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The head and tail conjecture for alternating knots

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We investigate the coefficients of the highest and lowest terms (also called the head and the tail) of the colored Jones polynomial and show that they stabilize for alternating links and for adequate links. To do this we apply techniques from skein theory.

57M25, 57M27

1 Introduction

The normalized colored Jones polynomial $J_{N,L}(q)$ for a link L is a sequence of Laurent polynomials in the variable $q^{1/2}$, ie $J_{N,L} \in \mathbb{Z}[q^{1/2}, q^{-1/2}]$. This sequence is defined for $N \geq 2$ so that $J_{2,L}(q)$ is the ordinary Jones polynomial, and $J_{N,U} = 1$ where U is the unknot. For links L with an odd number of components (including all knots), $J_{L,N}$ is actually in $\mathbb{Z}[q, q^{-1}]$. For links with an even number of components, $q^{1/2} J_{L,N} \in \mathbb{Z}[q, q^{-1}]$.

In [4] and [5], Oliver Dasbach and Xiao-Song Lin showed that, up to sign, the first two coefficients and the last two coefficients of $J_{N,K}(q)$ do not depend on N for alternating knots. They also showed that the third (and third to last) coefficient does not depend on N so long as $N \geq 3$. This and computational data led them to conjecture that the k^{th} coefficient does not depend on N so long as $N \geq k$. The goal of this paper is to prove this conjecture for all alternating links. It is also known that this property of $J_{N,L}(q)$ does not hold for all knots. In [2], with Oliver Dasbach, we examined the case of the $(4,3)$ torus knot for which this property fails.

Definition For two Laurent series $P_1(q)$ and $P_2(q)$ we define

$$P_1(q) \doteq_n P_2(q)$$

if after multiplying $P_1(q)$ by $\pm q^{s_1}$ and $P_2(q)$ by $\pm q^{s_2}$, s_1 and s_2 some powers, to get power series $P_1'(q)$ and $P_2'(q)$ each with positive constant term, $P_1'(q)$ and $P_2'(q)$ agree mod q^n. For example $-q^{-4} + 2q^{-3} - 3 + 11q \doteq_5 1 - 2q + 3q^4$.

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Another way of phrasing the above definition is that $P_1(q) \doteq_n P_2(q)$ if and only if their first n coefficients agree up to sign.

In [2] we defined two power series, the head and tail of the colored Jones polynomial $H_L(q)$ and $T_L(q)$.

Definition The tail of the colored Jones polynomial of a link L – if it exists – is a series $T_L(q)$, with

$$T_L(q) \doteq_N J_{L,N}(q) \quad \text{for all } N.$$

Similarly, the head $H_L(q)$ of the colored Jones polynomial of L is the tail of $J_{L,N}(q^{-1})$, which is equal to the colored Jones polynomial of the mirror image of L.

Note that $T_L(q)$ exists if and only if $J_{L,N}(q) \doteq_N J_{L,N+1}(q)$ for all N. For example, for the first few colors N the colored Jones polynomial of the knot 6_2 multiplied by q^{2N^2-N-1} is:

$$\begin{aligned}
N=2:&\quad 1-2q+2q^2-2q^3+2q^4-q^5+q^6\\
N=3:&\quad 1-2q+4q^3-5q^4+6q^6+\cdots-q^{14}+3q^{15}-q^{16}-q^{17}+q^{18}\\
N=4:&\quad 1-2q+2q^3+q^4-4q^5-2q^6+\cdots-2q^{29}-3q^{30}+3q^{32}-q^{34}-q^{35}+q^{36}\\
N=5:&\quad 1-2q+2q^3-q^4+2q^5-6q^6+\cdots-2q^{53}-q^{54}+4q^{55}-q^{58}-q^{59}+q^{60}\\
N=6:&\quad 1-2q+2q^3-q^4-2q^7+q^8+\cdots-3q^{82}+3q^{84}+q^{85}-q^{88}-q^{89}+q^{90}\\
N=7:&\quad 1-2q+2q^3-q^4-2q^6+4q^7+\cdots+4q^{119}+q^{121}-q^{124}-q^{125}+q^{126}
\end{aligned}$$

This is exactly the property conjectured by Dasbach and Lin to hold for all alternating knots, and the subject of the main theorem of this paper.

Theorem 1 *If L is an alternating link, then $J_{L,N}(q) \doteq_N J_{L,N+1}(q)$.*

Because the mirror image of an alternating link is alternating, this theorem says that the head and the tail exists for all alternating links. Theorem 1 was simultaneously and independently proved by Stavros Garoufalidis and Thang Le in [6] using alternate methods.

We are also able to prove a more general theorem about A–adequate links.

Theorem 2 *If L is an A–adequate link, then $J_{L,N}(q) \doteq_N J_{L,N+1}(q)$.*

Because all alternating links are A–adequate, Theorem 2 implies Theorem 1.

Special cases of Theorem 2 were also proven in [1; 3]. In [1] it was shown that for a knot K, which can be expressed as the closure of a positive braid, $T_K(q) = 1$. In [3] Abhijit Champanerkar and Ilya Kofman show this for a knot expressed as a positive braid with a full twist, and also determine a sequence of coefficients beyond the first N.

