Induced quasicocycles on groups
with hyperbolically embedded subgroups

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Let G be a group, H a hyperbolically embedded subgroup of G, V a normed G–module, U an H–invariant submodule of V. We propose a general construction which allows to extend 1–quasicocycles on H with values in U to 1–quasicocycles on G with values in V. As an application, we show that every group G with a nondegenerate hyperbolically embedded subgroup has dim $H_b^2(G, \ell^p(G)) = \infty$ for $p \geq 1$. This covers many previously known results in a uniform way. Applying our extension to quasimorphisms and using Bavard duality, we also show that hyperbolically embedded subgroups are undistorted with respect to the stable commutator length.

1 Introduction

Let $\mathbb{F}$ be a subfield of $\mathbb{C}$. All modules in this paper are left, all vector spaces are over $\mathbb{F}$. For a discrete group G, by a *normed G–module* we mean a normed vector space V endowed with a (left) action of the group G by isometries. Given a subgroup $H \leq G$, by an H–submodule of a G–module V we mean any H–invariant subspace of V with the induced action of H.

Let V be a normed G–module. Recall that a map $q\colon G \to V$ is called a 1–*quasicocycle* if there exists a constant $\varepsilon > 0$ such that for every $f, g \in G$ we have

$$\|q(fg) - q(f) - fq(g)\| \leq \varepsilon.$$

The $\mathbb{F}$–vector space of all 1–quasicocycles on G with values in V is denoted by $QZ^1(G, V)$.

The study of 1–quasicocycles is partially motivated by the fact that the kernel of the comparison map $H_b^2(G, V) \to H^2(G, V)$ from the second bounded cohomology to the ordinary second cohomology with coefficients in V can be identified with the quotient $QZ^1(G, V)/(\ell^\infty(G, V) + Z^1(G, V))$, where $\ell^\infty(G, V)$ and $Z^1(G, V)$ are

the subspaces of uniformly bounded maps and cocycles, respectively. In the last decade, techniques based on 1–quasicocycles and bounded cohomology have led to new breakthroughs in the study of rigidity of group von Neumann algebras, measure equivalence and orbit equivalence of groups, and low dimensional topology (see Calegari [7], Chifan and Sinclair [9], Monod [19], Popa [24] and references therein).

The main goal of this paper is to address the following "extension problem": *Under what conditions can a 1–quasicocycle on a subgroup $H \leq G$ be extended to the whole group G?*

Below we describe few known results in this direction.

Example 1.1 (Counting quasimorphisms) If $V = \mathbb{R}$ with the trivial action of G, 1–quasicocycles on G with values in V are called *quasimorphisms*. The classical examples are *counting quasimorphisms* of free groups introduced by Brooks [6]. Let F be a free group with a basis S and let w be a reduced word in $S \cup S^{-1}$. Given an element $f \in F$, denote by $c_w(g)$ the number of disjoint copies of w in the reduced representative of g. Then $h_w = c_w - c_{w^{-1}}$ defines a quasimorphism $F \to \mathbb{R}$ (Brooks [6]). Observe that $h_w(g)$ extends the obvious cocycle (ie, homomorphism) $H \to \mathbb{R}$ of the cyclic subgroup $H = \langle w \rangle \leq F$ that sends w^n to n for all $n \in \mathbb{Z}$.

This construction was further developed by Epstein and Fujiwara [11] and later by Bestvina and Fujiwara [4], who generalized it to the cases of hyperbolic groups and groups acting weakly properly discontinuous on hyperbolic spaces, respectively.

Recall that a 1–quasicocycle $q \in QZ^1(G, V)$ is called *antisymmetric* if

$$q(g^{-1}) = -g^{-1}q(g)$$

for every $g \in G$. The next example is essentially due to Thom (cf [26, Lemma 5.1]).

Example 1.2 (Extending antisymmetric 1–quasicocycles to free products) Let $G = H_1 * H_2$, let V be a normed G–module, and let U_i be an H_i–submodule of G, $i = 1, 2$. Then any antisymmetric 1–quasicocycles $q_i \in QZ^1(H_i, U_i)$, $i = 1, 2$, can be naturally extended to a 1–quasicocycle $G \to V$ using the normal form of elements of free products. That is, suppose that $g \in G$ has the normal form

$$g = h_1 k_1 \cdots h_n k_n,$$

where $h_i \in H_1$, $k_i \in H_2$ for $i = 1, \ldots, n$, and $k_1, h_2, \ldots, k_{n-1}, h_n$ are nontrivial. Let

$$q(g) = q_1(h_1) + h_1 q_2(k_1) + h_1 k_1 q_1(h_2) + \cdots + h_1 k_1 \cdots h_n q_2(k_n).$$

Checking that $q \in QZ^1(G, V)$ is easy. It is essential here that q_1 and q_2 are antisymmetric (see Remark 4.8).

Example 1.3 (No general extension construction exists) It is well-known and easy to prove that every quasimorphism on an amenable group decomposes as a sum of a homomorphism and a bounded map (Calegari [7]). This easily implies that if G is amenable and $H = \langle h \rangle \leq [G, G]$ is an infinite cyclic subgroup, then the natural homomorphism $H \to \mathbb{R}$ defined by $h^n \mapsto n$ does not extend to any quasimorphism of G.

In this paper we prove an extension theorem which can be thought of as a generalization of Examples 1.1 and 1.2. In fact, our construction is similar to Example 1.2, but the proof is much more involved. We state here a simplified version of our main result and refer to Theorem 4.2 for the full generality. For a group G and a normed G–module V, let $QZ^1_{as}(G, V)$ denote the subspace of all antisymmetric 1–quasicocycles on G with coefficients in V.

Theorem 1.4 *Let G be a group, H a hyperbolically embedded subgroup of G, V a normed G–module, U an H–submodule of V. Then there exists a linear map*

$$\iota \colon QZ^1_{as}(H, U) \to QZ^1_{as}(G, V)$$

such that for any $q \in QZ^1_{as}(H, U)$, we have $\iota(q)|_H \equiv q$.

It is well-known and easy to prove that every 1–quasicocycle is antisymmetric up to a bounded perturbation (see Lemma 2.5). In the notation of Theorem 1.4, this gives the following.

Corollary 1.5 *There exists a linear map $\varkappa \colon QZ^1(H, U) \to QZ^1(G, V)$ such that for any $q \in QZ^1(H, U)$, $\varkappa(q)|_H \in QZ^1(H, U)$ and*

$$\sup_{h \in H} \| \varkappa(q)(h) - q(h) \| < \infty.$$

The notion of a hyperbolically embedded subgroup of a group was introduced in Dahmani, Guirardel and Osin [10] and encompasses many examples of algebraic and geometric nature. We discuss some of them here and refer to the next section and the same work for the definition and details.

(a) Let G be any group and let $H \leq G$ be a finite subgroup or $H = G$. Then H is hyperbolically embedded in G. In what follows these cases are referred to as *degenerate.*

(b) Let G be a group hyperbolic relative to a collection of peripheral subgroups $\{H_\lambda\}_{\lambda \in \Lambda}$. Then every peripheral subgroup is hyperbolically embedded in G. In particular, if $G = H_1 * H_2$, then H_1 and H_2 are hyperbolically embedded in G.

(c) Let G be a relatively hyperbolic group and let g be a loxodromic element. Then g is contained in the unique maximal virtually cyclic subgroup $E(g)$ of G and $E(g)$ is hyperbolically embedded in G (Osin [22]). In particular, this holds for every infinite order element g of a hyperbolic group G.

(d) More generally, let G be a group acting on a hyperbolic space and containing a loxodromic element g that satisfies the Bestvina–Fujiwara WPD condition (see Bestvina and Fujiwara [4] or Dahmani et al [10] for the definition). Then g is contained in the unique maximal virtually cyclic subgroup $E(g)$ of G and $E(g)$ is hyperbolically embedded in G (Dahmani et al [10, Theorem 6.8]). This general result applies in the following cases: (d_1) G is the mapping class group of a punctured closed orientable surface and g is a pseudo-Anosov element (Bestvina and Fujiwara [4]); (d_2) $G = \mathrm{Out}(F_n)$ and g is a fully irreducible automorphism (Bestvina and Feighn [3]).

(e) Similarly to the previous example, let G be a group acting properly on a proper CAT(0) space and let g be a rank 1 element. Then g is contained in the unique maximal virtually cyclic subgroup $E(g)$ of G and $E(g)$ is hyperbolically embedded in G (Sisto [25]).

Example 1.6 (cf Bestvina and Fujiwara [4], Calegari and Fujiwara [8]) Let us illustrate our theorem by extending quasimorphisms in the case when G and g are as in examples (c), (d) or (e) above. It is well known and easy to prove that every infinite virtually cyclic group is either finite-by-(infinite cyclic) or finite-by-(infinite dihedral). If $E(g)$ is of the former type, there exists a homomorphism $q\colon E(g) \to \mathbb{R}$ that extends the natural map $g^n \to n$. By our theorem, q extends to a quasimorphism of G, which can be thought of as a generalization of the Brooks' counting quasimorphism. In particular, such quasimorphisms can always be constructed if G has no involutions.

On the other hand, if $E(g)$ is finite-by-(infinite dihedral), then it is easy to show that there exists $a \in G$ and $n \in \mathbb{N}$ such that

$$a^{-1}g^n a = g^{-n}.$$

This equality implies that every quasimorphism $E(g) \to \mathbb{R}$ is bounded. Thus no analogue of the counting quasimorphism exists in this case.

In Section 3, we develop the main idea in the construction of our extension, which is the notion of separating cosets of a subgroup H which is hyperbolically embedded in G. This allows use to associate a canonical, finite set of H–cosets to each $g \in G$, and to each such coset a finite collection of $h \in H$. This is essentially what is given by the normal forms of elements in Example 1.2, and we are then able to extend quasicocycles

in a similar manner. The main technical tool in proving that our extension actually gives a quasicocycle is the decomposition of the separating cosets of a triangle in Lemma 3.9.

In Section 5, we obtain some other corollaries of our main result. Recall that the class $\mathcal{C}_{\text{reg}}$ of Monod–Shalom is the class of groups for which $H_b^2(G, \ell^2(G)) \neq 0$. This definition was proposed as cohomological characterization of the notion of "negative curvature" in group theory (Monod and Shalom [20]). Monod and Shalom [21] develop a rich rigidity theory with respect to measure equivalence and orbit equivalence of actions of groups in $\mathcal{C}_{\text{reg}}$. These results have a variety of applications to measurable group theory, ergodic theory and von Neumann algebras.

Another similar class of groups is the class $\mathcal{D}_{\text{reg}}$ introduced by Thom [26]. $G \in \mathcal{D}_{\text{reg}}$ if G is nonamenable and there exists some $q \in QZ^1(G, \ell^2(G))$ which is unbounded. Thom proved rigidity results about the subgroup structure of groups in $\mathcal{D}_{\text{reg}}$ and showed that this class is closely related to $\mathcal{C}_{\text{reg}}$. However neither inclusion is known to hold between these two classes.

Let $\mathcal{X}$ denote the class of groups with nondegenerate hyperbolically embedded subgroups. Using Corollary 1.5 and the fact that every group $G \in \mathcal{X}$ contains a virtually free (but not virtually cyclic) hyperbolically embedded subgroup (Dahmani et al [10]), we prove the following.

