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Character algebras of decorated $\mathrm{SL}_2(C)$–local systems

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Let $\mathcal{S}$ be a connected and locally 1–connected space, and let $\mathcal{M} \subset \mathcal{S}$. A *decorated* $\mathrm{SL}_2(\mathbb{C})$*–local system* is an $\mathrm{SL}_2(\mathbb{C})$–local system on $\mathcal{S}$, together with a chosen element of the stalk at each component of $\mathcal{M}$.

We study the decorated $\mathrm{SL}_2(\mathbb{C})$–*character algebra* of $(\mathcal{S},\mathcal{M})$: the algebra of polynomial invariants of decorated $\mathrm{SL}_2(\mathbb{C})$–local systems on $(\mathcal{S},\mathcal{M})$. The character algebra is presented explicitly. The character algebra is shown to correspond to the $\mathbb{C}$–algebra spanned by collections of oriented curves in $\mathcal{S}$ modulo local topological rules.

As an intermediate step, we obtain an invariant-theory result of independent interest: a presentation of the algebra of $\mathrm{SL}_2(\mathbb{C})$–invariant functions on $\mathrm{End}(\mathbb{V})^m \oplus \mathbb{V}^n$, where $\mathbb{V}$ is the tautological representation of $\mathrm{SL}_2(\mathbb{C})$.

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1 Introduction

1.1 Character algebras

If $\mathcal{S}$ is a connected and locally 1–connected space and G is a group, then G acts on the set $\mathrm{Hom}(\pi_1(\mathcal{S}),G)$ by conjugation, and there is a well-known bijection

$$\left\{\begin{matrix} G\text{–local systems on } \mathcal{S} \\ \text{up to equivalence}\end{matrix}\right\} \xrightarrow{\sim} \left\{\begin{matrix} \text{homomorphisms } \pi_1(\mathcal{S}) \to G \\ \text{up to equivalence}\end{matrix}\right\}$$

This bijection takes a G–local system on $\mathcal{S}$ to its monodromy representation (see Section 2.2). If G is an algebraic group over a field k and $\pi_1(\mathcal{S})$ is finitely generated, then the *character variety* $\mathrm{Char}(\mathcal{S},G)$ is an affine k–variety that parametrizes closed G–orbits in $\mathrm{Hom}(\pi_1(\mathcal{S}),G)$. The character variety is the spectrum of the *character algebra* $\mathcal{O}\mathrm{Char}(\mathcal{S},G)$ of G–invariant regular functions on the k–variety $\mathrm{Hom}(\pi_1(\mathcal{S}),G)$.

Character varieties have received significant attention in the literature, and their geometry is related to the topology of $\mathcal{S}$ and the representation theory of G. One important

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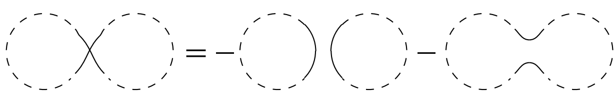
connection between the topology of a 3–manifold $\mathcal{S}$ and its character variety was discovered by Culler and Shalen in [5]. They demonstrate that affine curves in the character variety of $\mathrm{SL}_2(\mathsf{k})$–local systems on a 3–manifold $\mathcal{S}$ determine families of incompressible surfaces in $\mathcal{S}$.

We specialize to $G = \mathrm{SL}_2(\mathbb{C})$. In this case, Brumfiel and Hilden used general results of Procesi [9; 10] to give the following explicit presentation of $\mathcal{O}\mathrm{Char}(\mathcal{S}, \mathrm{SL}_2(\mathbb{C}))$.

Theorem 1.1.1 [3, Proposition 9.1.i] *The character algebra $\mathcal{O}\mathrm{Char}(\mathcal{S}, \mathrm{SL}_2(\mathbb{C}))$ is the commutative $\mathbb{C}$–algebra generated by $\{\chi_g \mid g \in \pi_1(\mathcal{S})\}$, with relations*

- $\chi_e = 2$, *and*
- $\chi_g \chi_h = \chi_{gh} + \chi_{g^{-1}h}$, *for all $g, h \in \pi_1(\mathcal{S})$.*

This algebra also has a purely topological description. The $(q = -1)$ *Kauffman skein algebra* $K_{-1}(\mathcal{S})$ is the commutative $\mathbb{C}$–algebra generated by homotopy classes of loops in $\mathcal{S}$ (that is, continuous images of S^1), with the following relations. Any contractible loop is equal to 2, and any loop with a self-intersection satisfies the following locally defined relation.[1]



A loop $l\colon S^1 \to \mathcal{S}$ defines a function $\chi_l \in \mathcal{O}\mathrm{Char}(\mathcal{S}, \mathrm{SL}_2(\mathbb{C}))$ that is defined by $\chi_l(\rho) = \mathrm{tr}(\rho(l))$. In this notation, Przytycki and Sikora have shown the following [11; 12].

Theorem 1.1.2 *The assignment $l \mapsto -\chi_l$ extends to a $\mathbb{C}$–algebra isomorphism $K_{-1}(\mathcal{S}) \to \mathcal{O}\mathrm{Char}(\mathcal{S}, \mathrm{SL}_2(\mathbb{C}))$.*

The purpose of the present paper is to generalize the previous two theorems to *decorated* $\mathrm{SL}_2(\mathbb{C})$*–local systems*. In the rest of the introduction, we define decorated $\mathrm{SL}_2(\mathbb{C})$–local systems, state the main theorems, and describe the organization of the paper.

1.2 Decorations

Fix a subset $\mathcal{M} \subseteq \mathcal{S}$, and let $\mathbb{V}$ be the tautological 2–dimensional representation of $\mathrm{SL}_2(\mathbb{C})$. A *decorated* $\mathrm{SL}_2(\mathbb{C})$*–local system* on $(\mathcal{S}, \mathcal{M})$ is a pair $(\mathcal{L}, d)$, where

- $\mathcal{L}$ is a $\mathrm{SL}_2(\mathbb{C})$–local system on $\mathcal{S}$, and

[1] An intersection between distinct loops will satisfy an identical relation; this is implied by the self-intersection relations.

- d is an element of the stalk $(\mathcal{L}\times_{\mathrm{SL}_2(\mathbb{C})}\mathbb{V})_{\mathcal{M}}$ at $\mathcal{M}$. Equivalently, it is a set $\{d_m\}$, where d_m is an element in the stalk $(\mathcal{L}\times_{\mathrm{SL}_2(\mathbb{C})}\mathbb{V})_m$ at m, for each connected component $m\subset\mathcal{M}$.

Like the undecorated case, decorated $\mathrm{SL}_2(\mathbb{C})$–local systems can be reduced to some *monodromy data*. In this case, this is a pair of maps

$$\pi_1(\mathcal{S})\to\mathrm{SL}_2(\mathbb{C}),\quad \pi_0(\widetilde{\mathcal{M}})\to\mathbb{V}$$

where $\widetilde{\mathcal{M}}$ is the preimage of $\mathcal{M}$ in the universal cover of $\mathcal{S}$. This data will be axiomatized as a morphism of *group actions* (defined in Section 2.4).

Analogous to the undecorated case, a *character function* is an $\mathrm{SL}_2(\mathbb{C})$–invariant regular function on the variety parametrizing the monodromy data. The character functions collectively form the *character algebra* $\mathcal{O}\mathrm{Char}(\mathcal{S},\mathcal{M})$, our central object of study.

The main result of the paper is the following presentation.

Theorem 3.5.2 *The character algebra* $\mathcal{O}\mathrm{Char}(\mathcal{S},\mathcal{M})$ *is the commutative* $\mathbb{C}$*–algebra generated by*

$$\{\chi_l\mid l\in\pi_1(\mathcal{S})\}\cup\{\chi_{(p,q)}\mid p,q\in\pi_0(\widetilde{\mathcal{M}})\}$$

with relations:

(1) $\chi_e=2$, *for* e *the identity in* G.

(2) $\chi_{(p,q)}=-\chi_{(q,p)}$

(3) $\chi_{(gp,gq)}=\chi_{(p,q)}$

(4) $\chi_g\chi_h=\chi_{gh}+\chi_{g^{-1}h}$

(5) $\chi_g\chi_{(p,q)}=\chi_{(gp,q)}+\chi_{(g^{-1}p,q)}$

(6) $\chi_{(p,q)}\chi_{(p',q')}=\chi_{(p,q')}\chi_{(p',q)}+\chi_{(p,p')}\chi_{(q,q')}$

The generators correspond to classes of oriented curves in $(\mathcal{S},\mathcal{M})$ (see Section 4). The relations in the theorem can be translated into simple manipulations of these curves, and so computations in the character algebra may be performed graphically.

1.3 The algebra of mixed $\mathrm{SL}_2(\mathbb{C})$–invariants

In proving Theorem 3.5.2, an invariant theory result of independent interest will be proven. For fixed $m,n\in\mathbb{N}$, the vector space $\mathrm{End}(\mathbb{V})^m\oplus\mathbb{V}^n$ is a $\mathbb{C}$–variety with an action of $\mathrm{SL}_2(\mathbb{C})$; denote its coordinate ring by $\mathcal{O}_{m,n}$. The invariant subalgebra

$\mathcal{O}^{\mathrm{SL}_2}_{m,n}$ is called the algebra of *mixed invariants*.[2] This algebra appears to have first been considered by Procesi in [9, Section 12].

Let $(\mathsf{A}_1, \dots, \mathsf{A}_m, v_1, \dots, v_n)$ denote an arbitrary element of $\mathrm{End}(\mathbb{V})^m \oplus \mathbb{V}^n$. For $1 \leq i \leq m$ and $1 \leq j,k \leq n$, define

$$\mathbf{X}_i(\mathsf{A}_1, \dots, \mathsf{A}_m, v_1, \dots, v_n) := \mathsf{A}_i \in \mathrm{End}(\mathbb{V}),$$
$$\mathbf{\Theta}_{j,k}(\mathsf{A}_1, \dots, \mathsf{A}_m, v_1, \dots, v_n) := v_j v_k^{\perp} \in \mathrm{End}(\mathbb{V}).$$

Theorem 6.4.1 *The algebra $\mathcal{O}^{\mathrm{SL}_2}_{m,n}$ of mixed invariants is the commutative $\mathbb{C}$–algebra generated by*

$$\{\mathrm{tr}(\mathbf{A}) \mid \forall \text{ words } \mathbf{A} \text{ in } \{\mathbf{X}_i, \mathbf{\Theta}_{i,j}\}\}$$

with relations:

- *(Procesi's F–relation) For words $\mathbf{A}, \mathbf{B}, \mathbf{C}$ in $\{\mathbf{X}_i, \mathbf{\Theta}_{i,j}\}$,*

$$\mathrm{tr}(\mathbf{ABC}) + \mathrm{tr}(\mathbf{CBA}) + \mathrm{tr}(\mathbf{A})\,\mathrm{tr}(\mathbf{B})\,\mathrm{tr}(\mathbf{C}) = \mathrm{tr}(\mathbf{B})\,\mathrm{tr}(\mathbf{AC}) + \mathrm{tr}(\mathbf{AB})\,\mathrm{tr}(\mathbf{C}) + \mathrm{tr}(\mathbf{A})\,\mathrm{tr}(\mathbf{BC}).$$

- *For a word $\mathbf{A}$ in the $\{\mathbf{X}_i, \mathbf{\Theta}_{i,j}\}$, and $1 \leq i,j \leq n$,*

$$\mathrm{tr}(\mathbf{A}\mathbf{\Theta}_{i,j}) = \mathrm{tr}(\mathbf{A}\mathbf{\Theta}_{j,i}) - \mathrm{tr}(\mathbf{A})\,\mathrm{tr}(\mathbf{\Theta}_{j,i}).$$

- *For words $\mathbf{A}, \mathbf{B}$ in $\{\mathbf{X}_i, \mathbf{\Theta}_{i,j}\}$, and $1 \leq i, j, i', j' \leq n$,*

$$\mathrm{tr}(\mathbf{A}\mathbf{\Theta}_{i,j}\mathbf{B}\mathbf{\Theta}_{i',j'}) = \mathrm{tr}(\mathbf{A}\mathbf{\Theta}_{i,j'})\,\mathrm{tr}(\mathbf{B}\mathbf{\Theta}_{i',j}).$$

1.4 Structure of paper

The first half of the paper concerns definitions and the presentation of results.

- Section 2 defines local systems, decorated local systems, and their monodromy data. Group actions that axiomatize the monodromy data are defined.
- Section 3 replaces the monodromy group action $(\pi_1(\mathcal{S}), \pi_0(\widetilde{\mathcal{M}}))$ with an arbitrary finitely generated group action (G, M), and defines the corresponding character algebra $\mathcal{O}\mathrm{Char}(G, M)$. Basic results on elementary character functions are given, as well as a statement of the main theorem (Theorem 3.5.2).
- Section 4 explores the topological presentation of the character algebra. A complete list of the rules for manipulating curves is presented, which is the topological translation of Theorem 3.5.2.

[2] It interpolates between the algebra of invariant functions on $\mathbb{V}^n$ (a subject of classical invariant theory) and the algebra of invariant functions on $\mathrm{End}(\mathbb{V})$ (a subject of modern interest).

The second half of the paper proves Theorem 3.5.2 by proving Theorem 6.4.1. It is rather technical and has an invariant-theory flavor.

- Section 5 introduces the prerequisites from invariant theory, and defines the algebras of mixed invariants $\mathcal{O}_{m,n}^{\mathrm{SL}_2}$ and mixed concomitants $\mathcal{E}_{m,n}^{\mathrm{SL}_2}$, which will be important intermediary objects. Several previously known results are presented as special cases of Theorem 6.4.1.
- Section 6 produces presentations of the algebras of mixed invariants and mixed concomitants (Theorems 6.4.1 and 6.4.2).
- Section 7 applies the presentations from the previous section to produce a presentation of the character algebra $\mathcal{O}\mathrm{Char}(\mathcal{S},\mathcal{M})$.

The paper concludes with an appendix on twisted character algebras, a variant useful for applications.

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2 Decorated $\mathrm{SL}_2(\mathbb{C})$–local systems

2.1 Spaces with marked regions

We begin by considering pairs $(\mathcal{S},\mathcal{M})$, where $\mathcal{S}$ is a connected and locally 1–connected space, and $\mathcal{M}\subset\mathcal{S}$ is a subset with finitely many connected components, each of which is path-connected. The marked subset $\mathcal{M}$ may be empty.

Subsequent constructions will only depend on the marked subset $\mathcal{M}$ up to homotopy equivalence in $\mathcal{S}$. In particular, if each component of $\mathcal{M}$ is contractible in $\mathcal{S}$, then each component of $\mathcal{M}$ can be replaced by a point. Regardless, $\pi_0(\mathcal{M})$ will denote the set of connected components of $\mathcal{M}$.

Choose a basepoint $p \in \mathcal{S}$, and let $\pi_1(\mathcal{S}) := \pi_1(\mathcal{S}, p)$ be the fundamental group at p. Fix a universal cover $\widetilde{\mathcal{S}} \to \mathcal{S}$ and let $\widetilde{\mathcal{M}}$ be the preimage of $\mathcal{M}$ in $\widetilde{\mathcal{S}}$.[3] The fundamental group $\pi_1(\mathcal{S})$ acts on $\widetilde{\mathcal{S}}$, $\widetilde{\mathcal{M}}$ and $\pi_0(\widetilde{\mathcal{M}})$.

2.2 G–local systems

A *G–local system* $\mathcal{L}$ on $\mathcal{S}$ is a locally constant sheaf of sets on $\mathcal{S}$ with a G–action, so that $\mathcal{L}$ is locally a free and transitive G–set (ie, a G–torsor). Two G–local systems are equivalent if there is a G–equivariant isomorphism of sheaves between them.

The choice of a basepoint p associates to every G–local system $\mathcal{L}$ its *monodromy representation* $\rho_{\mathcal{L}}\colon \pi_1(\mathcal{S}) \to G$. Two such representations are equivalent if they are G–conjugate. Then the following is standard (see, eg Szamuely [15, Theorem 2.5.15]).

Theorem 2.2.1 *The assignment $\mathcal{L} \to \rho_{\mathcal{L}}$ induces a bijection:*

$$\left\{ \begin{matrix} G\text{–local systems on } \mathcal{S} \\ \text{up to equivalence} \end{matrix} \right\} \xrightarrow{\sim} \left\{ \begin{matrix} \text{homomorphisms } \pi_1(\mathcal{S}) \to G \\ \text{up to equivalence} \end{matrix} \right\}$$

In particular, if $\mathcal{S}$ is simply connected, then every G–local system is trivial.

A G–local system $\mathcal{L}$ and a G–representation V determines an *induced vector bundle* $\mathcal{L} \times_G V$. This sheaf is $\mathcal{L}(\mathcal{U}) \times_G V \simeq V$ on simply connected sets $\mathcal{U} \subset \mathcal{S}$.

2.3 Decorated $\mathrm{SL}_2(\mathbb{C})$–local systems

Let $\mathbb{V}$ denote the standard $\mathrm{SL}_2(\mathbb{C})$–representation; it is 2–dimensional over $\mathbb{C}$. Fix an $\mathrm{SL}_2(\mathbb{C})$–invariant, skew-symmetric bilinear form ω on $\mathbb{V}$, though this will only be used later.

Definition 2.3.1 A *decorated* $\mathrm{SL}_2(\mathbb{C})$*–local system* on $(\mathcal{S}, \mathcal{M})$ is a pair $(\mathcal{L}, d)$, where

- $\mathcal{L}$ is a $\mathrm{SL}_2(\mathbb{C})$–local system on $\mathcal{S}$, and
- d is an element of the stalk $(\mathcal{L} \times_{\mathrm{SL}_2(\mathbb{C})} \mathbb{V})_{\mathcal{M}}$ at $\mathcal{M}$. Equivalently, it is a set $\{d_m\}$, where d_m is an element in the stalk $(\mathcal{L} \times_{\mathrm{SL}_2(\mathbb{C})} \mathbb{V})_m$ at m, for each connected component $m \subset \mathcal{M}$.