1.1 Plan of paper

In Section 2, we discuss definitions and basic results regarding adequate links and skein theory. In Section 3, we present the main lemma, which is a slight generalization of a lemma in [2] relating the lowest terms of the colored Jones polynomial to a certain skein theoretic graph. Finally, in Section 4, we present the proofs of Theorems 1 and 2 using the graph discussed in Section 3.

Acknowledgments

I would like to thank Oliver Dasbach for all his help and advice. I would also like to thank Pat Gilmer for teaching me all I know about skein theory.

2 Background

2.1 Alternating and adequate

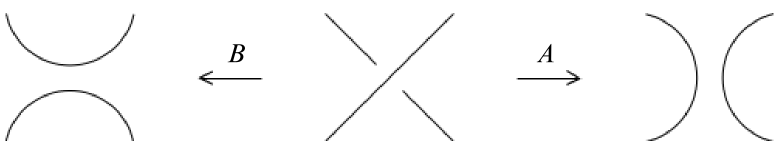


Figure 1: A and B smoothings

Given a link diagram D there are two ways to smooth each crossing, described in Figure 1. A state of the diagram is a choice of smoothing for each crossing. Two states are particularly important when dealing with the colored Jones polynomial; they are the all-A state S_A and the all-B state S_B. The all-A (respectively all-B) state is the state for which the A (B) smoothing is chosen for every crossing.

For a state S we can build a graph G_S called the state graph for S. The graph G_S has vertices the circles in S, and edges the crossings in D. Each edge connects the two vertices corresponding to the two circles that the crossing meets.

Definition A link diagram is A–adequate (B–adequate) if the state graph for S_A (S_B) has no loops.

A link diagram is adequate if it is both A and B–adequate.

A link is adequate if it has an adequate diagram.

The most important property of A–adequate diagrams is that the number of circles in S_A is a local maximum. In other words, any state that has only a single B smoothing will have one fewer circle than the all-A smoothing. Similarly for B–adequate diagrams, that the number of circles in S_B is a local maximum.

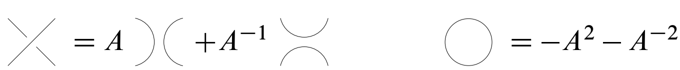
It is a well-known fact that all alternating links are adequate links. In particular, a reduced alternating diagram, that is an alternating diagram without any nugatory crossings, is an adequate diagram.

Another important fact about adequate diagrams is that parallels of A–adequate diagrams are also A–adequate diagrams. Given a diagram D, the r^{th} parallel of D denoted D^r is the diagram formed by replacing D with r parallel copies of D.

2.2 Skein theory

For a more detailed explanation of skein theory, see Lickorish [7] or Masbaum and Vogel [8].

The Kauffman bracket skein module, $S(M; R, A)$, of a 3–manifold M and ring R with invertible element A, is the free R–module generated by isotopy classes of framed links in M, modulo the submodule generated by the Kauffman relations:



If M has designated points on the boundary, then the framed links must include arcs that meet all of the designated points.

In this paper we will take $R = \mathbb{Q}(A)$, the field of rational functions in variable A with coefficients in $\mathbb{Q}$. As we are concerned with the lowest terms of a polynomial, we will need to express rational functions as Laurent series. This can always be done so that the Laurent series has a minimum degree.

Definition Let $f \in \mathbb{Q}(A)$, define $d(f)$ to be the minimum degree of f expressed as a Laurent series in A.

Note that $d(f)$ can be calculated without referring to the Laurent series. Any rational function f expressed as P/Q, where P and Q are both polynomials has $d(f) = d(P) - d(Q)$.

We will be concerned with two particular skein modules: $S(S^3; R, A)$, which is isomorphic to R under the isomorphism sending the empty link to 1, and $S(D^3; R, A)$, where D^3 has $2n$ designated points on the boundary. With these designated points, $S(D^3; R, A)$ is also called the Temperley–Lieb algebra TL_n.

We will give an alternate explanation for the Temperley–Lieb algebra. First, consider the disk D^2 as a rectangle with n designated points on the top and n designated points on the bottom. Let TLM_n be the set of all crossing-less matchings on these points, and define the product of two crossing-less matchings by placing one rectangle on top of the other and deleting any components that do not meet the boundary of the disk. With this product, TLM_n is a monoid, which we shall call the Temperley–Lieb monoid. It has generators h_i as in Figure 2, and following relations:

- $h_i h_i = h_i$
- $h_i h_{i\pm 1} h_i = h_i$
- $h_i h_j = h_j h_i$ if $|i - j| \geq 2$

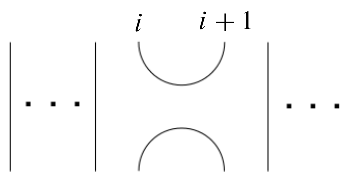


Figure 2: h_i

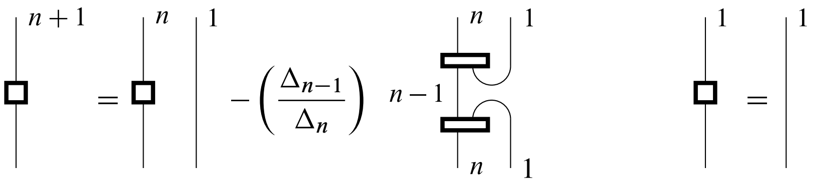
Any element in TL_n has the form $\sum_{M \in \mathrm{TLM}_n} c_M M$, where $c_M \in \mathbb{Q}(A)$. Multiplication in TL_n is slightly different from multiplication in TLM_n, because $h_i h_i = (-A^2 - A^{-2}) h_i$ in TL_n.

