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The axioms for n–angulated categories

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We discuss the axioms for an n–angulated category, recently introduced by Geiss, Keller and Oppermann in [1]. In particular, we introduce a higher "octahedral axiom", and show that it is equivalent to the mapping cone axiom for an n–angulated category. For a triangulated category, the mapping cone axiom, our octahedral axiom and the classical octahedral axiom are all equivalent.

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1 Introduction

Triangulated categories were introduced independently in algebraic geometry by Verdier [7; 8], based on ideas of Grothendieck, and in algebraic topology by Puppe [6]. These constructions have since played a crucial role in representation theory, algebraic geometry, commutative algebra, algebraic topology and other areas of mathematics (and even theoretical physics). Recently, Geiss, Keller and Oppermann introduced in [1] a new type of categories, called n–angulated categories, which generalize triangulated categories: the classical triangulated categories are the special case $n = 3$. These categories appear for instance when considering certain $(n-2)$–cluster tilting subcategories of triangulated categories. Conversely, certain n–angulated Calabi–Yau categories yield triangulated Calabi–Yau categories of higher Calabi–Yau dimension.

The four axioms for n–angulated categories are generalizations of the axioms for triangulated categories. In this paper, we discuss these axioms, inspired by works of Neeman [4; 5]. First, we show that the first two of the original axioms can be replaced by two alternative axioms. One of these alternative axioms requires that the collection of n–angles be closed under so-called weak isomorphisms, but not under direct sums and summands. The other axiom requires that the collection of n–angles be closed only under left rotations, but not right rotations. Second, we discuss the axioms that enable us to complete certain diagrams to morphisms of n–angles. The last of these axioms says that we can complete diagrams to morphisms of n–angles in such a way that the mapping cone is itself an n–angle. For triangulated categories (that is, when $n = 3$), this axiom is equivalent to the octahedral axiom, which was one of Verdier's

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original axioms. We show that this generalizes to n–angulated categories. Namely, we introduce a higher "octahedral axiom" for n–angulated categories, and show that this is equivalent to the mapping cone axiom. For $n = 3$, that is, for triangulated categories, our new axiom is almost the same as the classical octahedral axiom. In fact, it is apparently a bit weaker, but we show that they are equivalent. Therefore, for a triangulated category, the mapping cone axiom, our octahedral axiom and the classical octahedral axiom are all equivalent.

This paper is organized as follows. In Section 2, we recall the definition of n–angulated categories from [1], and in Section 3, we discuss the first two axioms. Finally, in Section 4, we introduce the higher octahedral axiom and prove our main theorem.

2 The axioms for n–angulated categories

Throughout Sections 2–4, we fix an additive category $\mathcal{C}$ with an automorphism $\Sigma\colon \mathcal{C} \to \mathcal{C}$, and an integer n greater than or equal to three. In this section, we recall the set of axioms for n–angulated categories as described in [1].

A sequence of objects and morphisms in $\mathcal{C}$ of the form

$$A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} A_n \xrightarrow{\alpha_n} \Sigma A_1$$

is called an n–Σ–*sequence*; we shall frequently denote such sequences by $A_\bullet$, $B_\bullet$ etc. The n–Σ–sequence $A_\bullet$ is *exact* if the induced sequence

$$\cdots \to \mathrm{Hom}_{\mathcal{C}}(B, A_1) \xrightarrow{(\alpha_1)_*} \mathrm{Hom}_{\mathcal{C}}(B, A_2) \xrightarrow{(\alpha_2)_*} \cdots$$
$$\cdots \xrightarrow{(\alpha_{n-1})_*} \mathrm{Hom}_{\mathcal{C}}(B, A_n) \xrightarrow{(\alpha_n)_*} \mathrm{Hom}_{\mathcal{C}}(B, \Sigma A_1) \to \cdots$$

of abelian groups is exact for every object $B \in \mathcal{C}$. The left and right *rotations* of $A_\bullet$ are the two n–Σ–sequences

$$A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \cdots \xrightarrow{\alpha_n} \Sigma A_1 \xrightarrow{(-1)^n \Sigma\alpha_1} \Sigma A_2,$$
$$\Sigma^{-1} A_n \xrightarrow{(-1)^n \Sigma^{-1}\alpha_n} A_1 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{n-2}} A_{n-1} \xrightarrow{\alpha_{n-1}} A_n,$$

respectively, and a *trivial* n–Σ–sequence is a sequence of the form

$$A \xrightarrow{1} A \to 0 \to \cdots \to 0 \to \Sigma A$$

or any of its rotations.

A *morphism* $A_\bullet \xrightarrow{\varphi} B_\bullet$ of n–Σ–sequences is a sequence $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_n)$ of morphisms in $\mathcal{C}$ such that the diagram

$$\begin{array}{ccccccccccc}
A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & A_3 & \xrightarrow{\alpha_3} & \cdots & \xrightarrow{\alpha_{n-1}} & A_n & \xrightarrow{\alpha_n} & \Sigma A_1 \\
\downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_3} & & & & \downarrow{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \Sigma\varphi_1} \\
B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & B_3 & \xrightarrow{\beta_3} & \cdots & \xrightarrow{\beta_{n-1}} & B_n & \xrightarrow{\beta_n} & \Sigma B_1
\end{array}$$

commutes. It is an *isomorphism* if $\varphi_1, \varphi_2, \dots, \varphi_n$ are all isomorphisms in $\mathcal{C}$, and a *weak isomorphism* if φ_i and φ_{i+1} are isomorphisms for some $1 \le i \le n$ (with $\varphi_{n+1} := \Sigma\varphi_1$). Note that the composition of two weak isomorphisms need not be a weak isomorphism. Also, note that if two n–Σ–sequences $A_\bullet$ and $B_\bullet$ are weakly isomorphic through a weak isomorphism $A_\bullet \xrightarrow{\varphi} B_\bullet$, then there does not necessarily exist a weak isomorphism $B_\bullet \to A_\bullet$ in the opposite direction.

Let $\mathcal{N}$ be a collection of n–Σ–sequences in $\mathcal{C}$. Then the pair $(\mathcal{C}, \mathcal{N})$ is a *pre–n–angulated category* if $\mathcal{N}$ satisfies the following three axioms:

- (N1) (a) $\mathcal{N}$ is closed under direct sums, direct summands and isomorphisms of n–Σ–sequences;
  - (b) for all $A \in \mathcal{C}$, the trivial n–Σ–sequence
  $$A \xrightarrow{1} A \to 0 \to \cdots \to 0 \to \Sigma A$$
  belongs to $\mathcal{N}$;
  - (c) for each morphism $\alpha\colon A_1 \to A_2$ in $\mathcal{C}$, there exists an n–Σ–sequence in $\mathcal{N}$ whose first morphism is α;
- (N2) an n–Σ–sequence belongs to $\mathcal{N}$ if and only if its left rotation belongs to $\mathcal{N}$;
- (N3) each commutative diagram
  $$\begin{array}{ccccccccccc}
A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & A_3 & \xrightarrow{\alpha_3} & \cdots & \xrightarrow{\alpha_{n-1}} & A_n & \xrightarrow{\alpha_n} & \Sigma A_1 \\
\downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & \vdots{\scriptstyle \varphi_3} & & & & \vdots{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \Sigma\varphi_1} \\
B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & B_3 & \xrightarrow{\beta_3} & \cdots & \xrightarrow{\beta_{n-1}} & B_n & \xrightarrow{\beta_n} & \Sigma B_1
\end{array}$$
  with rows in $\mathcal{N}$ can be completed to a morphism of n–Σ–sequences.

In this case, the collection $\mathcal{N}$ is a *pre–n–angulation* of the category $\mathcal{C}$ (relative to the automorphism Σ), and the n–Σ–sequences in $\mathcal{N}$ are *n–angles*. If, in addition, the collection $\mathcal{N}$ satisfies the following axiom, then it is an *n–angulation* of $\mathcal{C}$, and the category is *n–angulated*:

(N4) in the situation of (N3), the morphisms $\varphi_3, \varphi_4, \dots, \varphi_n$ can be chosen such that the mapping cone

$$A_2 \oplus B_1 \xrightarrow{\begin{bmatrix} -\alpha_2 & 0 \\ \varphi_2 & \beta_1 \end{bmatrix}} A_3 \oplus B_2 \xrightarrow{\begin{bmatrix} -\alpha_3 & 0 \\ \varphi_3 & \beta_2 \end{bmatrix}} \cdots$$
$$\cdots \xrightarrow{\begin{bmatrix} -\alpha_n & 0 \\ \varphi_n & \beta_{n-1} \end{bmatrix}} \Sigma A_1 \oplus B_n \xrightarrow{\begin{bmatrix} -\Sigma\alpha_1 & 0 \\ \Sigma\varphi_1 & \beta_n \end{bmatrix}} \Sigma A_2 \oplus \Sigma B_1$$

belongs to $\mathcal{N}$.

Note that in [1], it was not explicitly assumed that $\mathcal{N}$ be closed under isomorphisms, but it follows implicitly from closure under direct sums. Since closure under isomorphisms is a crucial part of many of our proofs, we have included it as a part of axiom (a). Note also that by [1, Proposition 1.5], every n–angle in a pre–n–angulated category is exact. Consequently, the composition of two consecutive morphisms in an n–angle is zero.

3 Axioms (N1) and (N2)

In this section, we discuss the first two defining axioms (N1) and (N2) for pre–n–angulated categories. It turns out that we may replace these axioms by the following ones:

(N1*) (a) if $A_\bullet \xrightarrow{\varphi} B_\bullet$ is a weak isomorphism of exact n–Σ–sequences with $A_\bullet \in \mathcal{N}$, then $B_\bullet$ belongs to $\mathcal{N}$;

(b) for all $A \in \mathcal{C}$, the trivial n–Σ–sequence

$$A \xrightarrow{1} A \to 0 \to \cdots \to 0 \to \Sigma A$$

belongs to $\mathcal{N}$;

(c) for each morphism $\alpha\colon A_1 \to A_2$ in $\mathcal{C}$, there exists an n–Σ–sequence in $\mathcal{N}$ whose first morphism is α;

(N2*) the left rotation of every n–Σ–sequence in $\mathcal{N}$ also belongs to $\mathcal{N}$.

