd}
M_{m/} \times K_{k/} \arrow[r] \arrow[d] \arrow[dr, phantom, "\nearrow"] & M \times K_{k/} \arrow[r] \arrow[d] \arrow[dr, phantom, "\nearrow"] & M \times K \arrow[d] \\
e \arrow[r, "m"'] & M \times e \arrow[r] & M \times K
\end{tikzcd}
\qquad
\begin{tikzcd}
(M \times K)_{(m,k)/} \arrow[r] \arrow[d] \arrow[dr, phantom, "\nearrow"] & M \times K \arrow[d] \\
e \arrow[r, "(m,k)"'] & M \times K
\end{tikzcd}$$

Now we conclude by observing that this pasting is naturally isomorphic to the square on the right-hand side which is $\mathbb{D}$–exact by (Der4). $\square$

Thus, whenever we want to establish a general result about the values $\mathbb{D}(M)$ of a derivator $\mathbb{D}$ we may assume that we are considering the underlying category of a derivator since we can always pass from $\mathbb{D}$ to $\mathbb{D}^M$.

We now show that the conclusion of Proposition 1.24 is actually equivalent to (Der4). Moreover, there is a further reformulation using the squares of the following form in Cat:

$$\begin{tikzcd} (u_1/u_2) \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"'] \arrow[dr, phantom, "\swArrow"] & J_1 \arrow[d, "u_1"] \\ J_2 \arrow[r, "u_2"'] & K_2 \end{tikzcd}$$

Here, the category (u_1/u_2) is the *comma category* where an object is a triple

$$(j_1, j_2, \alpha\colon u_1(j_1) \to u_2(j_2)), \quad j_1 \in J_1,\ j_2 \in J_2,$$

and the functors pr_i are the obvious projection functors. The arrow component of such an object defines the natural transformation depicted in the diagram. If we specialize to $J_1 = e$ or $J_2 = e$ we get back the diagrams showing up in the pointwise calculation of Kan extensions.

Proposition 1.26 *Let $\mathbb{D}$ be a prederivator which satisfies the axioms* (Der1)–(Der3). *Then the following three statements are equivalent:*

(1) *The prederivator $\mathbb{D}$ is a derivator, ie it also satisfies* (Der4).

(2) *The prederivator $\mathbb{D}$ satisfies base change for Grothendieck fibrations and opfibrations.*

(3) *The prederivator $\mathbb{D}$ satisfies base change for comma categories, ie the squares associated to comma categories are $\mathbb{D}$–exact.*

Proof By Proposition 1.24 we already know that (2) is implied by (1). The converse direction follows by similar but simpler arguments than the ones we used in the proof of Theorem 1.25. Since comma categories specialize to slice categories it is obvious that (3) implies (1). So we only have to prove that (1) implies (3). Using similar reduction arguments as in the last proof (including the behavior of base change with respect to pasting and the fact that isomorphisms are detected pointwise) it suffices to show that the following pasting is $\mathbb{D}$–exact for all objects $j_2 \in J_2$:

$$\begin{tikzcd} (u_1/u_2)_{/j_2} \arrow[r, "\mathrm{pr}"] \arrow[d, "p"'] \arrow[dr, phantom, "\swArrow"] & (u_1/u_2) \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"'] \arrow[dr, phantom, "\swArrow"] & J_1 \arrow[d, "u_1"] \\ e \arrow[r, "j_2"'] & J_2 \arrow[r, "u_2"'] & K \end{tikzcd}$$

Now, there is a canonical functor $R\colon J_{1/u_2(j_2)} \to (u_1/u_2)_{/j_2}$ which is defined by:

$$(j_1, u_1(j_1) \to u_2(j_2)) \longmapsto \big((j_1, u_1(j_1) \to u_2(j_2), j_2), j_2 \xrightarrow{\mathrm{id}} j_2\big)$$

This functor can be checked to be a right adjoint so that by Proposition 1.18 it suffices to show that the pasting in the following diagram is $\mathbb{D}$–exact:

$$\begin{tikzcd}
J_{1/u_2(j_2)} \arrow[r, "R"] \arrow[d, "p"'] \arrow[dr, phantom, "\swarrow"] & (u_1/u_2)_{/j_2} \arrow[r, "\mathrm{pr}"] \arrow[d, "p"'] \arrow[dr, phantom, "\swarrow"] & (u_1/u_2) \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"'] \arrow[dr, phantom, "\swarrow"] & J_1 \arrow[d, "u_1"] \\
e \arrow[r] & e \arrow[r, "j_2"'] & J_2 \arrow[r, "u_2"'] & K
\end{tikzcd}$$

But this pasting is precisely the square used to calculate homotopy Kan extensions along u_1 at $u_2(j_2)$ so that we can conclude by (Der4). $\square$

Let us now turn to the second important class of examples of derivators, namely the ones associated to nice model categories. This is included not only for the sake of completeness but also because our proof differs from the one given in [8]. Our proof is completely self-dual and is simpler in that it does not make use of the explicit description of the generating (acyclic) projective cofibrations of a diagram category associated to a cofibrantly generated model category. We restrict attention to the following situation.

Definition 1.27 A model category $\mathcal{M}$ is called *combinatorial* if it is cofibrantly generated and if the underlying category is locally presentable.

All we need from the theory of combinatorial model categories is the validity of the next theorem so that we could also work axiomatically with the conclusion of this theorem. Recall that the *projective model structure* on a diagram category is determined by the fact that the weak equivalences and the fibrations are defined levelwise. In the *injective model structure* this is the case for the weak equivalences and the cofibrations. We will denote the functor categories $\mathcal{M}^J$ endowed with the corresponding model structures by $\mathcal{M}^J_{proj}$ and $\mathcal{M}^J_{inj}$ respectively. The following statement about the projective model structures is a consequence of the lifting theorem of cofibrantly generated model structures along a left adjoint while the statement about the injective model structure was only proved more recently. Both results are for example established in [29, Proposition A.2.8.2].

Theorem 1.28 *Let $\mathcal{M}$ be a combinatorial model category and let J be a small category. The category $\mathcal{M}^J$ can be endowed with the projective and with the injective model structure.*

One point of these model structures is that certain adjunctions are now Quillen adjunctions for trivial reasons.

Lemma 1.29 *Let $\mathcal{M}$ be a combinatorial model category and let $u\colon J \to K$ be a functor. Then we have the following Quillen adjunctions*

$$(u_!, u^*)\colon \mathcal{M}^J_{proj} \to \mathcal{M}^K_{proj} \quad \text{and} \quad (u^*, u_*)\colon \mathcal{M}^K_{inj} \to \mathcal{M}^J_{inj}.$$

We now have almost everything at our disposal needed to establish the following result.

Proposition 1.30 *Let $\mathcal{M}$ be a combinatorial model category. Then the assignment*

$$\mathbb{D}_{\mathcal{M}}\colon \mathsf{Cat}^{\mathrm{op}} \to \mathsf{CAT}\colon J \mapsto \mathsf{Ho}(\mathcal{M}^J)$$

defines a strong derivator.

Proof The first two axioms are immediate while the existence of the homotopy Kan extension functors is guaranteed by the last lemma. Let us establish Kan's formula for homotopy right Kan extensions. For this purpose, let $u\colon J \to K$ be a functor and let $k \in K$ be an object. Consider the following diagram, which commutes up to natural isomorphism:

$$\begin{tikzcd} \mathcal{M}^{J_{k/}} \arrow[d, "\lim"'] \arrow[dr, phantom, "\cong"] & \mathcal{M}^J \arrow[l, "\mathrm{pr}^*"'] \arrow[d, "u_*"] \\ \mathcal{M} & \mathcal{M}^K \arrow[l, "k^*"] \end{tikzcd}$$

By the last lemma, the functors lim and u_* are right Quillen functors with respect to the injective model structures so it suffices to show the same for k^* and pr^*. Since weak equivalences are defined pointwise it suffices to show that in both cases injective fibrations are preserved. In the case of k^* we can use the adjunction $(k_!, k^*)$ and show that $k_!\colon \mathcal{M} \to \mathcal{M}^K$ preserves acyclic cofibrations. But an easy calculation with left Kan extensions shows that we have $k_!(X)_l \cong \coprod_{\mathsf{hom}_K(k,l)} X$. From this description it is immediate that $k_!$ preserves acyclic cofibrations. Similarly, it is enough to show that $\mathrm{pr}_!$ preserves injective acyclic cofibration which a special case of the following lemma. Finally, the derivator is strong by Lemma 1.9. □

Lemma 1.31 *Let $u\colon J \to K$ be a Grothendieck opfibration with discrete fibers and let $\mathcal{M}$ be a combinatorial model category. Then the functor $u^*\colon \mathcal{M}^K \to \mathcal{M}^J$ preserves injective fibrations.*

Proof By adjointness, it is enough to show that the left adjoint $u_!\colon \mathcal{M}^J \to \mathcal{M}^K$ preserves acyclic injective cofibrations. For this purpose, let $X \in \mathcal{M}^J$ and let $k \in K$. Then we make the following calculation:

$$u_!(X)_k \cong \mathrm{colim}_{J_{/k}}\, X \circ \mathrm{pr} \cong \mathrm{colim}_{J_k}\, X \circ \mathrm{pr} \circ c \cong \coprod_{j \in J_k} X_j$$

The first isomorphism is the pointwise formula for Kan extensions while the second one is given by the cofinality of right adjoints (Proposition 1.18) applied to the canonical functor $c\colon J_k \to J_{/k}$. Finally, the last isomorphism uses the fact that the Grothendieck opfibration has discrete fibers. From this explicit description of $u_!$ the claim follows immediately. □

The proof of the above theorem actually shows a bit more. Given a cofibrantly generated model category $\mathcal{M}$, the prederivator $\mathbb{D}_{\mathcal{M}}$ is what could be called a *cocomplete prederivator* (with the obvious meaning). But by far more is true. There is the following more general result which is due to Cisinski [8].

Theorem 1.32 *Let $\mathcal{M}$ be a model category and let J be a small category. Denote by W_J the class of levelwise weak equivalences in $\mathcal{M}^J$. Then the assignment*

$$\mathbb{D}_{\mathcal{M}}\colon \mathsf{Cat}^{\mathrm{op}} \to \mathsf{CAT}, \quad J \mapsto \mathcal{M}^J[W_J^{-1}]$$

defines a derivator.

2 The 2–category of derivators

2.1 Morphisms and natural transformations

Let $\mathbb{D}$ and $\mathbb{D}'$ be prederivators. A *morphism of prederivators* $F\colon \mathbb{D} \to \mathbb{D}'$ is a pseudonatural transformation between the 2–functors $\mathbb{D}$ and $\mathbb{D}'$ (see [6, Definition 7.5.2]). Spelling out this definition such a morphism is a pair $(F_\bullet, \gamma_\bullet^F)$ consisting of a collection of functors

$$F_J\colon \mathbb{D}(J) \to \mathbb{D}'(J), \quad J \in \mathsf{Cat},$$

and a family of natural isomorphisms $\gamma_u^F\colon u^* \circ F_K \to F_J \circ u^*$, $u\colon J \to K$, as indicated in

$$\begin{tikzcd} \mathbb{D}(K) \arrow[r, "F_K"] \arrow[d, "u^*"'] \arrow[dr, phantom, "\cong"] & \mathbb{D}'(K) \arrow[d, "u^*"] \\ \mathbb{D}(J) \arrow[r, "F_J"'] & \mathbb{D}'(J) \end{tikzcd}$$

satisfying certain coherence conditions. We will frequently suppress some indices to avoid awkward notation. Moreover, we will be sloppy and not distinguish between γ and γ^{-1} notationally. If all natural transformations γ are identities, we speak of a *strict morphism*. In general, given a morphism $F\colon \mathbb{D} \to \mathbb{D}'$ the functor $F_e\colon \mathbb{D}(e) \to \mathbb{D}'(e)$ is called the *underlying functor*.

Finally, *natural transformations* are given by modifications (see [6, Definition 7.5.3]). As an upshot we obtain the 2–category PDer of prederivators. The full sub-2–category spanned by derivators is denoted by Der, ie a morphism is just a morphism of prederivators whose domain and codomain is a derivator and similarly for natural transformations. Given two (pre)derivators $\mathbb{D}$ and $\mathbb{D}'$ let us denote the category of morphisms by $\mathsf{Hom}(\mathbb{D},\mathbb{D}')$.

Example 2.1 (1) The formation of represented prederivators defines a fully faithful 2–functor $y\colon \mathsf{CAT} \to \mathsf{PDer}^{\mathsf{strict}}$. In fact, this is a special case of the 2–categorical Yoneda lemma.

(2) The assignment which sends a prederivator $\mathbb{D}$ and a category M to $\mathbb{D}^M$ extends to a 2–functor which is part of an action on PDer:

$$(-)^{(-)}\colon \mathsf{Cat}^{\mathrm{op}} \times \mathsf{PDer} \to \mathsf{PDer}, \quad (M,\mathbb{D}) \mapsto \mathbb{D}^M$$

(3) Given a prederivator $\mathbb{D}$ and a small category M let us denote by $\mathbb{D}(-)^M$ the prederivator which sends K to $\mathbb{D}(K)^M$. The partial underlying diagram functors then assemble into a strict partial underlying diagram morphism of prederivators $\mathsf{dia}_{M,-}\colon \mathbb{D}^M \to \mathbb{D}(-)^M$.

2.2 Homotopy (co)limit preserving morphisms

Let $F\colon \mathbb{D} \to \mathbb{D}'$ be a morphism of derivators and let $u\colon J \to K$ be a functor. We thus have a natural transformation $Fu^* \to u^*F$ together with its inverse $u^*F \to Fu^*$. The calculus of mates hence gives rise to canonical natural transformations $\gamma^F_{u\,!}\colon u_!F \to Fu_!$ and $\gamma^F_{u\,*}\colon Fu_* \to u_*F$.

Definition 2.2 A morphism of derivators $F\colon \mathbb{D} \to \mathbb{D}'$ *preserves homotopy left Kan extensions along* $u\colon J \to K$ if the canonical natural transformation $u_!F \to Fu_!$ is an isomorphism. Similarly, F *preserves homotopy Kan extensions* if this is the case for all u.

Similarly, we speak of a morphism of derivators which preserves certain homotopy colimits and of course there are dual notions. In the case of represented derivators, the above comparison morphism is exactly the usual comparison morphism for colimits, so we recover the usual notion of a colimit-preserving functor. Axiom (Der4) implies that the following is true.