There is a special element in TL_n of fundamental importance to the colored Jones polynomial, called the Jones–Wenzl idempotent, denoted $f^{(n)}$. Diagrammatically this element is represented by an empty box with n strands coming out of it on two opposite sides. By convention an n next to a strand in a diagram indicates that the strand is replaced by n parallel ones.

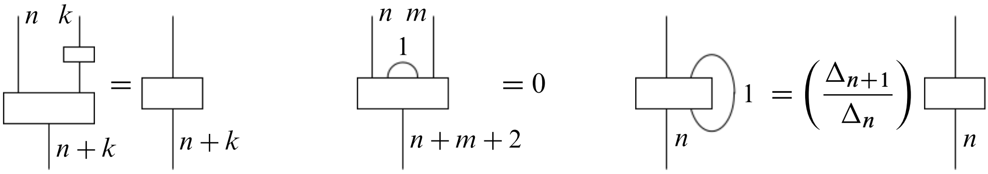
With

$$\Delta_n := (-1)^n \frac{A^{2(n+1)} - A^{-2(n+1)}}{A^2 - A^{-2}}$$

and $\Delta_n! := \Delta_n\Delta_{n-1}\cdots\Delta_1$, the Jones–Wenzl idempotent satisfies:



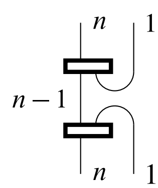
with the properties:



If $M \in \mathrm{TLM}_n$, define $f_M \in R$ as the coefficient of M in the expansion of the Jones–Wenzl idempotent. Thus $f^{(n)} = \sum_{M\in\mathrm{TLM}_n} f_M M$. If e is the identity element of TLM_n, then $f_e = 1$.

Lemma 3 *If $M \in \mathrm{TLM}_n$, then $d(f_M)$ is at least twice the minimum word length of M in terms of the h_i.*

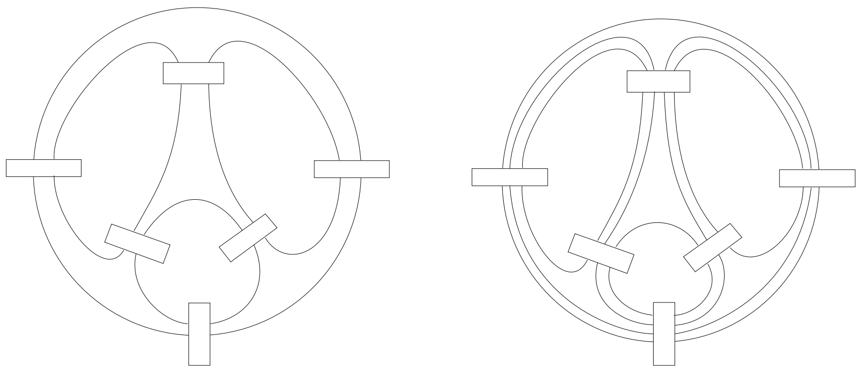
Proof This follows easily from the recursive definition of the idempotent by an inductive argument. The only issue is that terms of the form



may have a circle that needs to be removed. In this situation, the minimum degree of the coefficient is reduced by two, but the number of generators used is also reduced by one. □

Using Lemma 3 we can find a lower bound for the minimum degree of any element of $S(S^3; R, A)$ that contains the Jones–Wenzl idempotent. Before we do this, consider a crossing-less diagram S in the plane consisting of arcs connecting Jones–Wenzl idempotents. We will define what it means for such a diagram to be adequate in much the same way that a knot diagram can be A or B–adequate.

Construct a crossing-less diagram $\overline{S}$ from S by replacing each of the Jones–Wenzl idempotents in S by the identity of TL_n. Thus $\overline{S}$ is a collection of circles with no crossings. Consider the regions in $\overline{S}$ where the idempotents had previously been. S is adequate if no circle in $\overline{S}$ passes through any one of these regions more than once. Figure 3(a) shows an example of a diagram that is adequate and Figure 3(b) shows an example of a diagram that is not adequate. In both figures every arc is labeled 1.



(a) An adequate diagram (b) An inadequate diagram

Figure 3: Example of adequate and inadequate diagrams

If S is adequate, then the number of circles in $\overline{S}$ is a local maximum, in the sense that if the idempotents in S are replaced by other elements of TLM_n such that there is exactly one hook total in all of the replacements, then the number of circles in this diagram is one less than the number of circles in $\overline{S}$.

If the diagram S happens to have crossings in it, we can still construct the diagram $\overline{S}$, which is now a link diagram. Denote $D(S) := d(\overline{S})$.

Lemma 4 *If $S \in S(S^3; R, A)$ is expressed as a single crossing-less diagram containing the Jones–Wenzl idempotent, then $d(S) \geq D(S)$.*

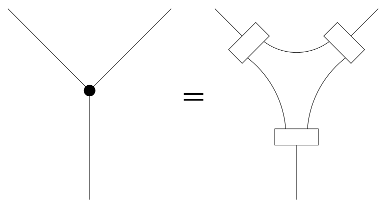
If the diagram for S is also an adequate diagram, then $d(S) = D(S)$.