In axiom (N1*), we do not require that $\mathcal{N}$ be closed under direct sums and summands. However, we do require that $\mathcal{N}$ be closed under weak isomorphisms (in one direction), and this is stronger than requiring that $\mathcal{N}$ be closed under isomorphisms. In axiom (N2*), we only require that $\mathcal{N}$ be closed under left rotations. This is sometimes done when considering triangulated categories, cf Keller and Vossieck [3].

Because of the new axiom (a), the exact $n-\Sigma$–sequences play an important role in the proofs to come. We therefore need to determine which properties a collection $\mathcal{N}$ of $n-\Sigma$–sequences must satisfy in order for all its elements to be exact. We do this in the following result.

Lemma 3.1 *If $\mathcal{N}$ is a collection of $n-\Sigma$–sequences satisfying the axioms* (b), (N2*) *and* (N3)*, then all the elements in $\mathcal{N}$ are exact.*

Proof Let

$$A_\bullet\colon\quad A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} A_n \xrightarrow{\alpha_n} \Sigma A_1$$

be an $n-\Sigma$–sequence in $\mathcal{N}$, and pick an integer $1 \le j \le n$. In the diagram

$$\begin{tikzcd}
A_j \arrow[r, "1"] \arrow[d, equal] & A_j \arrow[r] \arrow[d, "\alpha_j"] & 0 \arrow[r] \arrow[d, dashed] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d, dashed] & \Sigma A_j \arrow[d, equal] \\
A_j \arrow[r, "\alpha_j"] & A_{j+1} \arrow[r, "\alpha_{j+1}"] & A_{j+2} \arrow[r, "\alpha_{j+2}"] & \cdots \arrow[r, "(-1)^n\Sigma\alpha_{j-2}"] & \Sigma A_{j-1} \arrow[r, "(-1)^n\Sigma\alpha_{j-1}"] & \Sigma A_j
\end{tikzcd}$$

the two rows both belong to $\mathcal{N}$: the top row by (b), and the bottom row by (repeated use of) (N2*). Here we have made the conventions $\alpha_{-1} = (-1)^n\Sigma\alpha_{n-1}, \alpha_0 = (-1)^n\Sigma\alpha_n, \alpha_{n+1} = (-1)^n\Sigma\alpha_1, \alpha_{n+2} = (-1)^n\Sigma\alpha_2$. By (N3), we can complete the diagram to a morphism of $n-\Sigma$–sequences, hence the compositions

$$\alpha_2 \circ \alpha_1, \alpha_3 \circ \alpha_2, \ldots, \alpha_n \circ \alpha_{n-1}, (\Sigma\alpha_1) \circ \alpha_n$$

are all zero.

For objects $X, Y \in \mathcal{C}$, denote the abelian group $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ by (X,Y). Since all the possible compositions of morphisms from $A_\bullet$ are zero, the doubly infinite sequence

$$\cdots \to (B, \Sigma^{i-1}A_n) \xrightarrow{(\Sigma^{i-1}\alpha_n)_*} (B, \Sigma^i A_1) \xrightarrow{(\Sigma^i\alpha_1)_*} \cdots$$
$$\cdots \xrightarrow{(\Sigma^i\alpha_{n-1})_*} (B, \Sigma^i A_n) \xrightarrow{(\Sigma^i\alpha_n)_*} (B, \Sigma^{i+1}A_1) \to \cdots$$

of abelian groups and maps is a complex for every object $B \in \mathcal{C}$. Now pick an integer $1 \le i \le n$, and let f be an element in $\mathrm{Ker}(\Sigma^i\alpha_j)_*$. Then f is a morphism in $\mathrm{Hom}_{\mathcal{C}}(B, \Sigma^i A_j)$ with $(\Sigma^i\alpha_j) \circ f = 0$. Applying the automorphism Σ^{-i}, we obtain $\alpha_j \circ (\Sigma^{-i} f) = 0$, where $\Sigma^{-i} f$ is a morphism in $\mathrm{Hom}_{\mathcal{C}}(\Sigma^{-i}B, A_j)$. Now consider the diagram

$$\begin{tikzcd}
\Sigma^{-i}B \arrow[r] \arrow[d, "\Sigma^{-i}f"] & 0 \arrow[r] \arrow[d] & \cdots \arrow[r] & \Sigma^{1-i}B \arrow[r, "(-1)^n"] \arrow[d, dashed, "g"] & \Sigma^{1-i}B \arrow[d, "\Sigma^{1-i}f"] \\
A_j \arrow[r, "\alpha_j"] & A_{j+1} \arrow[r, "\alpha_{j+1}"] & \cdots \arrow[r, "(-1)^n\Sigma\alpha_{j-2}"] & \Sigma A_{j-1} \arrow[r, "(-1)^n\Sigma\alpha_{j-1}"] & \Sigma A_j
\end{tikzcd}$$

in which the two rows belong to $\mathcal{N}$ by (b) and (repeated use of) (N2*). By (N3), we can complete this diagram to a morphism of n–Σ–sequences, and in particular we obtain a morphism $g \in \mathrm{Hom}_{\mathcal{C}}(\Sigma^{1-i}B, \Sigma A_{j-1})$ with

$$(\Sigma\alpha_{j-1}) \circ g = \Sigma^{1-i} f.$$

Applying the automorphism Σ^{i-1} gives

$$f = (\Sigma^i \alpha_{j-1}) \circ (\Sigma^{i-1} g),$$

hence $f \in \mathrm{Im}(\Sigma^i \alpha_{j-1})_*$. This shows that the complex is exact, and so $A_\bullet$ is an exact n–Σ–sequence. □

We may now prove that axiom (N1) can be replaced with axiom (N1*).

Theorem 3.2 *If $\mathcal{N}$ is a collection of n–Σ–sequences satisfying the axioms* (N2) *and* (N3)*, then the following are equivalent:*

(1) *$\mathcal{N}$ satisfies* (N1)*;*

(2) *$\mathcal{N}$ satisfies* (N1*).

Proof The implication (1) $\Rightarrow$ (2) is part of [1, Lemma 1.4], hence we must prove that (1) follows from (2), ie that $\mathcal{N}$ satisfies (a) whenever it satisfies (N1*). Suppose therefore that $\mathcal{N}$ satisfies (N1*).

Since the collection $\mathcal{N}$ satisfies the axioms (b), (N2) and (N3), the n–Σ–sequences in $\mathcal{N}$ are exact by Lemma 3.1. Now let $A_\bullet$ and $B_\bullet$ be isomorphic n–Σ–sequences, with $A_\bullet$ in $\mathcal{N}$. Then $A_\bullet$ is exact, and so $B_\bullet$ must also be exact since it is isomorphic to $A_\bullet$. Since $A_\bullet$ and $B_\bullet$ are trivially weakly isomorphic through an isomorphism $A_\bullet \to B_\bullet$, the n–Σ–sequence $B_\bullet$ also belongs to $\mathcal{N}$. This shows that $\mathcal{N}$ is closed under isomorphisms.

Next, we show that $\mathcal{N}$ is closed under direct sums. Given two n–Σ–sequences

$$\begin{aligned}
A_\bullet\colon\quad & A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} A_n \xrightarrow{\alpha_n} \Sigma A_1,\\
B_\bullet\colon\quad & B_1 \xrightarrow{\beta_1} B_2 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_{n-1}} B_n \xrightarrow{\beta_n} \Sigma B_1,
\end{aligned}$$

in $\mathcal{N}$, the direct sum $A_\bullet \oplus B_\bullet$ is exact, since each of the sequences is exact by the above. Now use (c) to complete the first morphism in $A_\bullet \oplus B_\bullet$ to an n–Σ–sequence

$$A_1 \oplus B_1 \xrightarrow{\begin{bmatrix}\alpha_1 & 0\\ 0 & \beta_1\end{bmatrix}} A_2 \oplus B_2 \xrightarrow{\gamma_2} C_3 \xrightarrow{\gamma_3} \cdots \xrightarrow{\gamma_{n-1}} C_n \xrightarrow{\gamma_n} \Sigma A_1 \oplus \Sigma B_1$$

in $\mathcal{N}$. By (N3), the two commutative diagrams

$$\begin{tikzcd}
A_1 \oplus B_1 \arrow[r, "\begin{bmatrix}\alpha_1 & 0\\ 0 & \beta_1\end{bmatrix}"] \arrow[d, "{[1\ \ 0]}"] & A_2 \oplus B_2 \arrow[r, "\gamma_2"] \arrow[d, "{[1\ \ 0]}"] & C_3 \arrow[r, "\gamma_3"] \arrow[d, dashed, "\varphi_3"] & \cdots \arrow[r, "\gamma_{n-1}"] & C_n \arrow[r, "\gamma_n"] \arrow[d, dashed, "\varphi_n"] & \Sigma A_1 \oplus \Sigma B_1 \arrow[d, "{[1\ \ 0]}"] \\
A_1 \arrow[r, "\alpha_1"] & A_2 \arrow[r, "\alpha_2"] & A_3 \arrow[r, "\alpha_3"] & \cdots \arrow[r, "\alpha_{n-1}"] & A_n \arrow[r, "\alpha_n"] & \Sigma A_1,
\end{tikzcd}$$

$$\begin{tikzcd}
A_1 \oplus B_1 \arrow[r, "\begin{bmatrix}\alpha_1 & 0\\ 0 & \beta_1\end{bmatrix}"] \arrow[d, "{[0\ \ 1]}"] & A_2 \oplus B_2 \arrow[r, "\gamma_2"] \arrow[d, "{[0\ \ 1]}"] & C_3 \arrow[r, "\gamma_3"] \arrow[d, dashed, "\psi_3"] & \cdots \arrow[r, "\gamma_{n-1}"] & C_n \arrow[r, "\gamma_n"] \arrow[d, dashed, "\psi_n"] & \Sigma A_1 \oplus \Sigma B_1 \arrow[d, "{[0\ \ 1]}"] \\
B_1 \arrow[r, "\beta_1"] & B_2 \arrow[r, "\beta_2"] & B_3 \arrow[r, "\beta_3"] & \cdots \arrow[r, "\beta_{n-1}"] & B_n \arrow[r, "\beta_n"] & \Sigma B_1,
\end{tikzcd}$$