[3] With the assumptions on $\mathcal{S}$, a universal cover exists (Szamuely [15, Theorem 2.3.5]).

Note that (undecorated) $\mathrm{SL}_2(\mathbb{C})$–local systems are the special case when $\mathcal{M} = \varnothing$.

Decorated $\mathrm{SL}_2(\mathbb{C})$–local systems can be reduced to monodromy data by means of the universal cover. Given a decorated $\mathrm{SL}_2(\mathbb{C})$–local system $(\mathcal{L}, d)$ on $(\mathcal{S}, \mathcal{M})$, the universal covering map $\gamma\colon \widetilde{\mathcal{S}} \to \mathcal{S}$ induces a decorated $\mathrm{SL}_2(\mathbb{C})$–local system $(\gamma^*\mathcal{L}, \gamma^* d)$ on $(\widetilde{\mathcal{S}}, \widetilde{\mathcal{M}})$. Since $\widetilde{\mathcal{S}}$ is simply connected, $\gamma^*\mathcal{L}$ is a trivial $\mathrm{SL}_2(\mathbb{C})$–local system. A *global trivialization* of $\gamma^*\mathcal{L}$ is an isomorphism of $\mathrm{SL}_2(\mathbb{C})$–sets

$$\Gamma(\widetilde{\mathcal{S}}, \gamma^*\mathcal{L}) \simeq \mathrm{SL}_2(\mathbb{C}),$$

which, for any $\mathcal{U} \subset \widetilde{\mathcal{S}}$, defines isomorphisms of $\mathrm{SL}_2(\mathbb{C})$–sets

$$\Gamma(\mathcal{U}, \gamma^*\mathcal{L}) \simeq \mathrm{SL}_2(\mathbb{C}), \quad \Gamma(\mathcal{U}, \gamma^*\mathcal{L} \times_{\mathrm{SL}_2(\mathbb{C})} \mathbb{V}) \simeq \mathbb{V}.$$

A global trivialization identifies all the stalks in $\gamma^*\mathcal{L} \times_{\mathrm{SL}_2(\mathbb{C})} \mathbb{V}$ with $\mathbb{V}$, and so the decorations on $\widetilde{\mathcal{M}}$ are equivalent to a map $\rho_d\colon \pi_0(\widetilde{\mathcal{M}}) \to \mathbb{V}$.

The action of $\pi_1(\mathcal{S})$ on $\widetilde{\mathcal{S}}$ gives an action on global sections via the *monodromy map* $\rho\colon \pi_1(\mathcal{S}) \to SL_2(\mathbb{C})$. If two marked components $m, m' \in \mathcal{M}$ are related by the action of $l \in \pi_1(\mathcal{S})$, then their decorations must be related by the corresponding monodromy; that is, $\forall m \in \mathcal{M}$ and $l \in \pi_1(\mathcal{S})$,

$$\rho_d(lm) = \rho(l)\rho_d(m).$$

Changing the global trivialization of $\gamma^*\mathcal{L}$ by $g \in \mathrm{SL}_2(\mathbb{C})$ will send ρ to $g\rho g^{-1}$, and ρ_d to $g\rho_d$.

2.4 Group actions

This monodromy data (ρ, ρ_d) can be abstracted as follows. A *group action* is a pair (G, M), where G is a group and M is a G–set. A morphism of group actions is a pair of maps

$$(f_G, f_M)\colon (G, M) \to (G', M')$$

such that $f_G\colon G \to G'$ is a group homomorphism and $f_M\colon M \to M'$ is a function such that, for all $g \in G$ and $m \in M$,

$$f_M(gm) = f_G(g) f_M(m).$$

A group action (G, M) is *finitely generated* if G is finitely generated as a group and M has finitely many G–orbits.

Any element $g \in G$ determines an automorphism of group actions $(\mathrm{ad}_g, g\cdot)\colon (G, M) \to (G, M)$. Then G acts on the Hom $\to$ Hom set $\mathrm{Hom}((G', M'), (G, M))$ by postcomposition with the corresponding inner automorphism.

If $(\mathcal{L}, d)$ is a decorated $\mathrm{SL}_2(\mathbb{C})$–local system, choosing a global trivialization of $\gamma^*\mathcal{L}$ defines a morphism of group actions

$$(\rho, \rho_d)\colon (\pi_1(\mathcal{S}), \pi_0(\widetilde{\mathcal{M}})) \to (SL_2(\mathbb{C}), \mathbb{V}). \tag{2-1}$$

Two different global trivializations give morphisms related by the action of $\mathrm{SL}_2(\mathbb{C})$.

Proposition 2.4.1 *The map sending $(\mathcal{L}, d)$ to its monodromy data (ρ, ρ_d) is a bijection:*

$$\left\{ \begin{array}{c} \textit{decorated } \mathrm{SL}_2(\mathbb{C})\textit{–local systems} \\ \textit{on } (\mathcal{S}, \mathcal{M}) \textit{ up to equivalence} \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \textit{group action maps} \\ (\pi_1(\mathcal{S}), \pi_0(\widetilde{\mathcal{M}})) \to (SL_2(\mathbb{C}), \mathbb{V}) \\ \textit{up to the action of } \mathrm{SL}_2(\mathbb{C}) \end{array} \right\}$$

Proof We construct the inverse to the map $(\mathcal{L}, d) \mapsto (\rho, \rho_d)$. Given a representation

$$f\colon (\pi_1(\mathcal{S}), \pi_0(\widetilde{\mathcal{M}})) \to (SL_2(\mathbb{C}), \mathbb{V}),$$

the map $f_{\pi_1(\mathcal{S})}\colon \pi_1(\mathcal{S}) \to \mathrm{SL}_2(\mathbb{C})$ determines an $\mathrm{SL}_2(\mathbb{C})$–local system $\mathcal{L}$, by Theorem 2.2.1.

The map $f_{\pi_0(\widetilde{\mathcal{M}})}\colon \pi_0(\widetilde{\mathcal{M}}) \to \mathbb{V}$ determines an element of the stalk $\widetilde{d} \in (\mathcal{L} \times_{\mathrm{SL}_2(\mathbb{C})} \mathbb{V})_{\widetilde{\mathcal{M}}}$. For $U \subset \mathcal{S}$, sections of $\mathcal{L} \times_{\mathrm{SL}_2(\mathbb{C})} \mathbb{V}$ over U can be identified with $\pi_1(\mathcal{S})$–equivariant sections of $\gamma^{-1}U \subset \widetilde{\mathcal{S}}$. The group action condition implies that $\widetilde{d}$ is $\pi_1(\mathcal{S})$–equivariant, so it determines an element $d \in (\mathcal{L} \times_{\mathrm{SL}_2(k)} \mathbb{V})_{\mathcal{M}}$. This defines a decoration of $\mathcal{L}$. □

3 Character algebras

The decorated $\mathrm{SL}_2(\mathbb{C})$–local systems on $(\mathcal{S}, \mathcal{M})$ only depend on the finitely generated group action $(\pi_1(\mathcal{S}), \pi_0(\widetilde{\mathcal{M}}))$. This section will pass to the larger generality of an arbitrary finitely generated group action (G, M), with $(\pi_1(\mathcal{S}), \pi_0(\widetilde{\mathcal{M}}))$ demoted to the role of 'motivating example'.

3.1 The representation algebra

The first step is to endow the set of group action maps $(G, M) \to (\mathrm{SL}_2(\mathbb{C}), \mathbb{V})$ with the structure of an affine scheme.

Definition 3.1.1 The *representation algebra* of (G, M) is the $\mathbb{C}$–algebra $\mathcal{O}\mathrm{Rep}(G, M)$ such that, for any commutative $\mathbb{C}$–algebra A, there is a bijection (functorial in A):

$$\left\{ \begin{array}{c} \text{group action maps} \\ (G, M) \to (\mathrm{SL}_2(A), A \otimes \mathbb{V}) \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \mathbb{C}\text{–algebra maps} \\ \mathcal{O}\mathrm{Rep}(G, M) \to A \end{array} \right\}$$

The affine scheme $\mathrm{Rep}(G, M) := \mathrm{Spec}(\mathcal{O}\mathrm{Rep}(G, M))$ is called the *representation scheme*.

In particular, the $\mathbb{C}$–valued points of $\mathrm{Rep}(G, M)$ are in bijection with the set of group action maps $(G, M) \to (\mathrm{SL}_2(\mathbb{C}), \mathbb{V})$.[4] The proof of the existence and uniqueness of $\mathcal{O}\mathrm{Rep}(G, M)$ will be deferred to Proposition 7.1.3.

3.2 The character algebra

The group $\mathrm{SL}_2(\mathbb{C})$ acts naturally on $\mathcal{O}\mathrm{Rep}(G, M)$, and the set of $\mathrm{SL}_2(\mathbb{C})$–equivalence classes of maps $(G, M) \to (\mathrm{SL}_2(\mathbb{C}), \mathbb{V})$ is the set of $\mathrm{SL}_2(\mathbb{C})$–orbits in $\mathrm{Rep}(G, M)$.

Definition 3.2.1 The *character algebra* of (G, M) is $(\mathrm{SL}_2(\mathbb{C}), \mathbb{V})$ is the $\mathrm{SL}_2(\mathbb{C})$–invariant subalgebra of $\mathcal{O}\mathrm{Rep}(G, M)$,

$$\mathcal{O}\mathrm{Char}(G, M) := \mathcal{O}\mathrm{Rep}(G, M)^{\mathrm{SL}_2(\mathbb{C})} \subset \mathcal{O}\mathrm{Rep}(G, M).$$

The *character scheme* is the affine scheme $\mathrm{Char}(G, M) := \mathrm{Spec}(\mathcal{O}\mathrm{Char}(G, M))$.

A *character function* is an element of $\mathcal{O}\mathrm{Char}(G, M)$. A character function $\chi \in \mathcal{O}\mathrm{Char}(G, M)$ is a complex-valued function on $\mathrm{SL}_2(\mathbb{C})$–conjugacy classes of maps $(G, M) \to (\mathrm{SL}_2(\mathbb{C}), \mathbb{V})$. By Proposition 2.4.1, a character function

$$\chi \in \mathrm{Char}(\pi_1(\mathcal{S}), \pi_0(\widetilde{\mathcal{M}}))$$

determines a complex-valued invariant of decorated $\mathrm{SL}_2(\mathbb{C})$–local systems on $(\mathcal{S}, \mathcal{M})$.

Remark 3.2.2 The $\mathbb{C}$–valued points of $\mathrm{Char}(G, M)$ do not correspond to $\mathrm{SL}_2(\mathbb{C})$–orbits of $\mathbb{C}$–valued points in $\mathrm{Rep}(G, M)$. Orbits in $\mathrm{Rep}(G, M)$ may not be closed, and points in the orbit and its closure must go to the same point in $\mathrm{Char}(G, M)$.

3.3 Elementary character functions

There are two straight-forward constructions of character functions. First, let $g \in G$. For a map of group actions

$$\rho := (\rho_G, \rho_M)\colon (G, M) \to (\mathrm{SL}_2(\mathbb{C}), \mathbb{V}),$$

define

$$\chi_g(\rho) := \mathrm{tr}(\rho_G(g)). \tag{3-1}$$

[4] If (G, M) is finitely generated, $\mathcal{O}\mathrm{Rep}(G, M)$ is finitely generated over $\mathbb{C}$ (Proposition 7.1.3), and so the $\mathbb{C}$–valued points of $\mathrm{Rep}(G, M)$ are exactly the closed points of $\mathrm{Rep}(G, M)$ (e.g, [6, Corollary 13.12]).

For $a \in \mathrm{SL}_2(\mathbb{C})$,

$$\chi_g(a\cdot\rho) = \mathrm{tr}(a\rho(g)a^{-1}) = \mathrm{tr}(\rho(g)) = \chi_g(\rho)$$

and so χ_g defines a character function for all $g \in G$.

Next, let $p, q \in M$. Recall that ω is a fixed $\mathrm{SL}_2(\mathbb{C})$–invariant, skew-symmetric bilinear form on $\mathbb{V}$. For ρ an arbitrary representation, define

$$\chi_{(p,q)}(\rho) := \omega(\rho_M(p), \rho_M(q)). \tag{3-2}$$

For $a \in \mathrm{SL}_2(\mathbb{C})$,

$$\chi_{(p,q)}(a\cdot\rho) = \omega(a\rho_M(p), a\rho_M(q)) = \omega(\rho_M(p), \rho_M(q)) = \chi_{(p,q)}(\rho)$$

and so $\chi_{(p,q)}$ defines a character function for all $g \in G$.

3.4 Relations between elementary character functions

Proposition 3.4.1 *The following relations hold between character functions of the form χ_g and $\chi_{(p,q)}$:*

(1) $\chi_e = 2$, *for e the identity in G.*

(2) $\chi_{(p,q)} = -\chi_{(q,p)}$

(3) $\chi_{(gp,gq)} = \chi_{(p,q)}$

(4) $\chi_g\chi_h = \chi_{gh} + \chi_{g^{-1}h}$

(5) $\chi_g\chi_{(p,q)} = \chi_{(gp,q)} + \chi_{(g^{-1}p,q)}$

(6) $\chi_{(p,q)}\chi_{(p',q')} = \chi_{(p,q')}\chi_{(p',q)} + \chi_{(p,p')}\chi_{(q,q')}$

Proof Relation (1) follows from $\mathrm{tr}(\mathrm{Id}_{\mathbb{V}}) = \dim(\mathbb{V}) = 2$. Relation (2) follows from the anti-symmetry of ω. Relation (3) follows from the $\mathrm{SL}_2(\mathbb{C})$–invariance of ω. Relations (4) and (5) use the following lemma.

Lemma 3.4.2 *Let $g \in \mathrm{SL}_2(\mathbb{C})$. Then $g + g^{-1} = \mathrm{tr}(g)\cdot\mathrm{Id}_{\mathbb{V}}$.*

Proof of Lemma 3.4.2 The matrix g satisfies its own characteristic polynomial, that is,

$$g^2 - \mathrm{tr}(g)\cdot g + \mathrm{Id}_{\mathbb{V}} = 0.$$

Multiplying by g^{-1} gives the desired identity. □

Then Relation (4) is

$$\mathrm{tr}(g)\,\mathrm{tr}(h) = \mathrm{tr}(\mathrm{tr}(g)h) = \mathrm{tr}(gh + g^{-1}h) = \mathrm{tr}(gh) + \mathrm{tr}(g^{-1}h) \tag{3-3}$$

and Relation (5) is

$$\mathrm{tr}(g)\omega(p,q) = \omega(\mathrm{tr}(g)p,q) = \omega(gp + g^{-1}p,q) = \omega(gp,q) + \omega(g^{-1}p,q). \tag{3-4}$$

Relation (6) is equivalent to the 'quadratic Plücker relation'. To see this, choose ω–canonical coordinates x, y on $\mathbb{V}$, that is,

$$\omega(p,q) = x_p y_q - x_q y_p = \begin{vmatrix} x_p & x_q \\ y_p & y_q \end{vmatrix}, \quad \forall p,q \in \mathbb{V}.$$

Then Relation (6) becomes

$$\begin{vmatrix} x_p & x_q \\ y_p & y_q \end{vmatrix} \begin{vmatrix} x_{p'} & x_{q'} \\ y_{p'} & y_{q'} \end{vmatrix} = \begin{vmatrix} x_p & x_{q'} \\ y_p & y_{q'} \end{vmatrix} \begin{vmatrix} x_{p'} & x_q \\ y_{p'} & y_q \end{vmatrix} + \begin{vmatrix} x_p & x_{p'} \\ y_p & y_{p'} \end{vmatrix} \begin{vmatrix} x_q & x_{q'} \\ y_q & y_{q'} \end{vmatrix}$$

which can be verified directly. □

These relations formally imply several other relations.

Corollary 3.4.3 *The following relations also hold between character functions of the form χ_g and $\chi_{(p,q)}$:*

(7) $\chi_{p,p} = 0$

(8) $\chi_{g^{-1}} = \chi_g$

(9) $\chi_{hgh^{-1}} = \chi_g$

(10) $\chi_{g^i} = 2T_i(\chi_g/2)$, *where T_i is the i^{th} Chebyshev polynomial, defined by $T_0(x) = 1$, $T_1(x) = x$ and $T_{i+1}(x) = 2xT_i(x) - T_{i-1}(x)$.*

3.5 A presentation of $\mathcal{O}$Char(G, M)

The main result of this paper will be that the elementary character functions $\{\chi_g\}$ and $\{\chi_{(p,q)}\}$ generate $\mathcal{O}\mathrm{Char}(G, M)$, with all relations generated by those in Proposition 3.4.1. We formalize this as follows.

Definition 3.5.1 Let $H^+(G, M)$ denote[5] the commutative $\mathbb{C}$–algebra generated by symbols

- $[g]$, for all $g \in G$, and

[5] The notation $H^+(G, M)$ is chosen to match [3] and [12].

- $[p,q]$, for all $p,q \in M$,

with relations generated by:

(1) $[e]-2$

(2) $[p,q]+[q,p]$

(3) $[gp,gq]-[p,q]$

(4) $[g][h]-[gh]-[gh^{-1}]$

(5) $[g][p,q]-[gp,q]-[p,gq]$

(6) $[p,q][p',q']-[p,q'][p',q]-[p,p'][q,q']$

By Proposition 3.4.1, there is an algebra map

$$\chi\colon H^+(G,M) \to \mathcal{O}\mathrm{Char}(G,M)$$

such that $\chi([g]) = \chi_g$ and $\chi([p,q]) = \chi_{(p,q)}$.