Proof We can get an expansion of S by expanding each of the idempotents that appear in the diagram. Consider a single term T_1 in this expansion. Unless all of the idempotents have been replaced by the identity in this term, then there will be a hook somewhere in the diagram. By removing a single hook, we get a different term T_2 in the expansion. The number of circles in T_1 differs from the number of circles in T_2 by exactly one. Lemma 3 and the fact that removing a circle results in multiplying by $-A^2 - A^{-2}$ gives us a lower bound d_i of the degree of the term T_i $(i = 1, 2)$, that is, d_i is twice the number of hooks in T_i, minus twice the number of circles in T_i.

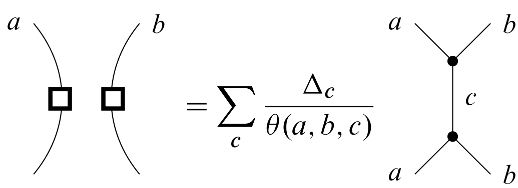
Because there are fewer hooks in T_2, this gives the inequality $d_2 \leq d_1 + 2 \pm 2$. This tells us that the lowest degree amongst terms in the expansion of S is the degree of the term with the idempotents replaced by the identity, $\overline{S}$.

If S is adequate, then for any term T_1 with only a single hook, T_2 will be $\overline{S}$, and thus T_2 will have one more circle than T_1. Therefore, $d(T_1) > d(\overline{S})$. This tells us that any term T in this expansion will have $d(T_1) > d(\overline{S})$, and thus $d(S) = d(\overline{S}) = D(S)$. □

We can use trivalent graphs to express elements in a skein module using the following correspondence:



Fusion is given by



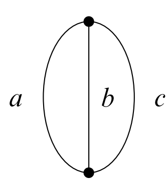
where the sum is over all c such that:

(1) $a + b + c$ is even.

(2) $|a - b| \leq c \leq a + b$.

To define $\theta(a,b,c)$, let a, b and c related as above and x, y and z be defined by $a = y + z$, $b = z + x$ and $c = x + y$ then

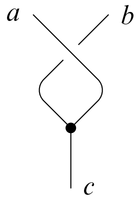
$$\theta(a,b,c) :=$$



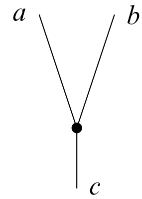
and one can show that

$$\theta(a,b,c) = \frac{\Delta_{x+y+z}!\,\Delta_{x-1}!\,\Delta_{y-1}!\,\Delta_{z-1}!}{\Delta_{y+z-1}!\,\Delta_{z+x-1}!\,\Delta_{x+y-1}!}.$$

Furthermore one has:



$$= (-1)^{(a+b-c)/2} A^{a+b-c+(a^2+b^2-c^2)/2}$$



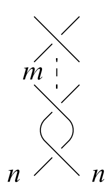
We are only interested in the list of coefficients of the colored Jones polynomial. In particular we consider polynomials up to powers of their variable. Up to a factor of $\pm A^s$ for some power s that depends on the writhe of the link diagram, the (unreduced) colored Jones polynomial $\widetilde{J}_{n,L}(A)$ of a link L can be defined as the value of the skein relation applied to the link were every component is decorated by an n together with the Jones–Wenzl idempotent. Recall that $A^{-4} = q$. To obtain the reduced colored Jones polynomial we have to divide $\widetilde{J}_{n,L}(A)$ by its value on the unknot. Thus:

$$J_{n+1,L}(q) := \left.\frac{\widetilde{J}_{n,L}(A)}{\Delta_n}\right|_{A=q^{-1/4}}$$

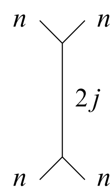
3 The main lemma

In this section we will relate the tail of the colored Jones polynomial to a certain trivalent graph viewed as an element of the Kauffman bracket skein module of $\mathbb{R}^3$. This construction was used in [2] to prove interesting properties of the head and the tail of the colored Jones polynomial.

Given a B–adequate diagram D of a link L, consider a negative twist region. Apply the identities of Section 2 to get the equation:



$$= \sum_{j=0}^{n} (\gamma(n,n,2j))^m \frac{\Delta_{2j}}{\theta(n,n,2j)}$$



Here $\gamma(a,b,c) := (-1)^{(a+b-c)/2} A^{a+b-c+(a^2+b^2-c^2)/2}$.

If we apply this equation to every maximal negative twist region, then we get an embedded trivalent graph called Γ. We get a colored graph $\Gamma_{n,(j_1,\dots,j_k)}$ where k is the number of maximal negative twist regions and $0 \le j_i \le n$ by coloring the edges coming from the i^{th} twist region by $2j_i$ and coloring all of the other edges by n. From the previous equation, it is clear that we get:

$$\tilde{J}_{n,L} \doteq \sum_{j_1,\dots,j_k=0}^{n} \prod_{i=1}^{k} (\gamma(n,n,2j_i))^m \prod_{i=1}^{k} \frac{\Delta_{2j_i}}{\theta(n,n,2j_i)} \Gamma_{n,(j_1,\dots,j_k)}$$

The following theorem is a useful tool to find properties of the head and tail of the colored Jones polynomial. In this paper, it will be used to prove the existence of the head and tail for all adequate links:

Theorem 5 *If D is a B–adequate diagram of the link L, and $\Gamma_{n,(n,\dots,n)}$ is the corresponding graph, then:*

$$\tilde{J}_{n,L} \doteq_{4(n+1)} \Gamma_{n,(n,\dots,n)}.$$

This theorem was proved for the case when D is a reduced alternating diagram in [2, Theorem 4.3]. The proof given there extends easily to B–adequate diagrams. We will present the proof again here with the modifications. In later sections, we shall denote $\Gamma_n := \Gamma_{n,(n,\dots,n)}$.