Corollary 1.7 *For any $G \in \mathcal{X}$, the dimension of the kernel of the comparison map $H_b^2(G, \ell^p(G)) \to H^2(G, \ell^p(G))$ is infinite. In particular, $\mathcal{X} \subseteq \mathcal{C}_{\text{reg}} \cap \mathcal{D}_{\text{reg}}$.*

This corollary recovers several previously known results in a uniform way. For example, this was known for hyperbolic groups (Mineyev, Monod and Shalom [17]) and more generally groups acting nonelementary and acylindrically on hyperbolic spaces (Hamenstädt [14]), groups acting properly on proper CAT(0) spaces and containing a rank 1 isometry (Hamenstädt [15]), and Out(F_n) for $n \geq 2$ (Hamenstädt [13]). All of these groups belong to $\mathcal{X}$ (Dahmani et al [10]).

At the final stage of our work we learned that Bestvina, Bromberg and Fujiwara [2] independently and simultaneously proved that the dimension of the kernel of the comparison map $H_b^2(G, E) \to H^2(G, E)$ is infinite for any group acting nonelementary on a hyperbolic space and containing a WPD loxodromic isometry and any uniformly convex Banach G–module E. In fact, the class of groups acting nonelementary on a hyperbolic space and containing a WPD loxodromic isometry coincides with our class $\mathcal{X}$ (see Dahmani et al [10, Theorem 6.8 and Corollary 6.10]). Thus the result of Bestvina, Bromberg and Fujiwara [2] is stronger than Corollary 1.7.

As another application, we show that hyperbolically embedded subgroups are undistorted with respect to the stable commutator length, scl. For the definition of scl we

refer to Section 5. Given a group G and a subgroup $H \leq G$ it is straightforward to see that $\mathrm{scl}_G(h) \leq \mathrm{scl}_H(h)$ for any $h \in [H, H]$, where scl_G and scl_H are the stable commutator lengths on $[G, G]$ and $[H, H]$, respectively.

On the other hand, recall that every torsion free group H can be embedded in a group G where every element is a commutator (see Lyndon and Schupp [16, Theorem 8.1] or Osin [23] for a finitely generated version of such an embedding). In particular, scl_G vanishes on G, while scl_H can be unbounded on $[H, H]$. Thus, in general, there is no upper bound on scl_H in terms of scl_G. In what follows, we say that H is *undistorted in G with respect to the stable commutator length* if there exists a constant B such that for every $h \in [H, H]$, we have $\mathrm{scl}_H(h) \leq B\mathrm{scl}_G(h)$.

Using Theorem 1.4 and the Bavard duality, we obtain the following.

Corollary 1.8 *Let G be a group, H a hyperbolically embedded subgroup of G. Then H is undistorted in G with respect to the stable commutator length.*

Even the following particular cases seem new. Recall that a subgroup $H \leq G$ is *almost malnormal* if $|H^g \cap H| < \infty$ for every $g \in G \setminus H$.

Corollary 1.9 *Every almost malnormal quasiconvex subgroup of a hyperbolic group is undistorted with respect to the stable commutator length. In particular, so is every finitely generated malnormal subgroup of a free group.*

In Section 5 we show that the almost malnormality condition can not be omitted even for free groups (see Remark 5.8).

2 Preliminaries

Notation and conventions In this paper we allow length functions and metrics to take infinite values. In particular, the word length $|\cdot|_S$ on a group G corresponding to a (not necessary generating) set S is defined by letting $|g|_S$ be the length of a shortest word in $S \cup S^{-1}$ representing g if $g \in \langle S \rangle$ and $|g|_S = \infty$ otherwise. The corresponding metric on G is denoted by d_S; thus $\mathrm{d}_S(f, g) = |f^{-1}g|_S$.

By a path p in a (Cayley) graph we always mean a combinatorial path; we denote the label of p by $\mathbf{Lab}(p)$ and we denote the origin and terminus of p by p_- and p_+ respectively.

For the rest of the paper, we will refer to 1–quasicocycles simply as quasicocycles.

Hyperbolically embedded subgroups Let G be a group, $\{H_\lambda\}_{\lambda \in \Lambda}$ a collection of subgroups of G. Let

$$(1) \qquad\qquad \mathcal{H} = \bigsqcup_{\lambda \in \Lambda} H_\lambda.$$

Given a subset $X \subseteq G$ such that G is generated by X together with the union of all H_λ's, we denote by $\Gamma(G, X \sqcup \mathcal{H})$ the Cayley graph of G whose edges are labeled by letters from the alphabet $X \sqcup \mathcal{H}$. That is, two vertices $g, h \in G$ are connected by an edge going from g to h and labeled by $a \in X \sqcup \mathcal{H}$ if and only if a represents the element $g^{-1}h$ in G. Note that some letters from $X \sqcup \mathcal{H}$ may represent the same element in G, in which case $\Gamma(G, X \sqcup \mathcal{H})$ has multiple edges corresponding to these letters.

We think of the Cayley graphs $\Gamma(H_\lambda, H_\lambda)$ as (complete) subgraphs of $\Gamma(G, X \sqcup \mathcal{H})$. For every $\lambda \in \Lambda$, we introduce a *relative metric* $\widehat{d}_\lambda \colon H_\lambda \times H_\lambda \to [0, +\infty]$ as follows. Given $h, k \in H_\lambda$, let $\widehat{d}_\lambda(h, k)$ be the length of a shortest path in $\Gamma(G, X \sqcup \mathcal{H})$ that connects h to k and has no edges in $\Gamma(H_\lambda, H_\lambda)$. If no such a path exists, we set $\widehat{d}_\lambda(h, k) = \infty$. Clearly $\widehat{d}_\lambda$ satisfies the triangle inequality. In case the collection consists of a single subgroup $H \leq G$, we denote the corresponding relative metric on H simply by $\widehat{d}$.

Definition 2.1 Let G be a group, X a (not necessary finite) subset of G. We say that a collection of subgroups $\{H_\lambda\}_{\lambda \in \Lambda}$ of G is *hyperbolically embedded in G with respect to X* (we write $\{H_\lambda\}_{\lambda \in \Lambda} \hookrightarrow_h (G, X)$) if the following conditions hold.

  (a) The group G is generated by X together with the union of all H_λ and the Cayley graph $\Gamma(G, X \sqcup \mathcal{H})$ is hyperbolic.

  (b) For every $\lambda \in \Lambda$, $(H_\lambda, \widehat{d}_\lambda)$ is a locally finite metric space; that is, any ball of finite radius in H_λ contains finitely many elements.

Further we say that $\{H_\lambda\}_{\lambda \in \Lambda}$ is hyperbolically embedded in G and write

$$\{H_\lambda\}_{\lambda \in \Lambda} \hookrightarrow_h G$$

if $\{H_\lambda\}_{\lambda \in \Lambda} \hookrightarrow_h (G, X)$ for some $X \subseteq G$.

Example 2.2 (a) Let G be any group. Then $G \hookrightarrow_h G$. Indeed take $X = \varnothing$. Then the Cayley graph $\Gamma(G, X \sqcup \mathcal{H})$ has diameter 1 and $d(h_1, h_2) = \infty$ whenever $h_1 \neq h_2$. Further, if H is a finite subgroup of a group G, then $H \hookrightarrow_h G$. Indeed $H \hookrightarrow_h (G, X)$ for $X = G$. These cases are referred to as *degenerate*. In what follows we are only interested in nondegenerate examples.

(b) Let $G = H \times \mathbb{Z}$, $X = \{x\}$, where x is a generator of $\mathbb{Z}$. Then $\Gamma(G, X \sqcup H)$ is quasiisometric to a line and hence it is hyperbolic. However $\widehat{d}(h_1, h_2) \leq 3$ for every $h_1, h_2 \in H$. Indeed let Γ_H denote the Cayley graph $\Gamma(H, H \setminus \{1\})$. In the shift $x\Gamma_H$ of Γ_H there is an edge (labeled by $h_1^{-1}h_2 \in H$) connecting $h_1 x$ to $h_2 x$, so there is a path of length 3 connecting h_1 to h_2 and having no edges in Γ_H (see Figure 1). Thus if H is infinite, then $H \not\hookrightarrow_h (G, X)$. Moreover, a similar argument shows that $H \not\hookrightarrow_h G$.

(c) Let $G = H * \mathbb{Z}$, $X = \{x\}$, where x is a generator of $\mathbb{Z}$. In this case $\Gamma(G, X \sqcup H)$ is quasiisometric to a tree (see Figure 1) and $\widehat{d}(h_1, h_2) = \infty$ unless $h_1 = h_2$. Thus $H \hookrightarrow_h (G, X)$.
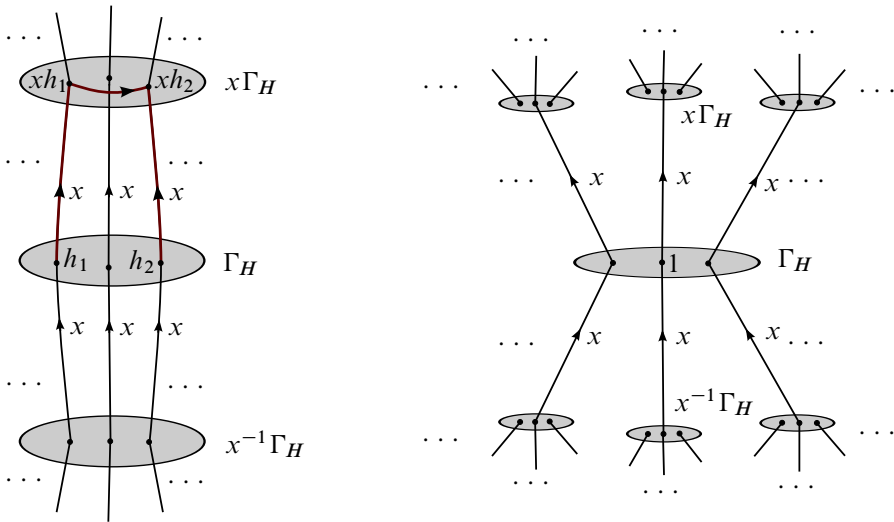

Figure 1: Cayley graphs $\Gamma(G, X \sqcup H)$ for $G = H \times \mathbb{Z}$ and $G = H * \mathbb{Z}$

It turns out that the relative metric $\widehat{d}_\lambda$ can be realized as a word metric with respect to some finite set.

Lemma 2.3 (Dahmani et al [10, Lemma 4.10]) *Let $\{H_\lambda\}_{\lambda \in \Lambda} \hookrightarrow_h G$. Then for each $\lambda \in \Lambda$, there exists a finite subset $Y_\lambda \subseteq H_\lambda$ such that $\widehat{d}_\lambda$ is bi-Lipschitz equivalent to the word metric with respect to Y_λ. That is, for $h_1, h_2 \in H_\lambda$, $\widehat{d}_\lambda(h_1, h_2)$ is finite if and only if $d_{Y_\lambda}(h_1, h_2)$ is, and the ratio $\widehat{d}_\lambda/d_{Y_\lambda}$ is uniformly bounded on $H_\lambda \times H_\lambda$ minus the diagonal.*

Components Let

$$\{H_\lambda\}_{\lambda \in \Lambda} \hookrightarrow_h (G, X).$$

Let q be a path in the Cayley graph $\Gamma(G, X \sqcup \mathcal{H})$. A (nontrivial) subpath p of q is called an H_λ–*subpath*, if the label of p is a word in the alphabet $H_\lambda \setminus \{1\}$. An H_λ–subpath p of q is an H_λ–*component* if p is not contained in a longer H_λ–subpath of q; if q is a loop, we require in addition that p is not contained in any longer H_λ–subpath of a cyclic shift of q. Further by a *component* of q we mean an H_λ–component of q for some $\lambda \in \Lambda$.