Proposition 2.3 *A morphism $F\colon \mathbb{D} \to \mathbb{D}'$ of derivators preserves homotopy left Kan extensions if and only if it preserves homotopy colimits.*

Proof Let us assume that F preserves homotopy colimits and let us consider a functor $u\colon J \to K$. We obtain the following pasting diagram in which the natural transformation on the right is the one we want to show to be an isomorphism:

$$\begin{tikzcd}
\mathbb{D}'(J_{/k}) \arrow[d, "p_!"'] \arrow[dr, phantom, "\Searrow"] & \mathbb{D}'(J) \arrow[l, "pr^*"'] \arrow[d, "u_!"'] \arrow[dr, phantom, "\Searrow"] & \mathbb{D}(J) \arrow[l, "F"'] \arrow[d, "u_!"] \\
\mathbb{D}'(e) & \mathbb{D}'(K) \arrow[l, "k^*"] & \mathbb{D}(K) \arrow[l, "F"]
\end{tikzcd}$$

Using axiom (Der4) and the fact that isomorphisms are detected pointwise it suffices to show that the pasting is an isomorphism. Lemma 1.14 implies that we have to show that the mate associated to the following left diagram is an isomorphism:

$$\begin{tikzcd}
\mathbb{D}'(J_{/k}) \arrow[dr, phantom, "\Swarrow"] & \mathbb{D}'(J) \arrow[l, "pr^*"'] \arrow[dr, phantom, "\Swarrow"] & \mathbb{D}(J) \arrow[l, "F"'] \\
\mathbb{D}'(e) \arrow[u, "p^*"] & \mathbb{D}'(K) \arrow[l, "k^*"] \arrow[u, "u^*"] & \mathbb{D}(K) \arrow[l, "F"] \arrow[u, "u^*"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{D}'(J_{/k}) \arrow[dr, phantom, "\Swarrow"] & \mathbb{D}(J_{/k}) \arrow[l, "F"'] \arrow[dr, phantom, "\Swarrow"] & \mathbb{D}(J) \arrow[l, "pr^*"'] \\
\mathbb{D}'(e) \arrow[u, "p^*"] & \mathbb{D}(e) \arrow[l, "F"] \arrow[u, "p^*"] & \mathbb{D}(K) \arrow[l, "k^*"] \arrow[u, "u^*"']
\end{tikzcd}$$

But, using the isomorphisms γ_{pr}^F and γ_k^F, this is equivalent to showing that the mate associated to the diagram on the right is an isomorphism which follows from our assumption on F and (Der4). $\square$

For convenience let us collect some closure properties of homotopy Kan extensions preserving morphisms all of which follow almost immediately from Lemma 1.14.

Proposition 2.4 *Let $\mathbb{D}, \mathbb{D}'$, and $\mathbb{D}''$ be derivators, let $u\colon I \to J$ and $v\colon J \to K$ be functors.*

(1) *The identity morphism $\mathrm{id}_{\mathbb{D}}\colon \mathbb{D} \to \mathbb{D}$ preserves homotopy left Kan extensions.*

(2) *If $F\colon \mathbb{D} \to \mathbb{D}'$ and $G\colon \mathbb{D}' \to \mathbb{D}''$ preserve homotopy left Kan extensions along u then so does the composition $G \circ F\colon \mathbb{D} \to \mathbb{D}''$.*

(3) *If $F\colon \mathbb{D} \to \mathbb{D}'$ preserves homotopy left Kan extensions along u and v then it preserves homotopy left Kan extensions along $v \circ u$.*

(4) *If $\tau\colon F \to G$ is a natural isomorphism of morphisms of derivators $\mathbb{D} \to \mathbb{D}'$ then F preserves homotopy left Kan extensions along u if and only if G does.*

Given two derivators $\mathbb{D}$ and $\mathbb{D}'$, denote by $\mathsf{Hom}_!(\mathbb{D}, \mathbb{D}')$ (respectively $\mathsf{Hom}_*(\mathbb{D}, \mathbb{D}')$) the full subcategory of $\mathsf{Hom}(\mathbb{D}, \mathbb{D}')$ spanned by the morphisms which preserve homotopy colimits (respectively homotopy limits). By the above proposition, these are replete subcategories giving rise to 2–categories $\mathsf{Der}_!$ and Der_*.

Proposition 2.5 *Let $\mathbb{D}$ be a derivator and let $v\colon L \to M$ be a functor. The morphism of derivators $v^*\colon \mathbb{D}^M \to \mathbb{D}^L$ preserves homotopy Kan extensions. In particular, this is the case for the evaluation morphisms $m^*\colon \mathbb{D}^M \to \mathbb{D}$.*

Proof By duality and Proposition 2.3 it is enough to show that v^* preserves homotopy limits. Thus, we have to show that for an arbitrary small category J the following square is $\mathbb{D}$–exact,

$$\begin{tikzcd} L \times J \arrow[r, "v\times\mathrm{id}"] \arrow[d, "\mathrm{pr}"'] \arrow[dr, phantom, "\nearrow"] & M \times J \arrow[d, "\mathrm{pr}"] \\ L \arrow[r, "v"'] & M, \end{tikzcd}$$

which follows from Proposition 1.24. □

Thus, this proposition tells us, in particular, that homotopy Kan extensions in the derivator $\mathbb{D}^M$ are calculated pointwise: For a functor $u\colon J \to K$ and an object $X \in \mathbb{D}^M(J)$ the canonical maps

$$\mathrm{HoLan}_u(X_m) \xrightarrow{\cong} (\mathrm{HoLan}_u\, X)_m \quad \text{and} \quad (\mathrm{HoRan}_u\, X)_m \xrightarrow{\cong} \mathrm{HoRan}_u(X_m)$$

are isomorphisms. There is a similar result in the absolute case, ie for homotopy (co)limits. These isomorphisms are well-behaved in the sense that the following diagram commutes

$$\begin{tikzcd} u_!m^*u^* \arrow[r, "\gamma_{u!}", "\cong"'] \arrow[d, "="'] & m^*u_!u^* \arrow[d, "\epsilon"] \\ u_!u^*m^* \arrow[r, "\epsilon"'] & m^* \end{tikzcd}$$

and dually for right Kan extensions. In fact, we just have to apply Lemma 1.14 to:

$$\begin{tikzcd} J \arrow[r, "m\times\mathrm{id}"] \arrow[d, "u"'] & M \times J \arrow[r, "\mathrm{id}\times u"] \arrow[d, "\mathrm{id}\times u"] & M \times K \arrow[d, "\mathrm{id}"] \\ K \arrow[r, "m\times\mathrm{id}"'] & M \times K \arrow[r, "\mathrm{id}"'] & M \times K \end{tikzcd} \quad = \quad \begin{tikzcd} J \arrow[r, "u"] \arrow[d, "u"'] & K \arrow[r, "m\times\mathrm{id}"] \arrow[d, "\mathrm{id}"] & M \times K \arrow[d, "\mathrm{id}"] \\ K \arrow[r, "\mathrm{id}"'] & K \arrow[r, "m\times\mathrm{id}"'] & M \times K \end{tikzcd}$$

This compatibility implies that, for $X \in \mathbb{D}^M(K)$, the counit $\epsilon\colon u_!u^*(X) \to X$ is an isomorphism in $\mathbb{D}^M(K)$ if and only if the counit $\epsilon\colon u_!u^*(X_m) \to X_m$ is an isomorphism in $\mathbb{D}(K)$ for all objects $m \in M$. For later reference, we collect the following convenient consequence for the case of a fully faithful functor $u\colon J \to K$.

Corollary 2.6 *Let $\mathbb{D}$ be a derivator, M a category and let $u\colon J \to K$ be a fully faithful functor. An object $X \in \mathbb{D}^M(K)$ lies in the essential image of $u_!\colon \mathbb{D}^M(J) \to \mathbb{D}^M(K)$ if and only if X_m lies in the essential image of $u_!\colon \mathbb{D}(J) \to \mathbb{D}(K)$ for all $m \in M$.*

The fact that homotopy Kan extensions in the derivator $\mathbb{D}^M$ are calculated pointwise (Proposition 2.5) can also be used to establish the following convenient result.

Corollary 2.7 *Let $F\colon \mathbb{D} \to \mathbb{D}'$ be a morphism of derivators and let $u\colon J \to K$ be a functor. Then F preserves homotopy left Kan extensions along u if and only if $F^M\colon \mathbb{D}^M \to \mathbb{D}'^M$ preserves homotopy left Kan extensions along u for all small categories M.*

Proof We have to show that like $\gamma_{u\,!}^F$ also $\gamma_{u\,!}^{F_M} = \gamma_{\mathrm{id}_M \times u!}^F$ is an isomorphism. Since isomorphisms can be detected pointwise and since m^* preserves homotopy left Kan extensions (by Proposition 2.5) this is equivalent to the fact that the pasting in the left diagram is a natural isomorphism:

$$
\begin{tikzcd}
\mathbb{D}'(J) \arrow[d, "u_!"'] & \mathbb{D}'^M(J) \arrow[l, "m^*"'] \arrow[d, "u_!"'] \arrow[dl, phantom, "\Searrow"] & \mathbb{D}^M(J) \arrow[l, "F^M"'] \arrow[d, "u_!"] \arrow[dl, phantom, "\Searrow"] \\
\mathbb{D}'(K) & \mathbb{D}'^M(K) \arrow[l, "m^*"] & \mathbb{D}^M(K) \arrow[l, "F^M"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{D}'(J) \arrow[d, "u_!"'] & \mathbb{D}(J) \arrow[l, "F"'] \arrow[d, "u_!"'] \arrow[dl, phantom, "\Searrow"] & \mathbb{D}^M(J) \arrow[l, "m^*"'] \arrow[d, "u_!"] \arrow[dl, phantom, "\Searrow"] \\
\mathbb{D}'(K) & \mathbb{D}(K) \arrow[l, "F"] & \mathbb{D}^M(K) \arrow[l, "m^*"]
\end{tikzcd}
$$

By the natural isomorphism $m^* \circ F^M \cong F \circ m^*$ (and strictly speaking Proposition 2.4) this is equivalent to the fact that the pasting in the right diagram is a natural isomorphism. But this follows from our assumption that F preserves homotopy left Kan extensions along u and the fact that m^* lies in $\mathsf{Hom}_!(\mathbb{D}^M, \mathbb{D})$. □

As a special case a morphism preserves initial objects or coproducts if and only if this is the case for the underlying functor. Let us briefly discuss adjunctions between derivators. As a first step there is the following result [27].

Lemma 2.8 *Let $L\colon \mathbb{D} \to \mathbb{D}'$ be a morphism of prederivators such that $L_K\colon \mathbb{D}(K) \to \mathbb{D}'(K)$ has a right adjoint R_K for each $K \in \mathsf{Cat}$. Then, there is a unique way to extend the $\{R_K\}$ to a* lax *morphism of prederivators $R\colon \mathbb{D}' \to \mathbb{D}$ such that the following diagram commutes for all functors $u\colon J \to K$, $X \in \mathbb{D}(K)$, and $Y \in \mathbb{D}'(K)$:*

$$
\begin{tikzcd}
\mathsf{hom}_{\mathbb{D}'(K)}(LX, Y) \arrow[r] \arrow[d, "u^*"'] & \mathsf{hom}_{\mathbb{D}(K)}(X, RY) \arrow[d, "u^*"] \\
\mathsf{hom}_{\mathbb{D}'(J)}(u^*LX, u^*Y) \arrow[d, "\gamma^L"'] & \mathsf{hom}_{\mathbb{D}(J)}(u^*X, u^*RY) \arrow[d, "\gamma^R"] \\
\mathsf{hom}_{\mathbb{D}'(J)}(Lu^*X, u^*Y) \arrow[r] & \mathsf{hom}_{\mathbb{D}(J)}(u^*X, Ru^*Y)
\end{tikzcd}
$$

Proof If we choose $X = RY$ and if we trace around the adjunction counit $\epsilon\colon LRY \to Y$ we see that we necessarily have $\gamma_u^R = (\gamma_u^L)_*^{-1}$. This actually defines a lax morphism of prederivators $R\colon \mathbb{D} \to \mathbb{D}'$ by Lemma 1.14. □

In general, we cannot deduce that the γ_u^R are isomorphisms, ie that $R\colon \mathbb{D}' \to \mathbb{D}$ is a *pseudo*-natural transformation. However, in the context of derivators we can again use Lemma 1.14 which guarantees that the transformations $u_! \circ L \to L \circ u_!$ and $u^* \circ R \to R \circ u^*$ are conjugate. From this we obtain the following result.

Proposition 2.9 *Let $L\colon \mathbb{D} \to \mathbb{D}'$ be a morphism of derivators which admits levelwise right adjoints and let $R\colon \mathbb{D}' \to \mathbb{D}$ be a lax morphism as in Lemma 2.8. The morphism L is a left adjoint morphism of derivators if and only if L preserves homotopy left Kan extensions if and only if R is a morphism of derivators. In particular, a morphism of derivators is an equivalence if and only if it is levelwise an equivalence of categories.*

Thus, together with Proposition 2.5 we obtain the following two classes of examples of adjunctions.

Example 2.10 (1) Let $\mathbb{D}$ be a derivator and let $v\colon L \to M$ be a functor. Then we have two adjunctions of derivators $(v_!, v^*)\colon \mathbb{D}^L \rightleftarrows \mathbb{D}^M$ and $(v^*, v_*)\colon \mathbb{D}^M \rightleftarrows \mathbb{D}^L$.

(2) Let $(F, U)\colon \mathcal{M} \to \mathcal{N}$ be a Quillen adjunction between combinatorial model categories. Then the formation of derived Quillen functors gives us two (in general nonstrict) morphisms of derivators $\mathbb{L}F\colon \mathbb{D}_{\mathcal{M}} \to \mathbb{D}_{\mathcal{N}}$ and $\mathbb{R}U\colon \mathbb{D}_{\mathcal{N}} \to \mathbb{D}_{\mathcal{M}}$. These are part of an adjunction of derivators $(\mathbb{L}F, \mathbb{R}U)\colon \mathbb{D}_{\mathcal{M}} \rightleftarrows \mathbb{D}_{\mathcal{N}}$. In particular, $\mathbb{L}F$ preserves homotopy left Kan extensions and $\mathbb{R}U$ preserves homotopy right Kan extensions.

In particular, a Quillen equivalence gives rise to a derived equivalence of derivators. This already makes more precise the statement that in addition to inducing an equivalence of homotopy categories a Quillen equivalence respects the entire "homotopy theory". Renaudin [38] has shown that the 2–category of locally presentable derivators and adjunctions is a bicategorical localization of the 2–category of combinatorial model categories and Quillen adjunctions at the class of Quillen equivalences.

