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Decomposition of loop spaces and periodic problem on π_*

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We provide a family of spaces localized at 2, whose stable homotopy groups are summands of their unstable homotopy groups. Applications to mod 2 Moore spaces are given.

55P35; 55Q10, 55Q52

1 Introduction

Homotopy theory is a central topic in the area of algebraic topology. Understanding the relationship between the stable homotopy groups and unstable homotopy groups is an important question in homotopy theory. Let X be a n–connected pointed space. Recall that the classical Freudenthal Suspension Theorem [3] states that the canonical map

$$\pi_k(X) \to \pi_k^s(X)$$

is an isomorphism if $k \leq 2n$ and an epimorphism if $k = 2n+1$. The Freudenthal Suspension Theorem relates the unstable homotopy groups to the stable homotopy groups.

Recently, a new interesting problem in this area has been proposed. Namely, find spaces X whose stable homotopy groups are summands of the unstable homotopy groups. Beben and Wu [1] gave examples of such spaces and applied their results to the Moore Conjecture. They showed that for a fixed odd prime p and some p–localization of a CW–complex of finite type X, there exists a sequence $\{l_n\}$ that converges to infinity such that $\Omega\Sigma^{l_n}X$ is a homotopy retract of ΩX. Hence $\pi_{k+1}(\Sigma^{l_n}X)$ is a retract of $\pi_k(X)$. Letting $\{l_n\}$ converge to infinity, the stable homotopy groups of X are seen to be summands of its unstable homotopy groups. Symbolically, the group $\pi_*^s(X)$ is a summand of $\pi_*(X)$. In this way, Beben and Wu reduced the aforementioned problem to finding spaces X together with a sequence $\{l_n\}$ that converges to infinity such that $\Omega\Sigma^{l_n}X$ is a retract of ΩX. In this article, we consider the case when $p=2$. Our results are given as follows.

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Theorem 1.1 *For every $1 \le i \le n$, let X_i be a path-connected 2–local CW–complex such that $\bar{H}_*(X_i;\mathbb{Z}/2\mathbb{Z})$ is of dimension 2 with generators u_i, v_i and $|u_i| < |v_i|$. Then*

$$\Omega\Sigma \wedge \bigwedge_{i=1}^{n}\left(\Sigma^{(3^k-1)/2(|u_i|+|v_i|)} X_i\right)$$

is a retract of $\Omega\Sigma \wedge \bigwedge_{i=1}^{n} X_i$ for every $k \ge 1$.

As a consequence of Theorem 1.1, the stable homotopy groups of certain 2–local spaces retract off their regular homotopy groups.

Corollary 1.2 *For every $1 \le i \le n$, let X_i be a path-connected 2–local CW–complex such that $\bar{H}_*(X_i;\mathbb{Z}/2\mathbb{Z})$ is of dimension 2 with generators u_i, v_i and $|u_i| < |v_i|$. Let*

$$b_k = \sum_{i=1}^{n}\left(\frac{3^k-1}{2}(|u_i|+|v_i|)\right)$$

and suppose that $\bigwedge_{i=1}^{n} X_i$ is $(m-1)$–connected. Then for large enough k such that $j \le b_k + 2m$, the group $\pi_j^s\left(\Sigma \wedge \bigwedge_{i=1}^{n} X_i\right)$ is a homotopy retract of $\pi_{j+b_k}\left(\Sigma \wedge \bigwedge_{i=1}^{n} X_i\right)$.

Theorem 1.1 deals with the case of finite wedge products of 2–cell CW–complexes. In the case of a single 2–cell CW–complex, this theorem can be strengthened by using a known decomposition of $\Omega\Sigma X$ [5; 6]. The strengthened theorem is given as follows.

Theorem 1.3 *Let X be a simply connected 2–local CW–complex such that the group $\bar{H}_*(X;\mathbb{Z}/2\mathbb{Z})$ is of dimension 2 with generators u, v and $|u| < |v|$. Then we have*

$$\Omega\Sigma X \simeq \prod_j \Omega\Sigma^{1+k_j(|u|+|v|)} X \times (\text{some other space})$$

where $2 < k_1 < k_2 < \cdots$ are all prime numbers greater than 2.