can be completed to morphisms of n–Σ–sequences, since the sequences involved are all in $\mathcal{N}$. This gives a weak isomorphism

$$\begin{tikzcd}
A_1 \oplus B_1 \arrow[r, "\begin{bmatrix}\alpha_1 & 0\\ 0 & \beta_1\end{bmatrix}"] \arrow[d, equal] & A_2 \oplus B_2 \arrow[r, "\gamma_2"] \arrow[d, equal] & C_3 \arrow[r, "\gamma_3"] \arrow[d, "\begin{bmatrix}\varphi_3\\ \psi_3\end{bmatrix}"] & \cdots \\
A_1 \oplus B_1 \arrow[r, "\begin{bmatrix}\alpha_1 & 0\\ 0 & \beta_1\end{bmatrix}"] & A_2 \oplus B_2 \arrow[r, "\begin{bmatrix}\alpha_2 & 0\\ 0 & \beta_2\end{bmatrix}"] & A_3 \oplus B_3 \arrow[r, "\begin{bmatrix}\alpha_3 & 0\\ 0 & \beta_3\end{bmatrix}"] & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots \arrow[r, "\gamma_{n-1}"] & C_n \arrow[r, "\gamma_n"] \arrow[d, "\begin{bmatrix}\varphi_n\\ \psi_n\end{bmatrix}"] & \Sigma A_1 \oplus \Sigma B_1 \arrow[d, equal] \\
\cdots \arrow[r, "\begin{bmatrix}\alpha_{n-1} & 0\\ 0 & \beta_{n-1}\end{bmatrix}"] & A_n \oplus B_n \arrow[r, "\begin{bmatrix}\alpha_n & 0\\ 0 & \beta_n\end{bmatrix}"] & \Sigma A_1 \oplus \Sigma B_1
\end{tikzcd}$$

of n–Σ–sequences. The top sequence belongs to $\mathcal{N}$ and is therefore exact, whereas the bottom sequence $A_\bullet \oplus B_\bullet$ is also exact. From (a) we conclude that $A_\bullet \oplus B_\bullet$ belongs to $\mathcal{N}$.

Finally, we show that $\mathcal{N}$ is closed under direct summands. Suppose therefore that $A_\bullet$ and $B_\bullet$ are n–Σ–sequences as above, that $B_\bullet$ belongs to $\mathcal{N}$ (hence $B_\bullet$ is exact), and that $A_\bullet$ is a direct summand of $B_\bullet$. Then there exists a diagram

$$\begin{tikzcd}
A_1 \arrow[r, "\alpha_1"] \arrow[d, "\varphi_1"] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "\varphi_2"] & A_3 \arrow[r, "\alpha_3"] \arrow[d, "\varphi_3"] & \cdots \arrow[r, "\alpha_{n-1}"] & A_n \arrow[r, "\alpha_n"] \arrow[d, "\varphi_n"] & \Sigma A_1 \arrow[d, "\Sigma\varphi_1"] \\
B_1 \arrow[r, "\beta_1"] \arrow[d, "\psi_1"] & B_2 \arrow[r, "\beta_2"] \arrow[d, "\psi_2"] & B_3 \arrow[r, "\beta_3"] \arrow[d, "\psi_3"] & \cdots \arrow[r, "\beta_{n-1}"] & B_n \arrow[r, "\beta_n"] \arrow[d, "\psi_n"] & \Sigma B_1 \arrow[d, "\Sigma\psi_1"] \\
A_1 \arrow[r, "\alpha_1"] & A_2 \arrow[r, "\alpha_2"] & A_3 \arrow[r, "\alpha_3"] & \cdots \arrow[r, "\alpha_{n-1}"] & A_n \arrow[r, "\alpha_n"] & \Sigma A_1
\end{tikzcd}$$

of morphisms $A_\bullet \xrightarrow{\varphi} B_\bullet$ and $B_\bullet \xrightarrow{\psi} A_\bullet$ of n–Σ–sequences, with $\psi_i \circ \varphi_i = 1_{A_i}$ for all i. For every object Z in $\mathcal{C}$, the sequence $\mathrm{Hom}_{\mathcal{C}}(Z, A_\bullet)$ of abelian groups and maps is a direct summand of the exact sequence $\mathrm{Hom}_{\mathcal{C}}(Z, B_\bullet)$, and is therefore itself exact. Consequently, the n–Σ–sequence $A_\bullet$ is exact. Now use (c) to complete the first morphism in $A_\bullet$ to an n–Σ–sequence

$$D_\bullet\colon\quad A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\delta_2} D_3 \xrightarrow{\delta_3} \cdots \xrightarrow{\delta_{n-1}} D_n \xrightarrow{\delta_n} \Sigma A_1$$

in $\mathcal{N}$ (in particular, $D_\bullet$ is exact). Using this sequence, we can obtain a diagram

$$\begin{tikzcd}
A_1 \arrow[r, "\alpha_1"] \arrow[d, "\varphi_1"] & A_2 \arrow[r, "\delta_2"] \arrow[d, "\varphi_2"] & D_3 \arrow[r, "\delta_3"] \arrow[d, dashed, "\theta_3"] & \cdots \arrow[r, "\delta_{n-1}"] & D_n \arrow[r, "\delta_n"] \arrow[d, dashed, "\theta_n"] & \Sigma A_1 \arrow[d, "\Sigma\varphi_1"] \\
B_1 \arrow[r, "\beta_1"] \arrow[d, "\psi_1"] & B_2 \arrow[r, "\beta_2"] \arrow[d, "\psi_2"] & B_3 \arrow[r, "\beta_3"] \arrow[d, "\psi_3"] & \cdots \arrow[r, "\beta_{n-1}"] & B_n \arrow[r, "\beta_n"] \arrow[d, "\psi_n"] & \Sigma B_1 \arrow[d, "\Sigma\psi_1"] \\
A_1 \arrow[r, "\alpha_1"] & A_2 \arrow[r, "\alpha_2"] & A_3 \arrow[r, "\alpha_3"] & \cdots \arrow[r, "\alpha_{n-1}"] & A_n \arrow[r, "\alpha_n"] & \Sigma A_1
\end{tikzcd}$$

whose rows are $D_\bullet$, $B_\bullet$ and $A_\bullet$. The top half of this diagram is a morphism $\theta\colon D_\bullet \to B_\bullet$, which we obtain from (N3), whereas the lower half is the morphism $\psi\colon B_\bullet \to A_\bullet$. Moreover, the composition $\psi \circ \theta\colon D_\bullet \to A_\bullet$ is a weak isomorphism, since $\psi_1 \circ \varphi_1 = 1_{A_1}$ and $\psi_2 \circ \varphi_2 = 1_{A_2}$. Since both $D_\bullet$ and $A_\bullet$ are exact, and $D_\bullet \in \mathcal{N}$, the sequence $A_\bullet$ belongs to $\mathcal{N}$ by (a). This shows that the collection $\mathcal{N}$ is closed under direct summands. We have now proved that $\mathcal{N}$ is closed under isomorphisms, direct sums and direct summands, which is axiom (a). $\square$

Next, we study the rotation axiom (N2). The following result shows that when we replace (N1) with (N1*), then we can also replace (N2) with the weaker version (N2*). In other words, in the rotation axiom we only need to require that the left rotation of an n–Σ sequence in $\mathcal{N}$ also belongs to $\mathcal{N}$.

Theorem 3.3 *If $\mathcal{N}$ is a collection of n–Σ–sequences satisfying the axioms* (N1*) *and* (N3)*, then the following are equivalent:*

1. *$\mathcal{N}$ satisfies* (N2)*;*
2. *$\mathcal{N}$ satisfies* (N2*).

Proof The implication (1) $\Rightarrow$ (2) is trivial. Thus assume $\mathcal{N}$ satisfies (N2*), and let

$$A_\bullet\colon\quad A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} A_n \xrightarrow{\alpha_n} \Sigma A_1$$

be an n–Σ–sequence in $\mathcal{N}$. By repeatedly applying (N2*), we obtain the n–Σ–sequence

$$A_n \xrightarrow{\alpha_n} \Sigma A_1 \xrightarrow{(-1)^n\Sigma\alpha_1} \cdots \xrightarrow{(-1)^n\Sigma\alpha_{n-2}} \Sigma A_{n-1} \xrightarrow{(-1)^n\Sigma\alpha_{n-1}} \Sigma A_n$$

in $\mathcal{N}$. Now use (c) to complete the morphism $\Sigma^{-1}A_n \xrightarrow{(-1)^n\Sigma^{-1}\alpha_n} A_1$ to an n–Σ–sequence

$$\Sigma^{-1}A_n \xrightarrow{(-1)^n\Sigma^{-1}\alpha_n} A_1 \xrightarrow{\beta_2} B_3 \xrightarrow{\beta_3} \cdots \xrightarrow{\beta_{n-1}} B_n \xrightarrow{\beta_n} A_n$$

in $\mathcal{N}$. By repeated use of (N2*), we obtain the n–Σ–sequence

$$A_n \xrightarrow{\alpha_n} \Sigma A_1 \xrightarrow{(-1)^n\Sigma\beta_2} \Sigma B_3 \xrightarrow{(-1)^n\Sigma\beta_3} \cdots \xrightarrow{(-1)^n\Sigma\beta_{n-1}} \Sigma B_n \xrightarrow{(-1)^n\Sigma\beta_n} \Sigma A_n$$

