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Centralizers of finite subgroups of the mapping class group

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In this paper, we study the action of finite subgroups of the mapping class group of a surface on the curve complex. We prove that if the diameter of the almost fixed point set of a finite subgroup H is big enough, then the centralizer of H is infinite.

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1 Introduction

Let S be an orientable surface of finite type with complexity at least 4, $\mathrm{Mod}(S)$ be the mapping class group of S, $C(S)$ be the curve complex of S and δ be the hyperbolicity constant of $C(S)$. (See Section 4 for the definitions of the above objects and references.) We prove the following theorem.

Main theorem *Let H be a finite subgroup of* $\mathrm{Mod}(S)$. *Let*

$$C_H = \{v \in C(S) : \mathrm{diam}(H \cdot v) \leq 6\delta\}.$$

There exists a constant D, depending only on the topological type of S, such that if $\mathrm{diam}(C_H) \geq D$, *then the centralizer of H in* $\mathrm{Mod}(S)$ *is infinite.*

We call points in C_H *almost fixed points* of H. Note that C_H is never empty. In fact, almost fixed points are very easy to find. Let $v \in C(S)$. Then any 1–quasicenter of the H–orbit of v is in C_H. (See Bridson and Haefliger [3, Chapter III.Γ, Lemma 3.3, p 460] for more detail.)

A motivation of the Main theorem is the following: Consider a sequence of homomorphisms $\{f_i\}$ from a finitely generated group G to $\mathrm{Mod}(S)$. This sequence of homomorphisms induce a sequence of actions of G on $C(S)$. Suppose that the translation lengths (with respect to some finite generating set of G) of these actions go to infinity. In this case, these actions of G on $C(S)$ converge to a nontrivial action of G on an $\mathbb{R}$–tree. The Main theorem provides some information about this action.

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Corollary 1.1 *Let T be the $\mathbb{R}$–tree obtained as above. Let K be the stabilizer in G of a nontrivial segment in T. Then there exists N, such that any finite subgroup H of $f_i(K)$ has infinite centralizer in $\mathrm{Mod}(S)$ for all $i \geq N$.*

The same phenomenon shows up when one considers the action of a hyperbolic group on its Cayley graph. We include the proof of the Main theorem for hyperbolic groups (Theorem 3.1) in this paper for the following reasons: First, even through experts in geometric group theory might know the proof for hyperbolic groups, as far as the author knows the proof is not in the literature. Second, since the two proofs are similar, while the mapping class group case requires many more tools (such as Masur and Minsky's theory of hierarchies) and is more technical, we think that the proof of the hyperbolic group case serves well as a warm-up.

The proofs of both Main theorems are based on a general fact proved in Section 2. Consider a "nice" finitely generated group G admitting a "nice" action on a infinite metric graph. Lemma 2.1 says if the cardinality of the set of almost fixed points (see Section 2 for definition) of a finite subgroup is big enough, then the centralizer of the finite subgroup is infinite.

In Section 3, we use the hyperbolicity of the Cayley graph of a hyperbolic group to show that having two almost fixed points far apart implies having a lot of points with small H–orbit. This is Lemma 3.2. Then we show that the action in this case is "nice" in the sense of Lemma 2.1 and the Main theorem for hyperbolic groups (Theorem 3.1) follows. In Section 4, we introduce the basic definitions we need to state the Main theorem and some tools we use in the proof of it. In Section 5, we prove the Main theorem for the mapping class group. The proof of the Main theorem for the mapping class group relies heavily on the theory of hierarchies. Readers who are not familiar with the theory of hierarchies should read Masur and Minsky [9]. In Section 6, we prove Corollary 1.1.

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2 The key lemma

Lemma 2.1 is a key fact we need in the proofs of the Main theorems, in both the hyperbolic case and the mapping class group case.

In order to state Lemma 2.1, we need to introduce some notation. Consider a finitely generated group G acting properly and cocompactly on an infinite locally finite metric graph K by isometries. Let H be a finite subgroup of G. Let a be a positive integer.

Suppose the cardinalities of finite subgroups of G are bounded above by some number C_0.

Let $K^{(0)}$ be the set of vertices of K and C_1 be the number of points in $K^{(0)}/G$.

For $p \in K$, let $B(p,a)$ denote the a–neighborhood of p in K and $\mathrm{card}_v(B(p,a))$ be the number of vertices in $B(p,a)$. Since K is locally finite, $\mathrm{card}_v(B(p,a))$ is finite. Since G acts on K cocompactly, there are only finitely many isometry types of $B(p,a)$. Hence $\{\mathrm{card}_v(B(p,a)) : p \in K^{(0)}\}$ is a finite set of finite numbers. Let C_2 be an upper bound for $\{\mathrm{card}_v(B(p,a)) : p \in K^{(0)}\}$.

Let $C_3 = \mathrm{Max}\{\mathrm{card}(\mathrm{stab}(p)) : p \in K^{(0)}\}$, where $\mathrm{stab}(p)$ is the stabilizer of p in G. Note that $\mathrm{card}(\mathrm{stab}(p))$ is finite for all $p \in K^{(0)}$ since the action of G on K is proper. On the other hand, since G acts on K cocompactly, $\mathrm{card}(\mathrm{stab}(p))$ only has finitely many different values. Therefore $\{\mathrm{card}(\mathrm{stab}(p)) : p \in K^{(0)}\}$ is a finite set of finite numbers. So C_3 exists.

Lemma 2.1 *Let $P_H = \{p \in K^{(0)} : \mathrm{diam}(H \cdot p) \leq a\}$. Then there exists a constant N, depending only on C_0, C_1, C_2, C_3, such that if $\mathrm{card}(P_H) \geq N$, the centralizer of H in G is infinite.*

Proof It suffices to take $N = ((C_0+1)(C_3)^{C_0}+1)C_1(C_2)^{C_0}$. Assume $\mathrm{card}(P_H) \geq N$. We show that in this case the centralizer of H is infinite.