A fundamental problem in homotopy theory is to compute the homotopy groups of a given space. We apply Theorem 1.3 to compute $\mathbb{Z}/8\mathbb{Z}$–summands of the homotopy groups of mod 2 Moore spaces. Let $\mathbb{R}\mathrm{P}^2$ be the projective plane. The n–dimensional mod 2 Moore space $P^n(2)$ is defined by $P^n(2) = \Sigma^{n-2}\mathbb{R}\mathrm{P}^2$ for $n \ge 2$. $P^n(2)$ can be viewed as the homotopy cofibre of the degree 2 map [2]: $S^{n-1} \to S^{n-1}$. That is to say, the cell complex $P^n(2)$ is obtained by attaching an n–cell to S^{n-1} and the attaching map is give by the degree 2 map.

This problem was studied earlier by Cohen and Wu [2]. They noted that a $\mathbb{Z}/8\mathbb{Z}$–summand of $\pi_*(P^{4n+1}(2))$ can be found in $\pi_{120n-14}(P^{4n+1}(2))$. They also asked whether $\pi_*(P^{4n+1}(2))$ has $\mathbb{Z}/8\mathbb{Z}$–summands in lower degrees. The following corollary of Theorem 1.3 answers Cohen and Wu's question in the affirmative.

Corollary 1.4 *The following homotopy equivalences exist for every $n \geq 1$:*

(i) $\Omega P^{4n}(2) \simeq \Omega P^{(8k+4)n-3k}(2) \times$ (some other space)
Thus $\pi_{(16k+8)n-6k-2}(P^{4n}(2))$ contains a $\mathbb{Z}/8\mathbb{Z}$–summand, for all $k \in \mathbb{Z}^{\geq 0}$ such that $k \equiv 2 \pmod 4$.

(ii) $\Omega P^{4n+1}(2) \simeq \Omega P^{(8k+4)n+1-k}(2) \times$ (some other space)
Thus $\pi_{(16k+8)n-2k}(P^{4n+1}(2))$ contains a $\mathbb{Z}/8\mathbb{Z}$–summand, for all $k \in \mathbb{Z}^{\geq 0}$ such that $k \equiv 3 \pmod 4$.

(iii) $\Omega P^{4n+2}(2) \simeq \Omega P^{(8k+4)n+2+k}(2) \times$ (some other space)
Thus $\pi_{(16k+8)n+2k+2}(P^{4n+3}(2))$ contains a $\mathbb{Z}/8\mathbb{Z}$–summand, for all $k \in \mathbb{Z}^{\geq 0}$ such that $k \equiv 0 \pmod 4$.

(iv) $\Omega P^{4n+3}(2) \simeq \Omega P^{(8k+4)n+3+3k}(2) \times$ (some other space)
Thus $\pi_{(16k+8)n+6k+4}(P^{4n+3}(2))$ contains a $\mathbb{Z}/8\mathbb{Z}$–summand, for all $k \in \mathbb{Z}^{\geq 0}$ such that $k \equiv 1 \pmod 4$.

In particular, there is a $\mathbb{Z}/8\mathbb{Z}$–summand in $\pi_{56n-6}(P^{4n+1}(2))$. This is of a degree lower than that given in [2].

This article is organized as follows. In Section 2, we introduce some notations and basic properties. The proofs of Theorem 1.1 and 1.3 are given in Section 3. The proofs of Corollary of 1.2 and 1.4 and some remarks are given in Section 4.

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2 Preliminary

Let X be a path-connected, 2–local CW–complex of finite type and let $X^{(n)}$ be the n–fold self smash product of the space X. Let S_k denote the symmetric group on k letters and let $\mathbb{Z}_{(2)}(S_n)$ denote the group ring over the 2 local integer $\mathbb{Z}_{(2)}$ generated by S_k.