in $\mathcal{N}$. By (N3), we may complete the diagram

$$\begin{tikzcd}
A_n \arrow[r, "\alpha_n"] \arrow[d, equal] & \Sigma A_1 \arrow[r, "(-1)^n\Sigma\beta_2"] \arrow[d, equal] & \Sigma B_3 \arrow[r, "(-1)^n\Sigma\beta_3"] \arrow[d, dashed, "\varphi_3"] & \cdots \\
A_n \arrow[r, "\alpha_n"] & \Sigma A_1 \arrow[r, "(-1)^n\Sigma\alpha_1"] & \Sigma A_2 \arrow[r, "(-1)^n\Sigma\alpha_2"] & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots \arrow[r, "(-1)^n\Sigma\beta_{n-1}"] & \Sigma B_n \arrow[r, "(-1)^n\Sigma\beta_n"] \arrow[d, dashed, "\varphi_n"] & \Sigma A_n \arrow[d, equal] \\
\cdots \arrow[r, "(-1)^n\Sigma\alpha_{n-2}"] & \Sigma A_{n-1} \arrow[r, "(-1)^n\Sigma\alpha_{n-1}"] & \Sigma A_n
\end{tikzcd}$$

and obtain a morphism of $n-\Sigma$–sequences. By applying the automorphism Σ^{-1} to the rows, and multiplying the maps with $(-1)^n$, we obtain a weak isomorphism

$$\begin{tikzcd}
\Sigma^{-1}A_n \arrow[r, "(-1)^n\Sigma^{-1}\alpha_n"] \arrow[d, equal] & A_1 \arrow[r, "\beta_2"] \arrow[d, equal] & B_3 \arrow[r, "\beta_3"] \arrow[d, "\Sigma^{-1}\varphi_3"] & \cdots \arrow[r, "\beta_{n-1}"] & B_n \arrow[r, "\beta_n"] \arrow[d, "\Sigma^{-1}\varphi_n"] & A_n \arrow[d, equal] \\
\Sigma^{-1}A_n \arrow[r, "(-1)^n\Sigma^{-1}\alpha_n"] & A_1 \arrow[r, "\alpha_1"] & A_2 \arrow[r, "\alpha_2"] & \cdots \arrow[r, "\alpha_{n-2}"] & A_{n-1} \arrow[r, "\alpha_{n-1}"] & A_n
\end{tikzcd}$$

of $n-\Sigma$–sequences. The top row belongs to $\mathcal{N}$ and is therefore exact by Lemma 3.1, whereas the bottom row is the right rotation of $A_\bullet$. Since $A_\bullet$ is exact, so is its right rotation, and from (a) we conclude that this right rotation also belongs to $\mathcal{N}$. □

Collecting the results in this section gives the following.

Theorem 3.4 *For a collection $\mathcal{N}$ of $n-\Sigma$–sequences, the following are equivalent:*

(1) $\mathcal{N}$ *satisfies* (N1), (N2) *and* (N3)*;*

(2) $\mathcal{N}$ *satisfies* (N1*), (N2) *and* (N3)*;*

(3) $\mathcal{N}$ *satisfies* (N1*), (N2*) *and* (N3)*.*

4 Axiom (N4)

For triangulated categories, it is a well known fact that Verdier's original octahedral axiom has several equivalent representations; see eg Holm and Jørgensen [2] for a discussion. It is natural to ask whether this also holds true for general n–angulated categories. We prove in this section that it does: we introduce a higher "octahedral axiom" (N4*) for n–angulated categories, and show that it is equivalent to axiom (N4).

What is the essence of the classical octahedral axiom for triangulated categories? It starts with three given triangles

$$\begin{aligned}
A_1 &\to A_2 \to A_3 \to \Sigma A_1,\\
A_1 &\to B_2 \to B_3 \to \Sigma A_1,\\
A_2 &\to B_2 \to C_3 \to \Sigma A_2,
\end{aligned}$$

that are connected, in that each pair of triangles share a common object. The axiom then guarantees the existence of two new morphisms, and from these new morphisms we obtain three things:

(1) a morphism of triangles;

(2) a new triangle, whose objects are objects in the three original triangles;

(3) commutativity relations between morphisms.

The reason why the axiom is called the "octahedral axiom" is that everything fits into an octahedron whose vertices are the objects, and where the edges are the morphisms.

The essence of the higher octahedral axiom for n–angulated categories that we now introduce is exactly the same. It starts with three given n–angles, and guarantees the existence of $3n-7$ new morphisms. From these new morphisms we obtain a morphism of n–angles, a new n–angle and a certain commutativity relation between morphisms.

(N4*) Given a commutative diagram

$$\begin{array}{ccccccccccccc}
A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & A_3 & \xrightarrow{\alpha_3} & \cdots & \xrightarrow{\alpha_{n-2}} & A_{n-1} & \xrightarrow{\alpha_{n-1}} & A_n & \xrightarrow{\alpha_n} & \Sigma A_1 \\
\| & & \downarrow{\scriptstyle \varphi_2} & & & & & & & & & & \| \\
A_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & B_3 & \xrightarrow{\beta_3} & \cdots & \xrightarrow{\beta_{n-2}} & B_{n-1} & \xrightarrow{\beta_{n-1}} & B_n & \xrightarrow{\beta_n} & \Sigma A_1 \\
 & & \downarrow{\scriptstyle \gamma_2} & & & & & & & & & & \\
 & & C_3 & & & & & & & & & & \\
 & & \downarrow{\scriptstyle \gamma_3} & & & & & & & & & & \\
 & & \vdots & & & & & & & & & & \\
 & & \downarrow{\scriptstyle \gamma_{n-2}} & & & & & & & & & & \\
 & & C_{n-1} & & & & & & & & & & \\
 & & \downarrow{\scriptstyle \gamma_{n-1}} & & & & & & & & & & \\
 & & C_n & & & & & & & & & & \\
 & & \downarrow{\scriptstyle \gamma_n} & & & & & & & & & & \\
 & & \Sigma A_2 & & & & & & & & & &
\end{array}$$

whose top rows and second column are n–angles, there exist $3n-7$ morphisms

$$\begin{aligned}
&A_i \xrightarrow{\varphi_i} B_i && (3 \le i \le n), \\
&A_i \xrightarrow{\psi_i} C_{i-1} && (4 \le i \le n), \\
&B_i \xrightarrow{\theta_i} C_i && (3 \le i \le n),
\end{aligned}$$

with the following two properties:

(a) the sequence $(1, \varphi_2, \varphi_3, \dots, \varphi_n)$ is a morphism of n–angles;

(b) the n–Σ–sequence

$$A_3 \xrightarrow{\begin{bmatrix}\alpha_3\\ \varphi_3\end{bmatrix}} A_4 \oplus B_3 \xrightarrow{\begin{bmatrix}-\alpha_4 & 0\\ \varphi_4 & -\beta_3\\ \psi_4 & \theta_3\end{bmatrix}} A_5 \oplus B_4 \oplus C_3 \xrightarrow{\mu_2} A_6 \oplus B_5 \oplus C_4 \xrightarrow{\mu_3} \cdots$$
$$\cdots \xrightarrow{\mu_{n-4}} A_n \oplus B_{n-1} \oplus C_{n-2} \xrightarrow{\eta} B_n \oplus C_{n-1} \xrightarrow{[\theta_n\ \gamma_{n-1}]} C_n \xrightarrow{\Sigma\alpha_2\circ\gamma_n} \Sigma A_3$$

is an n–angle where μ_i and η are the matrices

$$\mu_i = \begin{bmatrix} -\alpha_{i+3} & 0 & 0\\ (-1)^{i+1}\varphi_{i+3} & -\beta_{i+2} & 0\\ \psi_{i+3} & \theta_{i+2} & \gamma_{i+1}\end{bmatrix}, \quad \eta = \begin{bmatrix}(-1)^n\varphi_n & -\beta_{n-1} & 0\\ \psi_n & \theta_{n-1} & \gamma_{n-2}\end{bmatrix},$$

and $\gamma_n \circ \theta_n = \Sigma\alpha_1 \circ \beta_n$.

For small values of n, objects A_i, B_i, C_i with $i > n$ appearing in the axiom should be interpreted as zero objects (and so should objects C_i with $i < 3$). Specifically, when $n = 3$, that is, when $\mathcal{C}$ is a triangulated category, the triangle in (b) becomes

$$A_3 \xrightarrow{\varphi_3} B_3 \xrightarrow{\theta_3} C_3 \xrightarrow{\Sigma\alpha_2\circ\gamma_3} \Sigma A_3$$

and for $n = 4$, the 4–angle in (b) becomes

$$A_3 \xrightarrow{\begin{bmatrix}\alpha_3\\ \varphi_3\end{bmatrix}} A_4 \oplus B_3 \xrightarrow{\begin{bmatrix}\varphi_4 & -\beta_3\\ \psi_4 & \theta_3\end{bmatrix}} B_4 \oplus C_3 \xrightarrow{[\theta_4\ \gamma_3]} C_4 \xrightarrow{\Sigma\alpha_2\circ\gamma_4} \Sigma A_3.$$

Our aim is to prove that axiom (N4) may be replaced by the new axiom (N4*). In other words, we shall prove that if our category $\mathcal{C}$ is pretriangulated (that is, $\mathcal{C}$ satisfies (N1), (N2) and (N3)), then it satisfies (N4) if and only if it satisfies (N4*). In order to prove this, we need the following lemma.