Theorem 3.5.2 *The map* $\chi\colon H^+(G,M) \to \mathcal{O}\mathrm{Char}(G,M)$ *is an isomorphism.*

Proof Outline The proof will be contained in Sections 5–7, but we outline the idea here. If G is a free group on m generators and M is a free G–set of n orbits, then

$$\mathrm{Rep}(G,M) = \mathrm{SL}_2(\mathbb{C})^m \oplus \mathbb{V}^n,$$

as a subvariety of the affine space $\mathrm{End}(\mathbb{V})^m \oplus \mathbb{V}^n$. The invariant functions $\mathcal{O}_{m,n}^{\mathrm{SL}_2}$ on $\mathrm{End}(\mathbb{V})^m \oplus \mathbb{V}^n$ were studied in [9, Section 12] under the name of *mixed invariants*. We provide a presentation for the mixed invariants in Theorem 6.4.1, which can be used to present $\mathcal{O}\mathrm{Char}(G,M)$ when G and M are both free (see Remark 7.2.1).

When G and M are not free, a choice of m generators for G and n generators for M realizes $\mathrm{Rep}(G,M)$ as an SL_2–fixed subvariety of $\mathrm{End}(\mathbb{V})^m \oplus \mathbb{V}^n$, corresponding to the image of the generators in $\mathrm{SL}_2(\mathbb{C})$ and $\mathbb{V}$, respectively. The problem that arises is that the natural relations defining $\mathrm{Rep}(G,M)$ in $\mathrm{End}(\mathbb{V})^m \oplus \mathbb{V}^n$ are not SL_2–invariant, and so more relations might appear in the invariant subalgebra.

This is solved by considering the algebra $\mathcal{E}_{m,n}^{\mathrm{SL}_2}$ of SL_2–equivariant maps from $\mathrm{End}(\mathbb{V})^m \oplus \mathbb{V}^n$ to $\mathrm{End}(\mathbb{V})$, the algebra of *mixed concomitants*. When extended to this non-commutative algebra, the defining equations of $\mathrm{Rep}(G,M)$ become SL_2–invariant, and so it is possible to present $(\mathrm{End}(\mathbb{V}) \otimes \mathcal{O}\mathrm{Rep}(G,M))^{\mathrm{SL}_2}$; cf Theorem 7.3.2. Then, by taking traces of these elements, we present $\mathcal{O}\mathrm{Char}(G,M)$. □

Example 3.5.3 When $M = \varnothing$, then Relations (2), (3), (5), and (6) are vacuous, and the resulting presentation coincides with Brumfiel and Hilden's presentation (Theorem 1.1.1).

Example 3.5.4 When $G = \{e\}$ and $M = \{1, \dots, n\}$, then Relations (3), (4) and (5) are redundant, and the character algebra is the homogeneous coordinate ring of the Grassmannian $\mathrm{Gr}(2,n)$. The Plücker coordinates x_{ij} correspond to $\chi_{(i,j)}$.

Remark 3.5.5 These last two examples can be interpreted as saying that general character algebras $\mathcal{O}\mathrm{Char}(G, M)$ interpolate between Brumfiel and Hilden's $\mathrm{SL}_2(\mathbb{C})$–representation algebras $H^+(G)$ and the homogeneous coordinate rings of Grassmannians.

Example 3.5.6 For $G = \mathbb{Z}$ and $M = \{a, b\}$ (two G–invariant elements), the algebra $\mathcal{O}\mathrm{Char}(G, M)$ is generated by χ_1 and $\chi_{(a,b)}$, with the single relation $(\chi_1 + 2)\chi_{(a,b)} = 0$. The remaining elementary character functions can be expressed in terms of these two; for example, $\chi_i = 2T_i(\chi_1/2)$ and $\chi_{(b,a)} = -\chi_{(a,b)}$.

The corresponding scheme $\mathrm{Char}(G, M)$ is two affine lines crossing transversely. The two irreducible components of this scheme correspond to maps $\rho\colon (G, M) \to (\mathrm{SL}_2(\mathbb{C}), \mathbb{V})$ such that either

- $\rho_G(1) = \mathrm{Id}_{\mathbb{V}}$ (when $\chi_1 = \mathrm{tr}(\rho(1)) = 2$), or
- $\rho_M(a)$ and $\rho_M(b)$ are linearly dependent (when $\chi_{(a,b)} = \omega(\rho(a), \rho(b)) = 0$).

Since $\rho_M(a)$ and $\rho_M(b)$ are invariant vectors of $\rho_G(1)$, it is clear one of these two conditions must be satisfied.

4 A topological presentation of the character algebra

In this section, we return to the topological setting of $(G, M) = (\pi_1(\mathcal{S}), \pi_0(\widetilde{\mathcal{M}}))$, and hence to character functions for decorated $\mathrm{SL}_2(\mathbb{C})$–local systems. The elementary character functions naturally correspond to homotopy classes of oriented curves in $\mathcal{S}$, and the relations from the previous section can be graphically encoded.

4.1 Character functions of oriented curves

A *curve* in $(\mathcal{S}, \mathcal{M})$ will be a continuous map $c\colon \mathcal{C} \to \mathcal{S}$ from a connected, 1–dimensional manifold with boundary, such that any endpoints of $\mathcal{C}$ land in $\mathcal{M}$. An *orientation* of c is an orientation of $\mathcal{C}$.

There are two kinds of curves in $(\mathcal{S}, \mathcal{M})$:

- *loops*, which are modelled on the circle S^1, and
- *arcs*, which are modelled on the interval $[0,1]$ with $c(0), c(1) \in \mathcal{M}$.

A character function $\chi_c \in \mathcal{O}\mathrm{Char}(\mathcal{S},\mathcal{M})$ can be associated to any oriented curve c.

- An oriented loop l determines a conjugacy class of elements in $\pi_1(\mathcal{S})$. Since $\chi_l = \chi_{glg^{-1}}$, χ_l is a well-defined character function. If $(\mathcal{L}, d)$ is a decorated $\mathrm{SL}_2(\mathbb{C})$–local system, $\chi_l(\mathcal{L}, d)$ is the trace of the monodromy of $\mathcal{L}$ around l.
- An oriented arc a can be lifted to an oriented arc $\widetilde{a}$ in the universal cover $\widetilde{\mathcal{S}}$. The endpoints $p, q \in \widetilde{\mathcal{M}}$ determine a character function $\chi_a := \chi_{p,q}$; this is a well-defined function of a. If $(\mathcal{L}, d)$ is a decorated $\mathrm{SL}_2(\mathbb{C})$–local system, $\chi_a(\mathcal{L}, d)$ uses parallel transport along a to evaluate the skew-symmetric form ω at the decorations.

In both cases, the character function χ_c only depends on c up to homotopy keeping the endpoints in $\mathcal{M}$.

For a finite collection of oriented curves $\mathbf{c} = \{c_1, c_2, \ldots, c_i\}$, the associated character function is the product of the corresponding elementary character functions:

$$\chi_{\mathbf{c}} = \chi_{c_1}\chi_{c_2}\cdots\chi_{c_i}$$

4.2 Diagrammatic relations

Let $\mathrm{Curve}(\mathcal{S},\mathcal{M})$ denote the $\mathbb{C}$–algebra spanned by the set of finite collections of oriented curves in $(\mathcal{S},\mathcal{M})$ (up to homotopy). Then

$$\chi\colon \mathrm{Curve}(\mathcal{S},\mathcal{M}) \to \mathcal{O}\mathrm{Char}(\mathcal{S},\mathcal{M})$$

defines a map of algebras. Since the character algebra is generated by functions of the form χ_c, Theorem 3.5.2 shows that $\mathcal{O}\mathrm{Char}(\mathcal{S},\mathcal{M})$ is a quotient of $\mathrm{Curve}(\mathcal{S},\mathcal{M})$ with kernel determined by the relations in $\mathcal{O}\mathrm{Char}(\mathcal{S},\mathcal{M})$. The generators of this kernel can be interpreted as local manipulations. Figure 1 gives a complete set of rules for manipulating collections of curves without changing the corresponding character function.

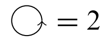
Remark 4.2.1 (1) The relation $\chi_{gp,gq} = \chi_{p,q}$ may appear to be missing; however, it is implicit in the definition of the character function of a curve.

(2) The second and third rule are formal consequences of the other rules.

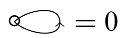
(3) While the last three rules appear to be virtually identical, there is an important distinction. The orientation of arcs in the last rule is arbitrary-seeming but necessary, whereas the orientation of loops in the fifth rule is truly arbitrary.

(4) Despite how the last three rules have been drawn, it is possible that the curves on the left hand side intersect along the chosen segments.

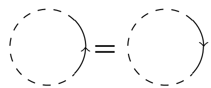
(1) A contractible loop is equal to 2 (Proposition 3.4.1, (1)):



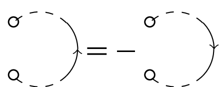
(2) A contractible arc is equal to 0 (Corollary 3.4.3, (7)):



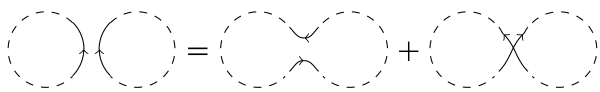
(3) An oriented loop is equal to its orientation-reversal (Corollary 3.4.3, (8)):



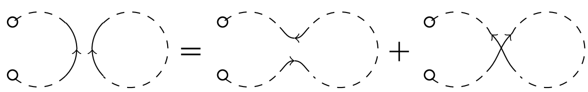
(4) An oriented arc is negative its orientation-reversal (Proposition 3.4.1, (2)):



(5) There are relations for pairs of curves, made by choosing nearby segments on each curve and summing over the two other ways to connect them.
Two loops (Proposition 3.4.1, (4)):



(6) An arc and a loop (Proposition 3.4.1, (5)):



(7) Two arcs (Proposition 3.4.1, (6)):

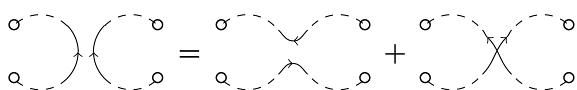


Figure 1: Diagrammatic rules for character functions of oriented curves

4.3 Weaknesses of this approach

While it might be tempting to regard this as the 'correct' way to topologically visualize the character algebra, there are two shortcomings of this approach.

- **Orientation-dependence** The function χ_c depends on the orientation of a curve c, but only up to a sign.
- **Non-local relations** The relations are not local in $\mathcal{S}$. For each crossing in a collection of curves, there is a relation, but the signs in that relation depend on

whether the crossing is between two distinct curves or the same curve, which is not local information.

In the undecorated case, the relations can be made local by sending a curve c to $-\chi_c$. This turns the last three rules into the *Kauffman skein relation at* $q=-1$ (Bullock [4] and Przytycki [12]). Such a fix will not be available in the decorated generality because of the orientation-dependence of the arcs.

A subsequent paper by the authors [8] will explore two methods for fixing these short-comings:

- Cleverly choosing a sign-correction $w(c)$ for each curve c, so that the map $c \to (-1)^{w(c)}\chi_c$ has the desired properties.
- Twisting the definition of decorated $\mathrm{SL}_2(\mathbb{C})$–local systems so that the corresponding character algebra $\widetilde{\mathcal{O}\mathrm{Char}}(\mathcal{S},\mathcal{M})$ can be canonically identified with a graphical algebra with the desired properties.

In both cases, the resulting graphical algebra will be the *Kauffman skein algebra* at $q=1$, where curves are allowed to have endpoints in $\mathcal{M}$. This algebra and its connections to Teichmüller theory have also been explored by Roger and Yang [13].

In this perspective, the first approach listed above is the decorated analog of Barrett's use of spin structures to flip signs in Kauffman skein algebras [2], while the second corresponds to his observation that the $q=1$ skein algebra corresponded to certain flat $\mathrm{SL}_2(\mathbb{C})$–connections on the frame bundle.

5 Invariants and concomitants on $\mathrm{End}(\mathbb{V})^m \oplus \mathbb{V}^n$

The rest of the paper is devoted to the proofs of Theorem 3.5.2. The main ingredient for the proof of Theorem 3.5.2 will be presentations for the algebras of invariants and matrix concomitants on the space $\mathrm{End}(\mathbb{V})^m \oplus \mathbb{V}^n$ (Theorems 6.4.1 and 6.4.2).

5.1 Preliminaries from invariant theory

This section collects the necessary results from invariant theory; a more substantial reference is [14].

Let G be a semisimple algebraic group over $\mathbb{C}$, and let A be a $\mathbb{C}$–algebra with an action of G. The subspace $A^G \subset A$ of G–invariant elements is a subalgebra, called the *invariant subalgebra*. Taking invariants is functorial; an $\mathrm{SL}_2(\mathbb{C})$–equivariant morphism of algebras $f\colon A \to B$ restricts to a morphism $f\colon A^G \to B^G$.

A basic tool for studying invariants of semisimple groups is the *Reynolds operator*.

Lemma 5.1.1 *Let G be a semisimple algebraic group over $\mathbb{C}$, and let A be a $\mathbb{C}$–algebra with an action of G. There is a $\mathbb{C}$–linear surjection $\gamma\colon A \dashrightarrow A^G$, called the Reynolds operator, such that:*

- $\gamma(a) = a$ *if* $a \in A^G \subset A$.
- $\gamma(ab) = a\gamma(b)$ and $\gamma(ba) = \gamma(b)a$ *if* $a \in A^G \subset A$.
- γ *commutes with G–equivariant maps.*

The main goal of invariant theory is usually to find a presentation of A^G. A typical approach is to write $A = B/I$, where B is an algebra with a G–action and I is a G–stable two-sided ideal. Then the following is standard.

Proposition 5.1.2 *If $A = B/I$, where A, B are $\mathbb{C}$–algebras with a G action and I is a G–stable two-sided ideal, then $A^G = B^G/I^G$.*

Proof The short exact sequence

$$I \xrightarrow{\iota} B \xrightarrow{\pi} A$$

of G–representations and their respective Reynolds operators fit into the following commutative diagram of $\mathbb{C}$–vector spaces:

$$\begin{tikzcd} I \arrow[r, hook, "\iota"] \arrow[d, "\gamma"'] & B \arrow[r, two heads, "\pi"] \arrow[d, "\gamma"'] & A \arrow[d, "\gamma"'] \\ I^G \arrow[r, "\iota^G"] & B^G \arrow[r, "\pi^G"] & A^G \end{tikzcd}$$

For any $a \in A^G \subset A$, choose a preimage $b \in B$. Then $\gamma(\pi(b)) = \pi^G(\gamma(b))$, and $\gamma(b) \in B^G$ is a preimage of a, so $\pi\colon B^G \to A^G$ is surjective.

Let $b \in B^G$ be in the kernel of π^G. As an element of B, b is in the kernel of π, and so $b \in I$. Since $b \in B^G$, it is G–invariant, and so $b \in I^G$. □

Finding generators for I^G can be difficult without the help of the following lemma.

Lemma 5.1.3 *If I is generated as an ideal of B by G–invariant elements $\{b_i\}$, then I^G is the ideal in B^G generated by $\{b_i\}$.*

Proof Let $a \in I^G$. Write $a = \sum_i c_i b_i$ for some $c_i \in B$. Then

$$a = \gamma(a) = \gamma\left(\sum_i c_i b_i\right) = \sum_i \gamma(c_i b_i) = \sum_i \gamma(c_i) b_i.$$

Since $\gamma(c_i) \in B^G$, the claim is proven. □

5.2 Three isomorphisms

The following three linear maps define isomorphisms of $\mathrm{SL}_2(\mathbb{C})$–representations that will be used repeatedly.

5.2.1 Transpose First, let $\mathbb{V}^\vee$ denote the dual vector space to $\mathbb{V}$. The invariant form ω defines the *transpose* map $\perp\colon \mathbb{V} \xrightarrow{\sim} \mathbb{V}^\vee$ by

$$v^\perp := \omega(v, -).$$

5.2.2 Outer product Next, define the *outer product* map $\Theta\colon \mathbb{V}\otimes\mathbb{V} \xrightarrow{\sim} \mathrm{End}(\mathbb{V})$ by

$$\Theta(v, w) := vw^\perp$$

(note that endomorphisms of the form $vw^\perp$ span $\mathrm{End}(\mathbb{V})$). It follows that

$$\mathrm{tr}(\Theta(v, w)) = w^\perp(v) = \omega(w, v) = -\omega(v, w). \tag{5-1}$$

5.2.3 Adjoint Finally, define the *adjoint* map $\iota\colon \mathrm{End}(\mathbb{V}) \to \mathrm{End}(\mathbb{V})$ by

$$(vw^\perp)^\iota := -wv^\perp.$$

Since endomorphisms of the form $vw^\perp$ span $\mathrm{End}(\mathbb{V})$, this completely determines ι.

Proposition 5.2.1 *The map ι has the following properties.*

1. *ι is an $\mathrm{SL}_2(\mathbb{C})$–equivariant anti-involution of the algebra* $\mathrm{End}(\mathbb{V})$.
2. *ι is the adjunction for the bilinear form ω, ie, $\omega(\mathsf{A}v, v') = \omega(v, \mathsf{A}^\iota v')$.*
3. $\mathsf{A} + \mathsf{A}^\iota = \mathrm{tr}(\mathsf{A})\cdot \mathrm{Id}_{\mathbb{V}}$. *Therefore, A is scalar iff A is ι–fixed.*
4. $\mathsf{A}\mathsf{A}^\iota = \mathsf{A}^\iota\mathsf{A} = \det(\mathsf{A})\cdot\mathrm{Id}_{\mathbb{V}}$. *Therefore, $\mathsf{A} \in \mathrm{SL}_2(\mathbb{C})$ iff $\mathsf{A}\mathsf{A}^\iota = \mathrm{Id}_{\mathbb{V}}$.*
5. *For e_1, e_2 an ω–canonical basis for $\mathbb{V}$ (ie, $\omega(e_1, e_2) = 1$), the action of ι on the corresponding matrices is:*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^\iota = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{5-2}$$

5.3 Mixed invariants and matrix concomitants

Let

$$\mathcal{O}_{m,n} := \mathcal{O}[\mathrm{End}(\mathbb{V})^m \oplus \mathbb{V}^n] = \mathrm{Sym}_{\mathbb{C}}(\mathrm{End}(\mathbb{V})^m \oplus \mathbb{V}^n)^\vee \tag{5-3}$$

denote the algebra of regular functions on the variety $\mathrm{End}(\mathbb{V})^m \oplus \mathbb{V}^n$. The group $\mathrm{SL}_2(\mathbb{C})$ acts on $\mathrm{End}(\mathbb{V})^m \oplus \mathbb{V}^n$ by conjugation on the first m factors and left action on the last n factors. This $\mathrm{SL}_2(\mathbb{C})$–action extends to an action on $\mathcal{O}_{m,n}$.

