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Homology equivalences of manifolds and zero-in-the-spectrum examples

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Working with group homomorphisms, a construction of manifolds is introduced, which preserves homology groups. The construction gives as special cases Quillen's plus construction with handles obtained by Hausmann, the existence of the one-sided h–cobordism of Guilbault and Tinsley, and the existence of homology spheres and higher-dimensional knots proved by Kervaire. We also use it to recover counter-examples to the zero-in-the-spectrum conjecture found by Farber and Weinberger, and by Higson, Roe and Schick.

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1 Introduction

The aim of this note is to propose a general surgery plus construction on manifolds. This is a manifold version of the generalized plus construction for CW–complexes found by the author in [20]. As applications, we give a unified approach to the plus construction with handles of Hausmann [7], the (mod L)–one-sided h–cobordism of Guilbault and Tinsley [5], the existence of homology spheres of Kervaire [13], and the existence of the higher-dimensional knots of Kervaire [12]. We also use it to recover some examples for the zero-in-the-spectrum conjecture found by Farber and Weinberger [3] and Higson, Roe and Schick [10]. First, we briefly review these existing works.

Let M be an n–dimensional ($n \geq 5$) closed manifold with fundamental group $\pi_1(M) = H$. Suppose that $\Phi\colon H \to G$ is a surjective group homomorphism of finitely presented groups with the kernel $\ker\Phi$ a perfect subgroup. Hausmann shows that Quillen's plus construction with respect to $\ker\Phi$ can by made by adding finitely many 2 and 3–handles to $M \times 1 \subset M \times [0,1]$ (cf Hausmann [7, Section 3] and the definition of φ_1 in Hausmann [8, page 115]). The resulting cobordism $(W; M, M')$ has W and M' the homotopy type of the Quillen plus construction M^+. In other words, the fundamental group $\pi_1(M') = \pi_1(W) = G$, and for any $\mathbb{Z}[G]$–module N, the inclusion map $M \hookrightarrow W$ induces homology isomorphisms $H_*(M; N) \cong H_*(W; N)$.

Guilbault and Tinsley [5] obtain a generalized manifold plus construction, as follows. Let (W, Y) be a connected CW–pair and L a normal subgroup of $\pi_1(Y)$. The inclusion

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$Y \hookrightarrow W$ is called a (mod L)–homology equivalence if it induces an isomorphism of fundamental groups and is a homology equivalence with coefficients in $\mathbb{Z}[\pi_1(Y)/L]$. A compact cobordism $(W; X, Y)$ is a (mod L)*–one-sided h–cobordism* if $X \hookrightarrow W$ induces a surjection of fundamental groups and $Y \hookrightarrow W$ is a (mod L)–homology equivalence. Let B be a closed n–manifold ($n \geq 5$) and $\alpha\colon \pi_1(B) \to G$ a surjective homomorphism onto a finitely presented group such that $\ker(\alpha)$ is strongly L'–perfect, ie, $\ker(\alpha) = [\ker(\alpha), L']$ for some group L', where $\ker(\alpha) \trianglelefteq L' \trianglelefteq \pi_1(B)$. For $L = L'/\ker(\alpha)$, Guilbault and Tinsley [5] show that there exists a (mod L)–one-sided h–cobordism $(W; B, A)$ such that $\pi_1(W) = G$ and $\ker(\pi_1(B) \to \pi_1(W)) = \ker(\alpha)$.

An n–dimensional homology sphere is a closed manifold M having the homology groups of the n–sphere, ie, $H_*(M) \cong H_*(S^n)$. Let π be a finitely presented group and $n \geq 5$. Kervaire shows that there exists an n–dimensional homology sphere M with $\pi_1(M) = \pi$ if and only if the homology groups satisfy $H_1(\pi; \mathbb{Z}) = H_2(\pi; \mathbb{Z}) = 0$.

For an integer $n \geq 1$, define an n–knot to be a differential imbedding $f\colon S^n \to S^{n+2}$ and the group of the n–knot f to be $\pi_1(S^{n+2} - f(S^n))$. Let G be a finitely presentable group. The weight $w(G)$ is the smallest integer k such that there exists a set of k elements $\alpha_1, \alpha_2, \ldots, \alpha_k \in G$ whose normal closure equals G. Kervaire [12] shows that a finitely presented group G is the group of a n–knot ($n \geq 5$) if and only if $H_1(G; \mathbb{Z}) = \mathbb{Z}$, $H_2(G; \mathbb{Z}) = 0$ and the weight of G is 1.

The zero-in-the-spectrum conjecture goes back to Gromov, who asked whether, for a closed, aspherical, connected and oriented Riemannian manifold M, there always exists some $p \geq 0$ such that zero belongs to the spectrum of the Laplace–Beltrami operator Δ_p acting on the square integrable p–forms on the universal covering $\widetilde{M}$ of M. Farber and Weinberger [3] show that the conjecture is not true if the condition that M is aspherical is dropped. More generally, Higson, Roe and Schick [10] show that for a finitely presented group G satisfying $H_0(G; C_r^*(G)) = H_1(G; C_r^*(G)) = H_2(G; C_r^*(G)) = 0$, there always exists a closed manifold Y of dimension n ($n \geq 6$) with $\pi_1(Y) = G$ such that Y is a counterexample to the conjecture if M is not required to be aspherical.

In this note, a more general construction is provided to preserve homology groups. For this, we have to introduce the notion of a finitely G–dense ring.

Definition 1.1 A finitely *G–dense ring* (R, ϕ) is a unital ring R together with a ring homomorphism $\phi\colon \mathbb{Z}[G] \to R$ such that, when R is regarded as a left $\mathbb{Z}[G]$–module via ϕ, for any finitely generated right $\mathbb{Z}[G]$–module M, finitely generated free right R–module F, and R–module surjection $f\colon M \otimes_{\mathbb{Z}[G]} R \twoheadrightarrow F$, the module F has a finite R–basis in $f(M \otimes 1)$.

This is an analog of G–dense rings defined in Ye [20]. Examples of finitely G–dense rings include the real reduced group C^*–algebra $C^*_{\mathbb{R}}(G)$, the real group von Neumann algebra $\mathcal{N}_{\mathbb{R}}G$, the real Banach algebra $l^1_{\mathbb{R}}(G)$, the rings $k = \mathbb{Z}/p$ (prime p) and $k \subseteq \mathbb{Q}$ a subring of the rationals (with trivial G–actions), the group ring $k[G]$ and so on.