3 Pointed derivators

3.1 Definition and basic examples

Since we are mainly interested in stable derivators, we turn immediately to the next richer structure, namely to pointed derivators. There are at least two ways to axiomatize a notion of a pointed derivator. From these two notions, we turn the "weaker one" into a definition. The "stronger one" will be referred to as a strongly pointed derivator, but we will show that these two notions actually coincide.

Definition 3.1 A derivator is *pointed* if the underlying category is pointed, ie admits a zero object.

Note that the pointedness is again only a property and not an additional structure. For a prederivator one would impose a slightly stronger condition: a prederivator is pointed if and only if all of its values and all restriction of diagram functors are pointed. In the case of a derivator these stronger properties follow immediately from the definition.

Proposition 3.2 *Let $\mathbb{D}$ be a pointed derivator and let $u\colon J \to K$ a functor in* Cat. *Then $\mathbb{D}(K)$ is also pointed and the functors $u_!, u^*, u_*$ are pointed. In particular, if a derivator $\mathbb{D}$ is pointed then this is also the case for $\mathbb{D}^M$ for every $M \in \mathsf{Cat}$.*

Thus a (pre)derivator is pointed if and only if it factors over the forgetful functor $\mathsf{CAT}_* \to \mathsf{CAT}$ from pointed categories to categories.

Example 3.3 (1) The represented prederivator $y(\mathcal{C})$ is pointed if and only if $\mathcal{C}$ is pointed.

(2) The derivator $\mathbb{D}_{\mathcal{M}}$ underlying a pointed combinatorial model category $\mathcal{M}$ is pointed.

(3) A derivator is pointed if and only if its dual is pointed. Similarly, a derivator $\mathbb{D}$ is pointed if and only if all of its shifts $\mathbb{D}^M$, $M \in \mathsf{Cat}$, are pointed.

We now mention the stronger axiom as used by Maltsiniotis in [32].

Definition 3.4 A derivator $\mathbb{D}$ is *strongly pointed* if it has the following two properties:

(1) For every sieve $j\colon J \to K$ the functor j_* has a right adjoint $j^!$:
$$(j_*, j^!)\colon \mathbb{D}(J) \rightleftarrows \mathbb{D}(K)$$

(2) For every cosieve $i\colon J \to K$ the functor $i_!$ has a left adjoint $i^?$:
$$(i^?, i_!)\colon \mathbb{D}(K) \rightleftarrows \mathbb{D}(J)$$

It is an immediate corollary of the definition that a strongly pointed derivator is pointed. In fact, one of the two additional properties is enough to ensure this. The converse will be established as Corollary 3.8.

Corollary 3.5 *If $\mathbb{D}$ is a strongly pointed derivator, then $\mathbb{D}$ is pointed.*

Proof It is enough to consider the cosieve $\varnothing_e\colon \varnothing \to e$. For an initial object $\varnothing_{e!}(0)$ in $\mathbb{D}(e)$ and an arbitrary $X \in \mathbb{D}(e)$, we then deduce

$$\mathsf{hom}_{\mathbb{D}(e)}(X, \varnothing_{e!}(0)) \cong \mathsf{hom}_{\mathbb{D}(\varnothing)}(\varnothing_e^{?} X, 0) = *,$$

so that $\varnothing_{e!}(0)$ is also terminal. □

We follow Heller [17] by introducing the following notation. Let $\mathbb{D}$ be a pointed derivator and let $u\colon J \to K$ be the inclusion of a full subcategory. The full, replete subcategory of $\mathbb{D}(K)$ spanned by the objects X vanishing on J, ie such that $u^*(X)=0$, is denoted by $\mathbb{D}(K,J) \subseteq \mathbb{D}(K)$.

Proposition 3.6 *Let $\mathbb{D}$ be a pointed derivator.*

(1) *Let $u\colon J \to K$ be a cosieve. Then $u_!$ induces an equivalence*

$$\mathbb{D}(J) \xrightarrow{\simeq} \mathbb{D}(K, K-J).$$

(2) *Let $u\colon J \to K$ be a sieve. Then u_* induces an equivalence*

$$\mathbb{D}(J) \xrightarrow{\simeq} \mathbb{D}(K, K-J).$$

This follows immediately from Proposition 1.23 and shows that the respective Kan extension functors are "extension by zero functors". We will make constant use of this result in the remainder of this paper.

Lemma 3.7 *Let $\mathbb{D}$ be a pointed derivator.*

(1) *Let $u\colon J \to K$ be a cosieve. Then the subcategory $\mathbb{D}(K,J) \subseteq \mathbb{D}(K)$ is coreflective, ie the inclusion functor ι admits a right adjoint.*

(2) *Let $u\colon J \to K$ be a sieve. Then the subcategory $\mathbb{D}(K,J) \subseteq \mathbb{D}(K)$ is reflective, ie the inclusion functor ι admits a left adjoint.*

Proof We will give the details for the proof of (2) and mention the necessary modifications for (1). So, let $u\colon J \to K$ be a sieve and let us construct the mapping cylinder category $\mathsf{cyl}(u)$. By definition, $\mathsf{cyl}(u)$ is the full subcategory of $K \times [1]$ spanned by the objects $(u(j),1)$ and $(k,0)$. Thus, it is defined by the following pushout diagram, where i_0 is the inclusion at 0:

$$\begin{tikzcd} J \arrow[r, "u"] \arrow[d, "i_0"'] & K \arrow[d] \\ J \times [1] \arrow[r] & \mathsf{cyl}(u) \end{tikzcd}$$

There are the natural functors

$$i\colon J \to \mathsf{cyl}(u), \quad j \mapsto (u(j),1) \quad \text{and} \quad s\colon K \to \mathsf{cyl}(u), \quad k \mapsto (k,0).$$

Moreover, id: $K \to K$ and $J \times [1] \xrightarrow{\mathrm{pr}} J \xrightarrow{u} K$ induce a unique functor $q\colon \mathsf{cyl}(u) \to K$. These functors satisfy the relations $q \circ i = u$, $q \circ s = \mathrm{id}_K$.

Consider now an object $X \in \mathbb{D}(\mathsf{cyl}(u), i(J))$ and let us calculate the value of $q_!(X)$ at some $u(j) \in K$. For this purpose, we show that the following pasting is homotopy exact:

$$\begin{tikzcd}
e \arrow[r, "{(i(j),1)}"] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "\swArrow"] & i(J)_{/i(j)} \times [1] \arrow[r, "\cong"] \arrow[d, "p"] \arrow[dr, phantom, "\swArrow"] & \mathsf{cyl}(u)_{/u(j)} \arrow[r, "\mathrm{pr}"] \arrow[d, "p"] \arrow[dr, phantom, "\swArrow"] & \mathsf{cyl}(u) \arrow[d, "q"] \\
e \arrow[r] & e \arrow[r] & e \arrow[r, "u(j)"'] & K
\end{tikzcd}$$

Since u is a sieve we have an isomorphism as depicted in the diagram. Moreover, $i(J)_{/i(j)} \times [1]$ has a terminal element so that the left two squares are homotopy exact by Proposition 1.18. Thus, we can conclude by (Der4) that the above pasting is homotopy exact and obtain $q_!(X)_{u(j)} \cong X_{i(j)} = 0$. The adjunction $(q_!, q^*)$ restricts to an adjunction $(q_!, q^*)\colon \mathbb{D}(\mathsf{cyl}(u), i(J)) \rightleftarrows \mathbb{D}(K, u(J))$.

Moreover, $s\colon K \to \mathsf{cyl}(u)$ is a sieve. Hence, by Proposition 3.6, we have an induced equivalence $(s_*, s^*)\colon \mathbb{D}(K) \xrightarrow{\simeq} \mathbb{D}(\mathsf{cyl}(u), \mathsf{cyl}(u) - s(K)) = \mathbb{D}(\mathsf{cyl}(u), i(J))$.

Putting these two adjunctions together we obtain the adjunction

$$(q_! \circ s_*, s^* \circ q^*)\colon \mathbb{D}(K) \rightleftarrows \mathbb{D}(\mathsf{cyl}(u), i(J)) \rightleftarrows \mathbb{D}(K, u(J)).$$

The relation $q \circ s = \mathrm{id}$ implies that the right adjoint of this adjunction is the inclusion ι as intended and the reflection is given by $r = q_! \circ s_*$.

The proof of (1) is similar. Instead of using $\mathsf{cyl}(u)$ one uses this time the mapping cylinder category $\mathsf{cyl}'(u)$, which is obtained by a similar pushout but using the inclusion i_1 instead of i_0. Let us denote the corresponding functors again by i, q, and s. Using a similar calculation of q_* and the fact that s is now a cosieve, we can construct a coreflection c. □

Corollary 3.8 *Let $\mathbb{D}$ be a pointed derivator. Then $\mathbb{D}$ is also strongly pointed.*

Proof Given a sieve $u\colon J \to K$ we have to show that u_* has a right adjoint. The inclusion $v\colon K - u(J) \to K$ of the complement is a cosieve. The above lemma applied to v thus gives us a coreflection $(\iota, c)\colon \mathbb{D}(K, K - u(J)) \rightleftarrows \mathbb{D}(K)$. Putting this together with the equivalence induced by u_* (guaranteed by Proposition 3.6) we obtain the desired adjunction:

$$(u_*, u^!)\colon\ \mathbb{D}(J) \underset{u^*}{\overset{u_*}{\rightleftarrows}} \mathbb{D}(K, K-u(J)) \underset{c}{\overset{\iota}{\rightleftarrows}} \mathbb{D}(K)$$

The proof in the case of a cosieve is, of course, the dual one. □

The proofs of the last two results were constructive. So, for later reference, let us give precise formulas for these additional adjoint functors. Let $\mathbb{D}$ be a pointed derivator and let $u\colon J \to K$ be a cosieve. Let us denote by $v\colon J' = K - u(J) \to K$ the sieve given by the complement. The adjunctions $(u^?, u_!)\colon \mathbb{D}(K) \rightleftarrows \mathbb{D}(J)$ and $(v_*, v^!)\colon \mathbb{D}(J) \rightleftarrows \mathbb{D}(K)$ are given by the following composite adjunctions respectively:

$$u^?\colon\ \mathbb{D}(K) \underset{s^*}{\overset{s_*}{\rightleftarrows}} \mathbb{D}(\mathsf{cyl}(v), i(J')) \underset{q^*}{\overset{q_!}{\rightleftarrows}} \mathbb{D}(K, v(J')) \underset{u_!}{\overset{u^*}{\rightleftarrows}} \mathbb{D}(J) :u_!$$

$$v_*\colon\ \mathbb{D}(J') \underset{v^*}{\overset{v_*}{\rightleftarrows}} \mathbb{D}(K, u(J)) \underset{q'_*}{\overset{q'^*}{\rightleftarrows}} \mathbb{D}(\mathsf{cyl}'(u), i'(J)) \underset{s'_!}{\overset{s'^*}{\rightleftarrows}} \mathbb{D}(K) :v^!$$

Here, $\mathsf{cyl}(v)$ is the mapping cylinder obtained from identifying the bottom $J' \times \{0\}$ of $J' \times [1]$ with the image of v, i is the inclusion in the cylinder, q is the projection and s is the canonical section of q. The notation in the second decomposition is similar, where the roles of 0 and 1 are interchanged.

3.2 Cocartesian and cartesian squares

We denote the category $[1] \times [1]$ by $\square$, ie $\square$ is the following poset considered as a category where we draw the first coordinate horizontally:

$$\begin{array}{ccc} (0,0) & \longrightarrow & (1,0) \\ \downarrow & & \downarrow \\ (0,1) & \longrightarrow & (1,1) \end{array}$$

For the treatment of cartesian and cocartesian squares, it is important to consider the following two inclusions of subcategories $i_\ulcorner\colon \ulcorner \to \square$ and $i_\lrcorner\colon \lrcorner \to \square$ which are given by the respective subposets:

$$\begin{array}{ccc} (0,0) & \longrightarrow & (1,0) \\ \downarrow & & \\ (0,1) & & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} & & (1,0) \\ & & \downarrow \\ (0,1) & \longrightarrow & (1,1) \end{array}$$

Definition 3.9 Let $\mathbb{D}$ be a derivator and let $X \in \mathbb{D}(\square)$.

(1) The square X is *cocartesian* if it lies in the essential image of $\mathrm{i}_{\ulcorner !}\colon \mathbb{D}(\ulcorner) \to \mathbb{D}(\square)$.

(2) The square X is *cartesian* if it lies in the essential image of $\mathrm{i}_{\lrcorner *}\colon \mathbb{D}(\lrcorner) \to \mathbb{D}(\square)$.

Lemma 1.21 implies that a square $X \in \mathbb{D}(\square)$ is cocartesian if and only if the canonical morphism $\epsilon_{(1,1)}\colon \mathrm{i}_{\ulcorner !}\,\mathrm{i}_{\ulcorner}{}^*(X)_{(1,1)} \to X_{(1,1)}$ is an isomorphism. There is a dual statement for cartesian squares. Our first aim in this section is to establish a "detection result" for (co)cartesian squares in larger diagrams [12] which will be used frequently later on. A *square in a category* J is a functor $\square \to J$ which is injective on objects.