Definition 5.3.1 The $\mathrm{SL}_2(\mathbb{C})$–invariant subalgebra $\mathcal{O}_{m,n}^{\mathrm{SL}_2} \subset \mathcal{O}_{m,n}$ will be called the algebra of *mixed invariants*.

The theory of mixed invariants and its presentations simultaneously generalizes the theory of invariant functions on $\mathbb{V}^n$ and the theory of invariant functions on $\mathrm{End}(\mathbb{V})^m$ (hence, 'mixed'). As such, there are many partial results; some of the history of this problem will be reviewed in Section 5.6.

Now, denote:

$$\mathcal{E}_{m,n} := \mathrm{End}(\mathbb{V}) \otimes \mathcal{O}_{m,n} \tag{5-4}$$

The multiplication in $\mathrm{End}(\mathbb{V})$ makes this into a non-commutative algebra over $\mathcal{O}_{m,n}$, where $\mathcal{O}_{m,n} \hookrightarrow \mathcal{E}_{m,n}$ as scalar matrices. The group $\mathrm{SL}_2(\mathbb{C})$ acts on $\mathcal{E}_{m,n}$ by

$$g \cdot (\mathsf{M} \otimes f) = g\mathsf{M}g^{-1} \otimes (g \cdot f), \quad \text{for } g \in \mathrm{SL}_2(\mathbb{C}), \mathsf{M} \in \mathrm{End}(\mathbb{V}), f \in \mathcal{O}_{m,n}.$$

The algebra $\mathcal{E}_{m,n}$ is equivalent to the algebra of regular functions from $\mathrm{End}(\mathbb{V})^m \oplus \mathbb{V}^n$ to $\mathrm{End}(\mathbb{V})$; from this perspective, the $\mathrm{SL}_2(\mathbb{C})$–action is by conjugation of the function.

Definition 5.3.2 The $\mathrm{SL}_2(\mathbb{C})$–invariant subalgebra $\mathcal{E}_{m,n}^{\mathrm{SL}_2} \subset \mathcal{E}_{m,n}$ will be called the algebra of *mixed matrix concomitants*.

While an interesting object in its own right, the mixed matrix concomitants are most useful as an intermediary in computing the mixed invariants and related algebras. Specifically, some relations will not be $\mathrm{SL}_2(\mathbb{C})$–invariant in $\mathcal{O}_{m,n}$, but will become invariant when extended to $\mathcal{E}_{m,n}$,[6] allowing Lemma 5.1.3 to be used.

Remark 5.3.3 (Notation for $\mathcal{E}_{m,n}$) Plain math font $\{A, B, \dots\}$ will be used to denote generic elements in $\mathcal{E}_{m,n}$, sans serif font $\{\mathsf{A}, \mathsf{B}, \dots\}$ will be used to denote the elements of $\mathrm{End}(\mathbb{V}) \subset \mathcal{E}_{m,n}$ (constant elements), and bold $\{\mathbf{A}, \mathbf{B}, \dots\}$ will be used to denote SL_2–invariant elements of $\mathcal{E}_{m,n}$. This can be very useful for visually distinguishing between otherwise identical-looking results like Lemma 6.2.1, Corollary 6.2.2 and Lemma 6.3.1.

[6] Or rather, the even part of $\mathcal{E}_{m,n}$; see Section 7.2.

5.4 Maps between $\mathcal{O}_{m,n}^{\mathrm{SL}_2}$ and $\mathcal{E}_{m,n}^{\mathrm{SL}_2}$

Results about algebras of mixed invariants and mixed matrix concomitants will be related by two maps, scalar inclusion and trace.

The scalar inclusion $\mathcal{O}_{m,n} \hookrightarrow \mathcal{E}_{m,n}$ is $\mathrm{SL}_2(\mathbb{C})$–equivariant, and so it induces a scalar inclusion of invariants:

$$- \cdot \mathrm{Id}_{\mathbb{V}} \colon \mathcal{O}_{m,n}^{\mathrm{SL}_2} \hookrightarrow \mathcal{E}_{m,n}^{\mathrm{SL}_2}$$

In this way, $\mathcal{E}_{m,n}^{\mathrm{SL}_2}$ is an algebra over $\mathcal{O}_{m,n}^{\mathrm{SL}_2}$.

The anti-involution ι on $\mathrm{End}(\mathbb{V})$ extends to an anti-involution of $\mathcal{E}_{m,n}$, and all the analogous properties in Proposition 5.2.1 remain true. In particular, for $A \in \mathcal{E}_{m,n}$, $A \in \mathcal{O}_{m,n} \cdot \mathrm{Id}_{\mathbb{V}}$ iff $A^\iota = A$, and so:

Proposition 5.4.1 *Under the scalar inclusion $\mathcal{O}_{m,n} \hookrightarrow \mathcal{E}_{m,n}$, the algebra of mixed invariants $\mathcal{O}_{m,n}^{\mathrm{SL}_2}$ is the subalgebra of $\mathcal{E}_{m,n}^{\mathrm{SL}_2}$ fixed by ι.*

The trace gives a linear map[7] $\mathrm{End}(\mathbb{V}) \dashrightarrow \mathbb{C}$, which induces an $\mathrm{SL}_2(\mathbb{C})$–equivariant, $\mathcal{O}_{m,n}$–module map $\mathrm{tr}\colon \mathcal{E}_{m,n} \dashrightarrow \mathcal{O}_{m,n}$. This then restricts to a $\mathcal{O}_{m,n}^{\mathrm{SL}_2}$–module map:

$$\mathrm{tr}\colon \mathcal{E}_{m,n}^{\mathrm{SL}_2} \dashrightarrow \mathcal{O}_{m,n}^{\mathrm{SL}_2}$$

Since $\mathrm{tr}(f \cdot \mathrm{Id}_{\mathbb{V}}) = 2f$, the map $2^{-1}\,\mathrm{tr}$ is a right inverse to scalar inclusion. It follows that tr surjects onto $\mathcal{O}_{m,n}^{\mathrm{SL}_2}$.

5.5 Elementary concomitants and invariants

For $1 \leq i \leq m$, let $\mathbf{X}_i \in \mathcal{E}_{m,n}$ denote the i^{th} *coordinate function*,

$$\mathbf{X}_i(\mathsf{A}_1, \mathsf{A}_2, \dots, \mathsf{A}_m, v_1, v_2, \dots, v_m) := \mathsf{A}_i. \tag{5-5}$$

Since the action of $\mathrm{SL}_2(\mathbb{C})$ is by conjugation, $\mathbf{X}_i \in \mathcal{E}_{m,n}^{\mathrm{SL}_2}$.

For $1 \leq i, j \leq n$, let $\mathbf{\Theta}_{i,j} \in \mathcal{E}_{m,n}$ denote $(i,j)^{\text{th}}$ *outer product function*,

$$\mathbf{\Theta}_{i,j}(\mathsf{A}_1, \mathsf{A}_2, \dots, \mathsf{A}_m, v_1, v_2, \dots, v_m) := \Theta(v_i, v_j) = v_i v_j^{\perp}. \tag{5-6}$$

Since Θ is $\mathrm{SL}_2(\mathbb{C})$–equivariant, $\mathbf{\Theta}_{i,j} \in \mathcal{E}_{m,n}^{\mathrm{SL}_2}$.

Because ι is $\mathrm{SL}_2(\mathbb{C})$–equivariant, if $\mathbf{A} \in \mathcal{E}_{m,n}^{\mathrm{SL}_2}$, then $\mathbf{A}^\iota \in \mathcal{E}_{m,n}^{\mathrm{SL}_2}$. In particular, $\mathbf{X}_i^\iota$ and $\mathbf{\Theta}_{i,j}^\iota$ are also matrix concomitants, although $\mathbf{\Theta}_{i,j}^\iota = -\mathbf{\Theta}_{j,i}$, so only $\mathbf{X}_i^\iota$ provides a new example of a matrix concomitant.

[7] Dashed arrows will be used to denote morphisms in a weaker category than their source and target; typically, linear maps between algebras which are not algebra maps.

Any of these concomitants, or more generally any word in these concomitants, can be made into an invariant by taking the trace.

5.6 Known results on $\mathcal{O}^{\mathrm{SL}_2}_{m,n}$ and $\mathcal{E}^{\mathrm{SL}_2}_{m,n}$

As a generalization of two well-known problems, there are many partial results on the structure of the algebras of mixed invariants and matrix concomitants.

The case $m = 0$ is classical; see Weyl [16] or Howe [7].

Theorem 5.6.1 (Invariants on $\mathbb{V}^n$) *The algebra $\mathcal{O}^{\mathrm{SL}_2}_{0,n}$ is generated by $\{\mathrm{tr}(\mathbf{\Theta}_{i,j})\}$, for $1 \le i, j \le n$, with relations generated by $\mathrm{tr}(\mathbf{\Theta}_{i,j}) = -\mathrm{tr}(\mathbf{\Theta}_{j,i})$ and*

$$\mathrm{tr}(\mathbf{\Theta}_{i,j})\,\mathrm{tr}(\mathbf{\Theta}_{i',j'}) = \mathrm{tr}(\mathbf{\Theta}_{i,j'})\,\mathrm{tr}(\mathbf{\Theta}_{i',j}) + \mathrm{tr}(\mathbf{\Theta}_{i,i'})\,\mathrm{tr}(\mathbf{\Theta}_{j,j'}).$$

For arbitrary-dimensional V, the problem of finding invariants on $\mathrm{End}(V)^m$ was first proposed by Artin in [1]. A complete presentation of invariants and matrix concomitants on $\mathrm{End}(V)^m$ was found by Procesi in [9].

We review the 2–dimensional case. The invariants are generated by traces of strings of coordinate functions, and the relations are generated by a single class of relation.

Theorem 5.6.2 [9, Theorems 1.3 and 4.5.a] *The algebra $\mathcal{O}^{\mathrm{SL}_2}_{m,0}$ is generated as a commutative algebra by $\mathrm{tr}(\mathbf{A})$, as $\mathbf{A}$ runs over all words in the coordinate functions $\{\mathbf{X}_i\}$. The relations are generated by Procesi's F–relation:*

$$\begin{aligned}\text{(5-7)}\quad & \mathrm{tr}(\mathbf{ABC}) + \mathrm{tr}(\mathbf{CBA}) + \mathrm{tr}(\mathbf{A})\,\mathrm{tr}(\mathbf{B})\,\mathrm{tr}(\mathbf{C}) \\ &= \mathrm{tr}(\mathbf{B})\,\mathrm{tr}(\mathbf{AC}) + \mathrm{tr}(\mathbf{AB})\,\mathrm{tr}(\mathbf{C}) + \mathrm{tr}(\mathbf{A})\,\mathrm{tr}(\mathbf{BC})\end{aligned}$$

as $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ run over all words in $\{\mathbf{X}_i\}$.

The algebra of matrix concomitants $\mathcal{E}^{\mathrm{SL}_2}_{m,0}$ can then be generated, over $\mathcal{O}^{\mathrm{SL}_2}_{m,0}$, by the coordinate functions, and the relations are again generated by a single class of relation.

Theorem 5.6.3 [9, Theorems 2.1 and 4.5.b] *The algebra $\mathcal{E}^{\mathrm{SL}_2}_{m,0}$ is generated, as an algebra over $\mathcal{O}^{\mathrm{SL}_2}_{m,0}$, by the coordinate functions $\{\mathbf{X}_i\}$. The relations are generated by Procesi's G-relation:*

$$\text{(5-8)}\qquad \mathbf{AB} + \mathbf{BA} - \mathrm{tr}(\mathbf{A})\mathbf{B} - \mathrm{tr}(\mathbf{B})\mathbf{A} - \mathrm{tr}(\mathbf{AB}) + \mathrm{tr}(\mathbf{A})\,\mathrm{tr}(\mathbf{B}) = 0$$

as $\mathbf{A}$ and $\mathbf{B}$ run over all words in $\{\mathbf{X}_i\}$.

Inspired by later results of Procesi, Brumfiel and Hilden produced a different presentation of $\mathcal{E}^{\mathrm{SL}_2}_{m,n}$ (as a $\mathbb{C}$–algebra), which is specific to $\mathrm{SL}_2(\mathbb{C})$.

Theorem 5.6.4 [3, Proposition 9.1.i] *The algebra $\mathcal{E}^{\mathrm{SL}_2}_{m,0}$ is generated by $\mathbf{X}_i$ and $\mathbf{X}_i^\iota$, with the relations generated by*

$$(\mathbf{A}+\mathbf{A}^\iota)\mathbf{B} = \mathbf{B}(\mathbf{A}+\mathbf{A}^\iota)$$

as $\mathbf{A}$ and $\mathbf{B}$ run over all words in $\{\mathbf{X}_i, \mathbf{X}_i^\iota\}$.

As one would expect, ι denotes the anti-involution in the algebra generated by $\{\mathbf{X}_i, \mathbf{X}_i^\iota\}$ that interchanges $\mathbf{X}_i$ and $\mathbf{X}_i^\iota$. Since $\mathbf{A}+\mathbf{A}^\iota = \mathrm{tr}(\mathbf{A})\cdot \mathrm{Id}_{\mathbb{V}}$, the theorem says that the defining relation amongst the matrix concomitants is that the trace is central.

An interesting aspect of this presentation is that the algebra of matrix concomitants is produced directly, and then the algebra of invariants is found as the ι–fixed subalgebra.

6 Presentations of $\mathcal{O}^{\mathrm{SL}_2}_{m,n}$ and $\mathcal{E}^{\mathrm{SL}_2}_{m,n}$

This section provides presentations of the algebras of mixed invariants and mixed matrix concomitants that will be instrumental in proving Theorem 3.5.2. The problem of presenting these algebras in the mixed generality was first studied by Procesi [9, Section 12] (for more general groups than $\mathrm{SL}_2(\mathbb{C})$), who found a generating set and outlined a brute force method for computing the relations.[8] Our approach incorporates the $\mathrm{SL}_2(\mathbb{C})$–specific approach of Brumfiel and Hilden, and this specialization makes the problem tractable.

6.1 The map ν

Consider the quadratic map of SL_2–varieties:

$$\mathbb{V}^n \to \mathrm{End}(\mathbb{V})^{n^2}$$
$$(v_1, v_2, \dots, v_n) \mapsto (v_1 v_1^\perp, v_1 v_2^\perp, v_1 v_3^\perp, \dots, v_n v_{n-2}^\perp, v_n v_{n-1}^\perp, v_n v_n^\perp)$$

This induces a $\mathrm{SL}_2(\mathbb{C})$–equivariant map on coordinate rings:

$$\nu\colon \mathcal{O}_{n^2,0} \to \mathcal{O}_{0,n} \tag{6-1}$$

Tensoring this map with the identity map on $\mathcal{O}_{m,0}$ or $\mathcal{E}_{m,0}$ gives

$$\nu\colon \mathcal{O}_{m+n^2,0} \to \mathcal{O}_{m,n}, \quad \nu\colon \mathcal{E}_{m+n^2,0} \to \mathcal{E}_{m,n}.$$

[8] However, as Procesi himself notes, producing a nice generating set for the relations via this method would likely be difficult or impossible.

Proposition 6.1.1 *The invariant maps*

$$\nu\colon \mathcal{O}^{\mathrm{SL}_2}_{m+n^2,0} \to \mathcal{O}^{\mathrm{SL}_2}_{m,n} \quad \text{and} \quad \nu\colon \mathcal{E}^{\mathrm{SL}_2}_{m+n^2,0} \to \mathcal{E}^{\mathrm{SL}_2}_{m,n}$$

are surjective.

Proof In [9, Theorem 12.1], Procesi produces sets of generators for the algebra of mixed invariants, showing that they correspond to traces of products of matrices and values of ω on pairs of vectors. Since these are in the image of ν, surjectivity follows. The analogous statement for mixed concomitants follows from the set of generators in [9, Theorem 12.2.d]. □

Remark 6.1.2 Since the previous section provides a presentation of $\mathcal{O}^{\mathrm{SL}_2}_{m+n^2,0}$ and of $\mathcal{E}^{\mathrm{SL}_2}_{m+n^2,0}$, this proposition reduces the problem of presenting the mixed invariants and matrix concomitants to the problem of understanding the kernel of ν.

6.2 The kernel of ν

Under ν, the first m coordinate functions in $\mathcal{E}_{m+n^2,0}$ go to the coordinate functions in $\mathcal{E}_{m,n}$. The last n^2 coordinate functions in $\mathcal{E}_{m+n^2,0}$ go to the outer product functions in $\mathcal{E}_{m,n}$, so by abuse of notation, the last n^2 coordinate functions in $\mathcal{E}_{m+n^2,0}$ will be denoted by $\mathbf{\Theta}_{i,j}$, for $1 \le i, j \le n$.

Lemma 6.2.1 *The kernel of the map* $\nu\colon \mathcal{O}_{n^2,0} \to \mathcal{O}_{0,n}$ *is generated by*

- $\mathrm{tr}(\mathsf{A}\mathbf{\Theta}_{i,j}) + \mathrm{tr}(\mathsf{A}\mathbf{\Theta}^{\iota}_{j,i})$, *for* $\mathsf{A} \in \mathrm{End}(\mathbb{V})$ *and* $1 \le i, j \le n$,
- $\mathrm{tr}(\mathsf{A}\mathbf{\Theta}_{i,j}\mathsf{B}\mathbf{\Theta}_{i',j'}) - \mathrm{tr}(\mathsf{A}\mathbf{\Theta}_{i,j'})\,\mathrm{tr}(\mathsf{B}\mathbf{\Theta}_{i',j})$, *for* $\mathsf{A}, \mathsf{B} \in \mathrm{End}(\mathbb{V})$ *and* $1 \le i, j \le n$.