Lemma 4.1 *Suppose $\mathcal{C}$ is n–angulated, and let*

$$\begin{tikzcd}
A_1 \arrow[r, "\alpha_1"] \arrow[d, equal] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "\varphi_2"] & A_3 \arrow[r, "\alpha_3"] & \cdots \arrow[r, "\alpha_{n-1}"] & A_n \arrow[r, "\alpha_n"] & \Sigma A_1 \arrow[d, equal] \\
A_1 \arrow[r, "\beta_1"] & B_2 \arrow[r, "\beta_2"] & B_3 \arrow[r, "\beta_3"] & \cdots \arrow[r, "\beta_{n-1}"] & B_n \arrow[r, "\beta_n"] & \Sigma A_1
\end{tikzcd}$$

be a commutative diagram whose rows are n–angles. Apply axiom (N4) *and complete the diagram to a morphism*

$$\begin{tikzcd}
A_1 \arrow[r, "\alpha_1"] \arrow[d, equal] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "\varphi_2"] & A_3 \arrow[r, "\alpha_3"] \arrow[d, dashed, "\varphi_3"] & \cdots \arrow[r, "\alpha_{n-1}"] & A_n \arrow[r, "\alpha_n"] \arrow[d, dashed, "\varphi_n"] & \Sigma A_1 \arrow[d, equal] \\
A_1 \arrow[r, "\beta_1"] & B_2 \arrow[r, "\beta_2"] & B_3 \arrow[r, "\beta_3"] & \cdots \arrow[r, "\beta_{n-1}"] & B_n \arrow[r, "\beta_n"] & \Sigma A_1
\end{tikzcd}$$

of n–angles, in such a way that the mapping cone is also an n–angle. Then the n–Σ–sequence

$$A_2 \xrightarrow{\begin{bmatrix} -\alpha_2 \\ \varphi_2 \end{bmatrix}} A_3 \oplus B_2 \xrightarrow{\begin{bmatrix} \alpha_3 & 0 \\ \varphi_3 & \beta_2 \end{bmatrix}} A_4 \oplus B_3 \xrightarrow{\begin{bmatrix} \alpha_4 & 0 \\ -\varphi_4 & \beta_3 \end{bmatrix}} \cdots$$

$$\cdots \xrightarrow{\begin{bmatrix} \alpha_{n-1} & 0 \\ (-1)^n \varphi_{n-1} & \beta_{n-2} \end{bmatrix}} A_n \oplus B_{n-1} \xrightarrow{[\,(-1)^{n+1}\varphi_n \;\; \beta_{n-1}\,]} B_n \xrightarrow{\Sigma\alpha_1 \circ \beta_n} \Sigma A_2$$

is an n–angle.

Proof The mapping cone is the middle n–Σ–sequence in the direct sum diagram

$$\begin{tikzcd}[column sep=huge]
A_2 \arrow[r, "\begin{bmatrix} -\alpha_2 \\ \varphi_2 \end{bmatrix}"] \arrow[d, "\begin{bmatrix} 1 \\ 0 \end{bmatrix}"] & A_3 \oplus B_2 \arrow[r, "\begin{bmatrix} \alpha_3 & 0 \\ \varphi_3 & \beta_2 \end{bmatrix}"] \arrow[d, "\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}"] & \cdots \\
A_2 \oplus A_1 \arrow[r, "\begin{bmatrix} -\alpha_2 & 0 \\ \varphi_2 & \beta_1 \end{bmatrix}"] \arrow[d, "\begin{bmatrix} 1 & \alpha_1 \end{bmatrix}"] & A_3 \oplus B_2 \arrow[r, "\begin{bmatrix} -\alpha_3 & 0 \\ \varphi_3 & \beta_2 \end{bmatrix}"] \arrow[d, "\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}"] & \cdots \\
A_2 \arrow[r, "\begin{bmatrix} -\alpha_2 \\ \varphi_2 \end{bmatrix}"] & A_3 \oplus B_2 \arrow[r, "\begin{bmatrix} \alpha_3 & 0 \\ \varphi_3 & \beta_2 \end{bmatrix}"] & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=huge]
\cdots \arrow[r, "\begin{bmatrix} \alpha_{n-1} & 0 \\ (-1)^n \varphi_{n-1} & \beta_{n-2} \end{bmatrix}"] & A_n \oplus B_{n-1} \arrow[r, "{[(-1)^{n+1}\varphi_n \;\; \beta_{n-1}]}"] \arrow[d, "\begin{bmatrix} (-1)^{n+1} & 0 \\ 0 & 1 \end{bmatrix}"] & B_n \arrow[r, "\Sigma\alpha_1 \circ \beta_n"] \arrow[d, "\begin{bmatrix} -\beta_n \\ 1 \end{bmatrix}"] & \Sigma A_2 \arrow[d, "\begin{bmatrix} 1 \\ 0 \end{bmatrix}"] \\
\cdots \arrow[r, "\begin{bmatrix} -\alpha_{n-1} & 0 \\ \varphi_{n-1} & \beta_{n-2} \end{bmatrix}"] & A_n \oplus B_{n-1} \arrow[r, "\begin{bmatrix} -\alpha_n & 0 \\ \varphi_n & \beta_{n-1} \end{bmatrix}"] \arrow[d, "\begin{bmatrix} (-1)^{n+1} & 0 \\ 0 & 1 \end{bmatrix}"] & \Sigma A_1 \oplus B_n \arrow[r, "\begin{bmatrix} -\Sigma\alpha_1 & 0 \\ 1 & \beta_n \end{bmatrix}"] \arrow[d, "\begin{bmatrix} 0 & 1 \end{bmatrix}"] & \Sigma A_2 \oplus \Sigma A_1 \arrow[d, "\begin{bmatrix} 1 & \Sigma\alpha_1 \end{bmatrix}"] \\
\cdots \arrow[r, "\begin{bmatrix} \alpha_{n-1} & 0 \\ (-1)^n \varphi_{n-1} & \beta_{n-2} \end{bmatrix}"] & A_n \oplus B_{n-1} \arrow[r, "{[(-1)^{n+1}\varphi_n \;\; \beta_{n-1}]}"] & B_n \arrow[r, "\Sigma\alpha_1 \circ \beta_n"] & \Sigma A_2 .
\end{tikzcd}$$

Therefore, by axiom (a), the top (bottom) row is also an n–angle. □

Now we prove that axioms (N4) and (N4*) are equivalent. We do this in two steps, showing first that axiom (N4) implies axiom (N4*).

Theorem 4.2 *If $\mathcal{N}$ is a collection of n–Σ–sequences in $\mathcal{C}$ satisfying axioms* (N1), (N2), (N3) *and* (N4), *then it also satisfies* (N4*).

Proof Suppose we are given a commutative diagram

$$\begin{tikzcd}
A_1 \arrow[r, "\alpha_1"] \arrow[d, equal] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "\varphi_2"] & A_3 \arrow[r, "\alpha_3"] & \cdots \arrow[r, "\alpha_{n-2}"] & A_{n-1} \arrow[r, "\alpha_{n-1}"] & A_n \arrow[r, "\alpha_n"] & \Sigma A_1 \arrow[d, equal] \\
A_1 \arrow[r, "\beta_1"] & B_2 \arrow[r, "\beta_2"] & B_3 \arrow[r, "\beta_3"] & \cdots \arrow[r, "\beta_{n-2}"] & B_{n-1} \arrow[r, "\beta_{n-1}"] & B_n \arrow[r, "\beta_n"] & \Sigma A_1
\end{tikzcd}$$

where the two rows are n–angles, and in addition an n–angle

$$A_2 \xrightarrow{\varphi_2} B_2 \xrightarrow{\gamma_2} C_3 \xrightarrow{\gamma_3} \cdots \xrightarrow{\gamma_{n-1}} C_n \xrightarrow{\gamma_n} \Sigma A_2.$$

Apply axiom (N4) and complete the given diagram to a morphism $(1, \varphi_2, \varphi_3, \ldots, \varphi_n)$ of n–angles, in such a way that the mapping cone is an n–angle. Then the first part of axiom (N4*) is already satisfied.

By Lemma 4.1, the n–Σ–sequence

$$A_2 \xrightarrow{\begin{bmatrix} -\alpha_2 \\ \varphi_2 \end{bmatrix}} A_3 \oplus B_2 \xrightarrow{\begin{bmatrix} \alpha_3 & 0 \\ \varphi_3 & \beta_2 \end{bmatrix}} A_4 \oplus B_3 \xrightarrow{\begin{bmatrix} \alpha_4 & 0 \\ -\varphi_4 & \beta_3 \end{bmatrix}} \cdots$$
$$\cdots \xrightarrow{\begin{bmatrix} \alpha_{n-1} & 0 \\ (-1)^n \varphi_{n-1} & \beta_{n-2} \end{bmatrix}} A_n \oplus B_{n-1} \xrightarrow{[(-1)^{n+1}\varphi_n \;\; \beta_{n-1}]} B_n \xrightarrow{\Sigma\alpha_1 \circ \beta_n} \Sigma A_2$$

is an n–angle. Then by axiom (N4) again, we have that there exist morphisms $\psi_i \colon A_i \to C_{i-1}$ $(4 \le i \le n)$ and $\theta_i \colon B_i \to C_i$ $(3 \le i \le n)$ such that the mapping cone of the morphism

$$\begin{tikzcd}
A_2 \arrow[r, "\begin{bmatrix} -\alpha_2 \\ \varphi_2 \end{bmatrix}"] \arrow[d, equal] & A_3 \oplus B_2 \arrow[r, "\begin{bmatrix} \alpha_3 & 0 \\ \varphi_3 & \beta_2 \end{bmatrix}"] \arrow[d, "{[0 \;\; 1]}"] & A_4 \oplus B_3 \arrow[r, "\begin{bmatrix} \alpha_4 & 0 \\ -\varphi_4 & \beta_3 \end{bmatrix}"] \arrow[d, "{[\psi_4 \;\; \theta_3]}"] & \cdots \\
A_2 \arrow[r, "\varphi_2"] & B_2 \arrow[r, "\gamma_2"] & C_3 \arrow[r, "\gamma_3"] & \cdots
\end{tikzcd}$$
$$\begin{tikzcd}
\cdots \arrow[r, "\begin{bmatrix} \alpha_{n-1} & 0 \\ (-1)^n \varphi_{n-1} & \beta_{n-2} \end{bmatrix}"] & A_n \oplus B_{n-1} \arrow[r, "{[(-1)^{n+1}\varphi_n \;\; \beta_{n-1}]}"] \arrow[d, "{[\psi_n \;\; \theta_{n-1}]}"] & B_n \arrow[r, "\Sigma\alpha_1 \circ \beta_n"] \arrow[d, "\theta_n"] & \Sigma A_2 \arrow[d, equal] \\
\cdots \arrow[r, "\gamma_{n-2}"] & C_{n-1} \arrow[r, "\gamma_{n-1}"] & C_n \arrow[r, "\gamma_n"] & \Sigma A_2
\end{tikzcd}$$