Proposition 3.10 *Let $i\colon \square \to J$ be a square in J and let $f\colon K \to J$ be a functor.*

(1) *Assume that the induced functor $\ulcorner \xrightarrow{\tilde{i}} \big(J - i(1,1)\big)_{/i(1,1)}$ has a left adjoint and that $i(1,1)$ does not lie in the image of f. Then for all $X = f_!(Y) \in \mathbb{D}(J)$, $Y \in \mathbb{D}(K)$, the induced square $i^*(X)$ is cocartesian.*

(2) *Assume that the induced functor $\lrcorner \xrightarrow{\tilde{i}} \big(J - i(0,0)\big)_{i(0,0)/}$ has a right adjoint and that $i(0,0)$ does not lie in the image of f. Then for all $X = f_*(Y) \in \mathbb{D}(J)$, $Y \in \mathbb{D}(K)$, the induced square $i^*(X)$ is cartesian.*

Proof We give a proof of (1). By assumption, f factors as $K \xrightarrow{\bar{f}} (J - i(1,1)) \xrightarrow{j} J$ so that our setup can be summarized by:

$$\begin{tikzcd}
 & \big(J - i(1,1)\big)_{/i(1,1)} \arrow[d, "\mathrm{pr}"] & \\
\ulcorner \arrow[ur, "\widetilde{\text{ß}}"] \arrow[r] \arrow[d, "\mathrm{i}_{\ulcorner}"'] & J - i(1,1) \arrow[d, "j"] & K \arrow[l, "\bar{f}"'] \arrow[dl, "f"] \\
\square \arrow[r, "i"'] & J &
\end{tikzcd}$$

We want to show that the adjunction counit $\epsilon\colon \mathrm{i}_{\ulcorner !}\,\mathrm{i}_{\ulcorner}{}^* \to \mathrm{id}$ is an isomorphism when applied to $i^* f_!(Y)$, $Y \in \mathbb{D}(K)$. But by Lemma 1.21 and Lemma 1.14 this is equivalent to showing that the base change morphism associated to the top pasting is an isomorphism when evaluated at $f_!(Y)$:

$$\begin{tikzcd}
\ulcorner \cong \ulcorner_{/(1,1)} \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\swarrow"] & \ulcorner \arrow[r, "\mathrm{i}_{\ulcorner}"] \arrow[d, "\mathrm{i}_{\ulcorner}"'] \arrow[dr, phantom, "\swarrow"] & \square \arrow[r, "i"] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "\swarrow"] & J \arrow[d, "\mathrm{id}"] \\
e \arrow[r, "(1,1)"'] & \square \arrow[r, "\mathrm{id}"'] & \square \arrow[r, "i"'] & J
\end{tikzcd}$$

$$\begin{tikzcd}
\ulcorner \arrow[r, "\widetilde{\ss}"] \arrow[d, "p"'] \arrow[dr, phantom, "\swarrow"] & J-i(1,1)_{/i(1,1)} \arrow[r, "\mathrm{pr}"] \arrow[d, "p"'] \arrow[dr, phantom, "\swarrow"] & J-i(1,1) \arrow[r, "j"] \arrow[d, "j"'] \arrow[dr, phantom, "\swarrow"] & J \arrow[d, "\mathrm{id}"] \\
e \arrow[r, "\mathrm{id}"'] & e \arrow[r, "i(1,1)"'] & J \arrow[r, "\mathrm{id}"'] & J
\end{tikzcd}$$

Using Lemma 1.14 again, this is equivalent to showing that the base change morphism associated to the bottom pasting gives an isomorphism when evaluated at $f_!(Y)$. But this is the case by (Der4) and Proposition 1.18, since $f_!(Y) \cong j_! \overline{f}_!(Y)$ lies in the essential image of $j_!$. $\square$

Typical applications of this proposition will be given when the categories under consideration are posets. For $n \geq 0$, we denote by $[n]$ the ordinal number $0 < \cdots < n$ considered as a category. Let $d^i\colon [n-1] \to [n]$, $0 \leq i \leq n$, be the unique monotone injection omitting i while $s^j\colon [n+1] \to [n]$, $0 \leq j \leq n$, is the unique monotone surjection hitting j twice. As usual, the images of these cosimplicial structure maps under a contravariant functor will be written as d_i and s_j respectively.

Lemma 3.11 *For every $0 \leq i \leq n-1$ we have an adjunction $(s^i, d^i)\colon [n] \rightleftarrows [n-1]$. In particular, we thus obtain the adjunctions*

$$(s^0, d^0)\colon [2]\times[1] \rightleftarrows [1]\times[1] \quad \textit{and} \quad (s^1, d^1)\colon [2]\times[1] \rightleftarrows [1]\times[1].$$

In the next proposition, we will consider squares in a derivator and some of its associated subdiagrams. To establish some short hand notation, let us denote by $d^i_v\colon [1] \to \square$ the face maps $\mathrm{id} \times d^i\colon [1] \to [1]\times[1] = \square$ in the "vertical direction" giving rise to "horizontal faces" and similarly in the other case. Images of these morphisms under contravariant functors will be written as d^h_i and d^v_i respectively.

Proposition 3.12 *Let $\mathbb{D}$ be a derivator.*

(1) *An object of $\mathbb{D}([1])$ is an isomorphism if and only if it lies in the essential image of the homotopy left Kan extension functor $0_!\colon \mathbb{D}(e) \to \mathbb{D}([1])$.*

(2) *Let $X \in \mathbb{D}(\square)$ be a square such that $d_1^v(X)$ is an isomorphism, ie we have $X_{0,0} \xrightarrow{\cong} X_{1,0}$. The square X is cocartesian if and only if also $d_0^v(X)$ is an isomorphism.*

Proof Statement (1) is a special case of Lemma 1.19 so let us establish (2). By (1) our assumption on X is equivalent to the adjunction counit $0_!0^*d_1^v(X) \to d_1^v(X)$ being an isomorphism. Using Lemma 1.21 and (Der4), we can reformulate this by saying that the base change morphism associated to the following pasting is an isomorphism when evaluated on X:

$$\begin{tikzcd}
e \cong e_{/1} \arrow[r, "\mathrm{pr}"] \arrow[d, "p"'] \arrow[dr, phantom, "\Swarrow"] & e \arrow[r, "0"] \arrow[d, "0"] \arrow[dr, phantom, "\Swarrow"] & {[1]} \arrow[r, "d_v^1"] \arrow[d, "\mathrm{id}"] \arrow[dr, phantom, "\Swarrow"] & \square \arrow[d, "\mathrm{id}"] \\
e \arrow[r, "1"'] & {[1]} \arrow[r, "\mathrm{id}"'] & {[1]} \arrow[r, "d_v^1"'] & \square
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
e \arrow[r, "{(0,0)}"] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "\Swarrow"] & \square \arrow[d, "\mathrm{id}"] \\
e \arrow[r, "{(1,0)}"'] & \square
\end{tikzcd}$$

We want to reformulate this in a way which is more convenient for this proof. For this purpose let us consider the following factorization of the *horizontal* face map:

$$d_h^1 = i_\ulcorner \circ j \colon [1] \xrightarrow{j} \ulcorner \xrightarrow{i_\ulcorner} \square$$

Now, our assumption that $d_1^v(X)$ is an isomorphism is equivalent to the counit $j_!j^*i_\ulcorner^*X \to i_\ulcorner^*X$ being an isomorphism. In fact, using Lemma 1.21 and (Der4), the claim about the counit can be equivalently restated by saying that the base change of the following pasting is an isomorphism when evaluated at X:

$$\begin{tikzcd}
e \cong [1]_{/(0,1)} \arrow[r, "\mathrm{pr}"] \arrow[d, "p"'] \arrow[dr, phantom, "\Swarrow"] & {[1]} \arrow[r, "j"] \arrow[d, "j"] \arrow[dr, phantom, "\Swarrow"] & \ulcorner \arrow[r, "i_\ulcorner"] \arrow[d, "\mathrm{id}"] \arrow[dr, phantom, "\Swarrow"] & \square \arrow[d, "\mathrm{id}"] \\
e \arrow[r, "{(0,1)}"'] & \ulcorner \arrow[r, "\mathrm{id}"'] & \ulcorner \arrow[r, "i_\ulcorner"'] & \square
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
e \arrow[r, "{(0,0)}"] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "\Swarrow"] & \square \arrow[d, "\mathrm{id}"] \\
e \arrow[r, "{(1,0)}"'] & \square
\end{tikzcd}$$

Thus, the claim follows from our previous reasoning. This in turn can be used to show that under our assumption the square X is cocartesian if and only if the base change associated to

$$\begin{tikzcd}
{[1]} \arrow[r, "j"] \arrow[d, "j"'] \arrow[dr, phantom, "\Swarrow"] & \ulcorner \arrow[r, "i_\ulcorner"] \arrow[d, "\mathrm{id}"] \arrow[dr, phantom, "\Swarrow"] & \square \arrow[d, "\mathrm{id}"] \\
\ulcorner \arrow[r, "\mathrm{id}"'] \arrow[d, "i_\ulcorner"'] \arrow[dr, phantom, "\Swarrow"] & \ulcorner \arrow[r, "i_\ulcorner"'] \arrow[d, "i_\ulcorner"] \arrow[dr, phantom, "\Swarrow"] & \square \arrow[d, "\mathrm{id}"] \\
\square \arrow[r, "\mathrm{id}"'] & \square \arrow[r, "\mathrm{id}"'] & \square
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
{[1]} \arrow[r, "d_h^1"] \arrow[d, "d_h^1"'] \arrow[dr, phantom, "\Swarrow"] & \square \arrow[d, "\mathrm{id}"] \\
\square \arrow[r, "\mathrm{id}"'] & \square
\end{tikzcd}$$

is an isomorphism at X. By Lemma 1.21 this is the case if and only if it is the case at $(1,1)$ which in turn is equivalent (by similar arguments as in the beginning of this proof) to the fact that $d_0^v(X)$ is an isomorphism. $\square$

We now discuss the composition and cancellation property of (co)cartesian squares. Recall from classical category theory that for a diagram in a category of the shape

$$\begin{tikzcd} X_{0,0} \arrow[r] \arrow[d] & X_{1,0} \arrow[r] \arrow[d] & X_{2,0} \arrow[d] \\ X_{0,1} \arrow[r] & X_{1,1} \arrow[r] & X_{2,1} \end{tikzcd}$$

the following holds: if the square on the left is a pushout, then the square on the right is a pushout if and only if the composite square is. The corresponding result in the theory of derivators is the content of the next proposition. The methods are similar to the ones used in the proof of Proposition 3.12 and the proof will hence be left to the reader. Moreover, since we only use horizontal face maps this time we again drop the additional index.

Proposition 3.13 *Let $\mathbb{D}$ be a derivator and let $X \in \mathbb{D}([2]\times[1])$.*

(1) *If $d_2(X) \in \mathbb{D}(\square)$ is cocartesian, then $d_0(X)$ is cocartesian if and only if $d_1(X)$ is cocartesian.*

(2) *If $d_0(X) \in \mathbb{D}(\square)$ is cartesian, then $d_2(X)$ is cartesian if and only if $d_1(X)$ is cartesian.*

Let us say that a morphism of derivators *preserves cocartesian squares* if it preserves homotopy left Kan extensions along $i_\ulcorner\colon \ulcorner \to \square$. There is the dual notion of a morphism which *preserves cartesian squares*. As an immediate consequence of Corollary 2.7 and Corollary 2.6 we have the following result.

Corollary 3.14 *Let $F\colon \mathbb{D} \to \mathbb{D}'$ be a morphism of derivators. Then F preserves cocartesian squares if and only if $F\colon \mathbb{D}^M \to \mathbb{D}'^M$ preserves cocartesian squares for all categories M. Moreover, an object $X \in \mathbb{D}^M(\square)$ is cocartesian if and only if the squares $X_m \in \mathbb{D}(\square)$ are cocartesian for all objects $m \in M$.*

Definition 3.15 Let $F\colon \mathbb{D} \to \mathbb{D}'$ be a morphism of derivators.

(1) The morphism F is *left exact* if it preserves cartesian squares and final objects.

(2) The morphism F is *right exact* if it preserves cocartesian squares and initial objects.

(3) The morphism F is *exact* if it is left exact and right exact.

It follows that a left exact morphism preserves, in particular, finite products and dually for a right exact morphism. In fact this follows from Proposition 1.23 together with an alternative description of products as pullbacks of diagrams which have the final object as value in the lower right corner. Of course left adjoint morphisms are right exact and dually for right adjoint morphisms.

3.3 Suspensions, loops, cones and fibers

Let $\mathbb{D}$ be a pointed derivator and let J be a category. In this subsection we want to construct the suspension and loop functors on $\mathbb{D}(J)$ and the cone and fiber functors on $\mathbb{D}(J\times[1])$. By Proposition 3.2, we can assume $J=e$.

Let us begin with the suspension functor Σ and the loop functor Ω. The "extension by zero functors" as given by Proposition 3.6 will again be crucial. Let us consider the following sequences of functors:

$$e \xrightarrow{(0,0)} \ulcorner \xrightarrow{i_{\ulcorner}} \square \xleftarrow{(1,1)} e, \qquad e \xrightarrow{(1,1)} \lrcorner \xrightarrow{i_{\lrcorner}} \square \xleftarrow{(0,0)} e.$$

Since $(0,0)\colon e\to\ulcorner$ is a sieve the homotopy right Kan extension functor $(0,0)_*$ gives us an "extension by zero functor" by Proposition 3.6 and similarly for the homotopy left Kan extension $(1,1)_!$ along the cosieve $(1,1)\colon e\to\lrcorner$.

Definition 3.16 Let $\mathbb{D}$ be a pointed derivator.

(1) The *suspension functor* Σ is given by

$$\Sigma\colon \mathbb{D}(e)\xrightarrow{(0,0)_*}\mathbb{D}(\ulcorner)\xrightarrow{i_{\ulcorner}{}_!}\mathbb{D}(\square)\xrightarrow{(1,1)^*}\mathbb{D}(e).$$

(2) The *loop functor* Ω is given by

$$\Omega\colon \mathbb{D}(e)\xrightarrow{(1,1)_!}\mathbb{D}(\lrcorner)\xrightarrow{i_{\lrcorner}{}_*}\mathbb{D}(\square)\xrightarrow{(0,0)^*}\mathbb{D}(e).$$

The motivation for these definitions should be clear from topology. Recall that given a pointed topological space X, the suspension ΣX is constructed by first taking two instances of the canonical inclusion into the (contractible!) cone $\mathsf{C}X$ and then forming the pushout:

$$\begin{array}{ccc} X & \longrightarrow & \mathsf{C}X \\ \downarrow & & \\ \mathsf{C}X & & \end{array} \qquad \begin{array}{ccc} X & \longrightarrow & \mathsf{C}X \\ \downarrow & & \downarrow \\ \mathsf{C}X & \longrightarrow & \Sigma X \end{array}$$

We can consider this diagram as a homotopy pushout. The above definition abstracts precisely this construction.