Two H_λ–components p_1, p_2 of a path q in $\Gamma(G, X \sqcup \mathcal{H})$ are called *connected* if there exists a path c in $\Gamma(G, X \sqcup \mathcal{H})$ that connects some vertex of p_1 to some vertex of p_2, and $\mathbf{Lab}(c)$ is a word consisting only of letters from $H_\lambda \setminus \{1\}$. In algebraic terms this means that all vertices of p_1 and p_2 belong to the same left coset of H_λ. Note also that we can always assume that c has length at most 1 as every nontrivial element of H_λ is included in the set of generators.

It is convenient to extend the metric $\widehat{\mathrm{d}}_\lambda$ defined above to the whole group G by assuming $\widehat{\mathrm{d}}_\lambda(f, g) := \widehat{\mathrm{d}}_\lambda(f^{-1}g, 1)$ if $f^{-1}g \in H_\lambda$ and $\widehat{\mathrm{d}}_\lambda(f, g) = \infty$ otherwise. One important technical tool is the following corollary of (a particular case of) Dahmani et al [10, Proposition 4.13].

Lemma 2.4 *There exists a constant $C > 0$ such that for any geodesic n–gon p in $\Gamma(G, X \sqcup \mathcal{H})$ and any isolated component a of p, we have $\widehat{\mathrm{d}}_\lambda(a_-, a_+) \leq Cn$.*

Proof Let $p = p_1 \cdots p_n$, where $p_1, \ldots, p_n$ are geodesic. For definiteness, suppose that a is a component of p_1, ie $p_1 = qar$. By Dahmani et al [10, Proposition 4.13] applied to the $(n+2)$–gon $qarp_2 \cdots p_n$, we have $\widehat{\mathrm{d}}_\lambda(a_-, a_+) \leq D(n+2) \leq 2Dn$, where D is a constant independent of n ($D = D(1, 0)$ in the notation of Dahmani et al [10, Proposition 4.13]). It remains to take any positive $C \geq 2D$. $\qquad\qquad\square$

Quasicocycles For a quasicocycle $q \in QZ^1(G, V)$ we define its *defect* $D(q)$ by

$$(2) \qquad\qquad D(q) = \sup_{f,g \in G} \|q(fg) - q(f) - fq(g)\|.$$

Note that

$$(3) \qquad\qquad \|q(1)\| = \|q(1 \cdot 1) - q(1) - 1q(1)\| \leq D(q).$$

We will use the following elementary fact.

Lemma 2.5 *Let G be a group, V a G–module. Then there exists a linear map*

$$\alpha\colon QZ^1(G,V) \to QZ^1_{as}(G,V)$$

such that for every $q \in QZ^1(G,V)$ we have

$$\sup_{g \in G} \|\alpha(q)(g) - q(g)\| < D(q).$$

Proof Take $\alpha(q)(g) = \frac{1}{2}(q(g) - gq(g^{-1}))$. Verifying all properties is straightforward. Indeed for every $g \in G$, we have

$$\begin{aligned}
\|\alpha(q)(g) - q(g)\| &= \tfrac{1}{2}\| -q(g) - gq(g^{-1})\| \\
&\leq \tfrac{1}{2}\|q(1) - q(g) - gq(g^{-1})\| + \tfrac{1}{2}\|q(1)\| \\
&\leq D(q),
\end{aligned}$$

where the last inequality uses (3). Further,

$$\alpha(q)(g^{-1}) = \tfrac{1}{2}(q(g^{-1}) - g^{-1}q(g)) = \tfrac{1}{2}g^{-1}(gq(g^{-1} - q(g)) = -g^{-1}\alpha(q)(g). \quad \square$$

Bounded cohomology Recall the definition of the bounded cohomology of a (discrete) group G with coefficients in an arbitrary normed G–module V. Let $C^n(G,V)$ be the vector space of n–cochains with coefficients in V, ie functions $G^n \to V$. The coboundary maps $d^n\colon C^n(G,V) \to C^{n+1}(G,V)$ are defined by the formula

$$d^n f(g_1,\ldots,g_{n+1}) = g_1 f(g_2,\ldots,g_{n+1})$$
$$+ \sum_{i=1}^{n}(-1)^i f(g_1,\ldots,g_{i-1},g_ig_{i+1},g_{i+2},\ldots,g_{n+1})$$
$$+ (-1)^{n+1}f(g_1,\ldots,g_n).$$

Let $Z^n(G,V)$ and $B^n(G,V)$ denote the cocycles and coboundaries of this complex respectively; that is, $Z^n(G,V) = \operatorname{Ker} d^n$ and $B^n(G,V) = \operatorname{Im} d^{n-1}$ for $n > 0$ and $B^0(G,V) = 0$. Recall that the ordinary cohomology groups are defined by

$$H^n(G,V) := Z^n(G,V)/B^n(G,V).$$

Restricting to the subspaces $C^n_b(G,V)$ of $C^n(G,V)$ consisting of functions whose image is bounded with respect to the norm on V, we get the complex of bounded cochains. Similarly let $Z^n_b(G,V)$ and $B^n_b(G,V)$ denote its cocycles and coboundaries. Then the group

$$H^n_b(G,V) := Z^n_b(G,V)/B^n_b(G,V)$$

is called the nth–*bounded cohomology group of G with coefficients in V*.

Note that there is a natural map $c\colon H_b^n(G, V) \to H^n(G, V)$ which is induced by the inclusion map of the cochain complexes. This map is called the *comparison map*, and the kernel of c is denoted $EH_b^n(G, V)$. The following lemma is proved in Monod [18] (see also Thom [26]) in the case when V is a Banach space. The same proof works in the general case. We briefly sketch the argument for convenience of the reader.

Lemma 2.6 *Let G be a discrete countable group, V a normed G–module. Then there exists an exact sequence*

$$0 \to \ell^\infty(G, V) + Z^1(G, V) \to QZ^1(G, V) \xrightarrow{\delta} H_b^2(G, V) \xrightarrow{c} H^2(G, V),$$

where $\ell^\infty(G, V)$ is the vector space of all uniformly bounded functions $G \to V$.

Proof We can identify $QZ^1(G, V)$ with the subspace of 1–cochains q for which d^1q is uniformly bounded, that is $d^1q \in C_b^2(G, V)$. Since $d^2 \circ d^1 \equiv 0$, d^1q is in fact a bounded 2–cocycle. Let $\delta\colon QZ^1(G, V) \to H_b^2(G, V)$ denote the composition of d^1 and the natural quotient map $Z_b^2(G, V) \to H_b^2(G, V)$. Then δq represents a trivial element of $H_b^2(G, V)$ if and only if $d^1q = d^1p$ for some bounded cochain p, which means $p \in \ell^\infty(G, V)$ and $q - p \in Z^1(G, V)$. Further if q is a bounded 2–cocycle and $[q]_b := q + B_b^2(G, V) \in H_b^2(G, V)$ is in the kernel of c, then $q = d^1f$ for some 1–cochain f, which means $f \in QZ^1(G, V)$ and $\delta f = [q]_b$. $\square$

3 Separating cosets

Throughout this section, we denote by G a group with hyperbolically embedded collection of subgroups $\{H_\lambda\}_{\lambda \in \Lambda} \hookrightarrow_h G$. Let X denote a subset of G such that $\{H_\lambda\}_{\lambda \in \Lambda} \hookrightarrow_h (G, X)$. We also keep the notation $\mathcal{H}$ and $\Gamma(G, X \sqcup \mathcal{H})$ introduced in the previous section. By $d_{X \cup \mathcal{H}}$ we denote the word metric on G with respect to the subset $X \sqcup \mathcal{H}$. By a coset of a subgroup we always mean a left coset.

We begin by introducing the notion of a separating coset for a pair of elements $f, g \in G$, which plays a crucial role in our construction.

Definition 3.1 We say that a path p in $\Gamma(G, X \sqcup \mathcal{H})$ penetrates a coset xH_λ for some $\lambda \in \Lambda$ if p decomposes as p_1ap_2, where p_1, p_2 are possibly trivial, $(p_1)_+ \in xH_\lambda$, and a is an H_λ–component of p. If, in addition, $\widehat{d}_\lambda(a_-, a_+) > 3C$, where C is the constant from Lemma 2.4, we say that p *essentially penetrates* xH_λ. Note that if p is geodesic, it penetrates every coset of H_λ at most once; in this case the vertices a_- and a_+ are called the *entrance and the exit points of p in xH_λ* and are denoted by $p_{\mathrm{in}}(xH_\lambda)$ and $p_{\mathrm{out}}(xH_\lambda)$, respectively.

Given two elements $f, g \in G$, we denote by $\mathcal{G}(f, g)$ the set of all geodesics in $\Gamma(G, X \sqcup \mathcal{H})$ going from f to g. Further we say that a coset xH_λ is (f, g)–separating if there exists a geodesic $p \in \mathcal{G}(f, g)$ that essentially penetrates xH_λ. For technical reasons we will also say xH_λ is (f, g)–separating whenever f and g are both elements of xH_λ and $f \neq g$; in this case we say xH_λ is *trivially* (f, g)–*separating*. The set of all (f, g)–separating cosets of H_λ is denoted by $S_\lambda(f, g)$.

The following lemma immediately follows from the definition and the facts that if $f, g, h \in G$ and $p \in \mathcal{G}(f, g)$, then $p^{-1} \in \mathcal{G}(g, f)$ and $hp \in \mathcal{G}(hf, hg)$.

Lemma 3.2 *For any $f, g, h \in G$ and any $\lambda \in \Lambda$, the following holds:*

  (a) $S_\lambda(f, g) = S_\lambda(g, f)$.

  (b) $S_\lambda(hf, hg) = \{hxH_\lambda \mid xH_\lambda \in S_\lambda(f, g)\}$.

The terminology in Definition 3.1 is justified by the first claim of following.

Lemma 3.3 *For any $\lambda \in \Lambda$, any $f, g \in G$ such that $f^{-1}g \notin H_\lambda$, and any (f, g)– separating coset xH_λ, the following hold:*

  (a) *Every path in $\Gamma(G, X \sqcup \mathcal{H})$ connecting f to g and composed of at most 2 geodesics penetrates xH_λ.*

  (b) *For any $p, q \in \mathcal{G}(f, g)$, we have*

$$\widehat{\mathrm{d}}_\lambda\big(p_{\mathrm{in}}(xH_\lambda), q_{\mathrm{in}}(xH_\lambda)\big) \leq 3C,$$
$$\widehat{\mathrm{d}}_\lambda\big(p_{\mathrm{out}}(xH_\lambda), q_{\mathrm{out}}(xH_\lambda)\big) \leq 3C.$$

Proof Let $xH_\lambda \in S_\lambda(f, g)$ be (f, g)–separating coset. Since $f^{-1}g \notin H_\lambda$, xH_λ is nontrivially separating. Thus there exists a geodesic $p \in \mathcal{G}(f, g)$ that essentially penetrates xH_λ; let a denote the corresponding H_λ–component of p. Let r be any other path in $\Gamma(G, X \sqcup \mathcal{H})$ connecting f to g and composed of at most 2 geodesics. If a is isolated in the loop pr^{-1}, we obtain $\widehat{\mathrm{d}}_\lambda(a_-, a_+) \leq 3C$ by Lemma 2.4. This contradicts the assumption that p essentially penetrates xH_λ. Hence a is not isolated in pr^{-1}. Since p is geodesic, a cannot be connected to a component of p. Therefore a is connected to a component of r, ie r penetrates xH_λ.