is an n–angle. In other words, the n–Σ–sequence

$$A_3 \oplus B_2 \oplus A_2 \xrightarrow{\begin{bmatrix} -\alpha_3 & 0 & 0 \\ -\varphi_3 & -\beta_2 & 0 \\ 0 & 1 & \varphi_2 \end{bmatrix}} A_4 \oplus B_3 \oplus B_2 \xrightarrow{\mu_1} A_5 \oplus B_4 \oplus C_3 \xrightarrow{\mu_2} \cdots$$

$$\cdots \xrightarrow{\mu_{n-4}} A_n \oplus B_{n-1} \oplus C_{n-2} \xrightarrow{\eta} B_n \oplus C_{n-1} \cdots \xrightarrow{\begin{bmatrix} -\Sigma\alpha_1 \circ \beta_n & 0 \\ \theta_n & \gamma_{n-1} \end{bmatrix}} \Sigma A_2 \oplus C_n$$

$$\xrightarrow{\begin{bmatrix} \Sigma\alpha_2 & 0 \\ -\Sigma\varphi_2 & 0 \\ 1 & \gamma_n \end{bmatrix}} \Sigma A_3 \oplus \Sigma B_2 \oplus \Sigma A_2$$

is an n–angle, where μ_i and η are the matrices

$$\mu_i = \begin{bmatrix} -\alpha_{i+3} & 0 & 0 \\ (-1)^{i+1}\varphi_{i+3} & -\beta_{i+2} & 0 \\ \psi_{i+3} & \theta_{i+2} & \gamma_{i+1} \end{bmatrix}, \quad \eta = \begin{bmatrix} (-1)^n\varphi_n & -\beta_{n-1} & 0 \\ \psi_n & \theta_{n-1} & \gamma_{n-2} \end{bmatrix}.$$

This n–angle is the middle n–Σ–sequence in the direct sum diagram shown in Figure 1. Note that since the composition of the last two morphisms in the middle n–angle is zero, the equality

$$\gamma_n \circ \theta_n = \Sigma\alpha_1 \circ \beta_n$$

holds, and this in turn implies the commutativity of the square Ω. Consequently, by (a), the top (bottom) n–Σ–sequence is an n–angle. This shows that the second part of axiom (N4*) is satisfied. □

We now prove the converse to Theorem 4.2, namely that the octahedral axiom (N4*) implies axiom (N4).

Theorem 4.3 *If $\mathcal{N}$ is a collection of n–Σ–sequences in $\mathcal{C}$ satisfying axioms* (N1), (N2), (N3) *and* (N4*), *then it also satisfies* (N4).

Proof Given a commutative diagram

$$\begin{tikzcd} A_1 \arrow[r, "\alpha_1"] \arrow[d, "\varphi_1"] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "\varphi_2"] & A_3 \arrow[r, "\alpha_3"] & \cdots \arrow[r, "\alpha_{n-1}"] & A_n \arrow[r, "\alpha_n"] & \Sigma A_1 \arrow[d, "\Sigma\varphi_1"] \\ B_1 \arrow[r, "\beta_1"] & B_2 \arrow[r, "\beta_2"] & B_3 \arrow[r, "\beta_3"] & \cdots \arrow[r, "\beta_{n-1}"] & B_n \arrow[r, "\beta_n"] & \Sigma B_1 \end{tikzcd}$$

$$
\begin{tikzcd}
A_3 \arrow[r, "\begin{bmatrix} \alpha_3 \\ \varphi_3 \end{bmatrix}"] \arrow[d, "\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}"] & A_4 \oplus B_3 \arrow[r, "\begin{bmatrix} -\alpha_4 & 0 \\ \varphi_4 & -\beta_3 \\ \psi_4 & \theta_3 \end{bmatrix}"] \arrow[d, "\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}"] & A_5 \oplus B_4 \oplus C_3 \arrow[d, equal] \\
A_3 \oplus B_2 \oplus A_2 \arrow[r, "\begin{bmatrix} -\alpha_3 & 0 & 0 \\ -\varphi_3 & -\beta_2 & 0 \\ 0 & 1 & \varphi_2 \end{bmatrix}"] \arrow[d, "\begin{bmatrix} -1 & 0 & \alpha_2 \end{bmatrix}"] & A_4 \oplus B_3 \oplus B_2 \arrow[r, "\mu_1"] \arrow[d, "\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \beta_2 \end{bmatrix}"] & A_5 \oplus B_4 \oplus C_3 \arrow[d, equal] \\
A_3 \arrow[r, "\begin{bmatrix} \alpha_3 \\ \varphi_3 \end{bmatrix}"] & A_4 \oplus B_3 \arrow[r, "\begin{bmatrix} -\alpha_4 & 0 \\ \varphi_4 & -\beta_3 \\ \psi_4 & \theta_3 \end{bmatrix}"] & A_5 \oplus B_4 \oplus C_3
\end{tikzcd}
$$

$$
\begin{tikzcd}
{} \arrow[r, "\mu_2"] & A_6 \oplus B_5 \oplus C_4 \arrow[r, "\mu_3"] \arrow[d, equal] & \cdots \arrow[r, "\mu_{n-4}"] & A_n \oplus B_{n-1} \oplus C_{n-2} \arrow[r, "\eta"] \arrow[d, equal] & B_n \oplus C_{n-1} \arrow[d, equal] \\
{} \arrow[r, "\mu_2"] & A_6 \oplus B_5 \oplus C_4 \arrow[r, "\mu_3"] \arrow[d, equal] & \cdots \arrow[r, "\mu_{n-4}"] & A_n \oplus B_{n-1} \oplus C_{n-2} \arrow[r, "\eta"] \arrow[d, equal] & B_n \oplus C_{n-1} \arrow[d, equal] \\
{} \arrow[r, "\mu_2"] & A_6 \oplus B_5 \oplus C_4 \arrow[r, "\mu_3"] & \cdots \arrow[r, "\mu_{n-4}"] & A_n \oplus B_{n-1} \oplus C_{n-2} \arrow[r, "\eta"] & B_n \oplus C_{n-1}
\end{tikzcd}
$$

$$
\begin{tikzcd}
{} \arrow[r, "\begin{bmatrix} \theta_n & \gamma_{n-1} \end{bmatrix}"] \arrow[d, phantom, "\Omega"] & C_n \arrow[r, "\Sigma\alpha_2 \circ \gamma_n"] \arrow[d, "\begin{bmatrix} -\gamma_n \\ 1 \end{bmatrix}"] & \Sigma A_3 \arrow[d, "\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}"] \\
{} \arrow[r, "\begin{bmatrix} -\Sigma\alpha_1 \circ \beta_n & 0 \\ \theta_n & \gamma_{n-1} \end{bmatrix}"] & \Sigma A_2 \oplus C_n \arrow[r, "\begin{bmatrix} \Sigma\alpha_2 & 0 \\ -\Sigma\varphi_2 & 0 \\ 1 & \gamma_n \end{bmatrix}"] \arrow[d, "\begin{bmatrix} 0 & 1 \end{bmatrix}"] & \Sigma A_3 \oplus \Sigma B_2 \oplus \Sigma A_2 \arrow[d, "\begin{bmatrix} -1 & 0 & \Sigma\alpha_2 \end{bmatrix}"] \\
{} \arrow[r, "\begin{bmatrix} \theta_n & \gamma_{n-1} \end{bmatrix}"] & C_n \arrow[r, "\Sigma\alpha_2 \circ \gamma_n"] & \Sigma A_3
\end{tikzcd}
$$

Figure 1

where the two rows are n–angles: we denote these by $A_\bullet$ and $B_\bullet$. We want to prove that we can complete the above diagram to a morphism of n–angles in such a way that the mapping cone of that morphism is again an n–angle.

From the given diagram we build the diagram

$$\begin{tikzcd}
A_1\oplus B_1 \arrow[r, "{\left[\begin{smallmatrix} 0 & 0 \\ (-1)^n\alpha_1 & 0 \\ 0 & -1 \end{smallmatrix}\right]}"] \arrow[d, equal] & B_2\oplus A_2\oplus B_1 \arrow[r, "{\left[\begin{smallmatrix} 0 & -\alpha_2 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right]}"] \arrow[d, "{[1\ \ -\varphi_2\ \ -\beta_1]}"] & A_3\oplus B_2 \arrow[r, "{[-\alpha_3\ \ 0]}"] & A_4 \arrow[r, "\alpha_4"] & \cdots \\
A_1\oplus B_1 \arrow[r, "{[(-1)^{n+1}\varphi_2\circ\alpha_1\ \ \beta_1]}"] & B_2 \arrow[r, "\beta_2"] \arrow[dl] & B_3 \arrow[r, "-\beta_3"] & B_4 \arrow[r, "-\beta_4"] & \cdots \\
0 \arrow[d] & \cdots \arrow[r, "\alpha_{n-2}"] & A_{n-1} \arrow[r, "\alpha_{n-1}"] & A_n \arrow[r, "{\left[\begin{smallmatrix} (-1)^n\alpha_n \\ 0 \end{smallmatrix}\right]}"] & \Sigma A_1\oplus\Sigma B_1 \arrow[d, equal] \\
\vdots \arrow[d] & \cdots \arrow[r, "-\beta_{n-2}"] & B_{n-1} \arrow[r, "{\left[\begin{smallmatrix} 0 \\ -\beta_{n-1} \end{smallmatrix}\right]}"] & \Sigma A_1\oplus B_n \arrow[r, "{\left[\begin{smallmatrix} -1 & 0 \\ (-1)^{n+1}\Sigma\varphi_1 & (-1)^{n+1}\beta_n \end{smallmatrix}\right]}"] & \Sigma A_1\oplus\Sigma B_1 \\
0 \arrow[d] \\
\Sigma A_2\oplus\Sigma B_1 \arrow[d, "{\left[\begin{smallmatrix} (-1)^n\Sigma\varphi_2 & (-1)^n\Sigma\beta_1 \\ (-1)^n & 0 \\ 0 & (-1)^n \end{smallmatrix}\right]}"] \\
\Sigma B_2\oplus\Sigma A_2\oplus\Sigma B_1
\end{tikzcd}$$

in which the top left square commutes. Let $X_\bullet$, $Y_\bullet$ and $Z_\bullet$ denote the three n–Σ–sequences