Proposition 3.17 *If $\mathbb{D}$ is a pointed derivator, then we have an adjunction*

$$(\Sigma, \Omega)\colon \mathbb{D}(e) \rightleftarrows \mathbb{D}(e).$$

Proof Let us denote by $\mathcal{M} \subset \mathbb{D}(\square), \mathcal{M}^{\ulcorner} \subset \mathbb{D}(\ulcorner)$ and $\mathcal{M}^{\lrcorner} \subset \mathbb{D}(\lrcorner)$ the respective full subcategories spanned by the objects X with $X_{1,0} \cong 0 \cong X_{0,1}$. The suspension and the loop functor can then be factored as:

$$\begin{tikzcd}
\Sigma\colon & \mathbb{D}(e) \arrow[r, "(0,0)_*", "\simeq"'] \arrow[d, equal] & \mathcal{M}^{\ulcorner} \arrow[r, "i_{\ulcorner !}"] \arrow[d, equal] & \mathcal{M} \arrow[r, "i_{\lrcorner}^*"] \arrow[d, equal] & \mathcal{M}^{\lrcorner} \arrow[r, "(1,1)^*", "\simeq"'] \arrow[d, equal] & \mathbb{D}(e) \arrow[d, equal] & \\
& \mathbb{D}(e) & \mathcal{M}^{\ulcorner} \arrow[l, "(0,0)^*"', "\simeq"] & \mathcal{M} \arrow[l, "i_{\ulcorner}^*"'] & \mathcal{M}^{\lrcorner} \arrow[l, "i_{\lrcorner *}"'] & \mathbb{D}(e) \arrow[l, "(1,1)_!"', "\simeq"] & :\Omega
\end{tikzcd}$$

The existence of the factorization is clear and the fact that the functors $(0,0)_*$ and $(1,1)_!$ restricted this way are equivalences follows from their fully faithfulness and Proposition 3.6. From this description, one sees immediately that we have an adjunction (Σ, Ω) which is, in fact, given as a composite adjunction of four adjunctions among which two are equivalences. $\square$

Using similar constructions, one can introduce *cone* and *fiber functors* for pointed derivators. Again, the definition is easily motivated from topology. If we consider a map of pointed spaces $f\colon X \to Y$ then the mapping cone $\mathsf{C}f$ of f is constructed by forming a pushout as indicated in the next diagram:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] & Y \\
\mathsf{C}X &
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\
\mathsf{C}X \arrow[r] & \mathsf{C}f
\end{tikzcd}$$

To axiomatize this in the context of a pointed derivator, let us consider the following morphisms of posets:

$$[1] \xrightarrow{\ i\ } \ulcorner \xrightarrow{\ i_{\ulcorner}\ } \square \xleftarrow{\ i_{\lrcorner}\ } \lrcorner \xleftarrow{\ j\ } [1]$$

Here, i is the sieve classifying the horizontal arrow while j is the cosieve classifying the vertical arrow. In particular, by Proposition 3.6, we have again extension by zero functors i_* and $j_!$.

Definition 3.18 Let $\mathbb{D}$ be a pointed derivator.

(1) The *cone functor* $\mathsf{Cone}\colon \mathbb{D}([1]) \to \mathbb{D}([1])$ is defined as the composition:

$$\mathsf{Cone}\colon \mathbb{D}([1]) \xrightarrow{i_*} \mathbb{D}(\ulcorner) \xrightarrow{i_{\ulcorner !}} \mathbb{D}(\square) \xrightarrow{j^*} \mathbb{D}([1])$$

(2) The *fiber functor* $\mathsf{Fiber}\colon \mathbb{D}([1]) \to \mathbb{D}([1])$ is defined as the composition:

$$\mathsf{Fiber}\colon \mathbb{D}([1]) \xrightarrow{j_!} \mathbb{D}(\lrcorner) \xrightarrow{i_{\lrcorner *}} \mathbb{D}(\square) \xrightarrow{i^*} \mathbb{D}([1])$$

(3) Let $\mathsf{C}\colon \mathbb{D}([1]) \to \mathbb{D}(e)$ be the functor obtained from the cone functor by evaluation at 1 and, similarly, let $\mathsf{F}\colon \mathbb{D}([1]) \to \mathbb{D}(e)$ be the functor obtained from the fiber functor by evaluation at 0.

Proposition 3.12 shows that the cone $\mathsf{C}f$ of an isomorphism f is the zero object 0. In general, the converse is only true in the stable situation (see Proposition 4.5). There is the following counterexample to the converse in the unstable situation.

Counterexample 3.19 Let $\mathcal{E}$ be an exact category in the sense of Quillen (see [37]). Moreover, let us assume $\mathcal{E}$ to have enough injectives but also that $\mathcal{E}$ is not Frobenius, ie the classes of injectives and projectives do not coincide. The stable category $\underline{\mathcal{E}}$ which is obtained from $\mathcal{E}$ by dividing out the maps factoring over injectives is a "suspended category" in the sense of [26]. Let now X be an object of $\mathcal{E}$ of injective dimension 1 and let $0 \to X \to I^0 = I \to I^1 = \Sigma X \to 0$ be an injective resolution of X. By definition of the suspended structure on $\underline{\mathcal{E}}$ (see [26] or [15, Chapter I]) the diagram

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "\mathrm{id}"'] & I \arrow[r, "v"] \arrow[d, "\mathrm{id}"] & \Sigma X \arrow[d, "\mathrm{id}"] \\
X \arrow[r] & I \arrow[r] & \Sigma X
\end{tikzcd}$$

gives rise to the distinguished triangle $X \xrightarrow{u} I \xrightarrow{v} \Sigma X \xrightarrow{\mathrm{id}} \Sigma X$. Since ΣX is trivial in the stable category $\underline{\mathcal{E}}$ the morphism u is an example of a morphism which is not an isomorphism but still has a vanishing cone. In the stable situation, ie in the Frobenius case, this counterexample cannot exist. In fact, the above resolution of X would split because ΣX is by assumption injective, hence projective, showing that the injective dimension of X is zero.

By methods similar to the ones in the proof of Proposition 3.17 one establishes the following result.

Proposition 3.20 *Let $\mathbb{D}$ be a pointed derivator. Then we have an adjunction:*

$$(\mathsf{Cone}, \mathsf{Fiber})\colon \mathbb{D}([1]) \rightleftarrows \mathbb{D}([1])$$

The above definitions can easily be extended (using Example 2.1) to morphisms of derivators. Thus, given a pointed derivator $\mathbb{D}$ we obtain, in particular, adjunctions of derivators

$$(\Sigma, \Omega)\colon \mathbb{D} \rightleftarrows \mathbb{D} \quad \text{and} \quad (\mathsf{Cone}, \mathsf{Fiber})\colon \mathbb{D}^{[1]} \rightleftarrows \mathbb{D}^{[1]}.$$

Since the construction of the above functors is based only on certain extension by zero functors and the formation of some (co)cartesian squares the following proposition is immediate.

Proposition 3.21 *Let $G\colon \mathbb{D} \to \mathbb{D}'$ be a morphism of pointed derivators.*

(1) *If G is left exact then there are canonical isomorphisms*

$$G \circ \Omega \to \Omega \circ G \quad \textit{and} \quad G \circ \mathsf{Fiber} \to \mathsf{Fiber} \circ G.$$

(2) *If G is right exact then there are canonical isomorphisms*

$$\Sigma \circ G \to G \circ \Sigma \quad \textit{and} \quad \mathsf{Cone} \circ G \to G \circ \mathsf{Cone}.$$

In [9] there is also an alternative description of some of the functors we just introduced. Using our explicit construction of the (co)exceptional inverse image functors at the end of Section 3.1 we can show the two approaches to be equivalent. So, let $\mathbb{D}$ be a pointed derivator and let us consider the cosieve $1\colon e \to [1]$ and the sieve $0\colon e \to [1]$. Corollary 3.8 implies that we have adjunctions:

$$(1^?, 1_!)\colon \mathbb{D}([1]) \rightleftarrows \mathbb{D}(e) \quad \text{and} \quad (0_*, 0^!)\colon \mathbb{D}(e) \rightleftarrows \mathbb{D}([1])$$

The formulas via the mapping cylinder constructions can be made very explicit in this case so that we have the following descriptions of the additional adjoints $1^?$ and $0^!$:

$$1^?\colon \mathbb{D}([1]) \underset{\simeq}{\xrightarrow{j_*}} \mathbb{D}(\ulcorner, (0,1)) \xrightarrow{\mathrm{pr}_{1!}} \mathbb{D}([1], 0) \underset{\simeq}{\xrightarrow{1^*}} \mathbb{D}(e)$$

$$0^!\colon \mathbb{D}([1]) \underset{\simeq}{\xrightarrow{j_!}} \mathbb{D}(\lrcorner, (1,0)) \xrightarrow{\mathrm{pr}_{1*}} \mathbb{D}([1], 1) \underset{\simeq}{\xrightarrow{0^*}} \mathbb{D}(e)$$

In both formulas, j denotes the functor classifying the horizontal arrow and the functors pr_1 are suitable restrictions of the projection on the first component $\square \to [1]$. It follows from Lemma 1.19 that in both cases the composition of the last two functors is naturally isomorphic to the homotopy colimit and homotopy limit functor respectively. A final application of (Der4) then implies the following result.

Proposition 3.22 *Let $\mathbb{D}$ be a pointed derivator then we have the following natural isomorphisms:*

$$\mathsf{C} \cong 1^?, \quad \Sigma \cong 1^? \circ 0_*, \quad \mathsf{F} \cong 0^! \quad \textit{and} \quad \Omega \cong 0^! \circ 1_!$$

In particular, we have adjunctions $(\mathsf{C}, 1_!)\colon \mathbb{D}([1]) \rightleftarrows \mathbb{D}(e)$ *and* $(0_*, \mathsf{F})\colon \mathbb{D}(e) \rightleftarrows \mathbb{D}([1])$.

4 Stable derivators

4.1 The additivity of stable derivators

In this subsection, we come to the central notion of a stable derivator. Similarly to the situation of a stable model category or a stable $(\infty, 1)$–category, one adds a "linearity condition" to the pointed situation. This notion was introduced by Maltsiniotis in [32] by forming a combination of the axioms of Grothendieck's derivators [14] and Franke's systems of triangulated diagram categories [12]. More details on the history can be found in Cisinski and Neeman [9].

Definition 4.1 A strong derivator is *stable* if it is pointed and if a square is cocartesian if and only if it is cartesian. These squares are called *bicartesian*.

The assumption on the derivator to be strong will be crucial in two situations in the construction of the canonical triangulated structures.

Example 4.2 (1) The derivator underlying a stable (combinatorial) model category is stable.

(2) Given an exact category $\mathcal{E}$ in the sense of Quillen [37] then the assignment

$$\mathbb{D}^b_{\mathcal{E}}\colon \quad \mathsf{Dir}_f^{\mathrm{op}} \to \mathsf{CAT}, \quad J \mapsto D^b(\mathcal{E}^J)$$

defines a stable derivator. Here, Dir_f is the 2–category of finite direct categories and $D^b(-)$ denotes the formation of the bounded derived category (see [25]).

(3) A derivator $\mathbb{D}$ is stable if and only if the dual derivator $\mathbb{D}^{\mathrm{op}}$ is stable.

Let us begin with the following convenient result.

Proposition 4.3 *Let $\mathbb{D}$ be a stable derivator and let M be a category. Then $\mathbb{D}^M$ is again stable.*

Proof It is immediate that a derivator $\mathbb{D}$ is strong if and only if $\mathbb{D}^M$ is strong for all categories M. Moreover, we know that $\mathbb{D}^M$ is pointed by Proposition 3.2. Thus, let us consider the (co)cartesian squares. For an object $X \in \mathbb{D}^M(\square)$, using Corollary 3.14, we have that X is cocartesian if and only if $X_m \in \mathbb{D}(\square)$ is cocartesian for all $m \in M$. Using the stability of $\mathbb{D}$ and the corresponding result for cartesian squares in $\mathbb{D}^M(\square)$ we are done. $\square$

We give immediately the expected result on the suspension and loop functors in this stable situation. Recall the definition of the categories $\mathcal{M}, \mathcal{M}^{\lrcorner}, \mathcal{M}^{\ulcorner}$, and the factorization of (Σ, Ω) in the case of a pointed derivator. Let us denote, in addition, by $\mathcal{M}^{\Sigma} \subset \mathcal{M}$ (respectively $\mathcal{M}^{\Omega} \subset \mathcal{M}$) the full subcategory spanned by the cocartesian (respectively cartesian) squares. With this notation, in the case of a *pointed* derivator, there is the following additional factorization of (Σ, Ω):

$$\begin{tikzcd}
\Sigma\colon & \mathbb{D}(e) \arrow[r, "(0,0)_*", "\simeq"'] \arrow[d, equal] & \mathcal{M}^{\ulcorner} \arrow[r, "i_{\ulcorner !}", "\simeq"'] \arrow[d, equal] & \mathcal{M}^{\Sigma} \arrow[r, "i_{\lrcorner}^*"] & \mathcal{M}^{\lrcorner} \arrow[r, "(1,1)^*", "\simeq"'] \arrow[d, equal] & \mathbb{D}(e) \arrow[d, equal] & \\
& \mathbb{D}(e) & \mathcal{M}^{\ulcorner} \arrow[l, "(0,0)^*"', "\simeq"] & \mathcal{M}^{\Omega} \arrow[l, "i_{\ulcorner}^*"'] & \mathcal{M}^{\lrcorner} \arrow[l, "i_{\lrcorner *}"', "\simeq"] & \mathbb{D}(e) \arrow[l, "(1,1)_!"', "\simeq"] & :\Omega
\end{tikzcd}$$

In this diagram, all but possibly the two restriction functors in the middle are equivalences. In the case of a *stable* derivator, we have $\mathcal{M}^{\Sigma} = \mathcal{M}^{\Omega}$ and these two restriction functors are also equivalences:

$$\begin{tikzcd}
\Sigma\colon & \mathbb{D}(e) \arrow[r, "(0,0)_*", "\simeq"'] \arrow[d, equal] & \mathcal{M}^{\ulcorner} \arrow[r, "i_{\ulcorner !}", "\simeq"'] \arrow[d, equal] & \mathcal{M}^{\Sigma} \arrow[r, "i_{\lrcorner}^*", "\simeq"'] \arrow[d, equal] & \mathcal{M}^{\lrcorner} \arrow[r, "(1,1)^*", "\simeq"'] \arrow[d, equal] & \mathbb{D}(e) \arrow[d, equal] & \\
& \mathbb{D}(e) & \mathcal{M}^{\ulcorner} \arrow[l, "(0,0)^*"', "\simeq"] & \mathcal{M}^{\Omega} \arrow[l, "i_{\ulcorner}^*"', "\simeq"] & \mathcal{M}^{\lrcorner} \arrow[l, "i_{\lrcorner *}"', "\simeq"] & \mathbb{D}(e) \arrow[l, "(1,1)_!"', "\simeq"] & :\Omega
\end{tikzcd}$$

This proves the first half of the next result. The second half can be proved in a similar way.

Proposition 4.4 *Let $\mathbb{D}$ be a stable derivator. Then we have equivalences of derivators*

$$(\Sigma, \Omega)\colon \mathbb{D} \xrightarrow{\simeq} \mathbb{D} \quad \textit{and} \quad (\mathsf{Cone}, \mathsf{Fiber})\colon \mathbb{D}^{[1]} \xrightarrow{\simeq} \mathbb{D}^{[1]}.$$

The following result is immediate from Proposition 3.12.

Proposition 4.5 *Let $\mathbb{D}$ be a stable derivator and let $X \in \mathbb{D}(\square)$. If two of the three following statements hold for the square X then so does the third one:*

(1) *The square X is cocartesian.*

(2) *The arrow $d_0^v X$ is an isomorphism.*

(3) *The arrow $d_1^v X$ is an isomorphism.*

In particular, an object $f \in \mathbb{D}([1])$ is an isomorphism if and only if the cone $\mathsf{C}f$ is zero.