Conventions Let π and G be two groups. Suppose that R is a $\mathbb{Z}[G]$–module and $BG, B\pi$ are the classifying spaces. For a group homomorphism $\alpha\colon \pi \to G$, we will denote by $H_*(G, \pi; R)$ the relative homology group $H_*(BG, B\pi; R)$ with coefficients in R. All manifolds are assumed to be connected smooth manifolds, until otherwise stated.

Our main result is the following.

Theorem 1.2 *Assume that G is a finitely presented group and (R, ϕ) is a finitely G–dense ring. Let X be a connected n–dimensional ($n \geq 5$) closed orientable manifold with fundamental group $\pi = \pi_1(X)$. Assume that $\alpha\colon \pi \to G$ is a group homomorphism of finitely presented groups such that the image $\alpha(\pi)$ is finitely presented and*

$$H_1(\alpha)\colon H_1(\pi; R) \to H_1(G; R) \text{ is injective, and}$$
$$H_2(\alpha)\colon H_2(\pi; R) \to H_2(G; R) \text{ is surjective.}$$

Suppose either that R is a principal ideal domain or that the relative homology group $H_1(G, \pi; R)$ is a stably free R–module. When 2 is not invertible in R, suppose that the manifold M is a spin manifold. Then there exists a closed R–orientable manifold Y with the following properties:

(i) *Y is obtained from X by attaching 1–handles, 2–handles and 3–handles, such that*

(ii) *$\pi_1(Y) \cong G$ and the inclusion map $g\colon X \to W$, the cobordism between X and Y, induces the same fundamental group homomorphism as α, and*

(iii) *for any integer $q \geq 2$ the map g induces an isomorphism of homology groups*

$$g_*\colon H_q(X; R) \xrightarrow{\cong} H_q(W; R). \tag{1-1}$$

Theorem 1.2 has the following applications.

(1) Let α be *surjective*.

- When $\ker\alpha$ is perfect and $R = \mathbb{Z}[G]$ the group ring, this is Quillen's plus construction with handles, which is obtained by Hausmann [7; 8] (see Corollary 4.1).

- When $\ker\alpha$ is strongly L'–perfect and $R=\mathbb{Z}[G/L]$, Theorem 1.2 implies the existence of (mod L)-one-sided h–cobordism obtained by Guilbault and Tinsley [4; 5] (see Corollary 4.2).

(2) Let $\pi=1$ (α is *injective).*

- When $X=S^n$ $(n\geq 5)$, $R=\mathbb{Z}$ and G is a superperfect group, Theorem 1.2 recovers the existence of homology spheres obtained by Kervaire [13] (see Corollary 4.4).
- When $X=S^n$ $(n\geq 5)$, $R=\mathbb{Z}$ and $H_1(G;\mathbb{Z})=\mathbb{Z}$, $H_2(G;\mathbb{Z})=0$, Theorem 1.2 recovers the existence of higher-dimensional knots obtained by Kervaire [12] (see Corollary 4.5).
- When $X=S^n$ $(n\geq 6)$ and $R=C^*_{\mathbb{R}}(G)$, the theorem yields the results obtained by Farber and Weinberger [3] and Higson, Roe and Schick [10] on the zero-in-the-spectrum conjecture (see Corollary 4.7).

For Bousfield's integral localization, Rodríguez and Scevenels [18] show that there is a topological construction that, while leaving the integral homology of a space unchanged, kills the intersection of the transfinite lower central series of its fundamental group. Moreover, this is the maximal subgroup that can be factored out of the fundamental group without changing the integral homology of a space. As another application of Theorem 1.2 with α surjective and $R=\mathbb{Z}$, we obtain a manifold version of Rodríguez and Scevenels' result.

Corollary 1.3 *Let $n\geq 5$ and X be a closed n–dimensional spin manifold with fundamental group π and N a normal subgroup of π. The following are equivalent.*

(i) *There exists a closed manifold Y obtained from X by adding 2–handles and 3–handles with $\pi_1(Y)=\pi/N$, such that for any $q\geq 0$ there is an isomorphism*
$$H_q(Y;\mathbb{Z})\cong H_q(X;\mathbb{Z}).$$

(ii) *The group N is normally generated by finitely many elements and is a relatively perfect subgroup of π, ie, $[\pi,N]=N$.*

The article is organized as follows. In Section 2, we introduce some basic facts about finitely G–dense rings, surgery theory, Poincaré duality with coefficients, and one-sided homology cobordism. The main theorem is proved in Section 3 and some applications are given in Section 4.

2 Preliminary results and basic facts

2.1 Finitely G–dense rings

Recall the concept of finitely G–dense rings from Definition 1.1 (see also Ye [21, Definition 1]). Compared with the definition of G–dense rings, we require all the modules in Definition 1.1 to be finitely generated. It is clear that a G–dense ring is finitely G–dense. The following lemma from Ye [21] gives some typical examples of finitely G–dense rings.

Lemma 2.1 (Ye [21, Lemma 2]) *Finitely G–dense rings include the real reduced group C^*–algebra $C^*_{\mathbb{R}}(G)$, the real group von Neumann algebra $\mathcal{N}_{\mathbb{R}}G$, the real Banach algebra $l^1_{\mathbb{R}}(G)$, the rings $k = \mathbb{Z}/p$ (prime p) and $k \subseteq \mathbb{Q}$ a subring of the rationals (with trivial G–actions), and the group ring $k[G]$.*

In a similar way to Ye's [20, Example 2.6], one can show that the ring of Gaussian integers $\mathbb{Z}[i]$ is not finitely G–dense for the trivial group G.

2.2 Basic facts on surgery

The proof of Theorem 1.2 is based on some facts in surgery theory. The following definition and lemmas can be found in Ranicki [17].