Further let $p, q \in \mathcal{G}(a, b)$ and $xH_\lambda \in S_\lambda(f, g)$. By part (a) we have $p = p_1 a p_2$ and $q = q_1 b q_2$, where $(p_1)_+ \in xH_\lambda$, $(q_1)_+ \in xH_\lambda$ and a, b are H_λ–components of p and q, respectively (see Figure 2). (Of course, p_i or q_i, $i = 1, 2$, can be trivial). Then a and b are connected. Let e be an edge or the trivial path connecting a_- to b_- and labeled by a letter from $H_\lambda \setminus \{1\}$. Applying Lemma 2.4 to the geodesic triangle $p_1 e q_1^{-1}$, we obtain $\widehat{\mathrm{d}}_\lambda(e_-, e_+) \leq 3C$, which gives us the first inequality in (b). The proof of the second inequality is symmetric. □

Corollary 3.4 *For any $f, g \in G$ and any $\lambda \in \Lambda$, we have $|S_\lambda(f, g)| \leq \mathrm{d}_{X \cup \mathcal{H}}(f, g)$. In particular, $S_\lambda(f, g)$ is finite.*
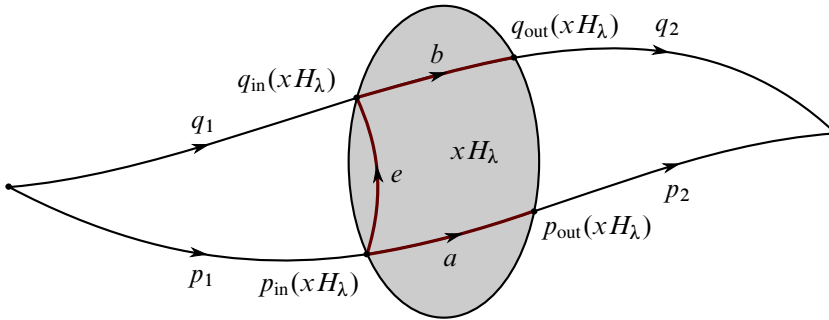

Figure 2

In this section we will use the following elementary observation several times.

Lemma 3.5 *Let p be a geodesic in $\Gamma(G, X \sqcup \mathcal{H})$. Suppose that p penetrates a coset $x H_\lambda$. Let p_0 be the initial subpath of p ending at $p_{\mathrm{in}}(x H_\lambda)$. Then $\ell(p_0) = \mathrm{d}_{X \cup \mathcal{H}}(p_-, x H_\lambda)$.*

Proof Clearly $\mathrm{d}_{X \cup \mathcal{H}}(p_-, x H_\lambda) \leq \ell(p_0)$. Suppose that $\mathrm{d}_{X \cup \mathcal{H}}(p_-, x H_\lambda) < \ell(p_0)$. Since $x H_\lambda$ has diameter 1 with respect to the metric $\mathrm{d}_{X \cup \mathcal{H}}$, we obtain

$$\mathrm{d}_{X \cup \mathcal{H}}(p_-, p_{\mathrm{out}}(x H_\lambda)) \leq \mathrm{d}_{X \cup \mathcal{H}}(p_-, x H_\lambda) + 1 < \ell(p_0) + 1.$$

However we obviously have $\ell(p_0) + 1 = \mathrm{d}_{X \cup \mathcal{H}}(p_-, p_{\mathrm{out}}(x H_\lambda))$. A contradiction. □

Definition 3.6 Given any $f, g \in G$, we define a relation $\preceq$ on the set $S_\lambda(f, g)$ as follows:

$$x H_\lambda \preceq y H_\lambda \quad \text{iff} \quad \mathrm{d}_{X \cup \mathcal{H}}(f, x H_\lambda) \leq \mathrm{d}_{X \cup \mathcal{H}}(f, y H_\lambda).$$

The next lemma is an immediate consequence of Lemma 3.3 and Lemma 3.5.

Lemma 3.7 *For any $f, g \in G$ and any $\lambda \in \Lambda$ with $f^{-1} g \notin H_\lambda$, $\preceq$ is a linear order on $S_\lambda(f, g)$ and every geodesic $p \in \mathcal{G}(f, g)$ penetrates all (f, g)–separating cosets according to the order $\preceq$. That is, $S_\lambda(f, g) = \{x_1 H_\lambda \preceq x_2 H_\lambda \preceq \cdots \preceq x_n H_\lambda\}$ for some $n \in \mathbb{N}$ and p decomposes as*

$$p = p_1 a_1 \cdots p_n a_n p_{n+1},$$

where a_i is an H_λ–component of p and $(p_i)_+ \in x_i H_\lambda$ for $i = 1, \ldots, n$ (see Figure 3).
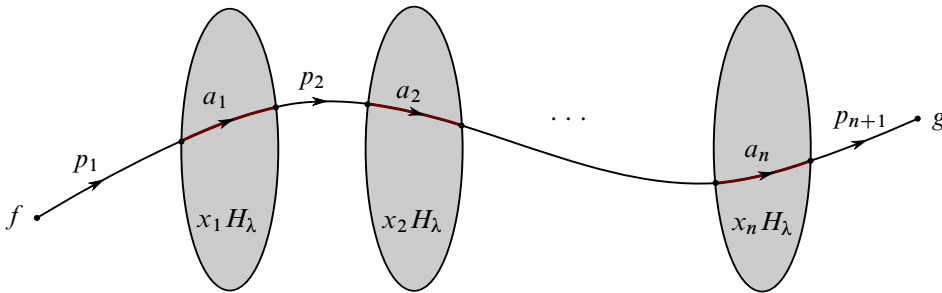
Figure 3

Given $f, g \in G$ and $xH_\lambda \in S_\lambda(f, g)$, we denote by $E(f, g; xH_\lambda)$ the set of ordered pairs of entrance-exit points of geodesics from $\mathcal{G}(f, g)$ in the coset xH_λ. That is,

$$E(f, g; xH_\lambda) = \{(p_{\text{in}}(xH_\lambda), p_{\text{out}}(xH_\lambda)) \mid p \in \mathcal{G}(f, g)\}.$$

Lemma 3.8 *For any $\lambda \in \Lambda$ and any $f, g, h, x \in G$, the following hold:*

 (a) $E(g, f; xH_\lambda) = \{(v, u) \mid (u, v) \in E(f, g; xH_\lambda)\}.$

 (b) $E(hf, hg; xH_\lambda) = \{(hu, hv) \mid (u, v) \in E(f, g; xH_\lambda)\}.$

 (c) $|E(f, g; xH_\lambda)| < \infty.$

Proof Parts (a) and (b) follow immediately from Lemma 3.2. To prove (c), note that if xH_λ trivially separates f and g, then $E(f, g; xH_\lambda) = \{(f, g)\}$. Further if xH_λ separates f and g nontrivially, fix any $(u, v) \in E(f, g; xH_\lambda)$. Then for any other $(u', v') \in E(f, g; xH_\lambda)$, we have $\widehat{\mathrm{d}}_\lambda(u, u') < 3C$ and $\widehat{\mathrm{d}}_\lambda(v, v') < 3C$ by part (b) of Lemma 3.3. Recall that $(H_\lambda, \widehat{\mathrm{d}}_\lambda)$ is a locally finite metric space by the definition of a hyperbolically embedded collection of subgroups. Hence $|E(f, g; xH_\lambda)| < \infty$. □

The main result of this section is the following.

Lemma 3.9 *For any $f, g, h \in G$ and any $\lambda \in \Lambda$, the set of all (f, g)–separating cosets of H_λ can be decomposed as*

$$S_\lambda(f, g) = S' \sqcup S'' \sqcup F,$$

where:

 (a) $S' \subseteq S_\lambda(f, h) \setminus S_\lambda(h, g)$ *and for every* $xH_\lambda \in S'$ *we have* $E(f, g; xH_\lambda) = E(f, h; xH_\lambda).$

 (b) $S'' \subseteq S_\lambda(h, g) \setminus S_\lambda(f, h)$ *and for every* $xH_\lambda \in S''$ *we have* $E(f, g; xH_\lambda) = E(h, g; xH_\lambda).$

 (c) $|F| \leq 2.$

Proof First, if $|S_\lambda(f, g)| \leq 2$ the statement is trivial, so we can assume $|S_\lambda(f, g)| > 2$. Let

$$S_\lambda(f, g) = \{x_1 H_\lambda \preceq x_2 H_\lambda \preceq \cdots \preceq x_n H_\lambda\}.$$

We fix any geodesics $q \in \mathcal{G}(h, g)$ and $r \in \mathcal{G}(f, h)$. By the first claim of Lemma 3.3, every coset from $S_\lambda(f, g)$ is penetrated by at least one of q, r. Without loss of generality we may assume that at least one of the cosets from $S_\lambda(f, g)$ is penetrated by r. Let $x_i H$ be the largest coset (with respect to the order $\preceq$) that is penetrated by r. Thus if $i < n$, then $x_{i+1} H$ is penetrated by q. Let

$$
\begin{aligned}
S' &= \{x_j H_\lambda \mid 1 \leq j < i\}, \\
S'' &= \{x_j H_\lambda \mid i + 1 < j \leq n\}, \\
F &= S_\lambda(f, g) \setminus (S' \cup S'').
\end{aligned}
$$

Obviously $|F| \leq 2$. It remains to prove (a) and (b). We will prove (a) only, the proof of (b) is symmetric.

Fix any $1 \leq j < i$. Let p be any geodesic from $\mathcal{G}(f, g)$. By Lemma 3.7, p decomposes as

$$p = p_1 a_1 p_2 a_2 p_3,$$

where a_1, a_2 are H_λ–components of p, $(p_1)_+ \in x_j H_\lambda$, and $(p_2)_+ \in x_i H_\lambda$. Similarly by the choice of i, r decomposes as

$$r = r_1 b r_2,$$

where b is an H_λ–component of r and $(r_2)_- \in x_i H_\lambda$ (see Figure 4).