Proof We use the isomorphism $\xi\colon \mathbb{V}^n \otimes \mathbb{V}^n \simeq (\mathrm{End}(\mathbb{V})^{n^2})^\vee$, given by

$$\xi(v_i \otimes w_j) := v^\perp \mathbf{\Theta}_{i,j} w = \omega(v, \mathbf{\Theta}_{i,j} w) = \mathrm{tr}(w v^\perp \mathbf{\Theta}_{i,j}).$$

This induces an isomorphism of algebras:

$$\xi\colon \mathrm{Sym}^\bullet(\mathbb{V}^n \otimes \mathbb{V}^n) \to \mathcal{O}_{n^2,0}$$

The induced map

$$\nu'\colon \mathrm{Sym}^\bullet(\mathbb{V}^n \otimes \mathbb{V}^n) \simeq \mathcal{O}_{n^2,0} \xrightarrow{\nu} \mathcal{O}_{0,n} \simeq \mathrm{Sym}^\bullet(\mathbb{V}^n)$$

is the natural symmetrization map. These maps fit together into a commutative diagram:

$$\begin{tikzcd}
\mathcal{T}(\mathbb{V}^n \otimes \mathbb{V}^n) \arrow[r] \arrow[d] & \mathrm{Sym}^\bullet(\mathbb{V}^n \otimes \mathbb{V}^n) \arrow[r, "\xi"] \arrow[d, "\nu'"] & \mathcal{O}_{n^2,0} \arrow[d, "\nu"] \\
\mathcal{T}(\mathbb{V}^n) \arrow[r] & \mathrm{Sym}^\bullet(\mathbb{V}^n) \arrow[r, "\sim"] & \mathcal{O}_{0,n}
\end{tikzcd}$$

Here, $\mathcal{T}$ denotes the tensor algebra, $\mathcal{T}(\mathbb{V}^n \otimes \mathbb{V}^n) \to \mathcal{T}(\mathbb{V}^n)$ is the natural degree 2 embedding, and the maps from $\mathcal{T}$ to $\mathrm{Sym}^\bullet$ are the symmetrization maps.

By definition, the kernel of

$$\mathcal{T}(\mathbb{V}^n) \to \mathrm{Sym}^\bullet(\mathbb{V}^n)$$

is spanned, as j runs over all natural numbers, by elements of the form

$$(v_1)_{p_1} \otimes (v_2)_{p_2} \otimes \cdots \otimes (v_j)_{p_j} - (v_1)_{\sigma(p_1)} \otimes (v_2)_{\sigma(p_2)} \otimes \cdots \otimes (v_j)_{\sigma(p_j)} \tag{6-2}$$

for $v_i \in \mathbb{V}$, $1 \leq p_i \leq j$ and $\sigma \in \Sigma_j$, the symmetric group on j letters. Since $\mathcal{T}(\mathbb{V}^n \otimes \mathbb{V}^n) \to \mathcal{T}(\mathbb{V}^n)$ is an inclusion, it follows that the kernel of

$$\mathcal{T}(\mathbb{V}^n \otimes \mathbb{V}^n) \to \mathrm{Sym}^\bullet(\mathbb{V}^n)$$

is spanned by elements of the form (6-2). Since $\mathcal{T}(\mathbb{V}^n \otimes \mathbb{V}^n) \to \mathrm{Sym}^\bullet(\mathbb{V}^n \otimes \mathbb{V}^n)$ is surjective, the kernel of

$$\nu' \colon \mathrm{Sym}^\bullet(\mathbb{V}^n \otimes \mathbb{V}^n) \to \mathrm{Sym}^\bullet(\mathbb{V}^n)$$

is spanned by the image of elements of the form (6-2).

The symmetric group Σ_j is generated by simple transpositions, and so the kernel of the map ν' is generated by two kinds of elements:

- $v_i \otimes w_j - w_j \otimes v_i$, for $v, w \in \mathbb{V}$ and $1 \leq i, j \leq n$,
- $(v_i \otimes w_j)(v'_{i'} \otimes w'_{j'}) - (v_i \otimes w'_{j'})(v'_{i'} \otimes w_j)$, for $v, w, v', w' \in \mathbb{V}$ and $1 \leq i, j, i', j' \leq n$.

We then compute the image of these generators under ξ:

$$\begin{aligned}
\xi(v_i \otimes w_j - w_j \otimes v_i) &= \mathrm{tr}(wv^{\perp}\mathbf{\Theta}_{i,j}) - \mathrm{tr}(vw^{\perp}\mathbf{\Theta}_{j,i}) \\
&= \mathrm{tr}(wv^{\perp}\mathbf{\Theta}_{i,j}) + \mathrm{tr}(wv^{\perp}\mathbf{\Theta}^{\iota}_{j,i}) \\
&= \mathrm{tr}(wv^{\perp}(\mathbf{\Theta}_{i,j} + \mathbf{\Theta}^{\iota}_{j,i}))
\end{aligned}$$

Since products of the form $wv^\perp$ span $\mathrm{End}(\mathbb{V})$, this gives the first kind of generator:

$$\begin{aligned}\xi(v_i\otimes w_j)(v'_{i'}\otimes w'_{j'}) &= \mathrm{tr}(wv^\perp\mathbf{\Theta}_{i,j})tr(w'v'^\perp\mathbf{\Theta}_{i',j'})\\ &= \mathrm{tr}(w(v'^\perp\mathbf{\Theta}_{i',j'}w')v^\perp\mathbf{\Theta}_{i,j})\\ &= \mathrm{tr}(w'v^\perp\mathbf{\Theta}_{i,j}wv'^\perp\mathbf{\Theta}_{i',j'})\\ \xi(v_i\otimes w'_{j'})(v'_{i'}\otimes w_j) &= \mathrm{tr}(w'v^\perp\mathbf{\Theta}_{i,j'})\,\mathrm{tr}(wv'^\perp\mathbf{\Theta}_{i',j})\end{aligned}$$

Combining these gives the second class of relation. □

Corollary 6.2.2 *The kernel K of the map $\nu\colon \mathcal{O}_{m+n^2,0}\to\mathcal{O}_{m,n}$ is spanned over $\mathbb{C}$ by:*

- $\mathrm{tr}(A\mathbf{\Theta}_{i,j})+\mathrm{tr}(A\mathbf{\Theta}^\iota_{j,i})$*, for $A\in\mathcal{E}_{m+n^2,0}$ and $1\le i,j\le n$,*
- $\mathrm{tr}(A\mathbf{\Theta}_{i,j}B\mathbf{\Theta}_{i',j'})-tr(A\mathbf{\Theta}_{i,j'})\,\mathrm{tr}(B\mathbf{\Theta}_{i',j})$*, for $A,B\in\mathcal{E}_{m+n^2,0}$ and $1\le i,j,$ $i',j'\le n$.*

Proof The preceding lemma provides a generating set for this kernel. The kernel is then spanned by $\mathcal{O}_{m+n^2,0}$–multiples of these generators. In all cases, this coefficient may be pulled through the trace, and absorbed into the definition of A or B. □

6.3 The invariant kernel of ν

The next step is to find the kernel of the map ν when restricted to the invariants.

Lemma 6.3.1 *The kernel of the map $\nu\colon \mathcal{O}^{\mathrm{SL}_2}_{m+n^2,0}\to\mathcal{O}^{\mathrm{SL}_2}_{m,n}$ is spanned by:*

- $\mathrm{tr}(\mathbf{A}\mathbf{\Theta}_{i,j})+\mathrm{tr}(\mathbf{A}\mathbf{\Theta}^\iota_{j,i})$*, for $\mathbf{A}\in\mathcal{E}^{\mathrm{SL}_2}_{m+n^2,0}$ and $1\le i,j\le n$,*
- $\mathrm{tr}(\mathbf{A}\mathbf{\Theta}_{i,j}\mathbf{B}\mathbf{\Theta}_{i',j'})-\mathrm{tr}(\mathbf{A}\mathbf{\Theta}_{i,j'})\,\mathrm{tr}(\mathbf{B}\mathbf{\Theta}_{i',j})$*, for $\mathbf{A},\mathbf{B}\in\mathcal{E}^{\mathrm{SL}_2}_{m+n^2,0}$ and $1\le i,j\le n$.*

Proof Let $R_{i,j}$ denote the SL_2–subrepresentation of $\mathcal{O}^{\mathrm{SL}_2}_{m+n^2,0}$ spanned by

$$\mathrm{tr}(A\mathbf{\Theta}_{i,j})+\mathrm{tr}(A\mathbf{\Theta}^\iota_{j,i}), \tag{6-3}$$

for $A\in\mathcal{E}_{m+n^2,0}$. There is a SL_2–equivariant surjection $\mathcal{E}_{m+n^2,0}\to R_{i,j}$ that sends A to (6-3). The induced map on invariants is a surjection, and so $R^{\mathrm{SL}_2}_{i,j}$ is spanned by

$$\mathrm{tr}(\mathbf{A}\mathbf{\Theta}_{i,j})+\mathrm{tr}(\mathbf{A}\mathbf{\Theta}^\iota_{j,i}), \tag{6-4}$$

for $\mathbf{A}\in\mathcal{E}^{\mathrm{SL}_2}_{m+n^2,0}$.

Let $R_{i,j,i',j'}$ denote the SL_2–subrepresentation $\mathcal{O}^{\mathrm{SL}_2}_{m+n^2,0}$ spanned by

$$\mathrm{tr}(A\mathbf{\Theta}_{i,j}B\mathbf{\Theta}_{i',j'})-\mathrm{tr}(A\mathbf{\Theta}_{i,j'})\,\mathrm{tr}(B\mathbf{\Theta}_{i',j}), \tag{6-5}$$

for $A, B \in \mathcal{E}_{m+n^2,0}$. There is a SL_2–equivariant surjection

$$\mathcal{E}_{m+n^2,0} \otimes_{\mathcal{O}_{m+n^2,0}} \mathcal{E}_{m+n^2,0} \to R_{i,j,i',j'} \tag{6-6}$$

that sends $A \otimes B$ to (6-5). The induced map on invariants is a surjection.

Choose a basis v_1, v_2 for $\mathbb{V}$, and let e_{ij} denote the (i,j) elementary matrix in this basis.

Lemma 6.3.2 *The invariants $(\mathcal{E}_{m+n^2,0} \otimes_{\mathcal{O}_{m+n^2,0}} \mathcal{E}_{m+n^2,0})^{\mathrm{SL}_2}$ are spanned by $\mathbf{A} \otimes \mathbf{B}$ and $\sum_{i,j\in\{1,2\}} e_{ij}\mathbf{A} \otimes e_{ji}\mathbf{B}$.*

Proof Consider $\mathcal{O}_{m+n^2+2,0}$, where the two final coordinate functions are denoted $\mathbf{Y}_1$ and $\mathbf{Y}_2$. Then there is an SL_2–equivariant inclusion

$$\mathcal{E}_{m+n^2,0} \otimes_{\mathcal{O}_{m+n^2,0}} \mathcal{E}_{m+n^2,0} \hookrightarrow \mathcal{O}_{m+n^2+2,0} \tag{6-7}$$

that sends $A \otimes B$ to $\mathrm{tr}(A\mathbf{Y}_1 B\mathbf{Y}_2)$. The image of this map consists of the elements of $\mathcal{O}_{m+n^2+2,0}$, which are linear in $\mathbf{Y}_1$ and in $\mathbf{Y}_2$.

By Theorem 5.6.2, the subspace of $\mathcal{O}^{\mathrm{SL}_2}_{m+n^2+2,0}$ that is linear in $\mathbf{Y}_1$ and in $\mathbf{Y}_2$ is spanned by $\mathrm{tr}(\mathbf{A}\mathbf{Y}_1\mathbf{B}\mathbf{Y}_2)$ and $\mathrm{tr}(\mathbf{A}\mathbf{Y}_1)\,\mathrm{tr}(\mathbf{B}\mathbf{Y}_2)$, for $\mathbf{A}, \mathbf{B} \in \mathcal{E}^{\mathrm{SL}_2}_{m+n^2+2,0}$. Note that:

$$\sum_{i,j\in\{1,2\}} \mathrm{tr}(e_{ij}\mathbf{A}\mathbf{Y}_1 e_{ji}\mathbf{B}\mathbf{Y}_2) = \sum_{i,j\in\{1,2\}} (\mathbf{A}\mathbf{Y}_1)_{jj}(\mathbf{B}\mathbf{Y}_2)_{ii} = \mathrm{tr}(\mathbf{A}\mathbf{Y}_1)\,\mathrm{tr}(\mathbf{B}\mathbf{Y}_2)$$

It follows that the image of the span of

$$\{\mathbf{A}\otimes\mathbf{B}\} \bigcup \left\{ \sum_{i,j\in\{1,2\}} e_{ij}\mathbf{A} \otimes e_{ji}\mathbf{B} \right\}, \quad \text{for } \mathbf{A}, \mathbf{B} \in \mathcal{E}^{\mathrm{SL}_2}_{m+n^2+2,0}$$

under the map (6-7) spans $\mathcal{O}^{\mathrm{SL}_2}_{m+n^2+2,0}$. □

The map (6-6) sends $\mathbf{A}\otimes\mathbf{B}$ to

$$\mathrm{tr}(\mathbf{A}\mathbf{\Theta}_{i,j}\mathbf{B}\mathbf{\Theta}_{i',j'}) - \mathrm{tr}(\mathbf{A}\mathbf{\Theta}_{i,j'})\,\mathrm{tr}(\mathbf{B}\mathbf{\Theta}_{i',j}), \tag{6-8}$$

and $\sum_{i,j\in\{1,2\}} e_{ij}\mathbf{A} \otimes e_{ji}\mathbf{B}$ to

$$\begin{aligned}
&\sum_{i,j\in\{1,2\}} \mathrm{tr}(e_{ij}\mathbf{A}\mathbf{\Theta}_{i,j}e_{ji}\mathbf{B}\mathbf{\Theta}_{i',j'}) - \mathrm{tr}(e_{ij}\mathbf{A}\mathbf{\Theta}_{i,j})\,\mathrm{tr}(e_{ji}\mathbf{B}\mathbf{\Theta}_{i',j'}) \\
&\qquad = \sum_{i,j\in\{1,2\}} (\mathbf{A}\mathbf{\Theta}_{i,j})_{jj}(\mathbf{B}\mathbf{\Theta}_{i',j'})_{ii} - (\mathbf{A}\mathbf{\Theta}_{i,j})_{ji}(\mathbf{B}\mathbf{\Theta}_{i',j'})_{ij} \\
&\qquad = \mathrm{tr}(\mathbf{A}\mathbf{\Theta}_{i,j})\,\mathrm{tr}(\mathbf{B}\mathbf{\Theta}_{i',j'}) - \mathrm{tr}(\mathbf{A}\mathbf{\Theta}_{i,j}\mathbf{B}\mathbf{\Theta}_{i',j'}).
\end{aligned}$$

Notice that this is of the same form as (6-8), except with a minus sign and i' and j exchanged. It follows that $R^{\mathrm{SL}_2}_{i,j,i',j'}$ is spanned by elements of these two forms.

Corollary 6.2.2 states that the kernel K of ν is:

$$\sum_{1\le i,j\le n} R_{i,j} + \sum_{1\le i,j,i',j'\le n} R_{i,j,i',j'}$$

The Reynolds operator (Lemma 5.1.1) is a linear projection $\gamma\colon \mathcal{O}_{m+n^2,0} \dashrightarrow \mathcal{O}^{\mathrm{SL}_2}_{m+n^2,0}$, which sends subrepresentations to their invariant subspaces. Applying this to K gives:

$$\gamma\Bigg(\sum_{1\le i,j\le n} R_{i,j} + \sum_{1\le i,j,i',j'\le n} R_{i,j,i',j'}\Bigg) = \sum_{1\le i,j\le n} R^{\mathrm{SL}_2}_{i,j} + \sum_{1\le i,j,i',j'\le n} R^{\mathrm{SL}_2}_{i,j,i',j'}$$

Therefore, K^{SL_2} is spanned by elements of the form (6-4) and (6-8), as the $\mathbf{A}$ and $\mathbf{B}$ run over $\mathcal{E}_{m+n^2,0}$ and i,j,i',j' run over $1,\dots,n$. $\square$

Lemma 6.3.3 *The kernel of the map* $\nu\colon \mathcal{E}^{\mathrm{SL}_2}_{m+n^2,0} \to \mathcal{E}^{\mathrm{SL}_2}_{m,n}$ *is generated by:*

- $\mathbf{\Theta}_{i,j} + \mathbf{\Theta}^{\iota}_{j,i}$, *for* $1 \le i,j \le n$
- $\mathbf{\Theta}_{i,j}\mathbf{B}\mathbf{\Theta}_{i',j'} - \mathrm{tr}(\mathbf{B}\mathbf{\Theta}_{i',j})\mathbf{\Theta}_{i,j'}$ *or, equivalently,*

$$\mathbf{\Theta}_{i,j}\mathbf{B}\mathbf{\Theta}_{i',j'} - \mathbf{B}\mathbf{\Theta}_{i',j}\mathbf{\Theta}_{i,j'} - \mathbf{\Theta}^{\iota}_{i',j}\mathbf{B}^{\iota}\mathbf{\Theta}_{i,j'}$$

for $\mathbf{B} \in \mathcal{E}^{\mathrm{SL}_2}_{m+n^2,0}$ *and* $1 \le i,j,i',j' \le n$.