Definition 2.2 Given an n–manifold M and an embedding $S^i \times D^{n-i} \subset M$ $(-1 \leq i \leq n)$, define the n–manifold $M' = (M - S^i \times D^{n-i}) \cup D^{i+1} \times S^{n-i-1}$ obtained from M by an i–surgery. The *trace* of the surgery on $S^i \times D^{n-i} \subset M$ is the elementary $(n+1)$–dimensional cobordism $(W; M, M')$ obtained from $M \times [0,1]$ by attaching an $(i+1)$–handle $W = M \times [0,1] \cup \bigcup_{S^i \times D^{n-i} \times 1} D^{i+1} \times D^{n-i}$.

The following lemma gives homotopy relations between the surgery trace and the manifolds.

Lemma 2.3 *Let M, M' and W be the manifolds defined in Definition 2.2. There are homotopy equivalences*

$$M \cup e^{i+1} \simeq W \simeq M' \cup e^{n-i},$$

where attaching maps are induced by the embedding of handles.

For a manifold M, denote by $w(M)$ the first Stiefel–Whitney class of the tangent bundle over M. The following lemma is Ranicki's [17, Proposition 4.24].

Lemma 2.4 *Let $(W; M, M')$ be the trace of an n–surgery on an m–dimensional manifold M killing $x \in H_n(M)$.*

(1) *If $1 \le n \le m-2$, then W and M' have the same orientation type as M (which means that M' is orientable iff M is orientable).*

(2) *If $n = m-1$ and M is orientable, so are M' and W.*

(3) *If $n = m-1$ and M is nonorientable, then M' is orientable if and only if $x = w(M) \in H_{m-1}(M; \mathbb{Z}_2) = H^1(M; \mathbb{Z}_2)$.*

(4) *If $n = 0$ and M is nonorientable, then so are W and M'.*

2.3 Poincaré duality with coefficients

In this subsection, we collect some facts about the Poincaré duality with coefficients. For more details, see Wall's book [19, Chapter 2]. Let X be a finite CW–complex with a universal covering space $\widetilde{X}$. Denote by $C_*(\widetilde{X})$ the cellular chain complex of X and by $C^*(\widetilde{X})$ the chain complex $\mathrm{Hom}_{\mathbb{Z}\pi_1(X)}(C_*(\widetilde{X}), \mathbb{Z}\pi_1(X))$. We call a finite CW–complex X a simple Poincaré complex if for some positive integer n and a representative cycle $\xi \in C_n(\widetilde{X}) \otimes_{\mathbb{Z}_1(X)} \mathbb{Z}$, the cap product induces a chain homotopy equivalence

$$\xi \cap : C^*(\widetilde{X}) \to C_{n-*}(\widetilde{X})$$

and the Whitehead torsion is vanishing. Similarly, we can define Poincaré pair (Y, X) by a simple homotopy equivalence

$$\xi \cap : C^*(\widetilde{Y}) \to C_{n-*}(\widetilde{Y}, \widetilde{X}).$$

When X is an n–dimensional closed manifold, X is a simple Poincaré complex of formal dimension n. When X is a compact manifold with boundary ∂X, the pair $(X, \partial X)$ is a simple Poincaré pair (cf Wall [19, Theorem 2.1, page 23]).

Lemma 2.5 *Let M be an n–dimensional orientable compact manifold with boundary $\partial M = X \dot\cup Y$ for closed manifolds X and Y. Then for any integer $q \ge 0$ and any $\mathbb{Z}\pi_1(M)$–module R, there is an isomorphism*

$$H^q(M, X; R) \to H_{n-q}(M, Y; R).$$

Proof Since X is a Poincaré complex and $(M, X \dot\cup Y)$ is a Poincaré pair, the lemma can be proved by considering the long exact homology and cohomology sequences for the cofiber sequence of pairs

$$(X, \varnothing) \to (M, Y) \to (M, X \dot\cup Y)$$

using Poincaré duality for X and the pair $(M, X\dot{\cup}Y)$. When R is commutative, this is in Hatcher's textbook [6, Theorem 3.43]. The proof of general case is similar. □

2.4 One-sided R–homology cobordism

Recall from Guilbault and Tinsley [5] that a *one-sided h–cobordism* $(W; X, Y)$ is a compact cobordism between closed manifolds such that $Y \hookrightarrow W$ is a homotopy equivalence. Motivated by this, we can define one-sided homology cobordism.

Definition 2.6 Let $(W; X, Y)$ be a compact cobordism between closed manifolds and R a $\mathbb{Z}\pi_1(W)$–module. We call $(W; X, Y)$ a *one-sided R–homology cobordism* if the inclusion $Y \hookrightarrow W$ induces

$$\pi_1(Y) \xrightarrow{\cong} \pi_1(W)$$

and, for any integer $q \geq 0$ an isomorphism $H_q(Y; R) \cong H_q(W; R)$.

The following are some easy facts.

(1) When $R = \mathbb{Z}[\pi_1(W)]$, one-sided R–homology cobordism is the same as one-sided h–cobordism.

Proof By the Whitehead Theorem, the homotopy equivalence follows from the homology equivalence with coefficients in $\mathbb{Z}[\pi_1(W)]$ and the isomorphism of fundamental groups. □

(2) Let $(W; X, Y)$ be a one-sided R–homology cobordism. For any integer $q \geq 0$, the inclusion map induces an isomorphism $H^q(W; R) \cong H^q(X; R)$.

Proof This follows directly from Poincaré duality with coefficients as in the previous subsection. □

(3) For a one-sided h–cobordism $(W; X, Y)$, the inclusion map $X \hookrightarrow W$ is a Quillen plus construction.

Proof Since $Y \hookrightarrow W$ is a homotopy equivalence, we have that for any integer $q \geq 0$, the relative cohomology group $H^q(W, Y; \mathbb{Z}[\pi_1(W)]) = 0$. By Poincaré duality, the inclusion map $X \hookrightarrow W$ induces homology isomorphism with coefficients in $\mathbb{Z}[\pi_1(W)]$. According to Berrick [1, 4.3 xi], the inclusion map is then a Quillen plus construction. □

(4) Let R be a principal ideal domain and $(W; X, Y)$ a one-sided R–homology cobordism. Then for any integer $q \geq 0$, there is an isomorphism $H_q(X; R) \cong H_q(Y; R)$.

Proof When the inclusion map $Y \hookrightarrow W$ induces an R–homology equivalence, it also induces an R–cohomology equivalence by the universal coefficients theorem. By Poincaré duality, X has the same homology as W, as well as Y. $\square$

3 Proof of Theorem 1.2

In this section, we will prove Theorem 1.2. First, we need some facts about finitely presented groups.