Since $(r_2)_-$ and $(p_2)_+$ belong to the same coset of H_λ, there exists a path e in $\Gamma(G, X \sqcup \mathcal{H})$ of length at most 1 such that $e_- = (p_2)_+$ and $e_+ = (r_2)_-$. By Lemma 3.5, we have $\ell(p_1 a_1 p_2) = \ell(r_1)$. Hence the path $t = p_1 a_1 p_2 e r_2$ has the same length as r, ie $t \in \mathcal{G}(f, h)$. Also,

$$\tag{4} p_{\text{in}}(x_j H_\lambda) = t_{\text{in}}(x_j H_\lambda),$$

$$\tag{5} p_{\text{out}}(x_j H_\lambda) = t_{\text{out}}(x_j H_\lambda).$$

So far all our arguments were valid for any $p \in \mathcal{G}(f, g)$. Since $x_j H_\lambda \in S_\lambda(f, g)$, there exists $p \in \mathcal{G}(f, g)$ that essentially penetrates $x_j H_\lambda$, ie $\widehat{\mathrm{d}}_\lambda((a_1)_-, (a_1)_+) > 3C$ in the above notation. In this case t also essentially penetrates $x_j H_\lambda$. Thus $x_j H \in S_\lambda(f, h)$. Moreover since we have (4) and (5) for every $p \in \mathcal{G}(f, g)$, we obtain $E(f, g; x_j H_\lambda) = E(f, h; x_j H_\lambda)$.
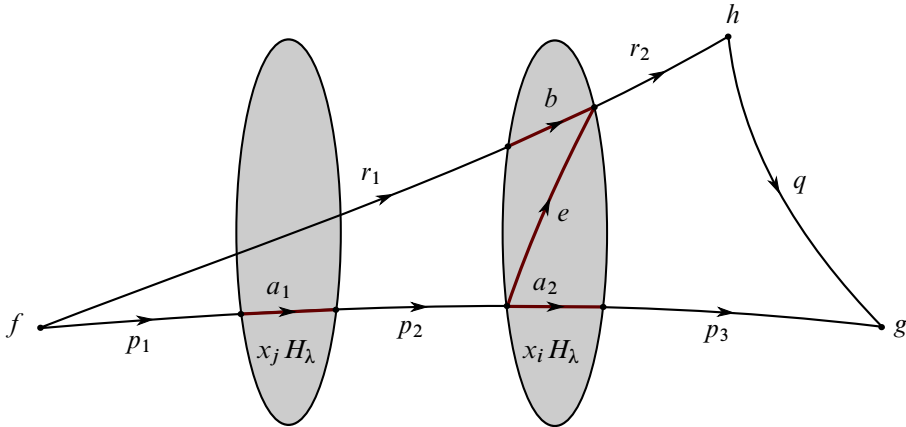
Figure 4

To complete the proof of (a) it remains to show that $x_j H_\lambda \notin S_\lambda(h, g)$. Clearly $g \notin x_j H_\lambda$, or p would not be geodesic, so $x_j H_\lambda$ does not trivially separate g and h. Thus, if $x_j H_\lambda \in S_\lambda(h, g)$ there must be a geodesic from h to g which essentially penetrates $x_j H_\lambda$. Hence by Lemma 3.3, every geodesic from h to g penetrates $x_j H_\lambda$, which means q penetrates $x_j H_\lambda$. Then using Lemma 3.5, the fact that every coset of H_λ has diameter 1 with respect to the metric $d_{X \cup \mathcal{H}}$, and the triangle inequality, we obtain

$$\begin{aligned}
\ell(q) &= d_{X \cup \mathcal{H}}(h, x_j H_\lambda) + 1 + d_{X \cup \mathcal{H}}(g, x_j H_\lambda) \\
&> d_{X \cup \mathcal{H}}(h, x_i H_\lambda) + 1 + d_{X \cup \mathcal{H}}(g, x_i H_\lambda) \\
&\geq \ell(r_2) + d_{X \cup \mathcal{H}}\big((r_2)_-, (p_3)_-\big) + \ell(p_3) \\
&\geq d_{X \cup \mathcal{H}}(h, g).
\end{aligned}$$

Since one of the inequalities is strict, this contradicts the assumption that q is geodesic.
$\square$

4   Extending quasicocycles

We keep all assumptions and notation from the previous section. For each $\lambda \in \Lambda$, let

$$\mathcal{F}_\lambda = \{h \in H_\lambda \mid h \in H_\mu \text{ for some } \mu \neq \lambda\}.$$

In particular, if $\{H_\lambda\}_{\lambda \in \Lambda}$ consists of a single subgroup H, the corresponding subset $\mathcal{F} = \varnothing$.

It follows from Lemma 2.4 that every $h \in F_\lambda$ satisfies $\widehat{d}_\lambda(1, h) \leq 2C$, where C is the constant from Lemma 2.4. Indeed for every such h there is a loop $e_1 e_2$ in $\Gamma(G, X \sqcup \mathcal{H})$,

where e_1 is an edge labeled by $h \in H_\lambda$ and e_2 is an edge labeled by the copy of h in H_μ for some $\mu \in \Lambda$. Since the metric space $(H_\lambda, \widehat{d}_\lambda)$ is locally finite by the definition of a hyperbolically embedded collection of subgroups, we obtain the following.

Lemma 4.1 $|\mathcal{F}_\lambda| < \infty$ *for all $\lambda \in \Lambda$.*

Also, for $q_\lambda \in QZ^1(H_\lambda, U_\lambda)$, let

(6) $$K_\lambda = \max\{\|q_\lambda(g)\| : \widehat{d}_\lambda(1, g) < 15C\}.$$

Observe that the constant K_λ is well-defined by local finiteness of $(H_\lambda, \widehat{d}_\lambda)$.

We can now state our main extension theorem in its full generality. Recall that for a quasicocycle q, $D(q)$ denotes its defect defined by (2).

Theorem 4.2   *Let G be a group, $\{H_\lambda\}_{\lambda \in \Lambda}$ a hyperbolically embedded collection of subgroups of G, V a normed G–module. For each $\lambda \in \Lambda$, let U_λ be an H_λ–submodule of G. Then there exists a linear map*

$$\iota: \bigoplus_{\lambda \in \Lambda} QZ^1_{as}(H_\lambda, U_\lambda) \to QZ^1_{as}(G, V)$$

such that for any $q = (q_\lambda)_{\lambda \in \Lambda} \in \bigoplus_{\lambda \in \Lambda} QZ^1_{as}(H_\lambda, U_\lambda)$ the following hold:

(a)   *For any $\lambda \in \Lambda$ and any $h \in H_\lambda \setminus \mathcal{F}_\lambda$, we have $\iota(q)(h) = q_\lambda(h)$. In particular,*

$$\sup_{h \in H_\lambda} \|\iota(q)(h) - q_\lambda(h)\| < \infty.$$

(b) $D(\iota(q)) \leq \sum_\lambda (54K_\lambda + 66D(q_\lambda)).$

Notice that the sum in part (b) is finite because $q_\lambda \equiv 0$ for all but finitely many λ, and thus $K_\lambda = D(q_\lambda) = 0$ for all but finitely many λ. If G contains a singe hyperbolically embedded subgroup, Theorem 4.2 obviously reduces to Theorem 1.4 mentioned in the introduction. Using Lemma 2.5, one can also obtain a general version of Corollary 1.7. We leave this to the reader.

Throughout the rest of the section, we use the notation of Theorem 4.2. Although our proof can be entirely written in the language of quasicocycles, the following concept helps making some arguments more symmetric and easier to comprehend. In the definition below, we write $s(a) = t(a)$ for two partial maps $s, t: A \to B$ if the value $s(a)$ is defined if and only if $t(a)$ is, and these values are equal whenever defined.

Definition 4.3 A *partial bicombing of G with coefficients in V* is a partial map $r: G \times G \to V$. We say that

(a) r is *G–equivariant* if $hr(f,g) = r(hf,hg)$ for any $f,g,h \in G$.

(b) r is *antisymmetric* if $r(f,g) = -r(g,f)$ for any $f,g \in G$.

(c) r has *bounded area* if there exists a constant A such that for any $f,g,h \in G$ for which $r(f,g)$, $r(g,h)$, and $r(h,f)$ are defined, we have

(7)
$$\|r(f,g) + r(g,h) + r(h,f)\| \leq A.$$

The infimum of all A satisfying (7) is called the *area* of r and is denoted by $A(r)$.

Let us fix $\lambda \in \Lambda$. Given $q_\lambda \in QZ_{as}^1(H_\lambda, U_\lambda)$, we define a partial map $r_\lambda: G \times G \to V$ by
$$r_\lambda(f,g) = f q_\lambda(f^{-1}g).$$

Thus $r_\lambda(f,g)$ is defined if and only if f and g belong to the same coset xH_λ.

Lemma 4.4 *The partial map $r_\lambda: G \times G \to V$ is an antisymmetric equivariant partial bicombing of G of area*

(8)
$$A(r_\lambda) \leq D(q_\lambda).$$

Proof Equivariance of r_λ is obvious and antisymmetry follows immediately from antisymmetry of q_λ. By equivariance it suffices to verify the bounded area condition for the a triple $1, g, h \in G$. We have
$$\|r_\lambda(1,g) + r_\lambda(g,h) + r_\lambda(h,1)\| = \|q_\lambda(g) + gq_\lambda(g^{-1}h) - q_\lambda(h)\| \leq D(q_\lambda). \quad \square$$

Corollary 4.5 *For any $n \in \mathbb{N}$, any $x \in G$, and any $g_0, \ldots, g_n \in xH_\lambda$, we have*
$$\left\| r_\lambda(g_0, g_n) - \sum_{i=1}^n r_\lambda(g_{i-1}, g_i) \right\| \leq (n-1)D(q_\lambda).$$

Proof The claim follows from antisymmetry, the definition of area, and (8) by induction. $\quad \square$

Our next goal is to construct a globally defined antisymmetric bounded area G–equivariant bicombing $\widetilde{r}_\lambda: G \times G \to V$ that extends r_λ. To this end, for each $f, g \in G$ and each coset xH_λ, we define the average
$$R_{\mathrm{av}}(f,g;xH_\lambda) = \frac{1}{|E(f,g;xH_\lambda)|} \sum_{\substack{(u,v) \in \\ E(f,g;xH_\lambda)}} r_\lambda(u,v).$$

If $xH_\lambda \notin S_\lambda(f, g)$, we assume $R_{\mathrm{av}}(f, g; xH_\lambda) = 0$. Note that $R_{\mathrm{av}}(f, g; xH_\lambda)$ is well-defined since $E(f, g; xH_\lambda) < \infty$ by part (c) of Lemma 3.8.

Lemma 4.6 *For any $f, g, h, x \in G$, the following hold:*

(a) $R_{\mathrm{av}}(f, g; xH_\lambda) = -R_{\mathrm{av}}(g, f; xH_\lambda)$.

(b) $R_{\mathrm{av}}(hf, hg; hxH_\lambda) = R_{\mathrm{av}}(f, g; xH_\lambda)$.

(c) *For any $(u, v) \in E(f, g; xH_\lambda)$, we have*

(9) $$\|r_\lambda(u, v) - R_{\mathrm{av}}(f, g; xH_\lambda)\| \le 2D(q_\lambda) + 2K_\lambda.$$

Proof The first claim follows from parts (a) of Lemma 3.8 and antisymmetry of r_λ. The second claim follows from parts (b) of Lemma 3.8 and the equivariance of r_λ.

To prove (c), note that for any $(u', v') \in E(f, g; xH_\lambda)$, we have

$$\max\{\widehat{\mathrm{d}}_\lambda(u, u'), \widehat{\mathrm{d}}_\lambda(v, v')\} \le 3C$$

by Lemma 3.3. Thus, using the triangle inequality and applying Corollary 4.5 to elements $u, u', v', v \in xH_\lambda$, we obtain

$$\|r_\lambda(u, v) - r_\lambda(u', v')\| \le \|r_\lambda(u, v) - r_\lambda(u, u') - r_\lambda(u', v') - r_\lambda(v', v)\|$$
$$+ \|r_\lambda(u, u')\| + \|r_\lambda(v', v)\|$$
$$\le 2D(q_\lambda) + 2K_\lambda.$$

This obviously implies (9). $\square$

Let

$$\widetilde{r}_\lambda(f, g) = \sum_{xH_\lambda \in S_\lambda(f, g)} R_{\mathrm{av}}(f, g; xH_\lambda).$$

Note that $\widetilde{r}_\lambda$ is well-defined as $S_\lambda(f, g)$ is finite for any $f, g \in G$ by Corollary 3.4.