By definition, C_1 is the number of G–orbits in $K^{(0)}$. By the pigeonhole principle, there are at least

$$r_1 = \frac{N}{C_1}$$

points of P_H in the same orbit. Choose a subset $P = \{p_1, \dots, p_{r_1}\}$ of P_H so that all elements of P are in the same G–orbit. Choose $g_i \in G$ so that $g_i \cdot p_1 = p_i$ for $2 \leq i \leq r_1$. Note that g_i^{-1} induces an isometry from $B(p_i,a)$ to $B(p_1,a)$.

Let $H = \{h_1, \dots, h_d\}$. First, we consider the action of h_1. For any $p_i \in P$, we have $h_1 \cdot p_i \in B(p_i,a)$ by the definition of P_H. Therefore, $g_i^{-1} \cdot h_1 \cdot p_i \in B(p_1,a)$. Since $\mathrm{card}_v(B(p_1,a)) \leq C_2$, by the pigeonhole principle, there exists $v_1 \in B(p_1,a)$ such that for at least r_1/C_2 many i, $g_i^{-1} \cdot h_1 \cdot p_i = v_1$. Let I_1 be the subset of $\{1, \dots, r_1\}$ such that for any $i \in I_1$, we have $g_i^{-1} \cdot h_1 \cdot p_i = v_1$, which is equivalent to $h_1 \cdot p_i = g_i \cdot v_1$.

Now consider h_2. As above, by the pigeonhole principle, there exists $v_2 \in B(p_1, a)$, and a subset I_2 of I_1 with $\mathrm{card}(I_2) \geq r_1/(C_2)^2$, such that $h_2 \cdot p_i = g_i \cdot v_2$ for all $i \in I_2$.

Repeating this process for all the elements of H, we have

$$h_t \cdot p_i = g_i \cdot v_t$$

for all $1 \leq t \leq d$ and all $i \in I_d$, where $I_d \subset I_{d-1} \subset \cdots \subset I_1$ and

$$r_2 = \mathrm{card}(I_d) \geq \frac{r_1}{(C_2)^d}.$$

Fix an element $b \in I_d$. For any $i \in I_d$, we have

$$h_1 \cdot g_i \cdot g_b^{-1} \cdot p_b = h_1 \cdot g_i \cdot p_1 = h_1 \cdot p_i = g_i \cdot v_1 = g_i \cdot g_b^{-1} \cdot h_1 \cdot p_b.$$

Therefore we have

$$h_1^{-1} \cdot g_b \cdot g_i^{-1} \cdot h_1 \cdot g_i \cdot g_b^{-1} \in \mathrm{stab}(p_b).$$

We know that $\mathrm{card}(\mathrm{stab}(p_b)) \leq C_3$. Now applying the pigeonhole principle again, we know that there exists a subset I_d^1 of I_d with $\mathrm{card}(I_d^1) \geq (r_2 - 1)/C_3$, such that for any $i, j \in I_d^1$,

$$h_1^{-1} \cdot g_b \cdot g_i^{-1} \cdot h_1 \cdot g_i \cdot g_b^{-1} = h_1^{-1} \cdot g_b \cdot g_j^{-1} \cdot h_1 \cdot g_j \cdot g_b^{-1},$$

which is equivalent to

$$g_j \cdot g_i^{-1} \cdot h_1 = h_1 \cdot g_j \cdot g_i^{-1}.$$

Repeating this process for all the elements of H, we get a subset I_d^d of I_d, with $\mathrm{card}(I_d^d) \geq (r_2 - 1)/(C_3)^d$, such that for any $i, j \in I_d^d$, any $1 \leq t \leq d$,

$$g_j \cdot g_i^{-1} \cdot h_t = h_t \cdot g_j \cdot g_i^{-1}.$$

Fix $c \in I_d^d$. Then for all $i \in I_d^d$, all $h_t \in H$, we have

$$g_c \cdot g_i^{-1} \cdot h_t = h_t \cdot g_c \cdot g_i^{-1}.$$

Hence $g_c \cdot g_i^{-1}$ centralizes H for all $i \in I_d^d$. Therefore, there are at least $\mathrm{card}(I_d^d)$ elements in the centralizer of H. But since $N = ((C_0 + 1)(C_3)^{C_0} + 1)C_1(C_2)^{C_0}$, we have

$$r_1 = \frac{N}{C_1} = ((C_0 + 1)(C_3)^{C_0} + 1)(C_2)^{C_0}.$$

Therefore, since $d \leq C_0$, we have

$$r_2 \geq \frac{r_1}{(C_2)^d} \geq (C_0 + 1)(C_3)^{C_0} + 1.$$

So, again using the fact that $d \le C_0$, we have

$$\mathrm{card}(I_d^d) \ge \frac{r_2 - 1}{(C_3)^d} \ge C_0 + 1.$$

So there are at least $C_0 + 1$ elements in the centralizer of H, but any finite subgroup of G has cardinality at most C_0, so the centralizer of H must be infinite. □

3 Main theorem and proof: the hyperbolic group case

We use the convention that a δ*–hyperbolic space* is a geodesic metric space in which all geodesics triangles are δ–thin. (See [3, Chapter III.H, Definition 1.16, p 408] for more detail.)

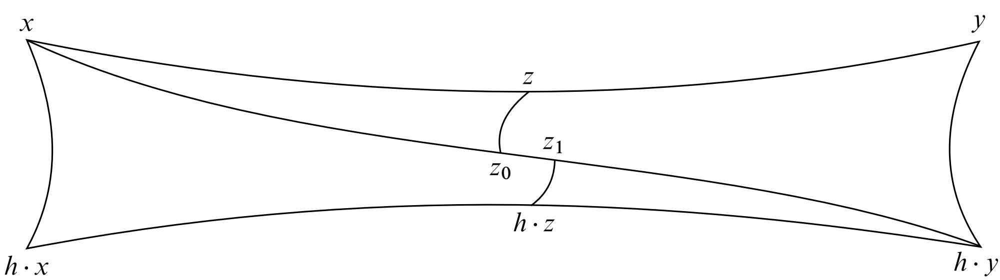
Theorem 3.1 *Let G be a hyperbolic group with $\{g_1, \dots, g_n\}$ as a generating set. Let K_G be the Cayley graph of G with respect to the given generating set. Let δ be the hyperbolicity constant for K_G. Let H be a finite subgroup of G. Let*

$$X_H = \{x \in K_G \,:\, \mathrm{diam}(H \cdot x) \le 6\delta\}.$$

There exists a constant D, depending only on δ and n, such that if $\mathrm{diam}(X_H) \ge D$*, then the centralizer of H in G is infinite.*

We call $x \in X_H$ *almost fixed points* of H.