Recall that a normal subgroup N of a group π is called *normally finitely generated* if there exists a finite set $S \subset N$ such that N is generated by elements of the form gsg^{-1} for $s \in S$ and $g \in \pi$. The following lemma gives an elementary characterization of when a normal subgroup is normally finitely generated. Since it is helpful for our later argument, we present a short proof here.

Lemma 3.1 *Let π be a finitely presented group and N a normal subgroup. Then N is normally finitely generated if and only if π/N is finitely presented.*

Proof The necessity of the condition is obvious. Conversely, choose a presentation of π/N with finitely many generators and finitely many relations. Let F_n be the free group with n generators and $f\colon F_n \to \pi$ a surjection with normally finitely generated kernel K. We can also assume that the generators of F_n are mapped surjectively to the generators of π/N by the composition of f with the quotient map $\pi \to \pi/N$. Here we use the fact that the condition that a group is finitely presented does not depend on the choice of a generator system (cf Ohshika [16, Proposition 1.3]). Since π/N is finitely presented, $f^{-1}(N)$ is normally finitely generated. Then $N = f(f^{-1}(N))$ is normally finitely generated. $\square$

In order to prove Theorem 1.2, we use the following lemma, which is a more general version of Hopf's exact sequence.

Lemma 3.2 (Higson, Roe and Schick [10, Lemma 2.2]) *Let G be a group and V be a left $\mathbb{Z}[G]$–module. For any CW–complex X with fundamental group G and universal covering space $\widetilde{X}$, there is an exact sequence*

$$H_2(\widetilde{X}) \otimes_{\mathbb{Z}[G]} V \to H_2(X; V) \to H_2(G; V) \to 0.$$

Proof of Theorem 1.2 We construct a manifold Y_1 whose fundamental group $\pi_1(Y) = G$ as follows. Fix a finite presentation

$$\langle x_1, x_2, \dots, x_k \mid y_1, y_2, \dots, y_l \rangle$$

of $\alpha(\pi)$. Extend the presentation of $\alpha(\pi)$ by generators and relations to yield a presentation

$$\langle x_1, x_2, \dots, x_k, g_1, g_2, \dots, g_u \mid y_1, y_2, \dots, y_l, r_1, r_2, \dots, r_v \rangle$$

of G by adding some generators and relations. For adding the generators $g_1, g_2, \dots, g_u$, let S_i^0 be a copy of the 0–sphere S^0 and

$$f_1\colon \coprod_{i=1}^{u} S_i^0 \times D^n \to X$$

be an embedding with disjoint image. Add 1–handles along f_1 to X. The resulting manifold is X_1 and denote by W_1 the surgery trace. We see that the manifold X_1 is actually the connected sum $X \sharp_{i=1}^{u} S^1 \times S^{n-1}$ and can have the same orientation type as X. Denote by e_i^j a copy of j–cells indexed by i. By Lemma 2.3, there are homotopy equivalences

$$X \cup \bigcup_{i=1}^{u} e_i^1 \simeq W_1 \simeq X_1 \cup \bigcup_{i=1}^{u} e_i^n.$$

According to the construction, the fundamental group of X_1 is

$$\pi_1(X_1) = \pi * F(g_1, g_2, \dots, g_u),$$

the free product of $\pi = \pi_1(X)$ and the free group of u generators. By Lemma 3.1, the kernel $\ker\alpha$ is normally generated by finitely many elements $z_1, z_2, \dots, z_p$. Denote by S the finite set $\{z_1, z_2, \dots, z_p, r_1, r_2, \dots, r_v\}$. Choose as usual a contractible open set U in X_1 as "base point". According to Whitney's theorem, any element in $\pi_1(X_1)$ is represented by an embedded 1–sphere. Since the manifold X_1 is orientable, the normal bundle of any embedded 1–sphere is trivial. For the elements in S, let S_λ^1 be a copy of the 1–sphere S^1 and let

$$f_2\colon \coprod_{\lambda \in S} S_\lambda^1 \times D^{n-1} \to X_1$$

be disjoint embeddings representing the corresponding elements in $\pi_1(X_1)$. Do surgery along f_2 by attaching 2–handles. The resulting manifold is X_2, and denote by W_2 the surgery trace. By Lemma 2.3 once again, there are homotopy equivalences

$$X_1 \cup \bigcup_{\lambda \in S} e_\lambda^2 \simeq W_2 \simeq X_2 \cup \bigcup_{\lambda \in S} e_\lambda^{n-1}.$$

Since $n \geq 4$, the fundamental group of X_2 is G. Let W' be the manifold obtained by gluing the two traces W_1 and W_2 together along X_1. There are homotopy equivalences

$$W' \simeq X \cup \bigcup_{i=1}^{u} e_i^1 \cup \bigcup_{\lambda \in S} e_\lambda^2 \simeq X_1 \cup \bigcup_{i=1}^{u} e_i^n \cup \bigcup_{\lambda \in S} e_\lambda^2 \simeq X_2 \cup \bigcup_{i=1}^{u} e_i^n \cup \bigcup_{\lambda \in S} e_\lambda^{n-1}.$$

This implies the fundamental group of W' is also G, since $n > 3$.

We consider the homology groups of the pair (W', X). Let $\widetilde{X}$ and $\widetilde{W}'$ be the universal covering spaces of X and W'. By Lemma 3.2, there is a commutative diagram

$$\begin{tikzcd}
H_2(\widetilde{X}) \otimes_{\mathbb{Z}G} R \arrow[r] \arrow[d] & H_2(\widetilde{W}') \otimes_{\mathbb{Z}G} R \arrow[d, "j_4"] & & & \\
H_2(X; R) \arrow[r, "j_2"] \arrow[d, "j_3"] & H_2(W'; R) \arrow[r, "j_1"] \arrow[d, "j_5"] & H_2(W', X; R) \arrow[r] & H_1(X'; R) \arrow[r] & H_1(W'; R) \\
H_2(\pi; R) \arrow[r, "\alpha_*"] & H_2(G; R) & & &
\end{tikzcd}$$

where the middle horizontal chain is the long exact sequence of homology groups for the pair (W', X) and the two vertical lines are the Hopf exact sequences as in Lemma 3.2. Notice that

$$H_1(X; R) \cong H_1(\pi; R) \to H_1(W'; R) \cong H_1(G; R)$$

is injective by assumption. This implies $j_1\colon H_2(W'; R) \to H_2(W', X; R)$ is surjective in the above diagram. Note that the map $\alpha_*\colon H_2(\pi; R) \to H_2(G; R)$ is surjective by assumption. By a diagram chase (for more details, see Ye [20, proof of Theorem 1.1]), there is a surjection

$$j_1 \circ j_4\colon H_2(\widetilde{W}') \otimes_{\mathbb{Z}G} R \to H_2(W', X; R).$$