Lemma 4.7 *The map $\widetilde{r}_\lambda \colon G \times G \to V$ is an antisymmetric G–equivariant bicombing of area*

(10) $$A(\widetilde{r}_\lambda) \le 66 D(q_\lambda) + 54 K_\lambda.$$

Proof Equivariance and antisymmetry of $\widetilde{r}_\lambda$ follow immediately from Lemma 3.2 and Lemma 4.6. In order to show that $\widetilde{r}_\lambda$ satisfies the bounded area condition, we need to estimate the norm of $\widetilde{r}_\lambda(f, g) + \widetilde{r}_\lambda(g, h) + \widetilde{r}_\lambda(h, f)$ uniformly on $f, g, h \in G$. Since $R_{\mathrm{av}}(f, g; xH_\lambda) = 0$ if $xH_\lambda \notin S_\lambda(f, g)$, we have

$$\widetilde{r}_\lambda(f, g) + \widetilde{r}_\lambda(g, h) + \widetilde{r}_\lambda(h, f) = \sum_{xH_\lambda \in G/H_\lambda} \rho(f, g, h; xH_\lambda),$$

where

$$\rho(f, g, h; xH_\lambda) := R_{\mathrm{av}}(f, g; xH_\lambda) + R_{\mathrm{av}}(g, h; xH_\lambda) + R_{\mathrm{av}}(h, f; xH_\lambda).$$

Of course, $\rho(f, g, h; xH_\lambda)$ is nontrivial only if $xH_\lambda \in S_\lambda(f, g) \cup S_\lambda(g, h) \cup S_\lambda(h, f)$.

Fix $f, g, h \in G$. We start by estimating $\rho(f, g, h; xH_\lambda)$ for cosets from $S_\lambda(f, g)$. Let $S_\lambda(f, g) = S' \sqcup S'' \sqcup F$ be the decomposition provided by Lemma 3.9. Suppose first that $xH_\lambda \in S'$. Then $xH_\lambda \in S_\lambda(f, h) = S_\lambda(h, f)$ and $E(f, g; xH_\lambda) = E(f, h; xH_\lambda)$ by Lemma 3.9. Hence

$$(11) \qquad R_{\mathrm{av}}(f, g; xH_\lambda) = R_{\mathrm{av}}(f, h; xH_\lambda) = -R_{\mathrm{av}}(h, f; xH_\lambda).$$

by Lemma 4.6 (a). On the other hand, Lemma 3.9 also states that $xH_\lambda \notin S_\lambda(h, g) = S_\lambda(g, h)$. Hence

$$(12) \qquad R_{\mathrm{av}}(g, h; xH_\lambda) = 0.$$

Summing up (11) and (12), we obtain $\rho(f, g, h; xH_\lambda) = 0$. Similarly,

$$\rho(f, g, h; xH_\lambda) = 0$$

for any $xH_\lambda \in S''$. Thus

$$(13) \qquad \sum_{\substack{xH_\lambda \in \\ S_\lambda(f, g)}} \rho(f, g, h; xH_\lambda) = \sum_{xH_\lambda \in F} \rho(f, g, h; xH_\lambda).$$

Fix a coset $xH_\lambda \in F$ and any $p \in \mathcal{G}(f, g)$, $q \in \mathcal{G}(h, g)$, $r \in \mathcal{G}(f, h)$. There are three cases to consider.

Case 1 $xH_\lambda \in S_\lambda(g, h) \cap S_\lambda(h, f)$. In this case we have $p = p_1 a p_2$, $q = q_1 c q_2$, $r = r_1 b r_2$, where a, c and b are H_λ–components of p, q and r, respectively, corresponding to the coset xH_λ (ie $a_\pm, b_\pm, c_\pm \in xH_\lambda$). Let e_1, e_2, e_3 be paths of lengths at most 1 labeled by elements of H_λ and connecting a_- to b_-, b_+ to c_-, and c_+ to a_+ (see Figure 5).

Since a geodesic in $\Gamma(G, X \sqcup \mathcal{H})$ can penetrate a coset of H_λ at most once, e_1 is either trivial or is an isolated component of a geodesic triangle (namely $p_1 e_1 r_1^{-1}$). The same holds true for e_1 and e_2. Hence by Lemma 2.4, we obtain

$$(14) \qquad \widehat{\mathrm{d}}_\lambda((e_i)_-, (e_i)_+) \le 3C, \quad i = 1, 2, 3.$$

In particular,

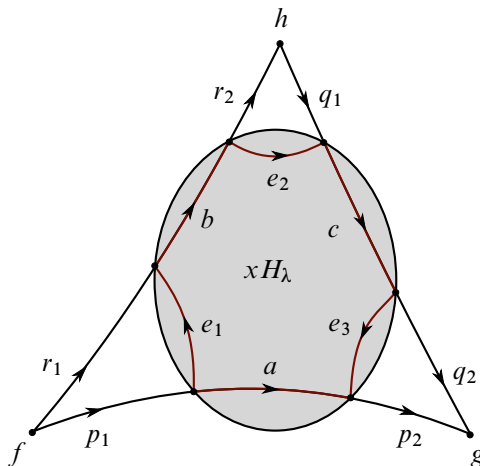$$(15) \qquad \|r_\lambda((e_i)_-, (e_i)_+)\| \le K_\lambda, \quad i = 1, 2, 3.$$

Figure 5

by the definition of K_λ (see (6)). Using the triangle inequality, applying Corollary 4.5 to the vertices of the hexagon $e_1 b e_2 c e_3 a^{-1}$, and using (15), we obtain

$$\|r_\lambda(a_-, a_+) + r_\lambda(b_+, b_-) + r_\lambda(c_+, c_-)\|$$

$$\leq \left\| r_\lambda(a_-, a_+) - r_\lambda(b_-, b_+) - r_\lambda(c_-, c_+) - \sum_{i=1}^{3} r_\lambda\big((e_i)_-, (e_i)_+\big) \right\|$$

$$+ \left\| \sum_{i=1}^{3} r_\lambda\big((e_i)_-, (e_i)_+\big) \right\|$$

$$\leq 5D(q_\lambda) + 3K_\lambda.$$

Now Lemma 4.6 (c) implies

$$(16) \quad \|\rho(f, g, h; xH_\lambda)\|$$

$$= \| R_{\mathrm{av}}(f, g; xH_\lambda) + R_{\mathrm{av}}(g, h; xH_\lambda) + R_{\mathrm{av}}(h, f; xH_\lambda) \|$$

$$\leq \| r_\lambda(a_-, a_+) + r_\lambda(c_+, c_-) + r_\lambda(b_+, b_-) \| + 6(D(q_\lambda) + K_\lambda)$$

$$\leq 11 D(q_\lambda) + 9 K_\lambda.$$

Case 2 $xH_\lambda \in S_\lambda(h, f) \setminus S_\lambda(g, h)$ or $xH_\lambda \in S_\lambda(g, h) \setminus S_\lambda(h, f)$. Since the proof in these cases is the same, we will only consider the case $xH_\lambda \in S_\lambda(h, f) \setminus S_\lambda(g, h)$. Let $p = p_1 a p_2$, $r = r_1 b r_2$, and e_1 be as in Case 1 and let e be the path of length at most 1 in $\Gamma(G, X \sqcup \mathcal{H})$ connecting b_+ to a_+ and labeled by an element of H_λ. There are two possibilities to consider.

(2a) First assume that e is isolated in the quadrilateral $ep_2q^{-1}r_2^{-1}$ (see Figure 6). Then we have $\widehat{d}_\lambda(e_-, e_+) \leq 4C$ by Lemma 2.4 and hence

$$\|r_\lambda(e_-, e_+)\| \leq K_\lambda.$$

Note that (15) remains valid for $i = 1$. Applying Corollary 4.5 to the vertices of the quadrilateral e_1bea^{-1} as in Case 1 we obtain

$$\|r_\lambda(a_-, a_+) + r_\lambda(b_+, b_-)\|$$
$$\leq \|r_\lambda(a_-, a_+) - r_\lambda\big((e_1)_-, (e_1)_+\big) - r_\lambda(b_-, b_+) - r_\lambda(e_-, e_+)\|$$
$$+ \|r_\lambda\big((e_1)_-, (e_1)_+\big)\| + \|r_\lambda(e_-, e_+)\|$$
$$\leq 3D(q_\lambda) + 2K_\lambda.$$

Since $xH_\lambda \notin S_\lambda(g, h)$, we have $R_{av}(g, h; xH_\lambda) = 0$. Finally Lemma 4.6 (c) implies

$$\tag{17} \|\rho(f, g, h; xH_\lambda)\| = \|R_{av}(f, g; xH_\lambda) + R_{av}(h, f; xH_\lambda)\|$$
$$\leq \|r_\lambda(a_-, a_+) + r_\lambda(b_+, b_-)\| + 4(D(q_\lambda) + K_\lambda)$$
$$\leq 7D(q_\lambda) + 6K_\lambda.$$

(2b) Suppose now that e is not isolated in the quadrilateral $ep_2q^{-1}r_2^{-1}$. Then e is connected to a component c of q. Let $q = q_1cq_2$ and let e_1 and e_2 be as in Case 1 (see Figure 5). Then (15) remains valid. In addition, we have $\widehat{d}_\lambda(c_-, c_+) \leq 3C$ as $xH_\lambda \notin S_\lambda(g, h)$ and hence q can not essentially penetrate xH_λ. Hence $\|r_\lambda(c_-, c_+)\| \leq K_\lambda$. The reader can easily verify that arguing as in Case 1 and then as in (17), we can obtain

$$\|r_\lambda(a_-, a_+) + r_\lambda(b_+, b_-)\| \leq 5A_\lambda + 4K_\lambda$$

and consequently

$$\tag{18} \|\rho(f, g, h; xH_\lambda)\| \leq 9D(q_\lambda) + 8K_\lambda.$$

Case 3 $xH_\lambda \notin S_\lambda(h, f) \cup S_\lambda(g, h)$. Let $p = p_1ap_2$ be as in Cases 1 and 2. There are three possibilities to consider.

(3a) a is an isolated component of $pq^{-1}r^{-1}$. In this case $\widehat{d}_\lambda(a_-, a_+) \leq 3C$.

(3b) a is connected to a component of exactly one of q, r. For definiteness, assume that a is connected to a component b of r. Then, in the notation of Case 2 (see Figure 6), e is isolated in $ep_2q^{-1}r_2^{-1}$ and we have $\widehat{d}_\lambda(e_-, e_+) \leq 4C$ by Lemma 2.4. As in Case 1, we have (14) for $i = 1$. Since $xH_\lambda \notin S_\lambda(h, f)$, r can not essentially
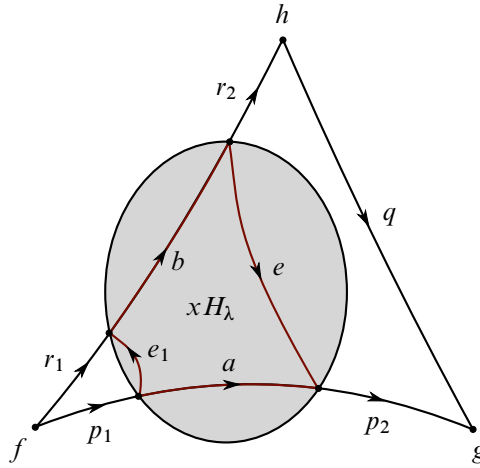
Figure 6

penetrate xH_λ. Thus $\widehat{d}_\lambda(b_-, b_+) \leq 3C$. Applying the triangle inequality to the quadrilateral $e_1 b e a^{-1}$, we obtain

$$\widehat{d}_\lambda(a_-, a_+) \leq 10C.$$

(3c) a is connected to a component b of r and a component c of q. Then in the notation of Case 1 and Figure 5, inequalities (14) remain valid and we also have $\widehat{d}_\lambda(b_-, b_+) \leq 3C$ and $\widehat{d}_\lambda(c_-, c_+) \leq 3C$ as in Case 3b. Applying the triangle inequality to the hexagon $e_1 b e_2 c e_3 a^{-1}$, we obtain

$$\widehat{d}_\lambda(a_-, a_+) \leq 15C.$$

Thus, in all Cases 3a)–3c) we have $\|r_\lambda(a_-, a_+)\| \leq K_\lambda$. Since $R_{\mathrm{av}}(g, h; xH_\lambda) = R_{\mathrm{av}}(h, f; xH_\lambda) = 0$ in this case, using Lemma 4.6 (c) we obtain

(19) $$\|\rho(f, g, h; xH_\lambda)\| = \|R_{\mathrm{av}}(f, g; xH_\lambda)\| \leq 2A(r_\lambda) + 3K_\lambda.$$

in Case 3.