Proof We consider $\mathcal{O}_{m+1,n}$ with final coordinate function denoted $\mathbf{Y}$. There is an inclusion $\mathcal{E}_{m,n} \to \mathcal{O}_{m+1,n}$ that sends A to $\mathrm{tr}(A\mathbf{Y})$. The image is the subspace of $\mathcal{O}_{m+1,n}$ is linear in $\mathbf{Y}$. We have a commutative diagram:

$$\begin{tikzcd}
\mathcal{E}^{\mathrm{SL}_2}_{m+n^2,0} \arrow[r, hook] \arrow[d, "\nu"] & \mathcal{O}^{\mathrm{SL}_2}_{m+n^2+1,0} \arrow[d, "\nu"] \\
\mathcal{E}^{\mathrm{SL}_2}_{m,n} \arrow[r, hook] & \mathcal{O}^{\mathrm{SL}_2}_{m+1,n}
\end{tikzcd}$$

Hence, the kernel of ν in $\mathcal{E}^{\mathrm{SL}_2}_{m+n^2,0}$ is the preimage of the kernel of ν in $\mathcal{O}^{\mathrm{SL}_2}_{m+n^2+1,0}$. If $\mathrm{tr}(\mathbf{A}\mathbf{\Theta}_{i,j}) + \mathrm{tr}(\mathbf{A}\mathbf{\Theta}^{\iota}_{j,i}) \in \mathcal{E}_{m+n^2+1,0}$ is linear in $\mathbf{Y}$, then $\mathbf{A} = \mathbf{B}\mathbf{Y}\mathbf{C}$, for some $\mathbf{B},\mathbf{C} \in \mathcal{E}_{m+n^2,0}$. Then:

$$\mathbf{C}(\mathbf{\Theta}_{i,j} + \mathbf{\Theta}^{\iota}_{j,i})\mathbf{B} \xrightarrow{\mathrm{tr}(-\mathbf{Y})} \mathrm{tr}(\mathbf{C}(\mathbf{\Theta}_{i,j} + \mathbf{\Theta}^{\iota}_{j,i})\mathbf{B}\mathbf{Y}) = \mathrm{tr}(\mathbf{A}\mathbf{\Theta}_{i,j}) + \mathrm{tr}(\mathbf{A}\mathbf{\Theta}^{\iota}_{j,i})$$

Thus, the preimage of $\mathrm{tr}(\mathbf{A}\mathbf{\Theta}_{i,j}) + \mathrm{tr}(\mathbf{A}\mathbf{\Theta}^{\iota}_{j,i})$ is in the ideal generated by $\mathbf{\Theta}_{i,j} + \mathbf{\Theta}^{\iota}_{j,i}$. A similar argument works for the other class of relations. $\square$

6.4 Presentations of mixed invariants and mixed concomitants

By Proposition 6.1.1, a presentation of $\mathcal{O}^{\mathrm{SL}_2}_{m,n}$ (resp. $\mathcal{E}^{\mathrm{SL}_2}_{m,n}$) can be produced by choosing a presentation of

$$\mathcal{O}^{\mathrm{SL}_2}_{m+n^2,0} \quad (\text{resp. } \mathcal{E}^{\mathrm{SL}_2}_{m+n^2,0})$$

and adding a relation coming from the generators for the kernel of ν.

Combining Procesi's presentation of $\mathcal{O}^{\mathrm{SL}_2}_{m+n^2,0}$ with the previous presentation of the kernel of ν gives the following.

Theorem 6.4.1 *The algebra $\mathcal{O}^{\mathrm{SL}_2}_{m,n}$ is generated as a commutative algebra by $\mathrm{tr}(\mathbf{A})$, as $\mathbf{A}$ runs over all words in the coordinate functions $\{\mathbf{X}_i, \mathbf{\Theta}_{i,j}\}$. The relations are generated by:*

- *Procesi's F–relation* (5-7),
- $\mathrm{tr}(\mathbf{A}\mathbf{\Theta}_{i,j}) = \mathrm{tr}(\mathbf{A}\mathbf{\Theta}_{j,i}) - \mathrm{tr}(\mathbf{A})\,\mathrm{tr}(\mathbf{\Theta}_{j,i})$, *for $\mathbf{A}$ a word in the $\{\mathbf{X}_i, \mathbf{\Theta}_{i,j}\}$ and $1 \leq i, j \leq n$,*
- $\mathrm{tr}(\mathbf{A}\mathbf{\Theta}_{i,j}\mathbf{B}\mathbf{\Theta}_{i',j'}) = \mathrm{tr}(\mathbf{A}\mathbf{\Theta}_{i,j'})\,\mathrm{tr}(\mathbf{B}\mathbf{\Theta}_{i',j})$, *for $\mathbf{A},\mathbf{B}$ words in $\{\mathbf{X}_i, \mathbf{\Theta}_{i,j}\}$ and $1 \leq i, j, i', j' \leq n$.*

Proof The generators and the first relation come from Theorem 5.6.2. The second two relations come from Lemma 6.3.1. Note that any occurrence of $\mathbf{X}_i^\iota$ or $\mathbf{\Theta}_{i,j}^\iota$ may be replaced by $\mathrm{tr}(\mathbf{X}_i) - \mathbf{X}_i$ or $\mathrm{tr}(\mathbf{\Theta}_{i,j}) - \mathbf{\Theta}_{i,j}$, so as to avoid using ι. In this way, the second relation is a reformulation of $\mathrm{tr}(\mathbf{A}\mathbf{\Theta}_{i,j}) = -\,\mathrm{tr}(\mathbf{A}\mathbf{\Theta}^\iota_{j,i})$. □

Combining Brumfiel and Hilden's presentation of $\mathcal{E}^{\mathrm{SL}_2}_{m+n^2,0}$ with the previous presentation of the kernel of ν gives the following.

Theorem 6.4.2 *The algebra $\mathcal{E}^{\mathrm{SL}_2}_{m,n}$ is generated by*

- *coordinate functions $\mathbf{X}_i$, for $1 \leq i \leq m$,*
- *adjoint coordinate functions $\mathbf{X}_i^\iota$, for $1 \leq i \leq m$,*
- *outer product functions $\mathbf{\Theta}_{i,j}$, for $1 \leq i, j \leq n$.*

The relations may be written two ways. Let ι be the anti-involution with $(\mathbf{X}_i)^\iota := \mathbf{X}_i^\iota$, and $(\mathbf{\Theta}_{i,j})^\iota := -\mathbf{\Theta}_{j,i}$. Then the relations are generated by:

(1) $(\mathbf{A}+\mathbf{A}^\iota)\mathbf{B} = \mathbf{B}(\mathbf{A}+\mathbf{A}^\iota)$, *for all $\mathbf{A},\mathbf{B}$ words in $\{\mathbf{X}_i, \mathbf{X}_i^\iota, \mathbf{\Theta}_{i,j}\}$,*

(2) $\mathbf{\Theta}_{i,j}\mathbf{A}\mathbf{\Theta}_{i',j'} = \mathbf{A}\mathbf{\Theta}_{i',j}\mathbf{\Theta}_{i,j'} - \mathbf{\Theta}_{j,i'}\mathbf{A}^\iota\mathbf{\Theta}_{i,j'}$, *for all $1 \leq i, j, i', j' \leq n$ and all words $\mathbf{A}$ in $\{\mathbf{X}_i, \mathbf{X}_i^\iota, \mathbf{\Theta}_{i,j}\}$.*

Let $\mathrm{tr}(\mathbf{A}) := \mathbf{A} + \mathbf{A}^\iota$. *Then the relations are generated by:*

(1) $\mathrm{tr}(\mathbf{A})\mathbf{B} = \mathbf{B}\,\mathrm{tr}(\mathbf{A})$, *for all* $\mathbf{A}, \mathbf{B}$ *words in* $\{\mathbf{X}_i, \mathbf{X}_i^\iota, \mathbf{\Theta}_{i,j}\}$,

(2) $\mathbf{\Theta}_{i,j}\mathbf{A}\mathbf{\Theta}_{i',j'} = \mathrm{tr}(\mathbf{A}\mathbf{\Theta}_{i',j})\mathbf{\Theta}_{i,j'}$ *for all* $1 \leq i, j, i', j' \leq n$ *and* $\mathbf{A}$ *a word in* $\{\mathbf{X}_i, \mathbf{X}_i^\iota, \mathbf{\Theta}_{i,j}\}$.

Proof Theorem 5.6.4 implies that $\mathcal{E}_{m,n}^{\mathrm{SL}_2}$ is generated by $\{\mathbf{X}_i, \mathbf{X}_i^\iota, \mathbf{\Theta}_{i,j}, \mathbf{\Theta}_{i,j}^\iota\}$, and that the trace of any element is central. Lemma 6.3.3 shows the rest of the relations are given by $\mathbf{\Theta}_{i,j} = -\mathbf{\Theta}_{j,i}^\iota$ and $\mathbf{\Theta}_{i,j}\mathbf{A}\mathbf{\Theta}_{i',j'} = \mathrm{tr}(\mathbf{A}\mathbf{\Theta}_{i',j})\mathbf{\Theta}_{i,j'}$. The relation $\mathbf{\Theta}_{i,j} = -\mathbf{\Theta}_{j,i}^\iota$ may be used to eliminate the variables $\mathbf{\Theta}_{i,j}^\iota$, which in turn eliminates the need for that relation. The remaining generators and relations give the theorem as stated. □

7 Presenting the character algebra

In this section we apply the results of the previous section to the decorated character algebra.

7.1 A presentation of the representation algebra

First, we will need a presentation of the representation algebra $\mathcal{O}\mathrm{Rep}(G, M)$, which is the coordinate ring of the representation scheme $\mathrm{Hom}((G, M), (\mathrm{SL}_2(\mathbb{C}), \mathbb{V}))$.

The definition of the representation algebra will require an explicit presentation of (G, M). Choose a finite generating set $\{g_i\}_{1\leq i\leq n} \subset G$ for G, and a finite set of G–orbit representatives $\{p_j\}_{1\leq j\leq n} \subset M$.[9]

Definition 7.1.1 The *representation algebra* $\mathcal{O}\mathrm{Rep}(G, M)$ is the quotient of $\mathcal{O}_{m,n}$ by the ideal generated by:

(1) $\det(\mathbf{X}_i) - 1$ for all $1 \leq i \leq n$,

(2) $\varphi(\mathbf{X}_{i_1}\mathbf{X}_{i_2}\cdots\mathbf{X}_{i_j} - \mathrm{Id}_{\mathbb{V}})$ for each relation $g_{i_1}g_{i_2}\cdots g_{i_j} = e$ between the generators in G, and each $\varphi \in \mathrm{End}(\mathbb{V})^\vee$,

(3) $\varphi((\mathbf{X}_{j_1}\mathbf{X}_{j_2}\cdots\mathbf{X}_{j_k} - \mathrm{Id}_{\mathbb{V}})\mathbf{v}_j)$ for each $1 \leq j \leq m$, each element $g_{j_1}g_{j_2}\cdots g_{j_k}$ of the stabilizer of p_j in G, and each $\varphi \in \mathbb{V}^\vee$.

Remark 7.1.2 If M is a free G–set, then the third class of relations is redundant. If G is a free group and M is a free G–set, then the second and third classes of relations are empty.

[9] For simplicity, we will assume the generating set $\{g_i\}$ is closed under inverses. That way, every element of g can be written as a inversion-free word in $\{g_i\}$.

Proposition 7.1.3 *Let A be a commutative $\mathbb{C}$–algebra. Then there is a natural, SL_2–equivariant bijection (functorial in A):*

$$\left\{ \begin{array}{c} \text{group action maps} \\ (G,M)\to(\mathrm{SL}_2(A), A\otimes\mathbb{V}) \end{array}\right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \mathbb{C}\text{–algebra maps} \\ \mathcal{O}\mathrm{Rep}(G,M)\to A \end{array}\right\}$$

Proof Given a map of group actions

$$f=(f_G,f_M)\colon (G,M)\to(\mathrm{SL}_2(A), A\otimes\mathbb{V}),$$

define a map $\alpha_f\colon \mathcal{O}_{m,n}\to A$ by:

$$\begin{aligned} \alpha_f(\varphi(\mathbf{X}_i)) &= \varphi(f_G(g_i)), \quad \text{for each } 1\le i\le m, \text{ and each } \varphi\in\mathrm{End}(\mathbb{V})^\vee, \\ \alpha_f(\varphi(\mathbf{v}_i)) &= \varphi(f_M(v_j)), \quad \text{for each } 1\le j\le n, \text{ and each } \varphi\in\mathbb{V}^\vee. \end{aligned}$$

Since $\det(f_G(g_i))=1$, the map α_f kills the elements $\det(\mathbf{X}_i)-1$ in $\mathcal{O}_{m,n}$. Since f is a map of group actions, α_f kills the remaining relations for $\mathcal{O}\mathrm{Rep}(G,M)$, and so α_f descends to a well-defined map $\mathcal{O}\mathrm{Rep}(G,M)\to A$.

Given a map $\alpha\colon \mathcal{O}\mathrm{Rep}(G,M)\to A$, define a map of group actions $f_\alpha=(f_G,f_M)$ as follows. Let $f_G(g_i)$ be the unique element in $\mathrm{End}(\mathbb{V})$ such that:

$$\varphi(f_G(g_i))=\alpha(\varphi(\mathbf{X}_i)), \quad \text{for each } \varphi\in\mathrm{End}(\mathbb{V})^\vee.$$

Let $f_M(v_j)$ be the unique element in $\mathbb{V}$ such that:

$$\varphi(f_M(v_j))=\alpha(\varphi(\mathbf{v}_j)), \quad \text{for each } \varphi\in\mathbb{V}^\vee.$$

Since α kills $\det(\mathbf{X}_i)-1$, $f_G(g_i)\in\mathrm{SL}_2(\mathbb{C})$. The remaining relations in $\mathcal{O}\mathrm{Rep}(G,M)$ imply that $f_\alpha\colon (G,M)\to(\mathrm{SL}_2(\mathbb{C}),\mathbb{V})$ is a map of group actions.

These two constructions are directly seen to be mutual inverses that are SL_2–equivariant. Functoriality is straight-forward. □

As a consequence of the above universal property, the algebra $\mathcal{O}\mathrm{Rep}(G,M)$ is independent of the choice of presentation for (G,M).

Corollary 7.1.4 *The $\mathbb{C}$–valued points of the scheme*

$$\mathrm{Rep}(G,M):=\mathrm{Spec}(\mathcal{O}\mathrm{Rep}(G,M))$$

are in bijection with $\mathrm{Hom}((G,M),(\mathrm{SL}_2(\mathbb{C}),\mathbb{V}))$.

7.2 Getting invariant relations

The aim is to present the SL_2–invariants in $\mathcal{O}\mathrm{Rep}(G, M)$. The previous section gives a presentation of $\mathcal{O}_{m,n}^{\mathrm{SL}_2}$. However, the relations of the second and third type given in Definition 7.1.1 are not SL_2–equivariant, and so Lemma 5.1.3 cannot be used. This is fixed by tensoring with $\mathrm{End}(\mathbb{V})$, and passing to the even subalgebra.

The defining surjection

$$\pi\colon \mathcal{O}_{m,n} \to \mathcal{O}\mathrm{Rep}(G, M)$$

gives a surjection

$$\pi_{\mathcal{E}}\colon \mathcal{E}_{m,n} \to \mathrm{End}(\mathbb{V}) \otimes \mathcal{O}\mathrm{Rep}(G, M).$$

Using $-I \in \mathrm{SL}_2(\mathbb{C})$, any $\mathrm{SL}_2(\mathbb{C})$–representation V splits into *even* and *odd* summands:

$$V^e := \{v \in V \mid (-I)\cdot v = v\}$$
$$V^o := \{v \in V \mid (-I)\cdot v = -v\}$$

The even part contains the $\mathrm{SL}_2(\mathbb{C})$–invariants; $V^{\mathrm{SL}_2} \subseteq V^e$.

One checks that $\mathrm{End}(\mathbb{V})$ is even and $\mathbb{V}$ is odd. It then follows that $\mathcal{O}^e_{m,n} \subset \mathcal{O}_{m,n}$ is the subalgebra of functions which have even degree in $\mathbb{V}^n$,

$$\mathcal{O}^e_{m,n} = \{f \in \mathcal{O}_{m,n} \mid f(\mathsf{A}_1, \dots, \mathsf{A}_n, v_1, \dots, v_m) = f(\mathsf{A}_1, \dots, \mathsf{A}_n, -v_1, \dots, -v_m)\}.$$

The even parts of $\mathcal{E}_{m,n}$ and $\mathcal{O}\mathrm{Rep}(G, M)$ are similarly equal to the subalgebras of functions even in the $\mathbb{V}^n$ part.

Let $\pi^e_{\mathcal{E}}$ denote the restricted surjection:

$$\pi^e_{\mathcal{E}}\colon \mathcal{E}^e_{m,n} \to \mathrm{End}(\mathbb{V}) \otimes \mathcal{O}\mathrm{Rep}(G, M)^e$$

Remark 7.2.1 (Technical necessities) Tensoring with $\mathrm{End}(\mathbb{V})$ and restricting to the even part are necessary so that the kernel of $\pi^e_{\mathcal{E}}$ is generated by SL_2–invariants. However, in simple cases this is unnecessary. If M is a free G–set, $\pi_{\mathcal{E}}$ already has an invariantly generated kernel. If G is a free group and M is a free G–set, π already has an invariantly generated kernel.

Lemma 7.2.2 *The kernel of* $\pi^e_{\mathcal{E}}$ *is generated by:*

(1) $\mathbf{X}_i\mathbf{X}_i^\iota - I$ *for all* $1 \le i \le n$,

(2) $\mathbf{X}_{i_1}\mathbf{X}_{i_2}\cdots\mathbf{X}_{i_j} - I$ *for each relation* $g_{i_1}g_{i_2}\cdots g_{i_j} = e$ *between the generators in* G,

(3) $(\mathbf{X}_{j_1}\mathbf{X}_{j_2}\cdots\mathbf{X}_{j_k} - I)\mathbf{\Theta}_{j,j'}$ *for each* $1 \leq j, j' \leq m$, *and each element* $g_{j_1}g_{j_2}\cdots g_{j_k}$ *of the stabilizer of* p_j.