Let us mention the following result which is immediate from Proposition 3.13 on the composition and the cancellation properties of (co)cartesian squares.

Proposition 4.6 *Let $\mathbb{D}$ be a stable derivator and let $X \in \mathbb{D}([2]\times[1])$. If two of the squares $d_0(X), d_1(X)$, and $d_2(X)$ are bicartesian, then so is the third one.*

The next aim is to establish the semiadditivity in the stable case, ie we want to show that the values admit finite biproducts. By Proposition 4.3, we can assume that $J = e$. We know already from Proposition 1.7 that the values of an arbitrary derivator admit finite coproducts and finite products.

Proposition 4.7 *Let $\mathbb{D}$ be a stable derivator and consider a functor $u\colon J \to K$. Then finite coproducts and finite products in $\mathbb{D}(J)$ are canonically isomorphic. Moreover, these are preserved by $u^*, u_!$, and u_*.*

Proof For the first part, it is again enough to show the result for the case $J = e$. Let us consider the inclusion $j_2\colon L_2 \to L_3$ of the left poset L_2 in the right poset L_3:

$$\begin{tikzcd} & (1,0) \arrow[r] & (2,0) \\ (0,1) \arrow[d] & & \\ (0,2) & & \end{tikzcd} \qquad \begin{tikzcd} (0,0) \arrow[r] \arrow[d] & (1,0) \arrow[r] & (2,0) \\ (0,1) \arrow[d] & & \\ (0,2) & & \end{tikzcd}$$

Moreover, let $j_1\colon e \sqcup e \to L_2$ be the map $(1,0) \sqcup (0,1)$ and let $j_3\colon L_3 \to [2]\times[2] = L$ be the obvious inclusion. Since j_1 is a sieve the homotopy Kan extension functor j_{1*} is an "extension by zero functor" by Proposition 3.6 and similarly for the homotopy Kan extension functor $j_{2!}$ associated to the cosieve j_2. Let us consider the functor:

$$\mathbb{D}(e)\times\mathbb{D}(e) \simeq \mathbb{D}(e\sqcup e) \xrightarrow{j_{1*}} \mathbb{D}(L_2) \xrightarrow{j_{2!}} \mathbb{D}(L_3) \xrightarrow{j_{3!}} \mathbb{D}(L)$$

The image $Q \in \mathbb{D}(L)$ of a pair $(X, Y) \in \mathbb{D}(e) \times \mathbb{D}(e)$ under this functor has as underlying diagram:

$$\mathsf{dia}_L(Q): \quad \begin{tikzcd} 0 \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & 0 \arrow[d] \\ Y \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & Y' \arrow[d] \\ 0 \arrow[r] & X' \arrow[r] & Z \end{tikzcd}$$

Let us denote the four inclusions of the smaller squares in L by $i_k, k = 1, \ldots, 4$, ie let us set

$$i_1 = d^2 \times d^2, \quad i_2 = d^0 \times d^2, \quad i_3 = d^2 \times d^0 \quad \text{and} \quad i_4 = d^0 \times d^0.$$

An application of Proposition 3.10 to these inclusions $i_k\colon \square \to L, k = 1, \ldots, 4$, and $f = j_3$ allows us to deduce that all squares are bicartesian. In fact, in all four cases, $i_k(1,1) \notin \mathrm{Im}(j_3)$ and we only have to check that the induced functors $\widetilde{i}_k\colon \ulcorner \to L - i_k(1,1)_{/i_k(1,1)}$ are right adjoints. For $k = 1$, this functor is an isomorphism while in the other three cases Lemma 3.11 applies. By Proposition 4.6, also the composite squares $(d_2 \times d_1)(Q)$ and $(d_1 \times d_2)(Q)$ are bicartesian. Hence, Proposition 3.12 ensures that we have isomorphisms $X \cong X'$ and $Y \cong Y'$. Similarly, the square $(d_1 \times d_1)(Q)$ is bicartesian and we obtain an isomorphism $Z \cong 0$. Thus, we see that B is simultaneously a coproduct of X and Y and a product of $X' \cong X$ and $Y' \cong Y$.

The fact that these biproducts are preserved by $u^*, u_!$, and u_* follows immediately since each of the three functors has an adjoint functor on at least one side. $\square$

A standard fact about (semi)additive categories thus implies the following.

Corollary 4.8 *Let $\mathbb{D}$ be a stable derivator and let J be a category. Every object of $\mathbb{D}(J)$ is canonically a commutative monoid object and a cocommutative comonoid object. In particular, the morphism set $\mathsf{hom}_{\mathbb{D}(J)}(X, Y), X, Y \in \mathbb{D}(J)$, carries canonically the structure of an abelian monoid.*

We will use the standard notation $\oplus$ for the biproduct. The next aim is to show that objects of the form ΩX are even abelian *group* objects and dually for ΣX. We give the proof in the case of ΩX in which case the constructions can be motivated by the process of concatenation of loops in topology. Let us begin with some preparations. Since the aim is to "model categorically" the concatenation and inversion of loops we have to consider finite direct sums of "loop objects". For the construction of the finite sums of loop objects there is the following conceptual approach which admits an obvious dualization. Let $\lrcorner_n$ be the poset with objects $e_0,\ldots,e_n$ and t and with ordering generated by $e_i \leq t, i=0,\ldots,n$. Let $\mathcal{F}in$ denote the category of the finite sets $\langle n\rangle=\{0,\ldots,n\}$ with all set-theoretic maps as morphisms between them. The assignment $\langle n\rangle \mapsto \lrcorner_n$ can be extended to a functor $\mathcal{F}in \to \mathsf{Cat}$ if we send $f\colon \langle k\rangle \to \langle n\rangle$ to $\lrcorner_f\colon \lrcorner_k \to \lrcorner_n$ with $\lrcorner_f(e_i)=e_{f(i)}$ and $\lrcorner_f(t)=t$. Since $t\colon e\to \lrcorner_n$ is a cosieve, $t_!\colon \mathbb{D}(e)\to\mathbb{D}(\lrcorner_n)$ gives us an "extension by zero functor". Define P_n as

$$P_n=\mathrm{Holim}_{\lrcorner_n}\circ t_!\colon \mathbb{D}(e)\to\mathbb{D}(\lrcorner_n)\to\mathbb{D}(e)$$

and note that we have a canonical isomorphism $P_1X\cong\Omega X$.

Lemma 4.9 *Let $\mathbb{D}$ be a pointed derivator. The above construction defines a bifunctor:*

$$P\colon \mathcal{F}in^{\mathrm{op}}\times\mathbb{D}(e)\to\mathbb{D}(e),\quad (\langle n\rangle,X)\mapsto P_nX$$

Proof The functoriality of P in the second variable is obvious so let us assume we are given a morphism $f\colon\langle k\rangle\to\langle n\rangle$. From such a morphism f we obtain the following diagram given on the left-hand side:

$$\begin{tikzcd}
e \arrow[r] \arrow[d, "t"'] & e \arrow[d, "t"] \\
\lrcorner_k \arrow[r, "\lrcorner_f"] \arrow[d, "p"'] & \lrcorner_n \arrow[d, "p"] \\
e \arrow[r] & e
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{D}(e) \arrow[d, "t_!"'] \arrow[dr, phantom, "\searrow"] & \mathbb{D}(e) \arrow[l] \arrow[d, "t_!"] \\
\mathbb{D}(\lrcorner_k) \arrow[d, "\mathrm{Holim}"'] \arrow[dr, phantom, "\nwarrow"] & \mathbb{D}(\lrcorner_n) \arrow[l, "\lrcorner_f^*"'] \arrow[d, "\mathrm{Holim}"] \\
\mathbb{D}(e) & \mathbb{D}(e) \arrow[l]
\end{tikzcd}$$

The formation of the corresponding mates gives rise to the pasting diagram on the right (note that we had to use both variants here). Using the fact that isomorphisms can be detected pointwise and (Der4) it is easy to check that the upper 2–cell is invertible. Thus we can define P_f as the following composition:

$$P_f\colon\quad P_n=\mathrm{Holim}_{\lrcorner_n}\circ t_!\longrightarrow \mathrm{Holim}_{\lrcorner_k}\circ\lrcorner_f^*\circ t_!\longrightarrow\mathrm{Holim}_{\lrcorner_k}\circ t_!=P_k$$

The functoriality of this construction follows from Lemma 1.14. □

Let us fix notation for some morphisms in $\mathcal{F}in$. Given a $(k+1)$–tuple $(i_0,i_1,\ldots,i_k)$ of elements of $\langle n\rangle$ let us denote by $(i_0i_1\ldots i_k)$ the corresponding morphism $\langle k\rangle\to\langle n\rangle$ which sends j to i_j. For $n\geq 1$ and $1\leq k\leq n$, we have thus the morphism $(k-1,k)\colon\langle 1\rangle\to\langle n\rangle$. So, for a pointed derivator $\mathbb{D}$ and an object $X\in\mathbb{D}(e)$, we obtain maps $(k-1,k)^*\colon P_nX\to P_1X\cong\Omega X$. These maps taken together define *Segal maps* and satisfy the "usual" Segal condition [40].

Lemma 4.10 *Let $\mathbb{D}$ be a pointed derivator and let $X \in \mathbb{D}(e)$. For $n \geq 1$ the Segal map is an isomorphism:*

$$s = s_n\colon P_n X \xrightarrow{\cong} \prod_{k=1}^{n} P_1(X) \cong \prod_{k=1}^{n} \Omega X$$

Proof We only check the case of $n = 2$. Let J be the poset obtained from $\lrcorner_2$ by adding two new elements ω_0 and ω_1 such that $\omega_0 \leq e_0, e_1$ and $\omega_1 \leq e_1, e_2$. Moreover, let us denote the resulting inclusion by $j\colon \lrcorner_2 \to J$. Under the obvious isomorphism $J \cong [1] \times \lrcorner$, we can consider the adjunction $(d^1 \times \mathrm{id}, s^0 \times \mathrm{id})\colon \lrcorner \rightleftarrows [1] \times \lrcorner$ as an adjunction $(L, R)\colon \lrcorner \rightleftarrows J$. By Proposition 1.18 we have a natural isomorphism between P_2 and

$$\mathbb{D}(e) \xrightarrow{t_!} \mathbb{D}(\lrcorner_2) \xrightarrow{j_*} \mathbb{D}(J) \xrightarrow{L^*} \mathbb{D}(\lrcorner) \xrightarrow{\mathrm{Holim}} \mathbb{D}(e).$$

But it is easy to see that the composition of the first three functors evaluated on X yields a diagram which vanishes at t and is isomorphic to ΩX at the two remaining arguments. It thus follows that we have an isomorphism $P_2(X) \cong \Omega X \times \Omega X$ induced by the Segal map. $\square$

It is a standard fact that Segal objects admit an associative *concatenation map* defined by:

$$\star\colon \Omega X \times \Omega X \xleftarrow[\cong]{s} P_2(X) \xrightarrow{(02)^*} \Omega X$$

Lemma 4.11 *Let $\mathbb{D}$ be a pointed derivator and let X be an object of $\mathbb{D}(e)$. The concatenation map $\star\colon \Omega X \times \Omega X \to \Omega X$ is an associative pairing on ΩX.*

Heading for the additive inverse of the identity on loop objects, let us consider the only nontrivial automorphism $\sigma\colon \langle 1\rangle \to \langle 1\rangle$ in $\mathcal{F}in$. Then $\lrcorner_\sigma\colon \lrcorner \to \lrcorner$ is the isomorphism interchanging the vertices $(1,0)$ and $(0,1)$. There is thus an induced automorphism $\sigma^* = (10)^*\colon \Omega X \to \Omega X$, which we call the *inversion of loops*.

Proposition 4.12 *Let $\mathbb{D}$ be a stable derivator and let $X \in \mathbb{D}(e)$. The inversion of loops map $\sigma^*\colon \Omega X \to \Omega X$ is an additive inverse to $\mathrm{id}_{\Omega X}$. In particular, $\Omega X \in \mathbb{D}(e)$ is an abelian* group *object.*

Proof By functoriality of the construction $P_\bullet X$, there is a right action of the symmetric group on three letters on $P_2 X$. We want to describe the corresponding action on $\Omega X \oplus \Omega X$ obtained by conjugation with the Segal map s. The strategy of the proof is then to use this action in order to relate the concatenation product and the addition of morphisms.

For different elements $i, j \in \langle 2\rangle$ let us denote by σ_{ij} the associated transposition. One checks that the following diagram commutes

$$\begin{tikzcd}
P_2X \arrow[r, "\sigma_{02}^*"] \arrow[d, "s"'] & P_2X \arrow[d, "s"] \\
\Omega X \oplus \Omega X \arrow[r, "\begin{pmatrix} 0 & \sigma^* \\ \sigma^* & 0 \end{pmatrix}"'] & \Omega X \oplus \Omega X,
\end{tikzcd}$$

where the arrows labeled by s are again Segal maps. From the equality of the maps

$$\sigma_{01} \circ (01) = (01) \circ \sigma \colon \langle 1\rangle \longrightarrow \langle 2\rangle$$

we conclude that the endomorphism of $\Omega X \oplus \Omega X$ corresponding to σ_{01} is a lower triangular matrix

$$s \circ \sigma_{01}^* \circ s^{-1} = \begin{pmatrix} \sigma^* & 0 \\ \alpha & \beta \end{pmatrix} \colon \Omega X \oplus \Omega X \longrightarrow \Omega X \oplus \Omega X$$

for some maps $\alpha, \beta \colon \Omega X \to \Omega X$. The fact that σ_{01} is an involution implies the relations:

$$\alpha\sigma^* + \beta\alpha = 0 \quad \text{and} \quad \beta^2 = \mathrm{id}$$

The aim is now to show that both maps α and β are identities which would in particular imply that σ^* is an additive inverse of $\mathrm{id}_{\Omega X}$.

From the relation $(02) = \sigma_{01} \circ (12)$ we immediately get $(02)^* = (12)^* \circ \sigma_{01}^* \colon P_2X \to \Omega X$. Using the matrix description of the map induced by σ_{01} we see that for two maps $f, g \colon \Omega X \to \Omega X$ there is the following formula for the concatenation product:

$$f \star g = \alpha f + \beta g \colon \Omega X \longrightarrow \Omega X$$

By Lemma 4.11 we know that the concatenation pairing is associative. If we compare the two expressions for $(0 \star 0) \star \mathrm{id}_{\Omega X}$ and $0 \star (0 \star \mathrm{id}_{\Omega X})$ we already obtain the first intended relation $\beta = \mathrm{id}_{\Omega X}$.