Lemma 3.2 *Let $x, y \in X_H$. Suppose $d(x, y) \ge 20\delta$. Let $[x, y]$ be a geodesic in K_G connecting x and y. Then for any vertex $z \in [x, y]$ such that $d(x, z) \ge 6\delta + 1$ and $d(z, y) \ge 6\delta + 1$, we have* $\mathrm{diam}(H \cdot z) \le 8\delta$.



Proof It suffices to prove that $d(h \cdot z, z) \le 8\delta$ for all $h \in H$.

Consider the geodesic triangle with edges

$$[x, y], \quad [x, h \cdot y], \quad [y, h \cdot y].$$

K_G is δ–hyperbolic, so the triangle satisfies the thin triangle condition. By the definition of z, we have $d(z,y) \geq 6\delta + 1$. Since $y \in X_H$, we have $d(y, h\cdot y) \leq 6\delta$ by the definition of X_H. Therefore, there is a point $z_0 \in [x, h\cdot y]$ such that $d(z, z_0) \leq \delta$ and $d(x, z_0) = d(x, z)$.

Now consider the triangle with edges

$$[x, h\cdot x], \quad [x, h\cdot y], \quad [h\cdot x, h\cdot y] = h\cdot [x,y].$$

As above, since $d(h\cdot z, h\cdot x) = d(x,z) \geq 6\delta + 1$ and $d(x, h\cdot x) \leq 6\delta$, there is a point $z_1 \in [x, h\cdot y]$ such that $d(h\cdot z, z_1) \leq \delta$ and $d(h\cdot y, z_1) = d(h\cdot y, h\cdot z)$. So we have

$$\begin{aligned} d(z_0, z_1) &= |d(x, z_0) + d(h\cdot y, z_1) - d(x, h\cdot y)| \\ &= |d(x,z) + d(h\cdot y, h\cdot z) - d(x, h\cdot y)| \\ &= |d(h\cdot x, h\cdot z) + d(h\cdot y, h\cdot z) - d(x, h\cdot y)| \\ &= |d(h\cdot x, h\cdot y) - d(x, h\cdot y)| \leq 6\delta. \end{aligned}$$

Now we know: $d(h\cdot z, z) \leq d(z, z_0) + d(h\cdot z, z_1) + d(z_0, z_1) \leq \delta + \delta + 6\delta = 8\delta$. □

Applying Lemma 2.1 to the action of G on K_G, we get the following lemma.

Lemma 3.3 *Let H and G be as in Theorem 3.1. Let*

$$P_H = \{x \in K_G \,:\, \operatorname{diam}(H\cdot x) \leq 8\delta\}.$$

There exists a constant N, depending only on δ and n, such that if $\operatorname{card}(P_H) \geq N$, then the centralizer of H in G is infinite.

Proof In order to apply Lemma 2.1, it suffices to show that in the current situation, C_0, C_1, C_2, C_3 are finite and they depend only on δ and n.

By [3, Chapter III.Γ, Theorem 3.2, p 459], there exists an upper bound, depending only on δ and n, for the cardinality of finite subgroups of G. So C_0 is finite and depends only on δ and n. We have $C_1 = 1$ since K_G/G has only one vertex. Also C_2 is finite and depends only on δ and n by the definition of Cayley graph. Finally, $C_3 = 1$ since the action is free. □

Proof of Theorem 3.1 Let $D = N + 12\delta + 4$, where N is the constant given by the previous lemma. Then D depends only on δ and n. Let $x, y \in X_H$ such that $d(x,y) \geq D$. Let $[x,y]$ be a geodesic connecting x, y. Let

$$B = \{z \in [x,y] \,:\, d(z,x) \geq 6\delta + 1, d(z,y) \geq 6\delta + 1\}.$$

Then $\operatorname{card}(B) \geq N$ and $B \subset P_H$, where P_H is as in the statement of Lemma 3.3. So $\operatorname{card}(P_H) \geq N$. Therefore, by Lemma 3.3, the centralizer of H in G is infinite. □

4 Mod(S): Background

Let $S = S_{\gamma,p}$ be an orientable surface of finite type, with genus γ and p punctures. The *complexity* of S is measured by $\xi(S) = 3\gamma(S) + p(S)$. In this paper, we only consider surfaces with $\xi \geq 4$. The only exception is the annulus, which only appears as a subsurface of S.

The *mapping class group* of S, denoted by $\mathrm{Mod}(S)$, is the group of orientation-preserving homeomorphisms of S modulo isotopy.

A *curve* on S will always mean the isotopy class of a simple closed curve that is not null-homotopic or homotopic into a puncture.

For surface S with $\xi \geq 5$, the *curve graph* $C(S)$ consists of a vertex for every curve, with edges joining pairs of distinct curves that have disjoint representatives on S. The curve graph is the 1–skeleton of the curve complex introduced by Harvey in [7], which is the flag complex associated to the curve graph.

When $\xi = 4$, the surface S is either a once-punctured torus $S_{1,1}$ or a four-times-punctured sphere $S_{0,4}$. We have an alternate definition for the curve graph $C(S)$: Vertices are still curves. Edges are given by pairs of distinct curves that have representatives that intersect once (for $S_{1,1}$) or twice (for $S_{0,4}$).

By assigning length 1 to each edge we make $C(S)$ into a metric graph. We use d_S to denote this metric. Masur and Minsky [8, Theorem 1.1] prove the following.