Summarizing (13), (16), (17), (18), (19), and taking into account that $|F| \leq 2$, we obtain

$$\left\| \sum_{xH_\lambda \in S_\lambda(f,g)} \rho(f, g, h; xH_\lambda) \right\| = \left\| \sum_{xH_\lambda \in F} \rho(f, g, h; xH_\lambda) \right\| \leq 22 D(q_\lambda) + 18 K_\lambda.$$

Repeating the same arguments for $S_\lambda(h, f)$ and $S_\lambda(g, f)$ and summing up, we obtain

$$\|\widetilde{r}_\lambda(f, g) + \widetilde{r}_\lambda(g, h) + \widetilde{r}_\lambda(h, f)\| \leq 66 D(q_\lambda) + 54 K_\lambda. \qquad \square$$

We are now ready to prove the main extension theorem.

Proof of Theorem 4.2 Let $q = (q_\lambda)_{\lambda \in \Lambda} \in \bigoplus_{\lambda \in \Lambda} QZ^1_{as}(H_\lambda, U_\lambda)$. For each $\lambda \in \Lambda$, let $\widetilde{r}_\lambda$ be the bicombing constructed above and let $\widetilde{q}_\lambda(g) = \widetilde{r}_\lambda(1, g)$. Then $\widetilde{q}_\lambda \in QZ^1_{as}(G, V)$. Indeed we have

$$
\begin{aligned}
(20) \qquad \|\widetilde{q}_\lambda(fg) - \widetilde{q}_\lambda(f) - f\widetilde{q}_\lambda(g)\| &= \|\widetilde{r}_\lambda(1, fg) - \widetilde{r}_\lambda(1, f) - f\widetilde{r}_\lambda(1, g)\| \\
&= \|\widetilde{r}_\lambda(1, fg) + \widetilde{r}_\lambda(f, 1) + \widetilde{r}_\lambda(fg, f)\| \\
&\le A(\widetilde{r}_\lambda).
\end{aligned}
$$

antisymmetry of $\widetilde{q}_\lambda$ follows from that of $\widetilde{r}_\lambda$.

Further we define

$$
\iota(q) = \sum_{\lambda \in \Lambda} \widetilde{q}_\lambda.
$$

Since q is supported on only finitely many λ, $\iota(q)$ is equal to a finite linear combination of quasicocycles, so $\iota(q) \in QZ^1_{as}(G, V)$. It is easy to see that the maps $QZ^1_{as}(H_\lambda, U_\lambda) \to QZ^1_{as}(G, V)$ defined by $q_\lambda \mapsto \widetilde{q}_\lambda$ are linear. Hence so is ι.

If $h \in H_\lambda \setminus \mathcal{F}_\lambda$, then $S_\lambda(1, h) = \{H_\lambda\}$ and $S_\mu(1, h) = \varnothing$ for any $\mu \ne \lambda$. Obviously $E(1, h; H_\lambda) = \{(1, h)\}$. Thus $\widetilde{r}_\lambda(1, h) = r_\lambda(1, h) = q_\lambda(h)$ and $\widetilde{r}_\mu(1, h) = 0$ whenever $\mu \ne \lambda$. Thus

$$
\iota(q)(h) = \sum_{\mu \in \Lambda} \widetilde{q}_\mu(h) = \sum_{\mu \in \Lambda} \widetilde{r}_\mu(1, h) = q_\lambda(h).
$$

This finishes the proof of (a). Part (b) follows from (20) and (10). $\square$

Remark 4.8 Our proof essentially uses the fact that the quasicocycles q_λ are antisymmetric. In fact, our approach provably fails for nonantisymmetric ones. This can be illustrated in the case when $G = F(x, y)$, the free group of rank 2, and $H = \langle x \rangle$. Indeed take $q \in QZ^1(H, \mathbb{R})$ defined by

$$
q(x^n) = \begin{cases} 1 & \text{if } n \ge 0, \\ 0 & \text{if } n < 0. \end{cases}
$$

Let $\widetilde{q}$ be the extension obtained as above using the subset $X = \{x, y\}$ of G. Take any $n \in \mathbb{N}$ such that $\widehat{d}(1, x^n) > 3C$ (in fact, $C = 0$ in this case, but we will not use this). Then it is straightforward to verify that $\widetilde{q}((yx^n)^k) = k$ while $\widetilde{q}((yx^n)^{-k}) = \widetilde{q}((x^{-n}y^{-1})^k) = 0$ for every $k \in \mathbb{N}$. This contradicts the quasicocycle identity as $k \to \infty$. A similar argument shows that the antisymmetry condition can not be dropped in Example 1.2.

5 Applications

Bounded cohomology   Our goal here is to prove Corollary 1.7. We begin with an auxiliary result.

Proposition 5.1   *Let G be a group, H a hyperbolically embedded subgroup of G, V a G–module, and U an H–submodule of V. Suppose that there exists a continuous projection $\pi\colon V \to U$. Then there is a linear map $\phi\colon QZ^1(H,U) \to EH_b^2(G,V)$ such that $\operatorname{Ker}\phi \subseteq \ell^\infty(H,U) + Z^1(H,U)$. In particular,*

$$\dim H_b^2(G,V) \geq \dim EH_b^2(G,V) \geq \dim EH_b^2(H,U).$$

Proof   We define ϕ to be the composition $\delta \circ \varkappa$, where $\varkappa$ is given by Corollary 1.5 and δ is the natural map $QZ^1(G,V) \to EH_b^2(G,V)$ (see Lemma 2.6). Note that if $\phi(q) = 0$ for some $q \in QZ^1(H,U)$, then

$$(21) \qquad\qquad\qquad \varkappa(q) = h + b,$$

where $b \in \ell^\infty(G,V)$ and $h \in Z^1(G,V)$. Since $\varkappa(q)(x) \in U$ for all $x \in H$, composing both sides of this equality with π and restricting to H we obtain

$$\varkappa(q)|_H = \pi \circ h|_H + \pi \circ b|_H.$$

Obviously $\pi \circ h|_H \in Z^1(H,U)$ and $\pi \circ b|_H \in \ell^\infty(H,U)$ since π is continuous. By Corollary 1.5, $(q - \varkappa(q)|_H) \in \ell^\infty(H,U)$, thus $q \in \ell^\infty(H,U) + Z^1(H,U)$. □

The next lemma is a simplification of Dahmani et al [10, Theorem 2.23].

Lemma 5.2   *Let $G \in \mathcal{X}$. Then for every $n \in \mathbb{N}$, there exists a subgroup $H_n \hookrightarrow_h G$ such that $H_n \cong F_n \times K$, where F_n is the free group of rank n and K is finite.*

We are now ready to prove Corollary 1.7.

Proof of Corollary 1.7   It is easy to see that the assumptions of Proposition 5.1 hold in the case $V = \ell^p(G)$ and $U = \ell^p(H)$. It is well known that $\dim EH_b^2(H) = \infty$ for every virtually free group which is not virtually cyclic (see. e.g., Hamenstädt [14]). To complete the proof it remains to note that every group $G \in \mathcal{X}$ contains a virtually free but not virtually cyclic hyperbolically embedded subgroup by Lemma 5.2. □

Stable commutator length Let G be a group, and let $g \in [G, G]$. The *commutator length* of g, denoted $\mathrm{cl}_G(g)$, is defined as the minimal number of commutators whose product is equal to g in G. The *stable commutator length* is defined by

$$\mathrm{scl}_G(g) = \lim_{n \to \infty} \frac{\mathrm{cl}_G(g^n)}{n}.$$

It is customary to extend scl_G to all elements g for which have some positive power $g^n \in [G, G]$ by letting $\mathrm{scl}_G(g) = \mathrm{scl}(g^n)/n$. Basic facts and theorems about stable commutator length can be found in Calegari [7].

Following Calegari [7], we will denote space of *quasimorphisms* on G by $\widehat{Q}(G)$. Recall that this is the same as $QZ^1(G, \mathbb{R})$, where $\mathbb{R}$ is considered as a G–module with the trivial action. Note that in this setting Theorem 1.4 says that any antisymmetric quasimorphism on H can be extended to a quasimorphism on G.

A quasimorphism φ on G is called *homogeneous* if for all $g \in G$ and all $n \in \mathbb{Z}$, $\varphi(g^n) = n\varphi(g)$. In particular, all homogeneous quasimorphisms are antisymmetric. We denote the subspace of homogeneous quasimorphisms by $Q(G)$. The connection between quasimorphisms and stable commutator length is provided by the Bavard Duality Theorem [1].

Theorem 5.3 (Bavard Duality Theorem) *For any $g \in [G, G]$, there is an equality*

$$(22) \qquad \mathrm{scl}_G(g) = \sup_{\varphi \in Q(G)} \frac{\varphi(g)}{2D(\varphi)},$$

where the supremum is taken over all homogeneous quasimorphisms of nonzero defect.

In fact, it is not hard to see that this supremum is always realized by some quasimorphism.

Given any quasimorphism φ, there is a standard way to obtain a homogeneous quasimorphism ψ, called the homogenization of φ. This is done by defining

$$\psi(g) = \lim_{n \to \infty} \frac{\varphi(g^n)}{n}.$$

Lemma 5.4 (Calegari [7, Corollary 2.59]) *Let $\varphi \in \widehat{Q}(G)$ with homogenization ψ. Then $D(\psi) \leq 2D(\varphi)$.*

Our plan for proving Corollary 1.8 will be to take an element $h \in H$ and apply Bavard Duality to find a homogeneous quasimorphism which realizes (22) with respect to scl_H. Then we can use Theorem 1.4 to extend this to a quasimorphism on all of G,

then apply Bavard Duality again to find a lower bound on $\mathrm{scl}_G(h)$. In order to do this we will need to understand the defect of the extended quasimorphism.

Let H be a group, and let $\xi\colon H \to H/[H,H] \otimes \mathbb{Q}$ be the natural map. A subset $Y \subseteq H$ will be called *nice* if Y can be decomposed as $Y = Y_1 \cup Y_2$ such that $\xi(Y_1)$ is linearly independent and $\xi|_{Y_2} \equiv 0$.