Instead of using $(02) = \sigma_{01} \circ (12)$, we can also use the relation $(02) = \sigma_{12} \circ (01) \colon \langle 1\rangle \to \langle 2\rangle$ to obtain a further description of the concatenation product. First, since

$$\sigma_{12} = \sigma_{02} \circ \sigma_{01} \circ \sigma_{02} \colon \langle 2\rangle \longrightarrow \langle 2\rangle$$

we obtain that the endomorphism on $\Omega X \oplus \Omega X$ induced by σ_{12}^* has the following matrix description:

$$s \circ \sigma_{12}^* \circ s^{-1} = \begin{pmatrix} cc\sigma^*\beta\sigma^* & \sigma^*\alpha\sigma^* \\ 0 & \sigma^* \end{pmatrix} \colon \Omega X \oplus \Omega X \longrightarrow \Omega X \oplus \Omega X$$

From this and the formula $(02)^* = (01)^* \circ \sigma_{12}^*$ we see that the concatenation product can also be written as:

$$f \star g = \sigma^* \beta \sigma^* f + \sigma^* \alpha \sigma^* g \colon \Omega X \longrightarrow \Omega X$$

A comparison of these two descriptions concludes the proof since we obtain $\alpha = \sigma^* \beta \sigma^* = \mathrm{id}_{\Omega X}$. □

Remark 4.13 Although we will not make use of this remark we want to emphasize the following. The proof of the last proposition shows that the addition on mapping spaces into loop objects coincides with the pairing induced by the concatenation of loops. Similarly, additive inverses are given by the inversion of loops. Thus for maps $f, g \colon U \to \Omega X$ we have:

$$f + g = f \star g \quad \text{and} \quad -f \overset{\text{def}}{=} \sigma^* f$$

A combination of this proposition, Proposition 4.7 on the semiadditivity of $\mathbb{D}(J)$ and the fact that (Σ, Ω) is a pair of inverse equivalences in the stable situation gives us immediately the following corollary.

Corollary 4.14 *If $\mathbb{D}$ is a stable derivator then $\mathbb{D}(J)$ is an additive category for an arbitrary J. Moreover, for an arbitrary functor $u \colon J \to K$, the induced functors $u^*, u_!$, and u_* are additive.*

4.2 The canonical triangulated structures

We can now attack the main result of this section, namely, that given a stable derivator $\mathbb{D}$ then the categories $\mathbb{D}(J)$ are canonically triangulated categories. Using Proposition 4.3, we can again assume without loss of generality that we are in the case $J = e$. The suspension functor of the triangulated structure will be the suspension functor $\Sigma \colon \mathbb{D}(e) \to \mathbb{D}(e)$ we constructed already. Thus, let us construct the class of distinguished triangles. For this purpose, let K denote the poset:

$$\begin{array}{ccccc} (0,0) & \longrightarrow & (1,0) & \longrightarrow & (2,0) \\ \downarrow & & & & \\ (0,1) & & & & \end{array}$$

Moreover, let $i_0 \colon [1] \to K$ be the map classifying the left horizontal arrow and let $i_1 \colon K \to [2] \times [1]$ be the inclusion. Let us denote the composition by

$$i \colon [1] \xrightarrow{i_0} K \xrightarrow{i_1} [2] \times [1].$$

Again, since i_0 is a sieve, i_{0*} gives us an extension by zero functor. Let us consider the functor:

$$T\colon \mathbb{D}([1]) \xrightarrow{i_{0*}} \mathbb{D}(K) \xrightarrow{i_{1!}} \mathbb{D}([2]\times[1])$$

We claim that the squares $d_0T(f), d_1T(f)$, and $d_2T(f) \in \mathbb{D}(\square)$ are then bicartesian for an arbitrary $f \in \mathbb{D}([1])$. Moreover, if the underlying diagram of f is $X \to Y$ then we have canonical isomorphisms $T(f)_{2,1} \cong \Sigma X$ and $T(f)_{1,1} \cong \mathsf{C}(f)$. In fact, by Proposition 4.6, it is enough to show the bicartesianness of $d_0T(f)$ and $d_2T(f)$. This can be done by two applications of the detection result Proposition 3.10 to $i_1\colon K \to J = [2]\times[1]$. It is easy to check (using Lemma 3.11 in one of the cases) that the assumptions of that proposition are satisfied. Since i_0 is a sieve, the underlying diagram of $d_1T(f)$ and $d_2T(f)$ respectively look like:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & T(f)_{2,1} \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & T(f)_{1,1} \end{tikzcd}$$

Moreover, by the proof of Proposition 4.4, $d_1T(f)$ lies in the essential image of

$$\mathbb{D}(e) \xrightarrow{(0,0)_*} \mathbb{D}(\ulcorner) \xrightarrow{i_{\ulcorner !}} \mathbb{D}(\square).$$

Hence, we have a canonical isomorphism $T(f)_{2,1} \cong \Sigma X$. Similarly, if we let $j\colon [1] \to \ulcorner$ denote the functor classifying the upper horizontal morphism $d_2T(f)$ then lies in the essential image of

$$\mathbb{D}([1]) \xrightarrow{j_*} \mathbb{D}(\ulcorner) \xrightarrow{i_{\ulcorner !}} \mathbb{D}(\square).$$

Hence, we also have a canonical isomorphism $T(f)_{1,1} \cong \mathsf{C}(f)$ as intended.

Thus, for $f \in \mathbb{D}([1])$, by first restricting $T(f)$ to $[3]$ in the expected way and then forming the underlying diagram in $\mathbb{D}(e)$, we obtain a triangle (T_f) in $\mathbb{D}(e)$ which is of the following form:

$$(T_f)\colon \quad X \longrightarrow Y \longrightarrow \mathsf{C}(f) \longrightarrow \Sigma X$$

Call a triangle in $\mathbb{D}(e)$ *distinguished* if it is isomorphic to (T_f) for some $f \in \mathbb{D}([1])$. Before we come to the main theorem let us recall the definition of a triangulated category. For more background on this theory see for example [35] or [39]. The form of the octahedron axiom given here is sufficient in order to obtain the usual form of the octahedron axiom. This observation was made in [26] (for a proof of it see [39]).

Definition 4.15 Let $\mathcal{T}$ be an additive category with a self-equivalence $\Sigma\colon \mathcal{T} \to \mathcal{T}$ and a class of so-called distinguished triangles $X \to Y \to Z \to \Sigma X$. The pair consisting of Σ and the class of distinguished triangles determines a *triangulated structure* on $\mathcal{T}$ if the following four axioms are satisfied. In this case, the triple consisting of the category, the endofunctor and the class of distinguished triangles is called a *triangulated category*.

(T1) For every $X \in \mathcal{T}$, the triangle $X \overset{\mathrm{id}}{\to} X \to 0 \to \Sigma X$ is distinguished. Every morphism in $\mathcal{T}$ occurs as the first morphism in a distinguished triangle and the class of distinguished triangles is replete, ie is closed under isomorphisms.

(T2) If the triangle $X \overset{f}{\to} Y \overset{g}{\to} Z \overset{h}{\to} \Sigma X$ is distinguished then also the rotated triangle $Y \overset{g}{\to} Z \overset{h}{\to} \Sigma X \overset{-f}{\to} \Sigma Y$ is distinguished.

(T3) Given two distinguished triangles and a commutative solid arrow diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "u"] & Y \arrow[r] \arrow[d, "v"] & Z \arrow[r] \arrow[d, dashed, "w"] & \Sigma X \arrow[d, "\Sigma u"] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X'
\end{tikzcd}$$

there exists a dashed arrow $w\colon Z \to Z'$ as indicated such that the extended diagram commutes.

(T4) For every pair of composable arrows $f_3\colon X \overset{f_1}{\to} Y \overset{f_2}{\to} Z$ there is a commutative diagram in which the rows and columns are distinguished triangles:

$$\begin{tikzcd}
X \arrow[r, "f_1"] \arrow[d, equal] & Y \arrow[r, "g_1"] \arrow[d, "f_2"'] & C_1 \arrow[r, "h_1"] \arrow[d] & \Sigma X \arrow[d, equal] \\
X \arrow[r, "f_3"'] & Z \arrow[r, "g_3"'] \arrow[d, "g_2"'] & C_3 \arrow[r, "h_3"'] \arrow[d] & \Sigma X \arrow[d, "\Sigma f_1"] \\
 & C_2 \arrow[r, equal] \arrow[d, "h_2"'] & C_2 \arrow[r, "h_2"'] \arrow[d, "\Sigma g_1 \circ h_2"] & \Sigma Y \\
 & \Sigma Y \arrow[r, "\Sigma g_1"'] & \Sigma C_1 &
\end{tikzcd}$$

Here is the important theorem about the canonical triangulated structures on the values of a stable derivator. The fact that these triangulations are compatible with the restriction and homotopy Kan extension functors will be discussed in Corollary 4.19.

Theorem 4.16 *Let $\mathbb{D}$ be a stable derivator and let J be a category. Endowed with the suspension functor $\Sigma\colon \mathbb{D}(J) \to \mathbb{D}(J)$ and the above class of distinguished triangles, $\mathbb{D}(J)$ is a triangulated category.*

Proof It suffices to do this for the case $J = e$. The additivity of $\mathbb{D}(e)$ is already given by Corollary 4.14. Moreover, in this stable setting, the suspension functor Σ is an equivalence.

(T1) The first part of axiom (T1) is settled by Proposition 4.5. Since $\mathbb{D}$ is strong every morphism in $\mathbb{D}(e)$ is up to isomorphism the underlying diagram of an object in $\mathbb{D}([1])$. The triangle associated to this object settles the second part of (T1). The last part of (T1) holds by definition of the class of distinguished triangles.

(T3) Axiom (T3) is settled similarly by reducing first to the situation of triangles of the form (T_f) for $f \in \mathbb{D}([1])$ and then applying the strength again.

(T2) We can again reduce to the case where the given distinguished triangle is (T_f) for some $f \in \mathbb{D}([1])$. Let us consider the category J given by the following full subposet of $[2] \times [2]$,

$$\begin{tikzcd}
(0,0) \arrow[r] \arrow[d] & (1,0) \arrow[r] \arrow[dd] & (2,0) \\
(0,1) \arrow[dr, bend right] & & \\
& (1,2), &
\end{tikzcd}$$

and let $i\colon [1] \to J$ be the functor classifying the upper left horizontal morphism. Then i is a sieve and i_* gives us thus an extension by zero functor. Moreover, let us denote by j the canonical inclusion of J in $K = [2]\times[2] - \{(0,2)\}$. For a given $f \in \mathbb{D}([1])$ let us consider $j_! i_*(f)$. Again, by a repeated application of Proposition 3.10 all squares in $j_! i_*(f)$ are bicartesian. If the diagram of f is $f\colon X \to Y$ then the underlying diagram of $j_! i_*(f)$ looks like:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] & Y \arrow[r] \arrow[d, "g"] & 0 \arrow[d] \\
0 \arrow[r] & \mathsf{C}f \arrow[r, "h"] \arrow[d] & \Sigma X \arrow[d] \\
& 0 \arrow[r] & \Sigma Y
\end{tikzcd}$$

In fact, the inclusion $(d^1 \times d^2)\colon \ulcorner \to K$ allows us to identify the value at $(2,1)$ with ΣX while the inclusion $(d^0 \times d^1)\colon \ulcorner \to K$ gives us an identification of the lower right corner with ΣY. However, this last inclusion differs from the usual one by the automorphism $\sigma\colon \ulcorner \to \ulcorner$. By Proposition 4.12, the induced map $\sigma^*\colon \Sigma Y \to \Sigma Y$ is $-\mathrm{id}_{\Sigma Y}$. Hence, using moreover the unique natural transformation of the two inclusions $(d^0 \times d^1) \to (d^1 \times d^2)\colon \ulcorner \to K$, we can identify the morphism $\Sigma X \to \Sigma Y$ as $-\Sigma f$ and this shows that the triangle (T_g) is as stated in the claim.

(T4) It remains to give a proof of the octahedron axiom. The proof of this will be split into two parts.

(1) In this part, we show that every "first half of an octahedron diagram" comes up to isomorphism from an object $F \in \mathbb{D}([2])$. Let us restrict attention to the upper left square

$$\begin{tikzcd} X \arrow[r, "f_1"] \arrow[d, equal] & Y \arrow[d, "f_2"] \\ X \arrow[r, "f_3"'] & Z \end{tikzcd}$$

of such a diagram. The strength of $\mathbb{D}$ guarantees that there is an object $F_1 \in \mathbb{D}([1])$ and an isomorphism $\mathsf{dia}\, F_1 \cong (f_1\colon X \to Y)$. Moreover, let us consider $p^*Z \in \mathbb{D}([1])$, where $p\colon [1] \to e$ is the unique functor. Then, we obtain a morphism $\phi\colon F_1 \to p^*Z$ as the image of f_2 under the two natural isomorphisms (we applied Lemma 1.19 to obtain the second one):

$$\mathsf{hom}_{\mathbb{D}(e)}(Y,Z) \cong \mathsf{hom}_{\mathbb{D}([1])}(F_1, 1_*Z) \cong \mathsf{hom}_{\mathbb{D}([1])}(F_1, p^*Z)$$

Considering this map $\phi\colon F_1 \to p^*Z$ as an object of $\mathbb{D}([1])^{[1]}$, a further application of the strength guarantees the existence of an object $Q \in \mathbb{D}(\square)$ such that $\mathsf{dia}_{[1],[1]}\, Q \cong (\phi\colon F_1 \to p^*Z)$:

$$\mathsf{dia}\, Q\colon \qquad \begin{tikzcd} X \arrow[r, "\phi_0"] \arrow[d, "f_1"'] & Z \arrow[d] \\ Y \arrow[r, "\phi_1"'] & Z \end{tikzcd}$$

If $i\colon [2] \to \square$ classifies the nondegenerate pair of composable arrows passing through the lower left corner $(0,1)$ then let us set $F = i^*Q \in \mathbb{D}([2])$. This F does the job.