Theorem 4.1 *$C(S)$ is an δ–hyperbolic metric space, where δ depends on S. Except when S is a sphere with 3 or fewer punctures, $C(S)$ has infinite diameter.*

When Y is an annulus with incompressible boundary in S, which is not homotopic into a puncture, $C(Y)$ is also defined. (See Masur and Minsky [9, Section 2.4] for the definition.)

Since elements in $\mathrm{Mod}(S)$ preserve disjointness of curves, $\mathrm{Mod}(S)$ acts on $C(S)$ by isometries. This action is cocompact since there are only finitely many curves on S up to homeomorphisms, but it is far from proper.

A *domain* Y in S will always mean an isotopy class of an incompressible, nonperipheral, connected open subsurface. Note that $\mathrm{Mod}(S)$ acts on the set of all domains of S.

The *marking graph* $\mathcal{M}(S)$ is a locally finite, connected graph whose vertices are complete markings on S and whose edges are elementary moves. A complete marking is a system of closed curves consisting of a base, which is a maximal simplex in the flag complex of $C(S)$, together with a choice of transversal curve for each element of the base, satisfying certain minimal intersection properties. (See [9, Section 2.5] for the exact definitions.) We make $\mathcal{M}(S)$ into a metric space by assigning length 1 to each edge. We use $d_{\mathcal{M}}$ to denote the metric on $\mathcal{M}(S)$. The marking graph $\mathcal{M}(S)$ admits an proper and cocompact action by $\mathrm{Mod}(S)$ by isometries.

Convention 4.2 For the rest of the paper, by an element $\nu \in C(S)$ or $\mu \in \mathcal{M}(S)$ we always mean a vertex of $C(S)$ or $\mathcal{M}(S)$ and similarly for a subset of $C(S)$ or $\mathcal{M}(S)$.

In [9], tight geodesics are defined to give some kind of local finiteness to deal with the fact that the curve graph is locally infinite. (See [9, Definition 4.2] for the definition.) A *hierarchy of tight geodesics* between any $\mu, \mu' \in \mathcal{M}(S)$ is a particular set of tight geodesics k, each in $C(W)$ for a subsurface $W \subset S$. The hierarchy is required to contain a tight geodesic in $C(S)$ between μ and μ', which is called the main geodesic of the hierarchy. μ and μ' are called the initial and terminal marking of the hierarchy. The subsurface $W \subset S$ is known as the domain of k. (See [9, Definition 4.4] for the exact definition of hierarchy.)

Let Y be a proper domain of S with $\xi \geq 4$ or an annular domain. Let $\pi_Y\colon C(S) \to \mathcal{P}(C(Y))$ be the subsurface projection defined in [9, Sections 2.3 and 2.4], where $\mathcal{P}(X)$ denote the set of finite subsets of X. Define $d_Y(A, B) \equiv d_Y(\pi_Y(A), \pi_Y(B))$ for sets or elements A and B in $C(S)$.

The geodesics in a hierarchy $\mathcal{H}$ behave well with subsurface projections of the initial and terminal markings of $\mathcal{H}$ in the following sense.

Lemma 4.3 [9, Lemma 6.2] *There exists constants M_1 and M_2 depending only on S such that the following is true: Let $I(\mathcal{H})$ and $T(\mathcal{H})$ be the initial and terminal marking of a hierarchy $\mathcal{H}$, respectively. If Y is any domain in S and $d_Y(I(\mathcal{H}), T(\mathcal{H})) \geq M_2$, then Y is the domain of a geodesic h in $\mathcal{H}$. Conversely if $h \in \mathcal{H}$ and Y is the domain of h, then $||h| - d_Y(I(\mathcal{H}), T(\mathcal{H}))| \leq 2M_1$.*

Hierarchies give rise to quasigeodesic paths between their initial and terminal markings, which are called hierarchy paths (see [9, Section 5]). Through these hierarchy paths hierarchies connect the geometry of $\mathcal{M}(S)$ with the geometry of $C(S)$ and $C(Y)$ for $Y \subset S$. The following lemma is one of the important connections we need.

Lemma 4.4 *Let $\mathcal{H}$ be a hierarchy. Let c be any positive number. Suppose that the lengths of all the geodesics in $\mathcal{H}$ are less than c. Then the distance between the initial marking and the terminal marking of $\mathcal{H}$ in $\mathcal{M}(S)$ is less than d, where d is a number depending only on c and the topological type of S.*

Proof By the above lemma, $d_Y(I(\mathcal{H}), T(\mathcal{H})) \le c + 2M_1$ for all domain Y of S. Now apply [9, Theorem 6.12] with $M = c + 2M_1 + 1$. □

The fellow traveler property of geodesics (Lemma 3.2) is crucial to proof of the Main theorem (for hyperbolic groups). In general hierarchy paths don't have this property. But when the main geodesics of two hierarchies fellow travel, [9, Lemma 6.7] gives us some control over their hierarchy paths.

The results about hierarchies in [9, Section 6] allow us to make some arguments from δ–hyperbolic geometry to work for $\mathcal{M}(S)$. This is the approach we are taking.

5 Proof of the Main theorem

In this section we prove the Main theorem for $\mathrm{Mod}(S)$. First, recall its statement.

Main theorem *Let H be a finite subgroup of* $\mathrm{Mod}(S)$. *Let*

$$C_H = \{\nu \in C(S) : \mathrm{diam}(H \cdot \nu) \le 6\delta\}.$$

There exists a constant D, depending only on the topological type of S, such that if $\mathrm{diam}(C_H) \ge D$, *then the centralizer of H in* $\mathrm{Mod}(S)$ *is infinite.*

We prove several lemmas before we prove the Main theorem.

Apply Lemma 2.1 to the action of $\mathrm{Mod}(S)$ on $\mathcal{M}(S)$. We get the following lemma.

Lemma 5.1 *Let a be any positive integer. Let H be a finite subgroup of* $\mathrm{Mod}(S)$. *Let* $P_H^a = \{\mu \in \mathcal{M}(S) : \mathrm{diam}(H \cdot \mu) \le a\}$. *There exists a constant N, depending only on S and a, such that if* $\mathrm{card}(P_H^a) \ge N$, *the centralizer of H is infinite.*

Proof In order to apply Lemma 2.1, it suffices to show that in the current situation, C_0, C_1, C_2, C_3 are finite and they depend only on S and a.