As $\widetilde{W}'$ is simply connected, the homology group $H_2(\widetilde{W}')$ is isomorphic to $\pi_2(W') \cong \pi_2(X_2)$. Notice that the homology group $H_2(W', X; R)$ can be taken to be a finitely generated free R–module as in Ye [20, proof of Theorem 1]. Since the ring R is a finitely G–dense ring in the sense of Definition 1.1, we can find a finite set S' of elements in $\pi_2(X_2)$ such that the image $j_1 \circ j_4(S')$ forms an R–basis for $H_2(W', X; R)$. Then there are maps $b_\lambda\colon S_\lambda^2 \to X_2$ with $\lambda \in S'$ such that for all $q \geq 2$, the composition of maps

$$H_q\Big(\bigvee_{\lambda \in S'} S_\lambda^2; R\Big) \to H_q(W'; R) \to H_q(W', X; R)$$

is an isomorphism.

We construct the manifolds Y and W as follows. Notice that an embedded 2–sphere in a k–dimensional ($k \geq 5$) orientable manifold M has trivial normal bundle if and only if it represents 0 in $\pi_1(\mathrm{SO}(k)) = \mathbb{Z}/2$ through the classifying map $M \to \mathrm{BSO}(k)$ (cf Milnor [15, page 45]). When 2 is invertible in the ring R, we can always choose the 2–spheres in S' to have trivial normal bundles. When 2 is not invertible in R, the manifold X is a spin manifold by assumption. This implies any embedded 2–sphere has a trivial normal bundle. Since $n \geq 5$, we can choose a map

$$f_3\colon \coprod_{\lambda \in S'} S^2_\lambda \times D^{n-2} \to X_2$$

to be a disjoint embedding whose components represent the elements b_λ. Do surgery along f_3 by attaching 3–handles. Let Y denote the resulting manifold and W_3 denote the surgery trace. Suppose that W is the manifold obtained by gluing W' and W_2 along X_2, which is a cobordism between X and Y. By Lemma 2.3, there are homotopy equivalences

$$X_2 \cup \bigcup_{\lambda \in S'} e^3_\lambda \simeq W_3 \simeq Y \cup \bigcup_{\lambda \in S'} e^{n-2}_\lambda$$

and

$$W \simeq W' \cup \bigcup_{\lambda \in S'} e^3_\lambda \simeq Y \cup \bigcup_{i=1}^{u} e^n_i \cup \bigcup_{\lambda \in S} e^{n-1}_\lambda \cup \bigcup_{\lambda \in S'} e^{n-2}_\lambda.$$

By the van Kampen Theorem, the fundamental group of Y is still G, since $n > 4$. Denoting by $H_*(\cdot)$ the homology groups $H_*(\cdot\,; R)$, we have the following commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & H_3(\bigvee D^3, \bigvee S^2) \arrow[r] \arrow[d] & H_2(\bigvee S^2, \mathrm{pt}) \arrow[r] \arrow[d] & H_2(\bigvee D^3, \mathrm{pt}) \arrow[r] \arrow[d] & H_2(\bigvee D^3, \bigvee S^2) \arrow[d] \\
\cdots \arrow[r] & H_3(W, W') \arrow[r] & H_2(W', X) \arrow[r] & H_2(W, X) \arrow[r] & H_2(W, W')
\end{tikzcd}$$

By a five lemma argument and the assumption that $\alpha_*\colon H_1(\pi; R) \to H_1(G; R)$ is injective, for any integer $q \geq 2$, the relative homology group $H_q(W, X; R) = 0$. This shows for any integer $q \geq 2$, the homology groups $H_q(X; R) \cong H_q(W; R)$ and proves the isomorphism in (1-1). $\square$

Remark 3.3 From the proof, we can see that for some special group homomorphism α and coefficients in R, the orientability or spin-ness of X in Theorem 1.2 can be dropped. For example, when α is surjective and $\ker(\alpha) < [\pi, \pi]$, we do not need X to be orientable. When $\ker(\alpha)$ is perfect (or weakly L–perfect for some normal group), the spin-ness of X can be dropped (cf Guilbault and Tinsley [5, proofs of Theorems 4.1 and 5.2]).

4 Applications

In this section, we give several applications of Theorem 1.2.

4.1 Surgery plus construction for manifolds

In this subsection, we get a manifold version of Quillen's plus construction by doing surgery. The following proposition is a special case of Theorem 1.2 when π has a normally finitely generated normal perfect subgroup P, $G = \pi/P$ and $R = \mathbb{Z}[G]$.

Corollary 4.1 *Let $n \geq 5$ be an integer and M a closed n–dimensional spin manifold. Suppose that P is a normal perfect subgroup in $\pi_1(M)$ normally generated by some finite elements. Then there exists a one-sided h–cobordism $(W; M, Y)$ such that $\pi_1(W) = \pi_1(M)/P$. More precisely, there exists a closed spin manifold Y with the following properties:*

(i) *Y is obtained from M by attaching 2–handles and 3–handles, such that*

(ii) *$\pi_1(Y) = \pi_1(M)/P$ and the inclusion map $g\colon M \to W$, the cobordism between X and Y, is Quillen's plus construction inducing epimorphism $\pi_1(M) \to \pi_1(M)/P$ of fundamental groups, and*