$$B_2\oplus A_2\oplus B_1 \xrightarrow{[1\ \ -\varphi_2\ \ -\beta_1]} B_2 \to 0 \to \cdots$$
$$\cdots \to 0 \to \Sigma A_2\oplus\Sigma B_1 \xrightarrow{\left[\begin{smallmatrix} (-1)^n\Sigma\varphi_2 & (-1)^n\Sigma\beta_1 \\ (-1)^n & 0 \\ 0 & (-1)^n \end{smallmatrix}\right]} \Sigma B_2\oplus\Sigma A_2\oplus\Sigma B_1,$$

$$A_1\oplus B_1 \xrightarrow{\left[\begin{smallmatrix} 0 & 0 \\ (-1)^n\alpha_1 & 0 \\ 0 & -1 \end{smallmatrix}\right]} B_2\oplus A_2\oplus B_1 \xrightarrow{\left[\begin{smallmatrix} 0 & -\alpha_2 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right]} A_3\oplus B_2 \xrightarrow{[-\alpha_3\ \ 0]} A_4 \xrightarrow{\alpha_4} \cdots$$
$$\cdots \xrightarrow{\alpha_{n-1}} A_n \xrightarrow{\left[\begin{smallmatrix} (-1)^n\alpha_n \\ 0 \end{smallmatrix}\right]} \Sigma A_1\oplus\Sigma B_1,$$

$$A_1 \oplus B_1 \xrightarrow{[(-1)^{n+1}\varphi_2\circ\alpha_1\ \beta_1]} B_2 \xrightarrow{\beta_2} B_3 \xrightarrow{-\beta_3} \cdots$$
$$\cdots \xrightarrow{-\beta_{n-2}} B_{n-1} \xrightarrow{\begin{bmatrix} 0 \\ -\beta_{n-1} \end{bmatrix}} \Sigma A_1 \oplus B_n \xrightarrow{\begin{bmatrix} -1 & 0 \\ (-1)^{n+1}\Sigma\varphi_1 & (-1)^{n+1}\beta_n \end{bmatrix}} \Sigma A_1 \oplus \Sigma B_1,$$

respectively. In order to apply (N4*) we need to prove that these n–Σ–sequences are n–angles.

It can easily be shown that $X_\bullet$ is isomorphic to the direct sum of the trivial n–angle on B_2 and the left rotations of the trivial n–angles on A_2 and B_1. Next, the n–Σ–sequence $Y_\bullet$ is isomorphic to the direct sum of the n–angle $A_\bullet$, the trivial n–angle on B_1 and the right rotation of the trivial n–angle on B_2. Similarly, the n–Σ–sequence $Z_\bullet$ is isomorphic to the direct sum of the n–angle $B_\bullet$ and the left rotation of the trivial n–angle on A_1. Hence, by (a) it follows that $X_\bullet$, $Y_\bullet$ and $Z_\bullet$ are n–angles.

Since $X_\bullet$, $Y_\bullet$ and $Z_\bullet$ are n–angles, we may apply axiom (N4*) to the above diagram. Consequently, there exist morphisms $\sigma_3, \sigma_4, \dots, \sigma_n$ and a morphism θ with the following three properties:

(1) the sequence $\big(1, \big[1\ \ -\varphi_2\ \ -\beta_1\big], \sigma_3, \sigma_4, \dots, \sigma_n\big)$ is a morphism $Y_\bullet \to Z_\bullet$ of n–angles;

(2) θ is a morphism $\Sigma A_1 \oplus B_n \to \Sigma A_2 \oplus \Sigma B_1$ with

$$\begin{bmatrix} (-1)^n\Sigma\varphi_2 & (-1)^n\Sigma\beta_1 \\ (-1)^n & 0 \\ 0 & (-1)^n \end{bmatrix} \circ \theta = \begin{bmatrix} 0 & 0 \\ (-1)^n\Sigma\alpha_1 & 0 \\ 0 & -1 \end{bmatrix} \circ \begin{bmatrix} -1 & 0 \\ (-1)^{n+1}\Sigma\varphi_1 & (-1)^{n+1}\beta_n \end{bmatrix};$$

(3) the n–Σ–sequence

$$A_3 \oplus B_2 \xrightarrow{\begin{bmatrix} -\alpha_3 & 0 \\ \sigma_{3,1} & \sigma_{3,2} \end{bmatrix}} A_4 \oplus B_3 \xrightarrow{\begin{bmatrix} -\alpha_4 & 0 \\ \sigma_4 & \beta_3 \end{bmatrix}} A_5 \oplus B_4 \xrightarrow{\begin{bmatrix} -\alpha_5 & 0 \\ -\sigma_5 & \beta_4 \end{bmatrix}} \cdots$$
$$\cdots \xrightarrow{\begin{bmatrix} -\alpha_{n-1} & 0 \\ (-1)^{n+1}\sigma_{n-1} & \beta_{n-2} \end{bmatrix}} A_n \oplus B_{n-1} \xrightarrow{\begin{bmatrix} (-1)^n\sigma_{n,1} & 0 \\ (-1)^n\sigma_{n,2} & \beta_{n-1} \end{bmatrix}} \Sigma A_1 \oplus B_n$$
$$\xrightarrow{\theta} \Sigma A_2 \oplus \Sigma B_1 \xrightarrow{\begin{bmatrix} (-1)^{n+1}\Sigma\alpha_2 & 0 \\ (-1)^n\Sigma\varphi_2 & (-1)^n\Sigma\beta_1 \end{bmatrix}} \Sigma A_3 \oplus \Sigma B_2$$

is an n–angle.

Observe that the n–angle $X_\bullet$ consists of the zero object at positions 3 through $n-1$. Therefore the morphisms ψ_i $(4 \le i \le n)$ and θ_i $(3 \le i \le n)$ given by (N4*) are all zero, except for θ_n, which we have called just θ.

From property (1) the diagram

$$
\begin{tikzcd}[ampersand replacement=\&]
A_1 \oplus B_1 \arrow[r, "{\left[\begin{smallmatrix} 0 & 0 \\ (-1)^n\alpha_1 & 0 \\ 0 & -1 \end{smallmatrix}\right]}"] \arrow[d, equal] \& B_2 \oplus A_2 \oplus B_1 \arrow[r, "{\left[\begin{smallmatrix} 0 & -\alpha_2 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right]}"] \arrow[d, "{[1\ \ -\varphi_2\ \ -\beta_1]}"] \& A_3 \oplus B_2 \arrow[r, "{[-\alpha_3\ \ 0]}"] \arrow[d, "\sigma_3"] \& A_4 \arrow[r, "\alpha_4"] \arrow[d, "\sigma_4"] \& \cdots \\
A_1 \oplus B_1 \arrow[r, "{[(-1)^{n+1}\varphi_2 \circ \alpha_1\ \ \beta_1]}"] \& B_2 \arrow[r, "\beta_2"] \& B_3 \arrow[r, "-\beta_3"] \& B_4 \arrow[r, "-\beta_4"] \& \cdots
\end{tikzcd}
$$

$$
\begin{tikzcd}[ampersand replacement=\&]
\cdots \arrow[r, "\alpha_{n-2}"] \& A_{n-1} \arrow[r, "\alpha_{n-1}"] \arrow[d, "\sigma_{n-1}"] \& A_n \arrow[r, "{\left[\begin{smallmatrix} (-1)^n\alpha_n \\ 0 \end{smallmatrix}\right]}"] \arrow[d, "\sigma_n"] \& \Sigma A_1 \oplus \Sigma B_1 \arrow[d, equal] \\
\cdots \arrow[r, "-\beta_{n-2}"] \& B_{n-1} \arrow[r, "{\left[\begin{smallmatrix} 0 \\ -\beta_{n-1} \end{smallmatrix}\right]}"] \& \Sigma A_1 \oplus B_n \arrow[r, "{\left[\begin{smallmatrix} -1 & 0 \\ (-1)^{n+1}\Sigma\varphi_1 & (-1)^{n+1}\beta_n \end{smallmatrix}\right]}"] \& \Sigma A_1 \oplus \Sigma B_1
\end{tikzcd}
$$

is commutative. Using the commutativity, we can conclude that

$$
\begin{aligned}
\begin{bmatrix} \sigma_{3,1} & \sigma_{3,2} \end{bmatrix} &= \begin{bmatrix} \varphi_3 & \beta_2 \end{bmatrix}, \\
\sigma_4 &= \varphi_4, \\
\sigma_5 &= -\varphi_5, \\
&\vdots \\
\sigma_{n-1} &= (-1)^{n-1}\varphi_{n-1}, \\
\begin{bmatrix} \sigma_{n,1} \\ \sigma_{n,2} \end{bmatrix} &= \begin{bmatrix} (-1)^{n+1}\alpha_n \\ (-1)^n\varphi_n \end{bmatrix},
\end{aligned}
$$

for some morphisms $\varphi_i\colon A_i \to B_i$ $(3 \le i \le n)$ making the sequence $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_n)$ into a morphism $\varphi\colon A_\bullet \to B_\bullet$ of n–angles.