By Nielsen Realization Theorem (see [13] for a proof in the case of punctured surfaces) every finite subgroup of $\mathrm{Mod}(S)$ can be realized as a subgroup of the isometry group of the surface with some hyperbolic structure. By Hurwitz's Automorphism Theorem, the size of the isometry group of a punctured hyperbolic surface is bounded above. (The bound is $84(g-1)$ when $g \ge 2$. When $g \le 1$, a similar argument as in [5, Section 7.2] gives an upper bound for the size of the isometry group.) Hence the orders of finite subgroups of $\mathrm{Mod}(S)$ are bounded above by a constant which depends only on the topological type of S. So C_0 is finite and depends only on S. By the construction of $\mathcal{M}(S)$, both C_1 and C_3 are finite and depend only on S. For the same reason, C_2 is finite and depends only on S and a. □

Lemma 5.2 *Suppose there exists a domain Y of S such that either $h(Y)=Y$ or $h(Y)$ and Y are disjoint for any $h\in H$. Then the centralizer of H is infinite.*

Proof Let A be the set of boundary components of Y and all the H–translates of Y. Then A is a set of pairwise disjoint curves. Let $T=\prod_{[\alpha]\in A} D_{[\alpha]}$, where $D_{[\alpha]}$ is the right Dehn twist around α.

Note that T has infinite order. We will prove the lemma by showing that T is in the centralizer of H. The idea is as follow: For any $h\in H$, we pick a representative $h_S\in \mathrm{Homeo}^+(S)$ of h and construct $T_h\in \mathrm{Homeo}^+(S)$ such that $h_S\cdot T_h=T_h\cdot h_S$. These elements then also commute in $\mathrm{Mod}(S)$. Then we note that for all $h\in H$, $T_h\simeq T$. Therefore $T=T_h$ in $\mathrm{Mod}(S)$.

For $h\in H$, h permutes the elements of A. Let

$$\big([\alpha_1^1],[\alpha_1^2],\dots,[\alpha_1^{j_1}]\big),\dots,\big([\alpha_n^1],[\alpha_n^2],\dots,[\alpha_n^{j_n}]\big)$$

be the decomposition of A into h–cycles. Then we have $h\cdot[\alpha_i^j]=[\alpha_i^{j+1}]$ and $h\cdot[\alpha_i^{j_i}]=[\alpha_i^1]$ for $1\le i\le n$.

For each $[\alpha]\in A$, pick a simple representative α such that representatives of different elements of A are disjoint. Pick a neighborhood $N(\alpha)$ for each α such that neighborhoods of different representatives are disjoint. It is easy to see that we can pick a representative $h_S\in\mathrm{Homeo}^+(S)$ of h such that the following are true for all $1\le i\le n$:

(1) h_S takes $N(\alpha_i^j)$ to $N(\alpha_i^{j+1})$ by homeomorphism for $j\le j_i-1$.

(2) h_S takes $N(\alpha_i^{j_i})$ to $N(\alpha_i^1)$ by homeomorphism.

(3) $(h_S)^{j_i}$ is the identity map on $N(\alpha_i^1)$ if $(h)^{j_i}$ preserves the two sides of $[\alpha_i^1]$.

(4) $(h_S)^{j_i}$ is a "π–rotation" on $N(\alpha_i^1)$ if $(h)^{j_i}$ flips the two sides of $[\alpha_i^1]$. Here the "π–rotation" map is an order 2 orientation-preserving map which flips the two boundary components of $N(\alpha_i^1)$.

Next, we define T_h. Let T_h be the identity map on $S-\bigcup_{[\alpha]\in A}N(\alpha)$. For all $1\le i\le n$, let T_h be a right Dehn twist $T_{\alpha_i^1}$ on $N(\alpha_i^1)$. For $2\le j\le j_i$, let T_h be $T_{\alpha_i^j}=(h_S)^{j-1}\cdot T_{\alpha_i^1}\cdot (h_S)^{1-j}$ on $N(\alpha_i^j)$.

On $S-\bigcup_{[\alpha]\in A}N(\alpha)$, T_h and h_S commute in $\mathrm{Mod}(S)$ since they commute in $\mathrm{Homeo}^+(S)$ as T_h is the identity.

Suppose $1 \le j \le j_i - 1$. On $N(\alpha_i^j)$ we have

$$h_S \cdot T_h = h_S \cdot (h_S)^{j-1} \cdot T_{\alpha_i^1} \cdot (h_S)^{1-j} = (h_S)^j \cdot T_{\alpha_i^1} \cdot (h_S)^{1-j},$$
$$T_h \cdot h_S = (h_S)^j \cdot T_{\alpha_i^1} \cdot (h_S)^{-j} \cdot h_S = (h_S)^j \cdot T_{\alpha_i^1} \cdot (h_S)^{1-j}.$$

So T_h and h_S also commute in $\mathrm{Homeo}^+(S)$, hence in $\mathrm{Mod}(S)$.

On $N(\alpha_i^{j_i})$, we have

$$h_S \cdot T_h = h_S \cdot (h_S)^{j_i-1} \cdot T_{\alpha_i^1} \cdot (h_S)^{1-j_i} = (h_S)^{j_i} \cdot T_{\alpha_i^1} \cdot (h_S)^{1-j_i},$$
$$T_h \cdot h_S = T_{\alpha_i^1} \cdot h_S.$$

If $(h_S)^{j_i}$ is the identity, then $(h_S)^{1-j_i} = h_S$. Again we see that T_h and h_S commute in $\mathrm{Homeo}^+(S)$, hence in $\mathrm{Mod}(S)$.