Proof Let I denote the two-sided ideal in $\mathcal{E}^e_{m,n}$ generated by the elements in the statement of the lemma, so that we wish to prove $I = \ker(\pi^e_{\mathcal{E}})$.

The kernel of $\pi_{\mathcal{E}}$ is spanned by elements of the form Ar, where $A \in \mathcal{E}_{m,n}$ and r runs over all the relations in Definition 7.1.1. Such an element is in $\mathcal{E}^e_{m,n}$ if both A and r are in $\mathcal{E}^e_{m,n}$, or if they are both in $\mathcal{E}^o_{m,n}$. The first two relations in Definition 7.1.1 are even, the third is odd. In the third case, we can find $\{\varphi_i\} \in \mathbb{V}^\vee$ and $A_i \in \mathcal{E}^e_{m,n}$ such that:

$$A\varphi((\mathbf{X}_{j_1}\mathbf{X}_{j_2}\cdots\mathbf{X}_{j_k} - I)\mathbf{v}_j) = \left[\sum_{1\leq i\leq m} A_i\varphi_i(\mathbf{v}_i)\right]\varphi((\mathbf{X}_{j_1}\mathbf{X}_{j_2}\cdots\mathbf{X}_{j_k} - I)\mathbf{v}_j)$$

For any $\varphi, \varphi' \in \mathbb{V}^\vee$, there is a $\varphi'' \in \mathrm{End}(\mathbb{V})^\vee$ such that

$$\varphi'(\mathbf{v}_i)\varphi((\mathbf{X}_{j_1}\mathbf{X}_{j_2}\cdots\mathbf{X}_{j_k} - I)\mathbf{v}_j) = \varphi''((\mathbf{X}_{j_1}\mathbf{X}_{j_2}\cdots\mathbf{X}_{j_k} - I)\mathbf{\Theta}_{j,i})$$

The kernel of $\pi^e_{\mathcal{E}}$ is then spanned by elements of the form (as A runs over ${}^e\mathcal{E}_{m,n}$):

(1) $A(\det(\mathbf{X}_i) - 1)$,

(2) $A\varphi\left(\mathbf{X}_{i_1}\mathbf{X}_{i_2}\cdots\mathbf{X}_{i_j} - I\right)$ for each relation in G, and each $\varphi \in \mathrm{End}(\mathbb{V})^\vee$,

(3) $A\varphi\left((\mathbf{X}_{j_1}\mathbf{X}_{j_2}\cdots\mathbf{X}_{j_k} - I)\mathbf{\Theta}_{j,i}\right)$ for each $1 \leq i, j \leq m$, each element $g_{j_1}g_{j_2}\cdots g_{j_k}$ of the stabilizer of p_j, and each $\varphi \in \mathrm{End}(\mathbb{V})^\vee$.

By Proposition 5.2.1, $\mathbf{X}_i\mathbf{X}_i^\iota - I = (\det(\mathbf{X}_i) - 1)I$. Then $I \subseteq \ker(\pi^e_{\mathcal{E}})$, since

$$I \in \mathrm{End}(\mathbb{V}) \otimes \mathrm{End}(\mathbb{V})^\vee.$$

Now, let $A \in \mathcal{E}^e_{m,n}$ and $\varphi \in \mathrm{End}(\mathbb{V})^\vee$. Assume first that $A = fvw^\perp$ and $\varphi = \mathrm{tr}(v'w'^\perp -)$, for $f \in \mathcal{O}^e_{m,n}$ and $v, w, v', w' \in \mathbb{V}$. Then, for $B \in I$,

$$A\varphi(B) = fvw^\perp\,\mathrm{tr}(v'w'^\perp B) = f(vw'^\perp)B(v'w^\perp) \in I.$$

By linearity, $A\varphi(B) \in I$ for general A and φ. Then the above spanning set for $\ker(\pi^e_{\mathcal{E}})$ is contained in I. Therefore, $I = \ker(\pi^e_{\mathcal{E}})$. □

The map $\pi^e_{\mathcal{E}}$ then has a kernel which is generated by SL_2–invariants. By Lemma 5.1.3, the kernel of this map when restricted to invariants is generated by the same set.

Corollary 7.2.3 *The kernel of* $\pi^{\mathrm{SL}_2}_{\mathcal{E}}\colon \mathcal{E}^{\mathrm{SL}_2}_{m,n} \to (\mathrm{End}(\mathbb{V}) \otimes \mathcal{O}\mathrm{Rep}(G, M))^{\mathrm{SL}_2}$ *is generated by:*

(1) $\mathbf{X}_i\mathbf{X}_i^{\iota} - I$ *for all* $1 \le i \le n$,

(2) $\mathbf{X}_{i_1}\mathbf{X}_{i_2}\cdots\mathbf{X}_{i_j} - I$ *for each relation* $g_{i_1}g_{i_2}\cdots g_{i_j} = e$ *between the generators in* G,

(3) $(\mathbf{X}_{j_1}\mathbf{X}_{j_2}\cdots\mathbf{X}_{j_k} - I)\mathbf{\Theta}_{j,j'}$ *for each* $1 \le j, j' \le m$, *and each element* $g_{j_1}g_{j_2}\cdots g_{j_k}$ *of the stabilizer of* p_j.

This corollary can be combined with the presentation of $\mathcal{E}_{m,n}^{\mathrm{SL}_2}$ (Theorem 6.4.2) to yield a presentation of $(\mathrm{End}(\mathbb{V}) \otimes \mathcal{O}\mathrm{Rep}(G, M))^{\mathrm{SL}_2}$.

Corollary 7.2.4 *The algebra*

$$(\mathrm{End}(\mathbb{V}) \otimes \mathcal{O}\mathrm{Rep}(G, M))^{\mathrm{SL}_2}$$

is generated by $\{\mathbf{X}_i, \mathbf{X}_i^{\iota}, \mathbf{\Theta}_{j,k}\}$ *for* $1 \le i \le m$ *and* $1 \le j, k \le m$, *with relations generated by*

(1) $\mathbf{X}_i\mathbf{X}_i^{\iota} - I$ *for all* $1 \le i \le n$,

(2) $\mathbf{X}_{i_1}\mathbf{X}_{i_2}\cdots\mathbf{X}_{i_j} - I$ *for each relation* $g_{i_1}g_{i_2}\cdots g_{i_j} = e$ *between the generators in* G,

(3) $(\mathbf{X}_{j_1}\mathbf{X}_{j_2}\cdots\mathbf{X}_{j_k} - I)\mathbf{\Theta}_{j,j'}$ *for each* $1 \le j, j' \le m$, *and each element* $g_{j_1}g_{j_2}\cdots g_{j_k}$ *of the stabilizer of* p_j,

(4) $(\mathbf{A} + \mathbf{A}^{\iota})\mathbf{B} = \mathbf{B}(\mathbf{A} + \mathbf{A}^{\iota})$, *for all* $\mathbf{A}, \mathbf{B}$ *words in* $\{\mathbf{X}_i, \mathbf{X}_i^{\iota}, \mathbf{\Theta}_{i,j}\}$,

(5) $\mathbf{\Theta}_{i,j}\mathbf{A}\mathbf{\Theta}_{i',j'} = \mathbf{A}\mathbf{\Theta}_{i',j}\mathbf{\Theta}_{i,j'} - \mathbf{\Theta}_{j,i'}\mathbf{A}^{\iota}\mathbf{\Theta}_{i,j'}$, *for all* $1 \le i, j, i', j' \le n$ *and* $\mathbf{A}$ *a word in* $\{\mathbf{X}_i, \mathbf{X}_i^{\iota}, \mathbf{\Theta}_{i,j}\}$.

7.3 A presentation of the matrix character algebra

There is a more natural presentation of the algebra $(\mathrm{End}(\mathbb{V}) \otimes \mathcal{O}\mathrm{Rep}(G, M))^{\mathrm{SL}_2}$ that is independent of the choice of presentation of (G, M). Let $\mathbb{C}G$ be the group ring of G, and let

$$\mathcal{T}_G M^2 := \mathcal{T}_{\mathbb{C}G}(\mathbb{C}M \otimes_{\mathbb{C}} \mathbb{C}M)$$

be the tensor algebra of $\mathbb{C}M \otimes \mathbb{C}M$ over $\mathbb{C}G$. For $g \in G$, the corresponding element in $\mathbb{C}G$ will be denoted $\mathbf{X}_g$; for $p, q \in M$, the corresponding element in $\mathbb{C}M \otimes \mathbb{C}M$ will be denoted $\mathbf{\Theta}_{p,q}$.

Lemma 7.3.1 *The tensor algebra* $\mathcal{T}_G M^2$ *is naturally isomorphic to the abstract algebra generated by* $\{\mathbf{X}_i, \mathbf{X}_i^{\iota}, \mathbf{\Theta}_{j,k}\}$ *for* $1 \le i \le m$ *and* $1 \le j, k \le m$, *with relations generated by:*

(1) $\mathbf{X}_i\mathbf{X}_i^{\iota} - I$ *for all* $1 \le i \le n$,

(2) $\mathbf{X}_{i_1}\mathbf{X}_{i_2}\cdots\mathbf{X}_{i_j} - I$ *for each relation* $g_{i_1}g_{i_2}\cdots g_{i_j} = e$ *between the generators in* G,

(3) $(\mathbf{X}_{j_1}\mathbf{X}_{j_2}\cdots\mathbf{X}_{j_k} - I)\mathbf{\Theta}_{j,j'}$ *for each* $1 \leq j, j' \leq m$, *and each element* $g_{j_1}g_{j_2}\cdots g_{j_k}$ *of the stabilizer of* p_j.

Proof First, note that Relation (1) in Corollary 7.2.3 means that $\mathbf{X}_i^\iota = \mathbf{X}_i^{-1}$. For $g \in G$, let $g_{k_1}g_{k_2}\cdots g_{k_l} = g$ be a word for g in the generating set $\{g_i, g_i^{-1}\}$. Then:

$$\mathbf{X}_g = \mathbf{X}_{k_1}\mathbf{X}_{k_2}\cdots\mathbf{X}_{k_l}$$

Relation (2) in Corollary 7.2.3 guarantees this element is independent of the choice of word for g. Similarly, for $p, q \in M$, let $g_{i_1}g_{i_2}\cdots g_{i_k}p_i = p$ and $g_{j_1}g_{j_2}\cdots g_{j_l}p_j = q$ be words for p and q. Then:

$$\mathbf{\Theta}_{p,q} = \mathbf{X}_{i_1}\mathbf{X}_{i_2}\cdots\mathbf{X}_{i_k}\mathbf{\Theta}_{i,j}\mathbf{X}_{j_l}^\iota\cdots\mathbf{X}_{j_2}^\iota\mathbf{X}_{j_l}^\iota$$

Relation (3) in Corollary 7.2.3 guarantees this element is independent of the choice of words for p and q. □

There is then a surjection

$$\mathcal{T}_G M^2 \to (\mathrm{End}(\mathbb{V}) \otimes \mathcal{O}\mathrm{Rep}(G, M))^{\mathrm{SL}_2} \tag{7-1}$$

whose kernel is generated by relations of type (4) and (5) in Corollary 7.2.3. The elements $\mathbf{X}_g$ and $\mathbf{\Theta}_{p,q}$ are identified with their image. As elements of $\mathrm{End}(\mathbb{V}) \otimes \mathcal{O}\mathrm{Rep}(G, M)$, they can be evaluated on a group action map ρ: $(G, M) \to (\mathrm{SL}_2(\mathbb{C}), \mathbb{V})$ to give elements of $\mathrm{End}(\mathbb{V})$. Specifically,

$$\mathbf{X}_g(\rho) = \rho(g), \quad \mathbf{\Theta}_{p,q}(\rho) = \Theta(\rho(p), \rho(q)) = \rho(p)\rho(q)^\perp.$$

The anti-involution ι: $\mathcal{T}_G M^2 \to \mathcal{T}_G M^2$ can be formally defined by

$$(\mathbf{X}_g)^\iota := \mathbf{X}_{g^{-1}}, \quad (\mathbf{\Theta}_{p,q})^\iota = -\mathbf{\Theta}_{q,p}.$$

Similarly, the linear map tr: $\mathcal{T}_G M^2 \dashrightarrow \mathcal{T}_G M^2$ can be formally defined, by

$$\mathrm{tr}(\mathbf{A}) := \mathbf{A} + \mathbf{A}^\iota.$$

Then, the following theorem is the fruit of all of the labor so far, and the source of all of the results to follow.

Theorem 7.3.2 *By the surjection* (7-1), *the algebra* $(\mathrm{End}(\mathbb{V}) \otimes \mathcal{O}\mathrm{Rep}(G, M))^{\mathrm{SL}_2}$ *is isomorphic to the quotient of* $\mathcal{T}_G M^2$ *by the two-sided ideal generated by*

(1) $\mathrm{tr}(\mathbf{A})\mathbf{B} - \mathbf{B}\,\mathrm{tr}(\mathbf{A})$, *for all* $\mathbf{A}, \mathbf{B} \in \mathcal{T}_G M^2$,

(2) $\boldsymbol{\Theta}_{p,q}\boldsymbol{\Theta}_{p',q'} - \mathrm{tr}(\boldsymbol{\Theta}_{p',q})\boldsymbol{\Theta}_{p,q'}$, *for all* $p, q, p', q' \in M$.

Proof Quotienting by just first three relations in Corollary 7.2.3 gives $\mathcal{T}_G M^2$, by the lemma. The remaining two relations are almost copied directly from Corollary 7.2.3; the only difference is that the Relation (2) no longer contains a generic word $\mathbf{A}$.

Note that it suffices to assume that such an $\mathbf{A}$ is a word in $\{\boldsymbol{\Theta}_{p,q}\}$, since any $\mathbf{X}_g$ may be absorbed into an outer product $\boldsymbol{\Theta}_{p,q}$. For $\mathbf{A} = \boldsymbol{\Theta}_{p',q'}$, using Relation (5):

$$\begin{aligned}\boldsymbol{\Theta}_{p,q}(\boldsymbol{\Theta}_{p',q'})\boldsymbol{\Theta}_{p'',q''} &= \mathrm{tr}(\boldsymbol{\Theta}_{p',q})\boldsymbol{\Theta}_{p,q'}\boldsymbol{\Theta}_{p'',q''}\\ &= \mathrm{tr}(\boldsymbol{\Theta}_{p',q})\,\mathrm{tr}(\boldsymbol{\Theta}_{p'',q'})\boldsymbol{\Theta}_{p,q''}\\ &= \mathrm{tr}((\boldsymbol{\Theta}_{p',q'})\boldsymbol{\Theta}_{p'',q})\boldsymbol{\Theta}_{p,q''}\end{aligned}$$

The same argument works for longer $\mathbf{A}$, and so every relation of type (5) in Corollary 7.2.3 can be deduced from Relation (2) in the statement of the theorem. □

7.4 A presentation of the character algebra

Recall the map

$$\chi\colon H^+(G, M) \to \mathcal{O}\mathrm{Rep}(G, M)^{\mathrm{SL}_2},$$

which we wish to show is an isomorphism. Composing with the inclusion

$$\mathcal{O}\mathrm{Rep}(G, M) \hookrightarrow \mathrm{End}(\mathbb{V}) \otimes \mathcal{O}\mathrm{Rep}(G, M)$$

gives

$$\chi\colon H^+(G, M) \to (\mathrm{End}(\mathbb{V}) \otimes \mathcal{O}\mathrm{Rep}(G, M))^{\mathrm{SL}_2}.$$

In this larger context, χ may be expressed in terms of the trace:

$$\chi([g]) = \mathrm{tr}(\mathbf{X}_g), \quad \chi([p, q]) = \mathrm{tr}(\boldsymbol{\Theta}_{q,p}).$$

The scalars $\mathbb{C} \subset \mathrm{End}(\mathbb{V})$ are characterized as the elements which are ι–invariant. By extension, the representation algebra

$$\mathcal{O}\mathrm{Rep}(G, M) \subset \mathrm{End}(\mathbb{V}) \otimes \mathcal{O}\mathrm{Rep}(G, M)$$

is the ι–invariant subalgebra, and so

$$\mathcal{O}\mathrm{Rep}(G, M)^{\mathrm{SL}_2} \subset (\mathrm{End}(\mathbb{V}) \otimes \mathcal{O}\mathrm{Rep}(G, M))^{\mathrm{SL}_2}$$

is the ι–invariant subalgebra. Since the trace map projects onto the ι–invariants, $\mathcal{O}\mathrm{Rep}(G, M)^{\mathrm{SL}_2}$ is the subalgebra of $(\mathrm{End}(\mathbb{V}) \otimes \mathcal{O}\mathrm{Rep}(G, M))^{\mathrm{SL}_2}$ that is the image of the trace.

Define a linear map $\tau\colon \mathcal{T}_G M^2 \dashrightarrow H^+(G,M)$ by $\tau(\mathbf{X}_g) := [g]$ and

$$\tau(\mathbf{\Theta}_{p_1,q_1}\mathbf{\Theta}_{p_2,q_2}\cdots\mathbf{\Theta}_{p_n,q_n}) := [q_1,p_2][q_2,p_3]\cdots[q_n,p_1].$$

The following proposition collects the important properties of τ.

Proposition 7.4.1 *Let* $\mathbf{A},\mathbf{B}\in \mathcal{T}_G M^2$.