For a rational function R, let $d(R)$ be the minimum degree of R considered as a power series. The theorem will now follow from the following three lemmas.

Lemma 6

$$d(\gamma(n,n,2n)) = d(\gamma(n,n,2(n-1))) - 4n$$
$$d(\gamma(n,n,2j)) \le d(\gamma(n,n,2(j-1)))$$

Lemma 7

$$d\left(\frac{\Delta_{2j}}{\theta(n,n,2j)}\right) = d\left(\frac{\Delta_{2(j-1)}}{\theta(n,n,2(j-1))}\right) - 2$$

Lemma 8 *If Γ is the graph coming from a B–adequate diagram, then:*

$$D(\Gamma_{n,(j_1,\dots,j_i,\dots,j_k)}) = D(\Gamma_{n,(j_1,\dots,j_i-1,\dots,j_k)}) \pm 2$$
$$d(\Gamma_{n,(n,\dots,n,\dots,n)}) = D(\Gamma_{n,(n,\dots,n-1,\dots,n)}) - 2$$

Proof of Lemma 6 $\gamma(n,n,2j) = \pm A^{n+n-2j+(n^2+n^2-(2j)^2)/2} = \pm A^{2n-2j+n^2-2j^2}$ Clearly $d(\gamma(n,n,2j))$ increases as j decreases. Furthermore:

$$d(\gamma(n,n,2n)) = -n^2$$
$$d(\gamma(n,n,2(n-1))) = 2n-2(n-1)+n^2-2(n-1)^2 = -n^2+4n \qquad \square$$

Proof of Lemma 7 To calculate $\theta(n,n,2j)$ note that in the previous formula for θ we get $x=j$, $y=j$ and $z=n-j$. Using this and the fact that $d(\Delta_n)=-2n$, we get:

$$\begin{aligned} d\left(\frac{\Delta_{2j}}{\theta(n,n,2j)}\right) &= d\left(\frac{\Delta_{2j}\,\Delta_{n-1}!\Delta_{n-1}!\Delta_{2j-1}!}{\Delta_{n+j}!\Delta_{j-1}!\Delta_{j-1}!\Delta_{n-j-1}!}\right) \\ &= d\left(\frac{\Delta_{2j}\,\Delta_{2j-1}\,\Delta_{n-j}}{\Delta_{n+j}\,\Delta_{j-1}\,\Delta_{j-1}}\right) + d\left(\frac{\Delta_{2(j-1)}!\Delta_{n-1}!\Delta_{n-1}!}{\Delta_{n+j-1}!\Delta_{j-2}!\Delta_{j-2}!\Delta_{n-j}!}\right) \\ &= -4j-2(2j-1)-2(n-j)+4(j-1)+2(n+j) \\ &\qquad\qquad\qquad + d\left(\frac{\Delta_{2(j-1)}}{\theta(n,n,2(j-1))}\right) \\ &= -2+d\left(\frac{\Delta_{2(j-1)}}{\theta(n,n,2(j-1))}\right) \end{aligned}$$

□

Proof of Lemma 8 Consider the graph $\Gamma_{n,(j_1,\dots,j_k)}$ viewed as an element in the skein module $S(S^3;\mathbb{Q}(A),A)$. We must compare

$$D(\Gamma_{n,(j_1,\dots,j_i,\dots,j_k)}) \quad \text{with} \quad D(\Gamma_{n,(j_1,\dots,j_i-1,\dots,j_k)}).$$

Recall that $D(S)$ is -2 times the number of circles in $\overline{S}$, where $\overline{S}$ is obtained from S by replacing the idempotents in the diagram by the identity in TL_m. For $\Gamma_{n,(j_1,\dots,j_i,\dots,j_k)}$ and $\Gamma_{n,(j_1,\dots,j_i-1,\dots,j_k)}$, the number of circles in each diagram differ by 1. Thus $D(\Gamma_{n,(j_1,\dots,j_i,\dots,j_k)})=D(\Gamma_{n,(j_1,\dots,j_i-1,\dots,j_k)})\pm 2$.

For $\Gamma_{n,(n,\dots,n)}$, replacing the idempotents with the identity results in the all B smoothing of the diagram D^n. Since D is a B–adequate diagram, so is D^n. For $\Gamma_{n,(n,\dots,n-1,\dots,n)}$, the replacement results in a smoothing of D^n with exactly one A smoothing. Thus the result of the replacement for $\Gamma_{n,(n,\dots,n)}$ will have one more circle than the result of the replacement for $\Gamma_{n,(n,\dots,n-1,\dots,n)}$, which give us

$$D(\Gamma_{n,(n,\dots,n,\dots,n)})=D(\Gamma_{n,(n,\dots,n-1,\dots,n)})-2.$$

Finally, since D^n is B–adequate, $\Gamma_{n,(n,\dots,n)}$ is adequate. Thus by Lemma 4,