Lemma 5.5 *Every finitely generated subgroup of H has a nice finite generating set.*

Proof Let H' be a finitely generated subgroup of H, and let X be a finite generating set of H'. Then $\xi(H')$ is a finitely generated subgroup of a torsion-free Abelian group, and hence $\xi(H')$ is a finitely generated free Abelian group. Let $\{v_1, \ldots, v_n\}$ be a basis for $\xi(H')$ as a free Abelian group and let $y_i \in H'$ be such that $\xi(y_i) = v_i$. Then for each $x \in X$, there exist integers $a_{x,1}, \ldots, a_{x,n}$ such that $\xi(x) = \sum_{i=1}^{n} a_{x,i} v_i$. Let $\hat{x} = x y_1^{-a_{x,1}}, \ldots, y_n^{-a_{x,n}}$. Now let $Y_1 = \{y_1, \ldots, y_n\}$, and let $Y_2 = \{\hat{x} \mid x \in X\}$. Then clearly $Y = Y_1 \cup Y_2$ is nice, and $\langle Y \rangle = \langle X \rangle = H'$. $\qquad\square$

Lemma 5.6 (Fuchs [12, Theorem 16.1]) *Let B be a subgroup of an Abelian group A, and let D be a divisible Abelian group. Then every homomorphism from $B \to D$ can be extended to a homomorphism from $A \to D$.*

The reason we are interested in nice subsets is the following lemma.

Lemma 5.7 *For any group H, any nice finite subset $Y \subseteq H$, and any $\varphi \in Q(H)$, there exists $\varphi' \in Q(H)$ such that $\varphi'|_{[H,H]} \equiv \varphi|_{[H,H]}$, $D(\varphi') = D(\varphi)$, and for all $y \in Y$,*

$$|\varphi'(y)| \le 2D(\varphi') \mathrm{scl}_H(y).$$

Proof Let $Y = Y_1 \cup Y_2$ be the decomposition given by the definition of a nice subset. If $y \in Y_2$, then there exists some n such that $y^n \in [H,H]$. Then for any $\varphi \in Q(H)$, Bavard Duality gives

$$(23) \qquad\qquad |\varphi(y)| \le 2D(\varphi) \mathrm{scl}_H(y).$$

Now, let $A = H/[H,H]$, and let B be the image of $\langle Y \rangle$ inside A. Let θ be the quotient map $\theta\colon H \to A$. Then by definition of nice subsets we can define a homomorphism $\alpha\colon B \to \mathbb{R}$ such that $\alpha(\theta(y)) = \varphi(y)$ for all $y \in Y_1$. Since $\mathbb{R}$ is divisible, Lemma 5.6 allows us to extend α to all of A. Composing α with θ gives a homomorphism $\beta\colon H \to \mathbb{R}$ which satisfies $\beta(y) = \varphi(y)$ for all $y \in Y_1$. Now we set $\varphi' = \varphi - \beta$. Since β vanishes on $[H,H]$, $\varphi'|_{[H,H]} \equiv \varphi|_{[H,H]}$. Since φ' is a shift of φ by a

homomorphism, $D(\varphi') = D(\varphi)$. Furthermore, combining the fact that $\varphi'(y) = 0$ for all $y \in Y_1$ with (23), we get that for all $y \in Y$,

$$|\varphi(y)| \leq 2D(\varphi') \operatorname{scl}_H(y). \qquad \square$$

We are now ready to prove Corollary 1.8.

Proof of Corollary 1.8 Let $H \hookrightarrow_h (G, X)$, and by Lemma 2.3 there exists Y' a finite subset of H such that the relative metric $\widehat{d}$ on H is bi-Lipschitz equivalent to the word metric with respect to Y'. By Lemma 5.5 the subgroup $\langle Y' \rangle$ has a nice finite generating set Y. Let d_Y be the word metric with respect to Y. Then d_Y is bi-Lipschitz equivalent to the relative metric $\widehat{d}$ on H, so there exists a constant L such that for all $f, g \in H$,

$$(24) \qquad d_Y(f, g) \leq L \widehat{d}(f, g).$$

Fix some $h \in [H, H]$, and let $\varphi \in Q(H)$ be the quasimorphism which realizes the Bavard Duality; that is, $\operatorname{scl}_H(h) = \varphi(h)/(2D(\varphi))$. Let φ' be the modified quasimorphism provided by Lemma 5.7.

Let $\iota \colon Q(H) \to \widehat{Q}(G)$ be map provided by Theorem 4.2. Then by part (b) of Theorem 4.2 we have

$$D(\iota(\varphi')) \leq 54K + 66D(\varphi'),$$

where K is defined by $K = \max\{|\varphi'(k)| : \widehat{d}(1, k) < 15C\}$. However, by (24) we get $K \leq \max\{|\varphi'(k)| : d_Y(1, k) < 15CL\}$. Inductively applying the definition of a quasimorphism along with Lemma 5.7, for any such k we get

$$|\varphi'(k)| \leq 15CL(D(\varphi') + 2D(\varphi') \max_{y \in Y}\{\operatorname{scl}_H(y)\}).$$

That is, we have bound K as a constant multiple of $D(\varphi')$. Thus there exists a constant M (which is independent of φ') such that

$$(25) \qquad D(\iota(\varphi')) \leq 54K + 66D(\varphi') \leq MD(\varphi').$$

Now, $\iota(\varphi')$ is a quasimorphism on G, and in order to apply Bavard Duality we homogenize $\iota(\varphi')$ to get a quasimorphism ψ, satisfying $D(\psi) \leq 2D(\iota(\varphi'))$. Then applying the definition of ψ, along with the homogeneity of φ' and the conditions of Theorem 1.4 gives

$$\psi(h) = \lim_{n \to \infty} \frac{\iota(\varphi')(h^n)}{n} = \varphi'(h) = \varphi(h).$$

Also, (25) and Lemma 5.4 show that $D(\psi) \leq 2D(\iota(\varphi')) \leq 2MD(\varphi') = 2MD(\varphi)$. Applying Bavard Duality again gives

$$\mathrm{scl}_G(h) \geq \frac{\psi(h)}{2D(\psi)} \geq \frac{\varphi(h)}{4MD(\varphi)} = \frac{1}{2M}\mathrm{scl}_H(h). \qquad \square$$

Proof of Corollary 1.9 If H is an almost malnormal quasiconvex subgroup of a hyperbolic group, then G is hyperbolic relative to H (Bowditch [5]). Hence H is hyperbolically embedded in G by Dahmani et al [10, Proposition 2.4] and the claim follows from Corollary 1.8. $\qquad \square$

Remark 5.8 Note that the malnormality condition can not be dropped in Corollary 1.9 even for free groups. For example, let $F = F(x, y, t)$ be the free group of rank 3 with basis $\{x, y, t\}$. In what follows we write a^b for $b^{-1}ab$ and $[a, b]$ for $a^{-1}b^{-1}ab$. Let $H = \langle x, y, x^t, y^t \rangle$ and let

$$h_k = [x, y]^{-k}[x^t, y^t]^k.$$

Since the subset $\{x, y, x^t, y^t\} \subseteq G$ is Nielsen reduced, the subgroup H is freely generated by x, y, x^t, y^t. Therefore $\mathrm{scl}_H(h_k) = k + 1/2$ (see Calegari [7, Example 2.100]). On the other hand, we have

$$\mathrm{scl}_G(h_k) = \mathrm{scl}_G([x, y]^{-k}([x, y]^k)^t) = \mathrm{scl}_G([[x, y]^k, t]) \leq 1.$$

Thus $\mathrm{scl}_H(h_k)/\mathrm{scl}_G(h_k) \to \infty$ as $k \to \infty$.

Acknowledgments

References

[1]   **C Bavard**, *Longueur stable des commutateurs*, Enseign. Math. 37 (1991) 109–150 MR1115747

[2]   **M Bestvina**, **K Bromberg**, **K Fujiwara**, *Bounded cohomology via quasi-trees* `arXiv: 1306.1542`

[3]   **M Bestvina**, **M Feighn**, *A hyperbolic* $\mathrm{Out}(F_n)$*–complex*, Groups Geom. Dyn. 4 (2010) 31–58   MR2566300

[4]   **M Bestvina**, **K Fujiwara**, *Bounded cohomology of subgroups of mapping class groups*, Geom. Topol. 6 (2002) 69–89   MR1914565

[5] **B H Bowditch**, *Relatively hyperbolic groups*, Internat. J. Algebra Comput. 22 (2012) 1250016, 66   MR2922380

[6] **R Brooks**, *Some remarks on bounded cohomology*, from: "Riemann surfaces and related topics", (I Kra, B Maskit, editors), Ann. of Math. Stud. 97, Princeton Univ. Press (1981) 53–63   MR624804

[7] **D Calegari**, *scl*, MSJ Memoirs 20, Mathematical Society of Japan, Tokyo (2009) MR2527432

[8] **D Calegari**, **K Fujiwara**, *Stable commutator length in word-hyperbolic groups*, Groups Geom. Dyn. 4 (2010) 59–90   MR2566301

[9] **I Chifan**, **T Sinclair**, *On the structural theory of* II_1 *factors of negatively curved groups*, Ann. Sci. Éc. Norm. Supér. (4) 46 (2013) 1–33

[10] **F Dahmani**, **V Guirardel**, **D Osin**, *Hyperbolically embedded subgroups and rotating families in groups acting on hyperbolic spaces*   `arXiv:1111.7048`

[11] **D B A Epstein**, **K Fujiwara**, *The second bounded cohomology of word-hyperbolic groups*, Topology 36 (1997) 1275–1289   MR1452851

[12] **L Fuchs**, *Abelian groups*, Publishing House of the Hungarian Academy of Sciences, Budapest (1958)   MR0106942

[13] **U Hamenstädt**, *Lines of minima in outer space*   `arXiv:0911.3620`

[14] **U Hamenstädt**, *Bounded cohomology and isometry groups of hyperbolic spaces*, J. Eur. Math. Soc. (JEMS) 10 (2008) 315–349   MR2390326

[15] **U Hamenstädt**, *Isometry groups of proper CAT(0)–spaces of rank one*, Groups Geom. Dyn. 6 (2012) 579–618   MR2961285

[16] **R C Lyndon**, **P E Schupp**, *Combinatorial group theory*, Ergeb. Math. Grenzgeb. 89, Springer, Berlin (1977)   MR0577064

[17] **I Mineyev**, **N Monod**, **Y Shalom**, *Ideal bicombings for hyperbolic groups and applications*, Topology 43 (2004) 1319–1344   MR2081428

[18] **N Monod**, *Continuous bounded cohomology of locally compact groups*, Lecture Notes in Mathematics 1758, Springer, Berlin (2001)   MR1840942

[19] **N Monod**, *An invitation to bounded cohomology*, from: "Proc. ICM, Vol. II", (M Sanz-Solé, J Soria, J L Varona, J Verdera, editors), Eur. Math. Soc., Zürich (2006) 1183–1211   MR2275641

[20] **N Monod**, **Y Shalom**, *Negative curvature from a cohomological viewpoint and cocycle superrigidity*, C. R. Math. Acad. Sci. Paris 337 (2003) 635–638   MR2030102

[21] **N Monod**, **Y Shalom**, *Orbit equivalence rigidity and bounded cohomology*, Ann. of Math. 164 (2006) 825–878   MR2259246

[22] **D V Osin**, *Elementary subgroups of relatively hyperbolic groups and bounded generation*, Internat. J. Algebra Comput. 16 (2006) 99–118   MR2217644

[23] **D Osin**, *Small cancellations over relatively hyperbolic groups and embedding theorems*, Ann. of Math. 172 (2010) 1–39   MR2680416

[24] **S Popa**, *Deformation and rigidity for group actions and von Neumann algebras*, from: "Proc ICM, Vol. I", (M Sanz-Solé, J Soria, J L Varona, J Verdera, editors), Eur. Math. Soc., Zürich (2007) 445–477   MR2334200

[25] **A Sisto**, *Contracting elements and random walks* `arXiv:1112.2666`

[26] **A Thom**, *Low degree bounded cohomology and L^2–invariants for negatively curved groups*, Groups Geom. Dyn. 3 (2009) 343–358   MR2486803

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