(iii) *Y has the homotopy type of Quillen's plus construction M^+.*

Proof We apply Theorem 1.2. Let $X = M$ and $\alpha\colon \pi = \pi_1(M) \to \pi/P$ be the quotient map. By Hilton and Stammbach [11], there is an exact sequence:

$$\begin{aligned} H_2(\pi; \mathbb{Z}[\pi/P]) \to H_2(\pi/P; \mathbb{Z}[\pi/P]) \to \mathbb{Z}[\pi/P] \otimes_{\mathbb{Z}[\pi/P]} P_{\mathrm{ab}} \\ \to H_1(\pi; \mathbb{Z}[\pi/P]) \to H_1(\pi/P; \mathbb{Z}[\pi/P]) \to 0 \end{aligned}$$

When $\mathbb{Z}[\pi/P] \otimes_{\mathbb{Z}[\pi/P]} P_{\mathrm{ab}} \cong \mathbb{Z} \otimes_{\mathbb{Z}} P_{\mathrm{ab}} = 0$, we can see

$$H_2(\pi; \mathbb{Z}[\pi/P]) \to H_2(\pi/P; \mathbb{Z}[\pi/P])$$

is surjective and $H_1(\pi; \mathbb{Z}[\pi/P]) \to H_1(\pi/P; \mathbb{Z}[\pi/P])$ is an isomorphism. Therefore, the conditions of group homomorphism α are satisfied. By Theorem 1.2, there exists a closed spin manifold Y and cobordism $(W; M, Y)$ such that for any integer $q \geq 0$, there is an isomorphism

$$H_q(M; \mathbb{Z}[G]) \cong H_q(W; \mathbb{Z}[G]).$$

This implies the inclusion map $g\colon X \to W$ is the Quillen plus construction (cf Berrick [1, 4.3 xi]). Therefore, for all integers q, the relative cohomology groups $H^q(W, X; \mathbb{Z}[G]) = 0$. According to the Poincaré duality in Lemma 2.5, for each integer $q \geq 0$, the relative homology group $H_q(W, Y; \mathbb{Z}[G]) = 0$ and there is an isomorphism

$$H_q(Y; \mathbb{Z}[G]) \cong H_q(W; \mathbb{Z}[G])$$

as well. This means the universal covering spaces of Y and W are homology equivalent and therefore also homotopy equivalent. Since Y and W have the same fundamental group, this implies the inclusion map $Y \to W$ is a homotopy equivalence. This finishes the proof. □

Corollary 4.1 was first obtained by Hausmann [7] (see also Hausmann [8] and Guilbault and Tinsley [4; 5]).

Recall the definition of (mod L)–one-sided h–cobordism by Guilbault and Tinsley [5]. We see that a (mod L)–one-sided h–cobordism is a one-sided $\mathbb{Z}[\pi_1(Y)/L]$–homology cobordism. The following result proved by Guilbault and Tinsley [5] is a corollary of Theorem 1.2 when $R = \mathbb{Z}[\pi_1(Y)/L]$. Note that $\mathbb{Z}[\pi_1(Y)/L]$ is a finitely $\pi_1(Y)$–dense ring.

Corollary 4.2 (Guilbault and Tinsley [5, Theorem 5.2]) *Let B be a closed n–manifold ($n \geq 5$) and $\alpha\colon \pi_1(B) \to G$ a surjective homomorphism onto a finitely presented group such that $\ker(\alpha)$ is strongly L'–perfect, ie, $\ker(\alpha) = [\ker(\alpha), L']$ for some group L', where $\ker(\alpha) \trianglelefteq L' \trianglelefteq \pi_1(B)$ and all loops representing elements in L' are orientation-preserving. Then for $L = L'/\ker(\alpha)$, there exists a (mod L)–one-sided h–cobordism $(W; B, A)$ such that $\pi_1(W) = G$ and $\ker(\pi_1(B) \to \pi_1(W)) = \ker(\alpha)$.*

Proof The proof is similar to that of Corollary 4.1. When $\ker(\alpha')$ is L'–perfect, we have that $\mathbb{Z}[G/L] \otimes_{\mathbb{Z}[G]} \ker(\alpha)_{\mathrm{ab}} = 0$. According to the 5–term exact sequence for group homology (cf Hilton and Stammbach [11, (8.2), page 202]),

$$\begin{aligned} H_2(\pi_1(B); \mathbb{Z}[G/L]) \xrightarrow{H_2(\alpha)} H_2(G; \mathbb{Z}[G/L]) \to \mathbb{Z}[G/L] \otimes_{\mathbb{Z}[G]} \ker(\alpha)_{\mathrm{ab}} \\ \longrightarrow H_1(\pi_1(B); k[G/L]) \xrightarrow{H_1(\alpha)} H_1(G; k[G/L]) \to 0, \end{aligned}$$

we can see that $H_2(\alpha)$ is surjective and $H_1(\alpha)$ is isomorphic. By Theorem 1.2 with $R = \mathbb{Z}[G/L]$ and the remark following it, there exists a cobordism $(W; A, B)$ such that $\pi_1(B) = \pi_1(W) = G$ and the inclusion $B \hookrightarrow W$ induces homology equivalence with coefficients in R. Considering the covering spaces of B and W with deck transformation group G/L, we can see that the inclusion $B \hookrightarrow W$ also induces a cohomology equivalence with coefficients R. By Poincaré duality in Lemma 2.5, the inclusion $A \hookrightarrow W$ induces homology equivalence with coefficients in R. This finishes the proof. □

4.2 Surgery preserving integral homology groups

In this subsection, we study the case when the integral homology groups of a manifold are preserved by doing surgery. Corollary 1.3 is a special case of Theorem 1.2 when $R = \mathbb{Z}$.