$$d(\Gamma_{n,(n,\dots,n,\dots,n)})=D(\Gamma_{n,(n,\dots,n,\dots,n)}).$$

□

4 Proof of the main theorem

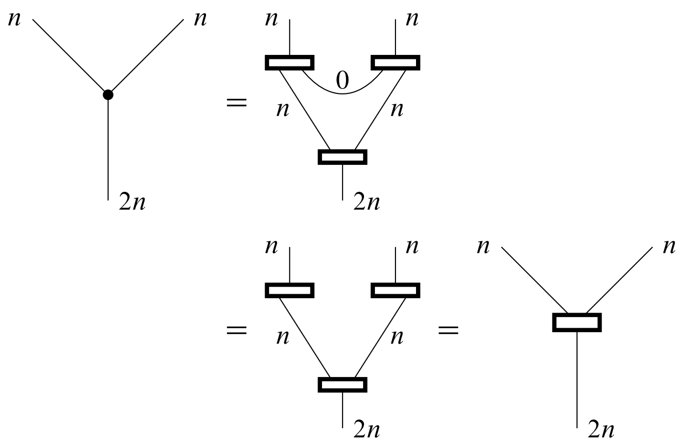
Using Theorem 5, the main theorem can be rewritten as follows:

Theorem 9 *If D is a B–adequate diagram for a link L and Γ_n its corresponding graph, then*

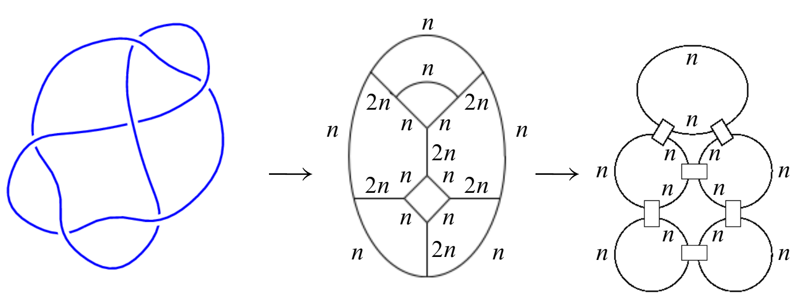
$$\Gamma_n \doteq_{4(n+1)} \Gamma_{n+1}.$$

Proof We will first prove Theorem 9 in the case of D being a reduced alternating diagram, and then we will show how the proof can be modified to apply to any B–adequate diagram in general.

Interpreting Γ_n as a skein element, we may use the following simplification:



In general, Γ_n will reduce to a collection of circles coming from the all-B smoothing S_B, "fused" together with the Jones–Wenzl Idempotent colored $2n$ for each maximal negative twist region. We will call this reduced form $S_B^{(n)}$.

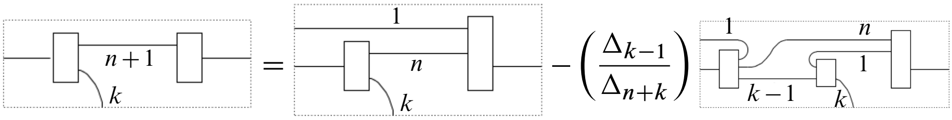


(a) The knot 6_2 (b) Γ_n (c) $S_B^{(n)}$

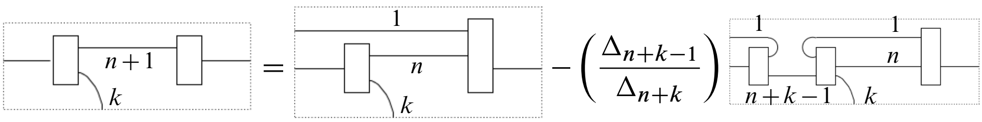
Figure 4: Example of the knot 6_2 along with Γ_n and $S_B^{(n)}$

We would now like to consider $S_B^{(n+1)}$ and show that we can reduce it to $S_B^{(n)}$ without affecting the lowest $4(n+1)$ terms. To do this we will first show a local relation that we will be able to use repeatedly.

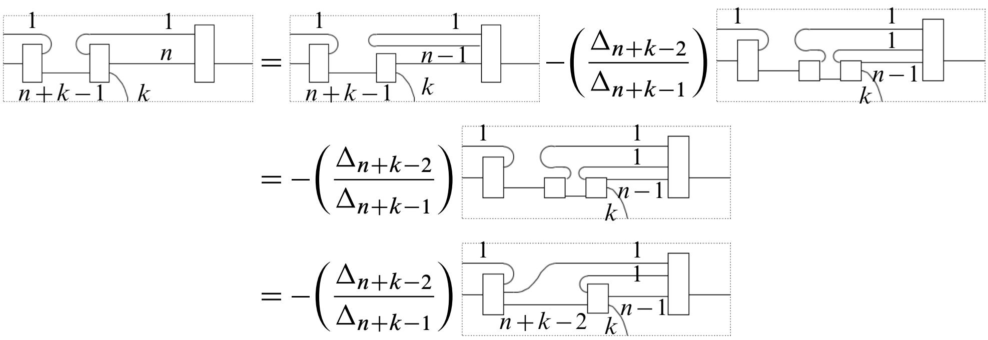
Lemma 10



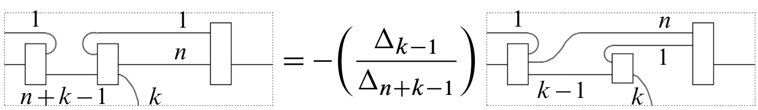
Proof Using the recursive formula for the Jones–Wenzl idempotent on the left of the left hand side of the identity, we get:



Applying the recursive formula again on the middle idempotent of the rightmost diagram, we get:



Now when we apply this recursive formula again, the first term will again be zero, and we can continue this process until we get:



□

Consider a circle s in S_B. The circle s appears in $S_B^{(n+1)}$, although it runs through several idempotents. The goal of the argument is to remove one copy of the circle s from the idempotents. Once this is done for each circle in S_B, then $S_B^{(n+1)}$ will have been reduced to $S_B^{(n)}$.