If $(h_S)^{j_i}$ is the "π–rotation" f, then $f \cdot (h_S)^{1-j_i} = h_S$. Therefore we have

$$h_S \cdot T_h = (h_S)^{j_i} \cdot T_{\alpha_i^1} \cdot (h_S)^{1-j_i} = f \cdot T_{\alpha_i^1} \cdot (h_S)^{1-j_i},$$
$$T_h \cdot h_S = T_{\alpha_i^1} \cdot h_S = T_{\alpha_i^1} \cdot f \cdot (h_S)^{1-j_i}.$$

One can easily check that $f \cdot T_{\alpha_i^1} = T_{\alpha_i^1} \cdot f$ in $\mathrm{Mod}(S)$. So T_h and h_S commute in $\mathrm{Mod}(S)$.

Finally, we note that T_h projects to T in $\mathrm{Mod}(S)$ and the proof of the lemma is complete. □

Let ν_0, ν_1 be C_H. The idea of the following lemma is the same as Lemma 3.2.

Lemma 5.3 *Suppose $d(\nu_0, \nu_1) \ge 20\delta$. Let $[\nu_0, \nu_1]$ be a geodesic in $C(S)$ connecting ν_0 and ν_1. Then for any vertex $b \in [\nu_0, \nu_1]$ such that $d_S(\nu_0, b) \ge 6\delta + 1$ and $d_S(b, \nu_1) \ge 6\delta + 1$, we have $\mathrm{diam}_S(H \cdot b) \le 8\delta$.*

Let μ_0, $\mu_1 \in \mathcal{M}(S)$ such that $\nu_0 \in \mathrm{base}(\mu_0)$, $\nu_1 \in \mathrm{base}(\mu_1)$. Let $\mathcal{H} = [\mu_0, \mu_1]$ be a hierarchy [9, Definition 4.4] with initial marking μ_0, terminal marking μ_1 and with the main geodesic connecting ν_0, ν_1. For $h \in H$, Let $\mathcal{H}_h$ be the h translate of $\mathcal{H}$.

Define B as follows:

$$B = \{b \in [\nu_0, \nu_H] : d_S(\nu_i, b) \ge 14\delta + 5, i = 0, 1\}.$$

Here $[\nu_0, \nu_H]$ is the main geodesic of $\mathcal{H}$. For any $b \in B$, $h \in H$, let μ_b be a marking compatible with a slice [9, Section 5] of $\mathcal{H}$ at b. Then $h \cdot \mu_b$ is a marking compatible with a slice of $\mathcal{H}_h$ at $h \cdot b$. Let $\mathcal{H}_b^h = [\mu_b, h \cdot \mu_b]$ be a hierarchy connecting μ_b and $h \cdot \mu_b$.

Lemma 5.4 *$\mathcal{H}_b^h$ is (K, M')–pseudoparallel* [9, Definition 6.5] *to $\mathcal{H}$, where K and M' depend only on S.*

Proof By Lemma 5.3, the main geodesic $[v_0, v_H]$ of $\mathcal{H}$ and the main geodesic $h \cdot [v_0, v_H]$ of $\mathcal{H}_h$ are $(8\delta + 2, 2\delta + 1)$–parallel [9, Definition 6.4] at b and $h \cdot b$ for all $b \in B$ and $h \in H$. Now apply [9, Lemma 6.7]. $\square$

Let M be the constant in [9, Theorem 3.1].

Lemma 5.5 *Let $b \in B$, $h \in H$. Suppose Y is the domain of a geodesic of $\mathcal{H}_b^h$. Then $d_Y(\mu_0, h \cdot \mu_0) \leq M$ and $d_Y(\mu_1, h \cdot \mu_1) \leq M$.*

Proof Let $[v_b, h \cdot v_b]$ be the main geodesic in $\mathcal{H}_b^h$. By Lemma 5.3, we have $d_S(v_b, h \cdot v_b) \leq 8\delta + 2$. Since Y is the domain of a geodesic in $\mathcal{H}_b^h$, it must be forward subordinate (see [9, Section 4.1] for the definition) to $[v_b, h \cdot v_b]$ at some vertex v. Let l be any boundary component of Y. Then $d_S(l, v) = 1$. Since $v_0 \in C_H$, we have $d_S(v_0, h \cdot v_0) \leq 6\delta$. Let $[v_0, h \cdot v_0]$ be a geodesic connecting v_0 and $h \cdot v_0$. Let v_i be a point on $[v_0, h \cdot v_0]$. By the triangle inequality,

$$\begin{aligned} d_S(v, v_i) &\geq d_S(v_0, v_b) - d_S(v, v_b) - d_S(v_i, v_0) \\ &\geq d_S(v_0, v_b) - d_S(v_b, h \cdot v_b) - d_S(v_0, h \cdot v_0) \\ &\geq (14\delta + 5) - (8\delta + 2) - 6\delta = 3. \end{aligned}$$

Then $d_S(l, v_i) \geq d_S(v, v_i) - d_S(l, v) \geq 3 - 1 = 2$. Therefore v_i intersects l. As a result, v_i intersects Y. And this is true for all $v \in [v_0, h' \cdot v_0]$. By [9, Theorem 3.1], $d_Y(h \cdot v_0, v_0) \leq M$. The exact same argument shows $d_Y(\mu_1, h \cdot \mu_1) \leq M$. $\square$

Now we are ready to prove the Main theorem.

Proof of Main theorem Recall that M is the constant in [9, Theorem 3.1]. Let M_1, M_2 be the constants in Lemma 4.3. Let K and M' be the constants in Lemma 5.4. Let $e = 2M + 8M_1 + M_2 + 2K + M'$. Let d be the constant given by Lemma 4.4 with $c = e + 2M_1$. Let N be the constant given by Lemma 5.1 with $a = d$. Let $D = N + 12\delta + 10$. Note that D depends only on the topological type of S.

We will show that the centralizer of H is infinite provided that $d_S(v_0, v_1) \geq D$.

The proof will break into 2 cases: If the length of geodesics of the hierarchies $\mathcal{H}_b^h$ are bounded for all $b \in B$, $h \in H$, then the distance between μ_b and $h \cdot \mu_b$ in $\mathcal{M}(S)$ are bounded. In this case, we have enough almost fixed points in $\mathcal{M}(S)$ and we can apply Lemma 5.1 to conclude that the centralizer of H in $\mathrm{Mod}(S)$ is infinite. On the other hand, if there is a "long" hierarchy $\mathcal{H}_b^h$, we are able to use an argument in Jing Tao's thesis [12] to show that there exists a subsurface Y of S such that elements of H either preserve Y or take Y completely off itself. Then we use Lemma 5.2 to complete the proof.