(2) In this second part, given an object $F \in \mathbb{D}([2])$, we construct an associated octahedron diagram in $\mathbb{D}(e)$. The pattern of this part of the proof is by now quite

familiar. Consider the category J given by the following full subposet of $[4]\times[2]$,

$$\begin{tikzcd}
(0,0) \arrow[r] \arrow[d] & (1,0) \arrow[r] \arrow[dd] & (2,0) \arrow[r] & (3,0) \arrow[dr, bend left] & \\
(0,1) \arrow[rrrr] \arrow[dr, bend right] & & & & (4,1) \\
& (1,2), & & &
\end{tikzcd}$$

and let $i\colon [2]\to J$ classify the two composable upper left morphisms. Moreover, let

$$j\colon J \longrightarrow K=[4]\times[2]-\{(4,0),(0,2)\}$$

be the canonical inclusion. Since i is a sieve, the homotopy right Kan extension functor i_* is an extension by zero functor. For $F\in\mathbb{D}([2])$ let us consider $D=j_!i_*(F)\in\mathbb{D}(K)$. If the underlying diagram of F is

$$X \xrightarrow{f_1} Y \xrightarrow{f_2} Z$$

then the underlying diagram of D is:

$$\begin{tikzcd}
X \arrow[r, "f_1"] \arrow[d] & Y \arrow[r, "f_2"] \arrow[d] & Z \arrow[r] \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \widehat{C}_1 \arrow[r] \arrow[d] & \widehat{C}_3 \arrow[r] \arrow[d] & SX \arrow[r] \arrow[d] & 0 \arrow[d] \\
& 0 \arrow[r] & \widehat{C}_2 \arrow[r] & SY \arrow[r] & S\widehat{C}_1
\end{tikzcd}$$

A repeated application of Proposition 3.10 guarantees that all squares in D are bicartesian. Hence the same is also true for all compound squares one can find in D. This allows us to find canonical isomorphisms $\widehat{C}_k\cong \mathsf{C}(f_k)$ if we set $f_3=f_2\circ f_1$. More precisely, the cone functor C has of course to be applied to $f_1=d_2(F)$, $f_2=d_0(F)$, and $f_3=d_1(F)\in\mathbb{D}([1])$. Similarly, we obtain isomorphisms $SX\cong\Sigma X$, $SY\cong\Sigma Y$, and $S\widehat{C}_1\cong\Sigma\widehat{C}_1$. Thus, one can extract an octahedron diagram in $\mathbb{D}(e)$ from the object D. $\square$

The next aim is to show that the functors belonging to a stable derivator can be canonically made into exact functors with respect to these structures. In the stable setting, Corollary 4.14 induces immediately the following one.

Corollary 4.17 *Let $F\colon \mathbb{D} \to \mathbb{D}'$ be a morphism of stable derivators, then*

$$F \text{ is left exact} \quad \Longleftrightarrow \quad F \text{ is exact} \quad \Longleftrightarrow \quad F \text{ is right exact.}$$

In particular, the components $F_J\colon \mathbb{D}(J) \to \mathbb{D}'(J)$ of an exact morphism are additive functors.

Exact morphisms are the "correct" morphisms for stable derivators. Some evidence for this is given by the next result.

Proposition 4.18 *Let $F\colon \mathbb{D} \to \mathbb{D}'$ be an exact morphism of stable derivators and let J be a category. The functor $F_J\colon \mathbb{D}(J) \to \mathbb{D}'(J)$ can be canonically endowed with the structure of an exact functor of triangulated categories.*

Proof By Proposition 4.3, we can assume without loss of generality that $J = e$. Moreover, by Proposition 3.21 we know that there is a canonical isomorphism $F \circ \Sigma \cong \Sigma \circ F$. The morphism F preserves composites of two cocartesian squares and in particular those which vanish at $(2,0)$ and $(0,1)$. Since these were used to define the class of distinguished triangles it follows that F together with the canonical isomorphism $F \circ \Sigma \cong \Sigma \circ F$ is exact. □

Corollary 4.19 *Let $\mathbb{D}$ be a stable derivator and let $u\colon J \to K$ be a functor. The induced functors $u^*\colon \mathbb{D}(K) \to \mathbb{D}(J)$ and $u_!, u_*\colon \mathbb{D}(J) \to \mathbb{D}(K)$ can be canonically endowed with the structure of exact functors.*

Proof Since we have adjunctions $(u_!, u^*)$ and (u^*, u_*), it suffices to show that u^* can be canonically endowed with the structure of an exact functor (see [34, page 463]). But $u^*\colon \mathbb{D}(K) \to \mathbb{D}(J)$ is the underlying functor of the exact morphism $u^*\colon \mathbb{D}^K \to \mathbb{D}^J$. □

5 Additive derivators as enhancement of pretriangulated categories

In this short section we introduce additive derivators and show that they can be considered as an enhancement of pretriangulated categories.

Definition 5.1 A derivator $\mathbb{D}$ is *additive* if the underlying category $\mathbb{D}(e)$ is additive.

Proposition 5.2 *If a derivator $\mathbb{D}$ is additive, then all categories $\mathbb{D}(J)$ are additive and for any functor $u\colon J \longrightarrow K$ the induced functors $u^*, u_!$, and u_* are additive. In particular, $\mathbb{D}$ is additive if and only if $\mathbb{D}^M$ is additive for all M. Moreover, $\mathbb{D}$ is additive if and only if $\mathbb{D}^{\mathrm{op}}$ is.*

In contrast to the above definition, let us call a prederivator additive if all values and all precomposition functors are additive. Thus the prederivator represented by a category is additive if and only if the representing category is additive.

Example 5.3 (1) Stable derivators are additive by Corollary 4.14.

(2) Let R be a ring and let $\mathsf{Ch}_{\geq 0}(R)$ denote the category of nonnegative chain complexes of left R–modules. Using the model structure from [10, Section 7] we obtain the derivator:

$$\mathbb{D}_R^{\geq 0} := \mathbb{D}_{\mathsf{Ch}_{\geq 0}(R)}\colon \mathsf{Cat}^{\mathrm{op}} \to \mathsf{CAT}, \quad J \mapsto \mathsf{Ho}(\mathsf{Ch}_{\geq 0}(R)^J) = D_{\geq 0}(R - \mathsf{Mod}^J)$$

Here, $D_{\geq 0}(-)$ denotes the formation of the nonnegative derived category of an abelian category. This derivator is additive but not stable.

For convenience let us sketch the definition of a right triangulated category (see [26; 4]). A *right triangulated category* consists of an additive category $\mathcal{A}$, an additive endofunctor Σ (which is not an equivalence in general!) and a class of so-called distinguished right triangles $X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$. This structure has to satisfy four axioms (RT1)–(RT4). The last three of them being precisely the same as for triangulated categories let us only recall the first one:

(RT1) For every $X \in \mathcal{A}$, the right triangle $0 \to X \stackrel{\mathrm{id}}{\to} X \to 0$ is distinguished. Every morphism in $\mathcal{A}$ occurs as the first morphism in a distinguished right triangle and the class of distinguished right triangles is replete, ie is closed under isomorphisms.

Let $\mathbb{D}$ be an additive derivator and let $X \in \mathbb{D}(J)$ for some small category J. Then the concatenation of loops $*\colon \Omega X \oplus \Omega X \to \Omega X$ and the inversion of loops $\sigma^*\colon \Omega X \to \Omega X$ turn ΩX into a group object of $\mathbb{D}(J)$. Moreover, given an object $U \in \mathbb{D}(J)$ and morphisms $f, g\colon U \to \Omega X$ then we have:

$$f + g = f * g \quad \text{and} \quad -f = \sigma^* f$$

The proof from the stable context also applies to additive derivators. This result is slightly nicer in the additive context: given an additive derivator we already had *both* an addition and a multiplication by -1 on the set of morphisms from U to ΩX and both of them can be interpreted geometrically by some "loop manipulation".

Now, given an additive derivator then the suspension functor $\Sigma\colon \mathbb{D}(J) \to \mathbb{D}(J)$ is additive since it is a left adjoint. Using precisely the same reasoning as in Section 4.2 we define a replete class of distinguished right triangles in $\mathbb{D}(J)$.

Theorem 5.4 *Let $\mathbb{D}$ be a strong, additive derivator and let J be a small category. Then the pair consisting of $\Sigma\colon \mathbb{D}(J) \to \mathbb{D}(J)$ and the above class of distinguished right triangles defines a right triangulated structure on $\mathbb{D}(J)$. Dually, the pair consisting of $\Omega\colon \mathbb{D}(J) \to \mathbb{D}(J)$ and the dually defined class of distinguished left triangles turns $\mathbb{D}(J)$ into a left triangulated category.*

The right triangle associated to an object $f \in \mathbb{D}([1])$ is denoted by $T_\Sigma(f)$ and dually. There is a stronger result since the left and the right triangulations are compatible in the sense of the following definition [3].

Definition 5.5 Let $\mathcal{A}$ be an additive category and let $(\Sigma, \Omega)\colon \mathcal{A} \rightleftarrows \mathcal{A}$ be an adjunction such that Σ and Ω are part of a right and left triangulation on $\mathcal{A}$ respectively. This quadruple is called a *pretriangulation* on $\mathcal{A}$ if the following properties are satisfied:

(PT1) Let us be given a right triangle and a left triangle in $\mathcal{A}$ as indicated in the next diagram. If we have morphisms α and β such that the square on the left is commutative then there is a morphism $\gamma\colon Z \to Y'$ such that the entire diagram commutes:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\alpha"'] & Y \arrow[r, "g"] \arrow[d, "\beta"'] & Z \arrow[r, "h"] \arrow[d, dashed, "\gamma"] & \Sigma X \arrow[d, "\epsilon\circ\Sigma\alpha"] \\
\Omega Z' \arrow[r, "f'"'] & X' \arrow[r, "g'"'] & Y' \arrow[r, "h'"'] & Z'
\end{tikzcd}$$

(PT2) Let us be given a right triangle and a left triangle in $\mathcal{A}$ as indicated in the next diagram. If we have morphisms α and β such that the square on the right is commutative then there is a morphism $\gamma\colon Y \to X'$ such that the entire diagram commutes:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\Omega\alpha\circ\eta"'] & Y \arrow[r, "g"] \arrow[d, dashed, "\gamma"'] & Z \arrow[r, "h"] \arrow[d, "\beta"] & \Sigma X \arrow[d, "\alpha"] \\
\Omega Z' \arrow[r, "f'"'] & X' \arrow[r, "g'"'] & Y' \arrow[r, "h'"'] & Z'
\end{tikzcd}$$

A *pretriangulated category* is an additive category together with a pretriangulation.

We have the following nice theorem about the values of a strong, additive derivator. In the proof we use the same notation as in the stable case (see Section 4.2).

Theorem 5.6 *Let $\mathbb{D}$ be a strong, additive derivator and let J be a small category. Then the adjunction $(\Sigma, \Omega)\colon \mathbb{D}(J) \rightleftarrows \mathbb{D}(J)$ together with the right and the left triangulated structure on $\mathbb{D}(J)$ guaranteed by Theorem 5.4 turn $\mathbb{D}(J)$ into a pretriangulated category.*

Proof By Proposition 5.2 we can assume $J = e$. Moreover, by duality it suffices to establish (PT1). We can assume that the first row is the underlying diagram of $T_\sigma(f)$ and that the second one is the underlying diagram of $T_\Omega(h')$. Let us not distinguish notationally between $f \in \mathbb{D}([1])$ and its underlying diagram $f\colon X \to Y$ in $\mathbb{D}(e)$ and similarly for other morphisms. We will construct the morphism γ in two steps. First, by the strength of our derivator $\mathbb{D}$ we can find a morphism $\phi\colon f \to f'$ in $\mathbb{D}([1])$ such that the underlying diagram of ϕ is precisely (α, β). From this we get a morphism $T_\sigma(\phi)\colon T_\sigma(f) \to T_\sigma(f')$ and it is easy to verify that under our identifications the morphism $T_\sigma(\phi)_{2,1}$ is just $\Sigma\alpha$.

For the second step observe that we have an isomorphism $T_\sigma(f')|_{[1]} \cong T_\Omega(h')|_{[1]}$ which induces by homotopy right Kan extension along i_0 a further isomorphism $T_\sigma(f')|_K \cong T_\Omega(h')|_K$. Now combining the adjunction $(i_{1!}, i_1{}^*)$ together with the canonical isomorphism $i_{1!}(T_\sigma(f')|_K) \cong T_\sigma(f')$ we obtain a morphism:

$$T_\sigma(f') \xrightarrow{\cong} i_{1!}(T_\sigma(f')|_K) \xrightarrow{\cong} i_{1!}(T_\sigma(h')|_K) \xrightarrow{\epsilon} T_\Omega(h')$$

This morphism evaluated at $(2,1)$ can be identified with $\epsilon\colon \Sigma\Omega Z' \to Z'$, the adjunction counit. Thus, if we define $\gamma\colon Z \to Y'$ to be the morphism $T_\sigma(f) \to T_\sigma(f') \to T_\Omega(h')$ evaluated at $(1,1)$ then we can conclude the proof. $\square$

These pretriangulations are canonical in the following sense. A *right exact* morphism between right triangulated categories is a pair consisting of an additive functor $F\colon \mathcal{A} \to \mathcal{A}'$ and a natural isomorphism $F \circ \Sigma \cong \Sigma \circ F$ which together send distinguished right triangles to distinguished right triangles. There is the obvious dual notion of a *left exact* morphism of left triangulated categories. Moreover, an *exact morphism between pretriangulated categories* is an additive functor which is endowed both with a right exact and a left exact structure. Finally, a *morphism of pretriangulated categories* $\mathcal{A}$ and $\mathcal{A}'$ is an adjunction $(L, R)\colon \mathcal{A} \rightleftarrows \mathcal{A}'$ such that the left adjoint is right exact and the right adjoint is left exact in the above sense. In the context of pretriangulated categories the exactness assumptions on L and R are *not* formal consequences of the adjointness since some choices where made earlier in the construction of the pretriangulations. However, for derivators the corresponding statement is true.

Proposition 5.7 *Let $\mathbb{D}$ and $\mathbb{D}'$ be strong, additive derivators and let $F\colon \mathbb{D} \to \mathbb{D}'$ be a morphism which preserves homotopy colimits. Then $F_J\colon \mathbb{D}(J) \to \mathbb{D}'(J)$ can be canonically turned into a right exact functor with respect to the canonical right triangulated structures on $\mathbb{D}(J)$ and $\mathbb{D}(J)'$.*

Corollary 5.8 *Let* $(L, R)\colon \mathbb{D} \rightleftarrows \mathbb{D}'$ *be an adjunction between strong, additive derivators. Then we obtain canonically morphisms* $(L_J, R_J)\colon \mathbb{D}(J) \rightleftarrows \mathbb{D}'(J)$ *of pretriangulated categories. Moreover, given a functor* $u\colon J \to K$ *between small categories, we obtain morphisms of pretriangulated categories* $(u_!, u^*)\colon \mathbb{D}(J) \rightleftarrows \mathbb{D}(K)$ *and* $(u^*, u_*)\colon \mathbb{D}(K) \rightleftarrows \mathbb{D}(J)$*. In particular,* $u^*\colon \mathbb{D}(K) \to \mathbb{D}(J)$ *is naturally an exact morphism of pretriangulated categories.*

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