Because the diagram D is alternating, the circle s bounds a disk which does not contain any of the other circles in S_B. This means that in $S_B^{(n+1)}$, the circle s looks like Figure 5. Here all of the arcs are labeled $n+1$.

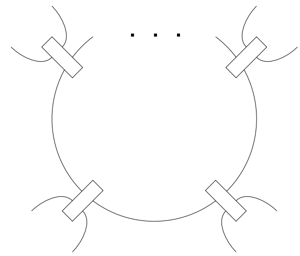
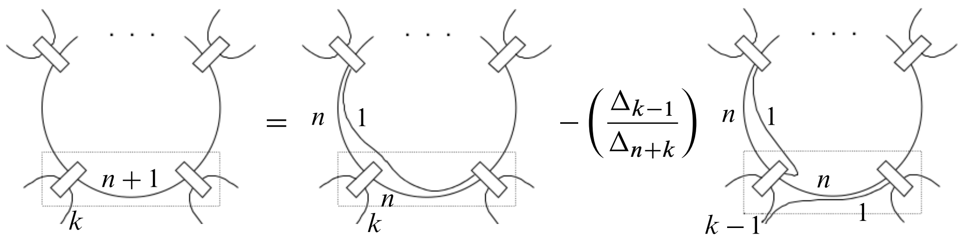


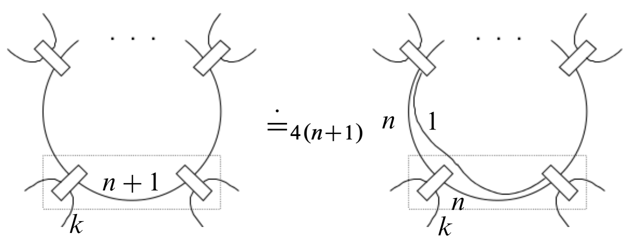
Figure 5: A circle from S_B seen in $S_B^{(n+1)}$

Apply Lemma 10 to get the following relation:



This argument will be applied to each circle in succession, so k is either n or $n+1$ depending on whether the argument has been applied to that circle yet. All non-labeled arcs are either n or $n+1$.

Now $S_B^{(n+1)}$ is expressed as the sum of two terms, and the claim is that the minimum degree of the second term is at least the minimum degree of the first term plus $4(n+1)$. Thus the equation simplifies to:



To see that the claim is true, first note that:

$$d\left(\frac{\Delta_{k-1}}{\Delta_{n+k}}\right) = 2(n+1).$$

Now we need to compare the degree of the two diagrams involved. By Lemma 4 we can get a lower bound for the minimum degrees of these diagrams, and since the knot diagram that these came from was B–adequate, the first term will be an adequate diagram, and thus, the lower bound will be equal to the actual minimum degree. Note that the element in TL_{n+2} shown in Figure 6 can be expressed as $h_{n+1}h_n\cdots h_1$. In the previous equation, this element appears in the rightmost term. When comparing this terms to the first term, each h_i merges two circles into one circle. Thus the number of circles in the diagrams differ by $n+1$. And, finally, a circle can be removed and replaced with a factor of $-A^2-A^{-2}$. This tells us that the difference in the minimum degrees of the diagrams is at least $2(n+1)$. Putting this together with the difference in degrees of the coefficients, the difference in minimum degrees of the terms themselves is at least $4(n+1)$.

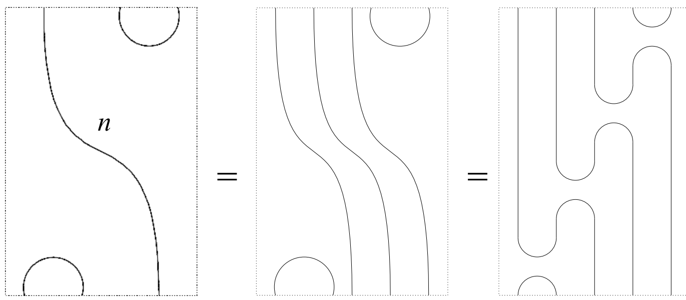


Figure 6: Multiple pictures expressing $h_{n+1}h_n\cdots h_1$

Apply this argument around the circle up to the final idempotent connected to that circle. Now the diagram looks like Figure 7.