Consider the action of $\mathbb{Z}_{(2)}(S_n)$ on $\Sigma X^{(n)}$ by permuting coordinates and taking the summations. For any $\delta \in \mathbb{Z}_{(2)}(S_n)$ we obtain a map

$$\tilde{\delta}\colon \Sigma X^{(n)} \to \Sigma X^{(n)}.$$

Let $V = \bar{H}_*(X;\mathbb{Z}/2\mathbb{Z})$, which is a graded $\mathbb{Z}/2\mathbb{Z}$–module. We use

$$\Sigma V^{\otimes n} = \bar{H}^*(S^1;\mathbb{Z}/2\mathbb{Z}) \otimes V^{\otimes n}$$

to denote the $\mathbb{Z}/2\mathbb{Z}$ reduced homology of $\Sigma X^{(n)}$. Therefore $\widetilde{\delta}$ induces a map

$$\widetilde{\delta}_*\colon \Sigma V^{\otimes n} \to \Sigma V^{\otimes n}$$

by permuting factors.

Define the *Dynkin–Specht–Wever* elements inductively. Start with $\beta_2 = 1-(1,2) \in \mathbb{Z}_{(2)}(S_2)$. Then

$$\beta_n = \beta_{n-1} \wedge \mathrm{id} - (1,2,\dots,n) \circ (\beta_{n-1} \wedge \mathrm{id}).$$

The element β_n induces a map $\widetilde{\beta}_n\colon \Sigma X^{(n)} \to \Sigma X^{(n)}$. Let ι_1 denote the generator of the mod 2 reduced homology of S^1. Then

$$\widetilde{\beta}_{n*}(\iota_1 \otimes x_1 \otimes \cdots \otimes x_n) = \iota_1 \otimes [[\dots[[x_1,x_2],\dots,x_{n-1}],x_n],$$

where $[[\dots[[x_1,x_2],\dots,x_{n-1}],x_n] \in (\mathbb{Z}/2\mathbb{Z})^{\otimes n}$ denotes the commutators. $\widetilde{\beta}_{n*} \circ \widetilde{\beta}_{n*} = n\widetilde{\beta}_{n*}$, following [2]. Hence if n is an odd integer, then

$$\tfrac{1}{n}\widetilde{\beta}_{n*} \circ \tfrac{1}{n}\widetilde{\beta}_{n*} = \tfrac{1}{n}\widetilde{\beta}_{n*}.$$

Denote by $\operatorname{hocolim}_f \Sigma X^{(n)}$ the mapping telescope of the following a sequence of maps:

$$\Sigma X^{(n)} \xrightarrow{f} \Sigma X^{(n)} \xrightarrow{f} \cdots$$

For an odd integer n, the elements

$$\tfrac{1}{n}\widetilde{\beta}_{n*} \quad \text{and} \quad \big(\mathrm{id}_{\Sigma X^{(n)}} - \tfrac{1}{n}\widetilde{\beta}_n\big)_*$$

are orthogonal idempotents. Since

$$\tfrac{1}{n}\widetilde{\beta}_{n*} \circ \big(\mathrm{id}_{\Sigma X^{(n)}} - \tfrac{1}{n}\widetilde{\beta}_n\big)_* \quad \text{and} \quad \big(\mathrm{id}_{\Sigma X^{(n)}} - \tfrac{1}{n}\widetilde{\beta}_n\big)_* \circ \tfrac{1}{n}\widetilde{\beta}_{n*}$$

are trivial, the following the composite is a homotopy equivalence:

$$\Sigma X^{(n)} \xrightarrow{\text{comult}} \Sigma X^{(n)} \vee \Sigma X^{(n)} \to \operatorname*{hocolim}_{\frac{1}{n}\widetilde{\beta}_n} \Sigma X^{(n)} \vee \operatorname*{hocolim}_{\mathrm{id}_{\Sigma X^{(n)}} - \frac{1}{n}\widetilde{\beta}_n} \Sigma X^{(n)}$$