Next, consider the morphism $\Sigma A_1 \oplus B_n \xrightarrow{\theta} \Sigma A_2 \oplus \Sigma B_1$. Using property (2), we see that

$$
\begin{aligned}
\begin{bmatrix} (-1)^n\Sigma\varphi_2 & (-1)^n\Sigma\beta_1 \\ (-1)^n & 0 \\ 0 & (-1)^n \end{bmatrix} \circ \theta &= \begin{bmatrix} 0 & 0 \\ (-1)^n\Sigma\alpha_1 & 0 \\ 0 & -1 \end{bmatrix} \circ \begin{bmatrix} -1 & 0 \\ (-1)^{n+1}\Sigma\varphi_1 & (-1)^{n+1}\beta_n \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ (-1)^{n+1}\Sigma\alpha_1 & 0 \\ (-1)^n\Sigma\varphi_1 & (-1)^n\beta_n \end{bmatrix}.
\end{aligned}
$$

Thus the morphism θ is given by the matrix

$$\theta = \begin{bmatrix} -\Sigma\alpha_1 & 0 \\ \Sigma\varphi_1 & \beta_n \end{bmatrix}.$$

Finally, from property (3) and what we have shown so far, the n–Σ–sequence

$$A_3 \oplus B_2 \xrightarrow{\begin{bmatrix} -\alpha_3 & 0 \\ \varphi_3 & \beta_2 \end{bmatrix}} A_4 \oplus B_3 \xrightarrow{\begin{bmatrix} -\alpha_4 & 0 \\ \varphi_4 & \beta_3 \end{bmatrix}} \cdots$$
$$\cdots \xrightarrow{\begin{bmatrix} -\alpha_n & 0 \\ \varphi_n & \beta_{n-1} \end{bmatrix}} \Sigma A_1 \oplus B_n \xrightarrow{\begin{bmatrix} -\Sigma\alpha_1 & 0 \\ \Sigma\varphi_1 & \beta_n \end{bmatrix}} \Sigma A_2 \oplus \Sigma B_1$$
$$\xrightarrow{\begin{bmatrix} (-1)^{n+1}\Sigma\alpha_2 & 0 \\ (-1)^n\Sigma\varphi_2 & (-1)^n\Sigma\beta_1 \end{bmatrix}} \Sigma A_3 \oplus \Sigma B_2$$

is an n–angle. Its right rotation

$$A_2 \oplus B_1 \xrightarrow{\begin{bmatrix} -\alpha_2 & 0 \\ \varphi_2 & \beta_1 \end{bmatrix}} A_3 \oplus B_2 \xrightarrow{\begin{bmatrix} -\alpha_3 & 0 \\ \varphi_3 & \beta_2 \end{bmatrix}} \cdots$$
$$\cdots \xrightarrow{\begin{bmatrix} -\alpha_n & 0 \\ \varphi_n & \beta_{n-1} \end{bmatrix}} \Sigma A_1 \oplus B_n \xrightarrow{\begin{bmatrix} -\Sigma\alpha_1 & 0 \\ \Sigma\varphi_1 & \beta_n \end{bmatrix}} \Sigma A_2 \oplus \Sigma B_1$$

is the mapping cone of φ, and this is an n–angle by axiom (N2). This completes the proof. □

Collecting Theorems 4.2 and 4.3 gives the following.

Theorem 4.4 *If $\mathcal{N}$ is a collection of n–Σ–sequences satisfying axioms* (N1), (N2) *and* (N3), *then the following are equivalent:*

(1) *$\mathcal{N}$ satisfies* (N4);

(2) *$\mathcal{N}$ satisfies* (N4*).

We now discuss the case when $n = 3$, that is, when our category $\mathcal{C}$ is a triangulated category. In this case, the classical octahedral axiom, which was introduced by Verdier in [7; 8], is the following:

(TR4) Given a commutative diagram

$$\begin{tikzcd}
A_1 \arrow[r, "\alpha_1"] \arrow[d, equal] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "\varphi_2"] & A_3 \arrow[r, "\alpha_3"] & \Sigma A_1 \arrow[d, equal] \\
A_1 \arrow[r, "\beta_1"] & B_2 \arrow[r, "\beta_2"] \arrow[d, "\gamma_2"] & B_3 \arrow[r, "\beta_3"] & \Sigma A_1 \\
& C_3 \arrow[d, "\gamma_3"] & & \\
& \Sigma A_2 & &
\end{tikzcd}$$

in which the top rows and second column are triangles. Then there exist morphisms $\varphi_3\colon A_3 \to B_3$ and $\theta_3\colon B_3 \to C_3$ with the following properties: the diagram

$$\begin{tikzcd}
A_1 \arrow[r, "\alpha_1"] \arrow[d, equal] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "\varphi_2"] & A_3 \arrow[r, "\alpha_3"] \arrow[d, "\varphi_3"] & \Sigma A_1 \arrow[d, equal] \\
A_1 \arrow[r, "\beta_1"] & B_2 \arrow[r, "\beta_2"] \arrow[d, "\gamma_2"] \arrow[dr, phantom, "\Theta"] & B_3 \arrow[r, "\beta_3"] \arrow[d, "\theta_3"] & \Sigma A_1 \\
& C_3 \arrow[r, equal] \arrow[d, "\gamma_3"] & C_3 \arrow[d, "\Sigma\alpha_2 \circ \gamma_3"] & \\
& \Sigma A_2 \arrow[r, "\Sigma\alpha_2"] & \Sigma A_3 &
\end{tikzcd}$$

is commutative, the third column is a triangle and $\gamma_3 \circ \theta_3 = \Sigma\alpha_1 \circ \beta_3$.

This is almost the same as our axiom (N4*): there is one difference. Namely, axiom (N4*) does *not* guarantee that the square Θ commutes. However, when $n = 3$ and we start with the diagram given in (TR4), then in the proof of Theorem 4.2 we obtain the commutative diagram

$$\begin{tikzcd}[ampersand replacement=\&]
A_2 \arrow[r, "\begin{bmatrix} -\alpha_2 \\ \varphi_2 \end{bmatrix}"] \arrow[d, equal] \& A_3 \oplus B_2 \arrow[r, "{[\varphi_3 \;\; \beta_2]}"] \arrow[d, "{[0 \;\; 1]}"] \& B_3 \arrow[r, "\Sigma\alpha_1 \circ \beta_3"] \arrow[d, "\theta_3"] \& \Sigma A_2 \arrow[d, equal] \\
A_2 \arrow[r, "\varphi_2"] \& B_2 \arrow[r, "\gamma_2"] \& C_3 \arrow[r, "\gamma_3"] \& \Sigma A_2 .
\end{tikzcd}$$

The commutativity of the middle square implies that the square Θ in (TR4) commutes. Therefore, we recover the original octahedral axiom (TR4) from axioms (N1), (N2), (N3) and (N4). Conversely, Neeman proves in [4, Theorem 1.8] that axioms (N1), (N2), (N3) and (TR4) together imply axiom (N4). Consequently, when $n = 3$ and the collection $\mathscr{N}$ of $3-\Sigma$–sequences satisfies axioms (N1), (N2) and (N3), then the following are equivalent:

(1) $\mathscr{N}$ satisfies (N4);
(2) $\mathscr{N}$ satisfies (TR4);
(3) $\mathscr{N}$ satisfies (N4*).

We end this section with a discussion of homotopy cartesian diagrams. Recall that when $n = 3$, then a commutative square

$$\begin{array}{ccc} A_1 & \xrightarrow{\alpha} & A_2 \\ \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} \\ B_1 & \xrightarrow{\beta} & B_2 \end{array}$$

is *homotopy cartesian* if there exists a triangle

$$A_1 \xrightarrow{\begin{bmatrix} -\alpha \\ \varphi_1 \end{bmatrix}} A_2 \oplus B_1 \xrightarrow{[\varphi_2\ \beta]} B_2 \xrightarrow{\partial} \Sigma A_1$$

for some morphism $B_2 \xrightarrow{\partial} \Sigma A_1$. Now let (TR4*) be the axiom which is the same as (TR4), but with the additional requirement that the commutative square

$$\begin{array}{ccc} A_2 & \xrightarrow{\alpha_2} & A_3 \\ \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_3} \\ B_2 & \xrightarrow{\beta_2} & B_3 \end{array}$$

is homotopy cartesian. Neeman shows in [4; 5] that (TR4) is equivalent to the stronger (TR4*). Consequently, the axioms (N4), (N4*), (TR4) and (TR4*) are all equivalent.

Now let $\mathcal{C}$ be n–angulated. Motivated by the above, we say that a commutative diagram

$$\begin{array}{ccccccccc} A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & \cdots & \xrightarrow{\alpha_{n-3}} & A_{n-2} & \xrightarrow{\alpha_{n-2}} & A_{n-1} \\ \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & & & \downarrow{\scriptstyle \varphi_{n-2}} & & \downarrow{\scriptstyle \varphi_{n-1}} \\ B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & \cdots & \xrightarrow{\beta_{n-3}} & B_{n-2} & \xrightarrow{\beta_{n-2}} & B_{n-1} \end{array}$$

is *homotopy cartesian* if the n–Σ–sequence

$$A_1 \xrightarrow{\begin{bmatrix} -\alpha_1 \\ \varphi_1 \end{bmatrix}} A_2 \oplus B_1 \xrightarrow{\begin{bmatrix} \alpha_2 & 0 \\ \varphi_2 & \beta_1 \end{bmatrix}} A_3 \oplus B_2 \xrightarrow{\begin{bmatrix} \alpha_3 & 0 \\ -\varphi_3 & \beta_2 \end{bmatrix}} \cdots$$

$$\cdots \xrightarrow{\begin{bmatrix} \alpha_{n-2} & 0 \\ (-1)^n \varphi_{n-2} & \beta_{n-3} \end{bmatrix}} A_{n-1} \oplus B_{n-2} \xrightarrow{[\,(-1)^{n+1}\varphi_{n-1} \;\; \beta_{n-2}\,]} B_{n-1} \xrightarrow{\partial} \Sigma A_1$$

is an n–angle for some morphism $B_{n-1} \xrightarrow{\partial} \Sigma A_1$. In the proof of Theorem 4.2, when we showed that axiom (N4*) follows from axiom (N4), we proved in addition that the commutative diagram

$$\begin{tikzcd}
A_2 \arrow[r, "\alpha_2"] \arrow[d, "\varphi_2"] & A_3 \arrow[r, "\alpha_3"] \arrow[d, "\varphi_3"] & \cdots \arrow[r, "\alpha_{n-2}"] & A_{n-1} \arrow[r, "\alpha_{n-1}"] \arrow[d, "\varphi_{n-1}"] & A_n \arrow[d, "\varphi_n"] \\
B_2 \arrow[r, "\beta_2"] & B_3 \arrow[r, "\beta_3"] & \cdots \arrow[r, "\beta_{n-2}"] & B_{n-1} \arrow[r, "\beta_{n-1}"] & B_n
\end{tikzcd}$$

is homotopy cartesian. In fact, that was precisely Lemma 4.1. Consequently, axiom (N4*) (and then also axiom (N4)) is equivalent to the stronger axiom which requires the above commutative diagram to be homotopy cartesian.

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