Case 1 For any $b \in B$, $h \in H$ and any subsurface Y of S supporting a geodesic of $\mathcal{H}_b^h$, we have $d_Y(\mu_b, h \cdot \mu_b) \leq e$.

Claim 1 *In Case 1, $d_{\mathcal{M}}(\mu_b, h \cdot \mu_b) \leq d$ for all $b \in B$, $h \in H$, where d is one of the numbers we used to define D.*

Proof By Lemma 4.3, the geodesic in Y has length at most $e + 2M_1$. Now the claim follows from Lemma 4.4 and the definition of d. □

Note that Claim 1 says that for any $b \in B$, μ_b is in P_H^d. Since $d_S(\nu_0, \nu_1) \geq D$, we have $|P_H^d| \geq |B| \geq D - 12\delta - 8 \geq N$. By Lemma 5.1 and the definition of N, the centralizer of H is infinite and the proof is complete in Case 1.

Case 2 There exists $b_l \in B$, $h_l \in H$, and a subsurface Y of S which supports a geodesic of $\mathcal{H}_{b_l}^{h_l}$, such that $d_Y(\mu_{b_l}, h_l \cdot \mu_{b_l}) \geq e$.

Claim 2 *In Case 2, $d_Y(\mu_0, \mu_1) \geq 2M + 4M_1 + M_2$.*

Proof Since we are in Case 2 we have $d_Y(\mu_{b_l}, h_l \cdot \mu_{b_l}) \geq e \geq M_2$. So by Lemma 4.3, Y is the domain of a geodesic of $\mathcal{H}_{b_l}^{h_l}$ of length at least

$$e - 2M_1 = 2M + 6M_1 + M_2 + 2K + M'.$$

In particular, this geodesic has length bigger than M'. By Lemma 5.4, $\mathcal{H}_{b_l}^{h_l}$ is (K, M')–pseudoparallel to $\mathcal{H}$. So Y is also the domain of a geodesic of $\mathcal{H}$, whose length is at least $2M + 6M_1 + M_2 + 2K + M' - 2K = 2M + 6M_1 + M_2 + M'$. Now applying Lemma 4.3 again, we know that

$$d_Y(\mu_0, \mu_1) \geq 2M + 6M_1 + M_2 + M' - 2M_1 \geq 2M + 4M_1 + M_2$$

as we claim. □

We prove the following key claim for Case 2 using an argument in [12, Lemma 3.3.4].

Claim 3 *In Case 2, for any $h \in H$, either $h(Y) = Y$ or $h(Y)$ and Y are disjoint.*

Proof Let $h \in H$. Applying Claim 2 and Lemma 5.5, we have

$$\begin{aligned} d_{h^{-1}(Y)}(\mu_0, \mu_1) &= d_Y(h \cdot \mu_0, h \cdot \mu_1) \\ &\geq d_Y(\mu_0, \mu_1) - d_Y(\mu_0, h \cdot \mu_0) - d_Y(\mu_1, h \cdot \mu_1) \\ &\geq 2M + 4M_1 + M_2 - M - M = 4M_1 + M_2 \geq M_2. \end{aligned}$$

So by Lemma 4.3, $h^{-1}(Y)$ is also a domain in $\mathcal{H}$. Suppose $h^{-1}(Y) \neq Y$. Then since $h^{-1}(Y)$ and Y have the same complexity, they are either disjoint from each other or they interlock (ie intersect but do not contain each other).

Suppose $h^{-1}(Y)$ and Y are not disjoint. Then by [9, Lemma 4.18], $h^{-1}(Y)$ and Y are time-ordered [9, Definition 4.16].

First suppose $Y \prec_t h^{-1}(Y)$(Here $\prec_t$ is the notation for time order). As in the proof of [9, Lemma 6.11], there exist a slice in $\mathcal{H}$ so that its associated compatible marking ν satisfies

$$d_Y(\nu, \mu_1) \leq M_1 \quad \text{and} \quad d_{h^{-1}(Y)}(\nu, \mu_0) \leq M_1.$$

Then since $d_{h^{-1}(Y)}(\nu, \mu_0) = d_Y(h \cdot \mu_0, h \cdot \nu)$, we have

$$d_Y(\mu_0, h \cdot \nu) \leq d_Y(\mu_0, h \cdot \mu_0) + d_Y(h \cdot \mu_0, h \cdot \nu) \leq M + M_1.$$

By Claim 2, we have

$$\begin{aligned} d_Y(\mu_1, h \cdot \nu) &\geq d_Y(\mu_0, \mu_1) - d_Y(\mu_0, h \cdot \nu) \\ &\geq 2M + 4M_1 + M_2 - (M + M_1) \geq 2M_1. \end{aligned}$$

Therefore, by [4, Lemma 1], we have

$$d_{h^{-1}(Y)}(\mu_0, h \cdot \nu) \leq 2M_1.$$

Hence we get

$$\begin{aligned} d_Y(\mu_0, h^2 \cdot \nu) &\leq d_Y(\mu_0, h \cdot \mu_0) + d_Y(h \cdot \mu_0, h^2 \cdot \nu) \\ &\leq M + d_{h^{-1}(Y)}(\mu_0, h \cdot \nu) \leq M + 2M_1. \end{aligned}$$

Then by Claim 2, we have

$$\begin{aligned} d_Y(\mu_1, h^2 \cdot \nu) &\geq d_Y(\mu_0, \mu_1) - d_Y(\mu_0, h^2 \cdot \nu) \\ &\geq 2M + 4M_1 + M_2 - (M + 2M_1) \geq 2M_1. \end{aligned}$$

Again by [4, Lemma 1], we have

$$d_{h^{-1}(Y)}(\mu_0, h^2 \cdot \nu) \leq 2M_1.$$

Iterating this argument, we get

$$\begin{aligned} d_Y(\mu_0, h^i \cdot \nu) &\le d_Y(\mu_0, h \cdot \mu_0) + d_Y(h \cdot \mu_0, h^i \cdot \nu) \\ &\le M + d_{h^{-1}(Y)}(\mu_0, h^{i-1} \cdot \nu) \le M + 2M_1. \end{aligned}$$

Since this is true for all $i \ge 0$ and h has finite order, we have

$$d_Y(\mu_0, \nu) \le M + 2M_1.$$

Hence, we get

$$d_Y(\mu_0, \mu_1) \le d_Y(\mu_0, \nu) + d_Y(\nu, \mu_1) \le M + 2M_1 + M_1 \le M + 3M_1,$$

contradicting Claim 2.