Proof of Corollary 1.3 We show that (ii) implies (i) first. By Hilton and Stammbach [11], there is an exact sequence

$$H_2(\pi;\mathbb{Z}) \to H_2(\pi/N;\mathbb{Z}) \to N/[\pi,N] \to H_1(\pi;\mathbb{Z}) \to H_1(\pi/N;\mathbb{Z}) \to 0. \tag{3}$$

When $N = [\pi, N]$, we have that the map $H_2(\pi;\mathbb{Z}) \to H_2(\pi/N)$ is surjective and $H_1(\pi;\mathbb{Z}) \to H_1(\pi/N)$ is an isomorphism. According to Theorem 1.2 with $R = \mathbb{Z}$, there exists a closed spin manifold Y obtained from X by adding 2–handles and 3–handles with $\pi_1(Y) = \pi_1(X)/N$. Let W be the cobordism between X and Y. Furthermore, we have for any $q \geq 0$ there is an isomorphism $H_q(X;\mathbb{Z}) \cong H_q(W;\mathbb{Z})$. According to the universal coefficients theorem, for all integers $q \geq 0$ the relative cohomology groups $H^q(W, X;\mathbb{Z}) = 0$. Therefore by Theorem 1.2(ii), for any integer $q \geq 0$ there is an isomorphism

$$H_q(Y;\mathbb{Z}) \cong H_q(X;\mathbb{Z}).$$

Conversely, suppose $(W; X, Y)$ is a cobordism with the boundary X and Y. Since Y is obtained from X by doing surgery below the middle dimension, we have isomorphisms $H_1(Y;\mathbb{Z}) \cong H_1(W;\mathbb{Z})$ and $H_2(Y;\mathbb{Z}) \cong H_2(W;\mathbb{Z}) \cong H_2(X;\mathbb{Z})$. Therefore the inclusion map $X \to W$ induces an isomorphism

$$H_1(\pi;\mathbb{Z}) = H_1(X;\mathbb{Z}) \cong H_1(W;\mathbb{Z}) = H_1(\pi_1(W);\mathbb{Z}).$$

According to the Hopf exact sequence (cf Lemma 3.2), there is a commutative diagram

$$\begin{array}{ccc}
H_2(X;\mathbb{Z}) & \twoheadrightarrow & H_2(\pi;\mathbb{Z}) \\
\downarrow & & \downarrow \\
H_2(W;\mathbb{Z}) & \twoheadrightarrow & H_2(\pi_1(Y);\mathbb{Z})
\end{array}$$

where the left vertical map is an isomorphism. This shows that the right vertical map is an epimorphism. According to the same exact sequence (3) above, we have $N = [\pi, N]$, which means N is relative perfect. Since $\pi_1(X)/N = \pi_1(Y)$ is finitely presented, N is normally finitely generated by Lemma 3.1. □

Corollary 1.3 is a manifold version of a result obtained by Rodríguez and Scevenels [18] for CW–complexes.

4.3 The fundamental groups of homology manifolds

In this subsection, we study the fundamental groups of manifolds with the same homology type as a 2–connected manifold.

Theorem 4.3 *Let G be a finitely presented group, R a subring of the rationals or the constant ring $\mathbb{Z}/p$ (prime p) and $n \geq 5$ an integer. Suppose M is a 2–connected manifold of dimension n. Then the following are equivalent:*

(i) *There exists an n–dimensional manifold Y obtained from M by adding 1–handles, 2–handles and 3–handles with $\pi_1(Y) = G$, and for any integer $q \geq 0$, we have*
$$H_q(Y; R) \cong H_q(M; R).$$

(ii) *The group G is R–superperfect, ie, $H_1(G; R) = 0$ and $H_2(G; R) = 0$.*

Proof We show that (i) implies (ii) first. By assumption, we have $H_2(M; R) = H_1(M; R) = 0$. If some manifold Y has the same homology groups with coefficients in R as M does, then $H_1(G; R) \cong H_1(M; R) = 0$. According to the Hopf exact sequence in Lemma 3.2, we get $H_2(G; R) = 0$.

Conversely, suppose that G is a finitely presented group with $H_2(G; R) = H_1(G; R) = 0$. Let $X = M$, $\pi = 1$, the trivial group, and $f\colon \pi \to G$ the obvious group homomorphism. Note that the 2–connected manifold X is always a spin manifold and R is a principal ideal domain. According to Theorem 1.2, we get a manifold Y with $\pi_1(Y) = G$ such that for any integer $q \geq 0$, there is an isomorphism $H_q(W; R) \cong H_q(M; R)$ for the cobordism W. According to the universal coefficient theorem, for all integers $q \geq 0$, the relative cohomology groups $H^q(W, M; R) = 0$. Therefore, for any integer $q \geq 0$, there is an isomorphism $H_q(Y; R) \cong H_q(M; R)$ by Theorem 1.2(ii). □

Recall that an n–dimensional R–homology sphere is an n–dimensional manifold Y such that $H_i(Y; R) = H_i(S^n; R)$ for any integer $i \geq 0$. The first part of the following result proved by Kervaire in [13] is a special case of Theorem 4.3 when $M = S^n$ and $R = \mathbb{Z}$.

Corollary 4.4 *Let G be a finitely presented group and $n \geq 5$ be an integer. Then there exists an n–dimensional homology sphere Y with $\pi_1(Y) = G$ if and only if G is superperfect, ie, $H_1(G; \mathbb{Z}) = 0$ and $H_2(G; \mathbb{Z}) = 0$. Moreover, such manifolds can taken to be in the same cobordism class as S^n.*

Hausmann and Weinberger [9] constructed a superperfect group G for which any 4–manifold Y with $\pi_1(Y) = G$ satisfies $\chi(Y) > 2$. As a consequence it follows that Theorem 4.3 and Corollary 4.4 do not extend to dimension four.

4.4 The fundamental groups of higher-dimensional knots

In this subsection, we study the fundamental groups of higher-dimensional knots. The following result proved by Kervaire [12] is a corollary of Theorem 1.2. (Recall the definition of weight $w(G)$ from the introduction.)

Corollary 4.5 *Given an integer $n \geq 3$, a finitely presentable group G is isomorphic to $\pi_1(S^{n+2} - f(S^n))$ for some differential embedding $f\colon S^n \to S^{n+2}$ if and only if the first homology group $H_1(G;\mathbb{Z}) = \mathbb{Z}$, the weight of G is 1, and the second homology group $H_2(G;\mathbb{Z}) = 0$.*