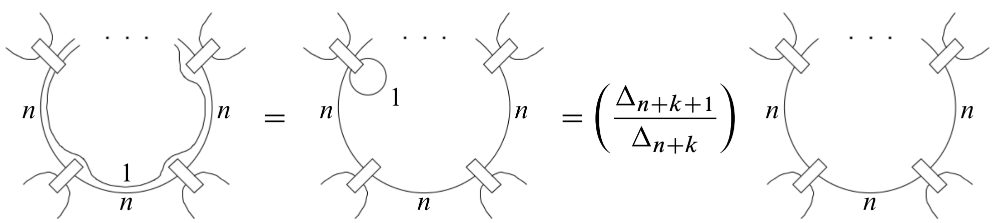


Figure 7: Reducing $S_B^{(n+1)}$ to $S_B^{(n)}$

We can rewrite the coefficient here:

$$\left(\frac{\Delta_{n+k+1}}{\Delta_{n+k}}\right)=(-1)\frac{A^{2(n+k+2)}-A^{-2(n+k+2)}}{A^{2(n+k+1)}-A^{-2(n+k+1)}}=(-A^{-2})\frac{A^{4(n+k+2)}-1}{A^{4(n+k+1)}-1}\doteq_{4(n+1)}1$$

Now applying this argument to every circle in $S_B^{(n+1)}$, we see that

$$S_B^{(n+1)}\doteq_{4(n+1)}S_B^{(n)},$$

and thus,

$$\Gamma_n \doteq_{4(n+1)} \Gamma_{n+1}.$$

This proves Theorem 9 in the case of reduced alternating diagrams. For the case when the diagram D is B–adequate, most of the proof still applies. The main difference is that Figure 5 is not accurate because a circle s in S_B might not bound a disk, and thus in $S^{(n+1)}$ may have idempotents that alternate which side of the circle it fuses to other circles. Figure 8 shows a non-alternating B–adequate diagram of the trefoil where the dotted circle is an example of such a circle s. We would still like to pull out one copy of s, but in this case we must be more careful while doing so.

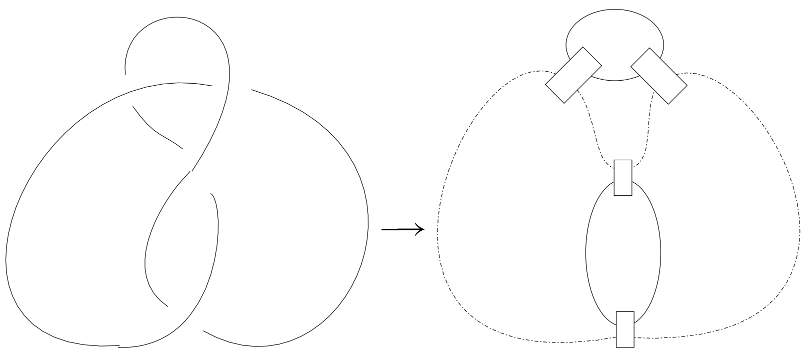


Figure 8: Example of a non-alternating B–adequate knot diagram

First we will modify the diagram as in Figure 9 by adding crossings along the circle s between any pair of idempotents which alternate which side of s is the outer side. The procedure is to modify the diagram so that the outer strand passes over all of the other copies of s, so that it is still the outer strand when it meets the next idempotent in line. Call this new diagram T. When expanding T by summing over all possible smoothings of the crossings, only one state is non-zero, and that state is $S^{(n+1)}$. Since this particular smoothing has an equal number of A and B smoothings, we get that $S^{(n+1)} = T$.

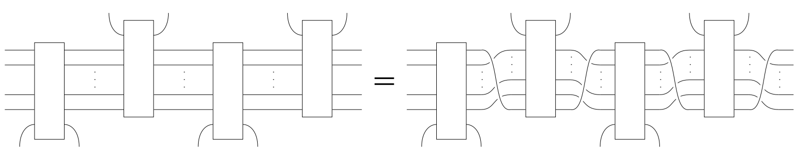
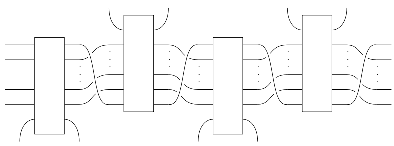
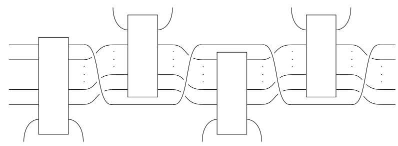


Figure 9: Modifying $S^{(n+1)}$ to get T

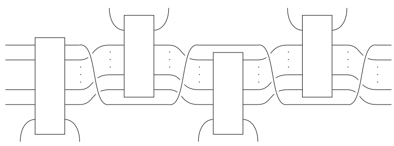
Now we will again apply the procedure to pull out one copy of s as in the alternating case, giving us the following equation:



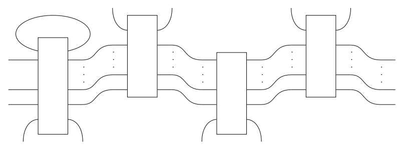
$$\doteq_{4(n+1)}$$



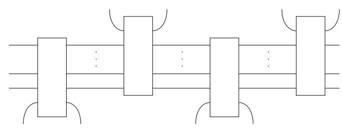
Finally, there is a strand that can be pulled over all other strands and idempotents until it is just a loop, which can be removed.



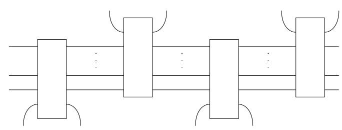
$$=$$



$$= \left(\frac{\Delta_{n+k+1}}{\Delta_{n+k}}\right)$$



$$\doteq_{4(n+1)}$$



This completes the argument. □

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