In the same way, we can show that $h^{-1}(Y) \prec_t Y$ cannot happen either. So $h^{-1}(Y)$ and Y are not time-ordered and hence are disjoint. Therefore, $h(Y)$ and Y are disjoint provided that $h(Y) \ne Y$ as required. □

Now we apply Lemma 5.2 to conclude that the centralizer of H is infinite. Therefore the proof of Main theorem is complete. □

6 Application

In this section we prove Corollary 1.1.

Let G be a finitely generated group with a generating set $\{g_1, \dots, g_n\}$. Let $\{f_i\}$ be a sequence of homomorphisms from G to $\mathrm{Mod}(S)$. The f_i induce a sequence of actions ρ_i of G on $C(S)$, where

$$\rho_i(g)(\nu) = f_i(g) \cdot \nu.$$

Let

$$d_i = \inf_{\nu \in C(S)} \Big(\max_{1 \le t \le n} d_S(\nu, f_i(g_t) \cdot \nu) \Big).$$

Suppose d_i goes to infinity as i goes to infinity. Then ρ_i subconverges to a nontrivial action ρ of G on an $\mathbb{R}$–tree T in the sense of Bestvina–Paulin. Replace ρ_i by a convergent subsequence, which we still denote by ρ_i.

Remark 6.1 In Paulin's original construction for hyperbolic groups, d_i goes to infinity as long as f_i are nonconjugate. This is not true for $\mathrm{Mod}(S)$.

Recall the statement of Corollary 1.1.

Corollary 6.2 *Let T be the $\mathbb{R}$–tree obtain as above. Let K be the stabilizer in G of a nontrivial segment in T. There exists N, such that any finite subgroup H of $f_i(K)$ has infinite centralizer in* $\mathrm{Mod}(S)$ *for all $i \geq N$.*

Proof Let $[x,y]$ be the nontrivial segment in T stabilized by K. Let $l = d_T(x,y)$ and $\epsilon \leq \frac{1}{10}l$. By the construction of T, for i large enough there exists $x_i, y_i \in C(S)$ such that for all $h \in K$ we have

$$\left|\frac{1}{d_i}d_S(x_i,y_i) - d_T(x,y)\right| \leq \epsilon,$$
$$\left|\frac{1}{d_i}d_S(x_i,f_i(h)\cdot x_i) - d_T(x,\rho(h)x)\right| \leq \epsilon,$$
$$\left|\frac{1}{d_i}d_S(y_i,f_i(h)\cdot y_i) - d_T(y,\rho(h)y)\right| \leq \epsilon.$$

(See [2, Proposition 3.6] for more detail.) Since $l = d_T(x,y)$ and h fixes $[x,y]$, we have

$$d_S(x_i,y_i) \geq d_i(l-\epsilon),$$
$$d_S(x_i,f_i(h)\cdot x_i) \leq d_i\epsilon,$$
$$d_S(y_i,f_i(h)\cdot y_i) \leq d_i\epsilon.$$

Therefore the $f_i(K)$–orbit of x_i has bounded diameter. Let C_{x_i} be a 1–quasicenter (see [3, Chapter III.Γ, Lemma 3.3, p 460] for the definition) of the $f_i(K)$–orbit of x_i. Then all the $f_i(K)$–translates C_{x_i} are also 1–quasicenter of the $f_i(K)$–orbit of x_i. Therefore by [3, Chapter III.Γ, Lemma 3.3, p 460],

$$d_S(C_{x_i}, f_i(h)\cdot C_{x_i}) \leq 4\delta + 2 \leq 6\delta.$$

Similarly, we have

$$d_S(C_{y_i}, f_i(h)\cdot C_{y_i}) \leq 4\delta + 2 \leq 6\delta.$$

So x_i, y_i are in $C_{f_i(K)}$, which is defined in the Main theorem.

By the definition of quasicenter, we have

$$d_S(C_{x_i}, x_i) \leq \mathrm{diam}(f_i(K)\cdot x_i) \leq d_i\epsilon,$$
$$d_S(C_{y_i}, y_i) \leq \mathrm{diam}(f_i(K)\cdot y_i) \leq d_i\epsilon,$$

and so

$$d_S(C_{x_i}, C_{y_i}) \geq d_i(l-\epsilon) - d_i\epsilon - d_i\epsilon \geq d_i(l-3\epsilon).$$

Therefore when i is large enough

$$d_S(C_{x_i}, C_{y_i}) \geq D,$$

where D is the constant in the Main theorem. Now applying the Main theorem to a finite subgroup H of $f_i(K)$, we know that H has infinite centralizer in $\mathrm{Mod}(S)$. $\square$

Suppose G splits over a finite segment stabilizer C. ($G = A *_C B$ if G splits as an amalgamated free product.) Then Corollary 6.2 allows one to construct homomorphisms from G to $\mathrm{Mod}(S)$ of the form $\varphi_i(a) = f_i(a)$ for $a \in A$ and $\varphi_i(b) = z^{-1} f_i(b) z$ for $b \in B$, where z is an element of $\mathrm{Mod}(S)$ which centralizes $f_i(C)$. We think that this type of homomorphisms might be useful when one tries to use the "shortening argument" (see Alibegović [1], Groves [6], Rips and Sela [10] and Sela [11]) to study $\mathrm{Hom}(G, \mathrm{Mod}(S))$.

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