Proof Suppose that for a finitely presentable group G, we have $H_1(G;\mathbb{Z}) = \mathbb{Z}$, $H_2(G;\mathbb{Z}) = 0$ and the weight $w(G) = 1$. Let $X = S^{n+2}$, $\pi = 1$, the trivial group, and $\alpha\colon \pi \to G$ the trivial group homomorphism. Notice that α induces an injection of the first homology groups and surjection of the second homology groups. By Theorem 1.2 with $R = \mathbb{Z}$, we get a closed manifold Y with $\pi_1(Y) = G$, obtained from S^{n+2} by attaching 1–handles, 2–handles and 3–handles. Suppose that W is the surgery trace. By Poincaré duality, for any integer $q \leq n+1$ the relative cohomology group $H^q(W, Y) = 0$. According to the universal coefficient theorem, we have that for each integer $2 \leq i \leq n+1$ there is an isomorphism

$$H_i(Y) \cong H_i(X) = 0.$$

Let $\gamma \in G$ be an element such that G is normally generated by γ and $\varphi\colon S^1 \to Y$ be a differential embedding representing γ. Extend φ to be an embedding $\varphi'\colon S^1 \times D^{n+1} \to Y$. Do surgery to Y along φ' to get a manifold M. It can be easily seen that M is simply connected and for each integer $1 \leq i \leq n$ the homology group $H_i(M) = 0$. Therefore, M is a $(n+2)$–sphere by the solution to the higher-dimensional Poincaré Conjecture (note that all manifolds are assumed to be smooth). Let $\phi\colon D^2 \times S^n \to M$ be the embedding and choose $f = \phi(0,\cdot)$ to be the embedding $S^n \to M$. It can be directly checked that

$$\pi_1(M - f(S^n)) \cong \pi_1(M - \phi(D^2 \times S^n)) \cong \pi_1(Y - \varphi'(S^1 \times D^{n+1})) \cong \pi_1(Y) = G.$$

This finishes the "if" part.

Conversely, suppose $f\colon S^n \to S^{n+2}$ is a differential embedding. According to Alexander duality, we have $H_2(S^{n+2} - f(S^n)) = 0$ and $H_1(S^{n+2} - f(S^n)) = H_1(G;\mathbb{Z}) = 0$.

By Hopf's theorem in Lemma 3.2 (with the coefficient $V = \mathbb{Z}$), the second homology group $H_2(G;\mathbb{Z}) = 0$. Let $\alpha\colon S^1 \to S^{n+2} - f(S^n)$ be an embedding such that $\alpha(S^1)$ bounds a small 2–disc in S^{n+2} that intersects $f(S^n)$ transversally at exactly one point. Then the group $\pi_1(S^{n+2} - f(S^n))$ is normally generated by the element represented by α. For more details, see Kervaire [12, proof of Lemma 2]. This proves the weight $w(G) = 1$ and finishes the proof. □

Corollary 4.5 is Theorem 1 in Kervaire [12]. Similarly, we can show that Theorem 3 in [12] concerning the fundamental groups of links is also a corollary of Theorem 1.2. That is, for an integer $n \geq 3$, a finitely presentable group G is isomorphic to $\pi_1(S^{n+2} - L_k)$ for some k disjointly embedded n–spheres L_k if and only if $H_1(G;\mathbb{Z}) = \mathbb{Z}^k$, $w(G) = k$ and $H_2(G;\mathbb{Z}) = 0$ (cf [12, Theorem 3]). The proof is of the same pattern as that of Corollary 4.5 and will be left to the reader.

4.5 Zero-in-the-spectrum conjecture

In the notation of the introduction, the *zero not belonging to the spectrum* of $\Delta = \Delta_*$ can also be expressed as the vanishing of $H_*(M; C_r^*(\pi_1(M)))$. The following is a version of the zero-in-the-spectrum conjecture using homology. For more details, we refer the reader to the book of Lück [14].

Conjecture 4.6 Let M be a closed, connected, oriented and aspherical manifold with fundamental group π. Then for some $i \geq 0$, $H_i(X; C_r^*(\pi)) \neq 0$.

If the condition that X is aspherical is dropped, the following corollary, which is a special case of Theorem 1.2 when $R = C_{\mathbb{R}}^*(G)$ and $\pi = 1$, shows the above conjecture is not true. This result is a generalization of the results obtained by Farber and Weinberger [3], and Higson, Roe and Schick [10]. Recall that the real C^*–algebra $C_{\mathbb{R}}^*(G)$ is a finitely G–dense ring.

Corollary 4.7 (Higson, Roe and Schick [10]) *Let G be a finitely presented group with the homology groups*

$$H_0(G; C_r^*(G)) = H_1(G; C_r^*(G)) = H_2(G; C_r^*(G)) = 0.$$

For every integer $n \geq 6$ there is a closed manifold M of dimension n such that $\pi_1(Y) = G$ and for each integer $n \geq 0$, the homology group $H_n(Y; C_r^*(G)) = 0$.

Proof According to Ye [20, Proposition 4.8], the vanishing of lower degree homology groups with coefficients in $C_r^*(G)$ is the same as that with coefficients in $C_{\mathbb{R}}^*(G)$. Then we have

$$H_0(G; C_{\mathbb{R}}^*(G)) = H_1(G; C_{\mathbb{R}}^*(G)) = H_2(G; C_{\mathbb{R}}^*(G)) = 0.$$

Note that the real C^*–algebra $C^*_{\mathbb{R}}(G)$ is a finitely G–dense ring. Let $\alpha\colon \pi = 1 \to G$, $R = C^*_{\mathbb{R}}(G)$ and $X = S^n$ in Theorem 1.2. By the long exact sequence of homology groups

$$\cdots \to H_1(1; C^*_{\mathbb{R}}(G)) \to H_1(G; C^*_{\mathbb{R}}(G)) \to H_1(G, 1; C^*_{\mathbb{R}}(G)) \to H_0(1; C^*_{\mathbb{R}}(G)) \to H_0(G; C^*_{\mathbb{R}}(G)) \to 0,$$

we have

$$H_1(G, \pi; C^*_{\mathbb{R}}(G)) = H_0(1; C^*_{\mathbb{R}}(G)) = C^*_{\mathbb{R}}(G),$$

which is a free $C^*_{\mathbb{R}}(G)$–module. Therefore, there exists a closed manifold Y by Theorem 1.2 such that for any integer $0 \leq q \leq [n/2]$, the homology group $H_q(Y; C^*_{\mathbb{R}}(G)) = 0$. According to the universal coefficients theorem and Poincaré duality for L^2–homology (cf Farber [2, Theorems 6.6 and 6.7]), we get that for any integer $q \geq 0$, the homology group $H_n(Y; C^*_r(G)) = 0$. $\square$

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