(1) $\tau(\mathbf{A}^\iota) = \tau(\mathbf{A})$

(2) $\tau(\mathbf{AB}) = \tau(\mathbf{BA})$

(3) $\tau(\mathrm{tr}(\mathbf{A})\mathbf{B}) = \tau(\mathbf{A})\tau(\mathbf{B})$

(4) *The map* τ *descends to a map* $\tau\colon (\mathrm{End}(\mathbb{V})\otimes\mathcal{O}\mathrm{Rep}(G,M))^{\mathrm{SL}_2} \dashrightarrow H^+(G,M)$.

Proof (1) One has $[g]+[g^{-1}] = [e][g] = 2[g]$ and so $[g^{-1}] = [g]$:

$$\tau(\mathbf{X}_g) = [g] = [g^{-1}] = \tau(\mathbf{X}_{g^{-1}})$$

Next, by definition, $[p,q] = -[q,p]$, and so:

$$\begin{aligned}\tau((\mathbf{\Theta}_{p_1,q_1}\mathbf{\Theta}_{p_2,q_2}\cdots\mathbf{\Theta}_{p_n,q_n})^\iota) &= \tau(\mathbf{\Theta}^\iota_{p_n,q_n}\mathbf{\Theta}^\iota_{p_{n-1},q_{n-1}}\cdots\mathbf{\Theta}^\iota_{p_1,q_1})\\ &= (-1)^n\tau(\mathbf{\Theta}_{q_n,p_n}\mathbf{\Theta}_{q_{n-1},p_{n-1}}\cdots\mathbf{\Theta}_{q_1,p_1})\\ &= (-1)^n[p_n,q_{n-1}][p_{n-1},q_{n-2}]\cdots[p_1,q_n]\\ &= [q_1,p_2][q_2,p_3]\cdots[q_n,p_1]\\ &= \tau(\mathbf{\Theta}_{p_1,q_1}\mathbf{\Theta}_{p_2,q_2}\cdots\mathbf{\Theta}_{p_n,q_n})\end{aligned}$$

(2) One has

$$[gh] = [g][h]-[h^{-1}g] = [h][g]-[g^{-1}h] = [hg]$$

and so $\tau(\mathbf{X}_g\mathbf{X}_h) = [gh] = [hg] = \tau(\mathbf{X}_h\mathbf{X}_g)$.

$$\begin{aligned}\tau(\mathbf{X}_g\mathbf{\Theta}_{p_1,q_1}\mathbf{\Theta}_{p_2,q_2}\cdots\mathbf{\Theta}_{p_n,q_n}) &= [q_1,p_2][q_2,p_3]\cdots[q_n,gp_1]\\ &= [q_1,p_2][q_2,p_3]\cdots[g^{-1}q_n,p_1]\\ &= \tau(\mathbf{\Theta}_{p_1,q_1}\mathbf{\Theta}_{p_2,q_2}\cdots\mathbf{\Theta}_{p_n,q_n}\mathbf{X}_g)\end{aligned}$$

$$\begin{aligned}&\tau(\mathbf{\Theta}_{p_1,q_1}\cdots\mathbf{\Theta}_{p_i,q_i}\mathbf{\Theta}_{p_{i+1},q_{i+1}}\cdots\mathbf{\Theta}_{p_n,q_n})\\ &\qquad= [q_1,p_2]\cdots[q_i,p_{i+1}][q_{i+1},p_{i+2}]\cdots[q_n,p_1]\\ &\qquad= [q_{i+1},p_{i+2}]\cdots[q_n,p_1][q_1,p_2]\cdots[q_i,p_{i+1}]\\ &\qquad= \tau(\mathbf{\Theta}_{p_{i+1},q_{i+1}}\cdots\mathbf{\Theta}_{p_n,q_n}\mathbf{\Theta}_{p_1,q_1}\cdots\mathbf{\Theta}_{p_i,q_i})\end{aligned}$$

Thus, $\tau(\mathbf{AB}) = \tau(\mathbf{BA})$.

(3) We check all the cases:

$$\tau(\mathrm{tr}(\mathbf{X}_g)\mathbf{X}_h) = \tau(\mathbf{X}_{gh} + \mathbf{X}_{g^{-1}h}) = [gh] + [g^{-1}h] = [g][h] = \tau(\mathbf{X}_g)\tau(\mathbf{X}_h)$$

$$\begin{aligned}\tau(\mathrm{tr}(\mathbf{X}_g)\mathbf{\Theta}_{p_1,q_1}\mathbf{\Theta}_{p_2,q_2}\cdots\mathbf{\Theta}_{p_n,q_n}) &= \tau((\mathbf{\Theta}_{gp_1,q_1} + \mathbf{\Theta}_{g^{-1}p_1,q_1})\mathbf{\Theta}_{p_2,q_2}\cdots\mathbf{\Theta}_{p_n,q_n}) \\ &= [q_1,p_2]\cdots[q_{n-1},p_n]([q_n,gp_1] + [q_n,g^{-1}p_1]) \\ &= [g][q_1,p_2]\cdots[q_{n-1},p_n][q_n,p_1] \\ &= \tau(\mathbf{X}_g)\tau(\mathbf{\Theta}_{p_1,q_1}\mathbf{\Theta}_{p_2,q_2}\cdots\mathbf{\Theta}_{p_n,q_n})\end{aligned}$$

$$\begin{aligned}&\tau(\mathrm{tr}(\mathbf{\Theta}_{p_1,q_1}\cdots\mathbf{\Theta}_{p_j,q_j})\mathbf{\Theta}_{p_{j+1},q_{j+1}}\cdots\mathbf{\Theta}_{p_n,q_n}) \\ &\quad= \tau(\mathbf{\Theta}_{p_1,q_1}\cdots\mathbf{\Theta}_{p_j,q_j}\mathbf{\Theta}_{p_{j+1},q_{j+1}}\cdots\mathbf{\Theta}_{p_n,q_n} \\ &\qquad\qquad + (-1)^j\mathbf{\Theta}_{q_j,p_j}\cdots\mathbf{\Theta}_{q_1,p_1}\mathbf{\Theta}_{p_{j+1},q_{j+1}}\cdots\mathbf{\Theta}_{p_n,q_n}) \\ &\quad= [q_1,p_2]\cdots[q_j,p_{j+1}]\cdots[q_n,p_1] + (-1)^j[p_j,q_{j-1}]\cdots[p_1,p_{j+1}]\cdots[q_n,q_j] \\ &\quad= [q_1,p_2]\cdots[q_{j-1},p_j][q_{j+1},p_{j+2}]\cdots[q_{n-1},p_n]([q_j,p_{j+1}][q_n,p_1] \\ &\qquad\qquad - [p_1,p_{j+1}][q_n,q_j]) \\ &\quad= ([q_1,p_2]\cdots[q_{j-1},p_j][q_j,p_1])([q_{j+1},p_{j+2}]\cdots[q_{n-1},p_n][q_n,p_j]) \\ &\quad= \tau(\mathbf{\Theta}_{p_1,q_1}\cdots\mathbf{\Theta}_{p_j,q_j})\tau(\mathbf{\Theta}_{p_{j+1},q_{j+1}}\cdots\mathbf{\Theta}_{p_n,q_n})\end{aligned}$$

(4) Let $\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D} \in \mathcal{T}_G M^2$. Then:

$$\begin{aligned}\tau(\mathbf{C}\,\mathrm{tr}(\mathbf{A})\mathbf{B}\mathbf{D}) &= \tau(\mathrm{tr}(\mathbf{A})\mathbf{B}\mathbf{D}\mathbf{C}) \\ &= \tau(\mathbf{A})\tau(\mathbf{B}\mathbf{D}\mathbf{C}) \\ &= \tau(\mathbf{A})\tau(\mathbf{D}\mathbf{C}\mathbf{B}) \\ &= \tau(\mathrm{tr}(\mathbf{A})\mathbf{D}\mathbf{C}\mathbf{B}) \\ &= \tau(\mathbf{C}\mathbf{B}\,\mathrm{tr}(\mathbf{A})\mathbf{D})\end{aligned}$$

Therefore, τ kills $\mathbf{C}(\mathrm{tr}(\mathbf{A})\mathbf{B} - \mathbf{B}\,\mathrm{tr}(\mathbf{A}))\mathbf{D}$. Next, for $\{p_i\},\{q_i\} \in M$,

$$\begin{aligned}&\tau(\mathbf{\Theta}_{p_1,q_1}\cdots\mathbf{\Theta}_{p_j,q_j}\mathbf{\Theta}_{p_{j+1},q_{j+1}}\cdots\mathbf{\Theta}_{p_n,q_n}) \\ &\quad= [q_1,p_2]\cdots[q_{j-1},p_j][q_j,p_{j+1}]\cdots[q_n,p_1] \\ &\quad= \tau(\mathbf{\Theta}_{p_{j+1},q_j})\tau(\mathbf{\Theta}_{p_1,q_1}\cdots\mathbf{\Theta}_{p_j,q_{j+1}}\cdots\mathbf{\Theta}_{p_n,q_n}) \\ &\quad= \tau(\mathbf{\Theta}_{p_{j+1},q_j})\tau(\mathbf{\Theta}_{p_j,q_{j+1}}\cdots\mathbf{\Theta}_{p_n,q_n}\mathbf{\Theta}_{p_1,q_1}\cdots\mathbf{\Theta}_{p_{j-1},q_j}) \\ &\quad= \tau(\mathrm{tr}(\mathbf{\Theta}_{p_{j+1},q_j})\mathbf{\Theta}_{p_j,q_{j+1}}\cdots\mathbf{\Theta}_{p_n,q_n}\mathbf{\Theta}_{p_1,q_1}\cdots\mathbf{\Theta}_{p_{j-1},q_j}) \\ &\quad= \tau(\mathbf{\Theta}_{p_1,q_1}\cdots\mathbf{\Theta}_{p_{j-1},q_j}\,\mathrm{tr}(\mathbf{\Theta}_{p_{j+1},q_j})\mathbf{\Theta}_{p_j,q_{j+1}}\cdots\mathbf{\Theta}_{p_n,q_n})\end{aligned}$$

Therefore, τ kills $\mathbf{A}(\mathbf{\Theta}_{p,q}\mathbf{\Theta}_{p',q'} - \mathrm{tr}(\mathbf{\Theta}_{q,p'})\mathbf{\Theta}_{p,q'})\mathbf{B}$. By Theorem 7.3.2, the map τ descends to a map $(\mathrm{End}(\mathbb{V}) \otimes \mathcal{O}\mathrm{Rep}(G,M))^{\mathrm{SL}_2} \dashrightarrow H^+(G,M)$. $\square$

We can now show that $\chi\colon H^+(G,M) \to \mathcal{O}\mathrm{Rep}(G,M)^{\mathrm{SL}_2}$ is an isomorphism.

Proof of Theorem 3.5.2 First, we show $\tau\chi(r) = 2r$ for all r by inducting on the length l of $r \in H^+(G,M)$. First, $\tau\chi(1) = \tau(1) = [e] = 2$. Next, let $r \in H^+(G,M)$ be a word of length $l-1$. Then:

$$\tau\chi([g]r) = \tau(\mathrm{tr}(\mathbf{X}_g)\chi(r)) = \tau(\mathbf{X}_g)\tau\chi(r) = [g]\tau\chi(r) = 2[g]r$$
$$\tau\chi([p,q]r) = \tau(\mathrm{tr}(\mathbf{\Theta}_{q,p})\chi(r)) = \tau(\mathbf{\Theta}_{q,p})\tau\chi(r) = [p,q]\tau\chi(r) = 2[p,q]r$$

Therefore, χ is an inclusion.

Next, we show that $\chi\tau(A) = \mathrm{tr}(A)$ for all $A \in \mathrm{End}(\mathbb{V}) \otimes \mathcal{O}\mathrm{Rep}(G,M)^{\mathrm{SL}_2}$:

$$\chi\tau(\mathbf{X}_g) = \chi([g]) = \mathrm{tr}(\mathbf{X}_g)$$
$$\begin{aligned}\chi\tau(\mathbf{\Theta}_{p_1,q_1}\mathbf{\Theta}_{p_2,q_2}\cdots\mathbf{\Theta}_{p_n,q_n}) &= \chi([q_1,p_2][q_2,p_3]\cdots[q_n,p_1]) \\ &= \mathrm{tr}(\mathbf{\Theta}_{p_2,q_1})\,\mathrm{tr}(\mathbf{\Theta}_{p_3,q_2})\cdots\mathrm{tr}(\mathbf{\Theta}_{p_1,q_n}) \\ &= \mathrm{tr}(\mathbf{\Theta}_{p_1,q_1}\mathbf{\Theta}_{p_2,q_2}\cdots\mathbf{\Theta}_{p_n,q_n})\end{aligned}$$

Therefore, the image of χ is $\mathcal{O}\mathrm{Rep}(G,M)^{\mathrm{SL}_2}$. $\square$

Appendix A Twisted character algebras

The preceding theory concerned group action maps $(G,M) \to (\mathrm{SL}_2(\mathbb{C}),\mathbb{V})$. A useful variant is to consider group action maps which are 'twisted' by a $\mathbb{Z}_2$–character of a central extension. This is important for applications to Teichmüller space, skein algebras and cluster algebras.

A.1 Twisting by a central extension

Let (G,M) be a group action. A *central extension* of (G,M) will be a group action (G',M'), together with a map

$$f = (f_G, f_M)\colon (G',M') \to (G,M)$$

such that

- f_G and f_M are surjective,
- the kernel K of f_G is central in G', and

- for each $m \in M$, the preimage $f_M^{-1}(m) \subset M'$ is a (non-empty) free K–orbit.

A central extension of groups $G' \to G$ determines a central extension of group actions up to non-canonical isomorphism.

Let $s\colon K \to \{\pm 1\}$ be a group homomorphism. Then an *s–twisted morphism* of (G, M) into $(\mathrm{SL}_2(\mathbb{C}), \mathbb{V})$ is a map of group actions $\rho\colon (G', M') \to (\mathrm{SL}_2(\mathbb{C}), \mathbb{V})$ such that, for all $k \in K$, $\rho(k) = s(k)$. If $s(K) = 1$, then an s–twisted morphism is equivalent to a(n untwisted) morphism $(G, M) \to (\mathrm{SL}_2(\mathbb{C}), \mathbb{V})$. Similarly, a splitting $h\colon (G, M) \to (G', M')$ (ie, $f \circ h$ is the identity) induces a bijection between s–twisted morphisms and untwisted morphisms by pulling back along h.

A.2 The twisted character algebra

The set of s–twisted morphisms of (G, M) into $(\mathrm{SL}_2(\mathbb{C}), \mathbb{V})$ has a natural variety structure. Define the *s–twisted representation algebra* as:

$$\mathcal{O}\mathrm{Rep}_s(G, M) := \mathcal{O}\mathrm{Rep}(G', M')/\langle g - s(g)\rangle_{g \in K}$$

This algebra has the following universal property; this follows from Proposition 7.1.3:

$$\left\{ \begin{matrix} \text{group action maps} \\ \rho\colon (G', M') \to (\mathrm{SL}_2(A), A \otimes \mathbb{V}) \\ \text{such that } \rho(g) = s(g), \forall g \in K \end{matrix} \right\} \xrightarrow{\sim} \left\{ \begin{matrix} \mathbb{C}\text{–algebra maps} \\ \mathcal{O}\mathrm{Rep}_s(G, M) \to A \end{matrix} \right\}$$

This algebra has an $\mathrm{SL}_2(\mathbb{C})$–action. Define the *twisted character algebra* to be the SL_2–invariant subalgebra,

$$\mathcal{O}\mathrm{Char}_s(G, M) := \mathcal{O}\mathrm{Rep}_s(G, M)^{\mathrm{SL}_2}.$$

The definition of $Rep_s(G, M)$ provides a surjection

$$\mathcal{O}\mathrm{Rep}(G', M') \to \mathcal{O}\mathrm{Rep}_s(G, M),$$

which induces a map

$$\mu\colon \mathcal{O}\mathrm{Char}(G', M') \to \mathcal{O}\mathrm{Char}_s(G, M).$$

Lemma A.2.1 *The map μ is surjective, and the kernel of μ is generated by elements of the form:*

- $\chi_{gh} - s(g)\chi_h$, *for $g \in K$ and $h \in G'$, and*
- $\chi_{(gp,q)} - s(g)\chi_{(p,q)}$, *for $g \in K$ and $p, q \in M'$.*

Proof outline This proof is in the same spirit as the presentation of the character algebra, so we only outline the details. First, the map on representation algebras induces a surjection

$$\mathrm{End}(\mathbb{V}) \otimes \mathcal{O}\mathrm{Rep}(G', M') \to \mathrm{End}(\mathbb{V}) \otimes \mathcal{O}\mathrm{Rep}_s(G, M)$$

One observes that the kernel is generated by $\mathbf{X}_g - s(g)I$ as g runs over K. These relations are SL_2–invariant, and so by Lemma 5.1.3, the kernel of

$$(\mathrm{End}(\mathbb{V}) \otimes \mathcal{O}\mathrm{Rep}(G', M'))^{\mathrm{SL}_2} \to (\mathrm{End}(\mathbb{V}) \otimes \mathcal{O}\mathrm{Rep}_s(G, M))^{\mathrm{SL}_2}$$

is also generated by $\mathbf{X}_g - s(g)\,\mathrm{Id}$ as g runs over K.

Since the corresponding character algebras are the ι–invariant subalgebras in this map, the kernel of μ is spanned by elements of the form:

$$\mathrm{tr}(\mathbf{A}(\mathbf{X}_g - s(g)\,\mathrm{Id})\mathbf{B}) = \mathrm{tr}((\mathbf{X}_g - s(g))\mathbf{BA}) = \mathrm{tr}(\mathbf{X}_g\mathbf{BA}) - s(g)\,\mathrm{tr}(\mathbf{BA})$$

From this, the theorem may be deduced directly. □

Then $\mathcal{O}\mathrm{Char}_s(G, M)$ can be presented by using Theorem 3.5.2, with the additional classes of relations $\{\chi_{gh} = s(g)\chi_h\}$ and $\{\chi_{(gp,q)} = s(g)\chi_{(p,q)}\}$.

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