This is because the induced map on the homology with coefficients in 2–local integers is an isomorphism. Hence $\operatorname{hocolim}_{\frac{1}{n}\widetilde{\beta}_n} \Sigma X^{(n)}$ retracts off $\Sigma X^{(n)}$. Let

$$\widetilde{L}_n(X) = \operatorname*{hocolim}_{\frac{1}{n}\widetilde{\beta}_n} \Sigma X^{(n)}.$$

Let $p\colon \Sigma X^{(n)} \to \widetilde{L}_n(X)$ be the projection and let $i\colon \widetilde{L}_n(X) \to \Sigma X^{(n)}$ be the canonical inclusion. Then $\frac{1}{n}\widetilde{\beta}_{n*}$ is identical to the composition:

$$H_*(\Sigma X^{(n)}; \mathbb{Z}/2\mathbb{Z}) \xrightarrow{p_*} H_*(\widetilde{L}_n(X); \mathbb{Z}/2\mathbb{Z}) \xrightarrow{i_*} H_*(\Sigma X^{(n)}; \mathbb{Z}/2\mathbb{Z}) \tag{1}$$

In the special case when X is a suspension, as in [6] we let

$$L_n(X) = \operatorname*{hocolim}_{\frac{1}{n}\beta_n} X^{(n)}.$$

In this case $\widetilde{L}_n(X) \simeq \Sigma L_n(X)$.

It is well known that the mod 2 reduced homology of $\mathbb{R}\mathrm{P}^2 \wedge \mathbb{R}\mathrm{P}^2$ contains a spherical class of degree 3. We generalize this fact to any path-connected 2–cell CW–complex.

Proposition 2.1 *Let X be a 2–local 2–cell path-connected CW–complex such that $\bar{H}_*(X; \mathbb{Z}/2\mathbb{Z})$ is of dimension 2 with generators u and v such that $|v| = m$ and $|u| = n$. Then there exists a map*

$$\alpha\colon S^{m+n} \to X \wedge X$$

such that the image in α_* of induced map on mod 2 homology is generated by $[u, v] \in H_*(X \wedge X; \mathbb{Z}/2\mathbb{Z})$.

Proof When $m = n$, X is just a wedge product of two spheres and the statement is trivial. Assume without loss of generality that $m > n$. Let $f\colon S^{m-1} \to S^n$ be the attaching map of the m–cell to the n–cell of X. Consider the homotopy cofibration

$$X \wedge S^{m-1} \xrightarrow{\mathrm{id}_X \wedge f} X \wedge S^n \to X \wedge X.$$

Take the $(2m-1)$–skeleton of $X \wedge X$ and note that $S^n \wedge S^{m-1} \simeq \mathrm{sk}_{2m-2}(X \wedge S^{m-1})$, hence we obtain a homotopy cofibration

$$S^n \wedge S^{m-1} \xrightarrow{i \wedge f} X \wedge S^n \to \mathrm{sk}_{2m-1}(X \wedge X).$$

Next we will show that $i \wedge f$ is null-homotopic. Since $i \wedge f$ is homotopic to the composite

$$S^n \wedge S^{m-1} \xrightarrow{\mathrm{id}_{S^n} \wedge f} S^n \wedge S^n \xrightarrow{i \wedge \mathrm{id}_{S^n}} X \wedge S^n,$$

we obtain the homotopy commutative diagram

$$\begin{tikzcd} S^n \wedge S^{m-1} \arrow[r, "\mathrm{id}_{S^n} \wedge f"] \arrow[d, "\tau'"] & S^n \wedge S^n \arrow[d, "\tau"] & \\ S^{m-1} \wedge S^n \arrow[r, "f \wedge \mathrm{id}_{S^n}"] & S^n \wedge S^n \arrow[r, "i \wedge \mathrm{id}_{S^n}"] & X \wedge S^n, \end{tikzcd}$$

where the bottom row is homotopy cofibration and τ' and τ are switching maps.

There are two cases: the map $S^n \wedge S^n \xrightarrow{\tau} S^n \wedge S^n$ has degree either 1 or -1. If $\deg(\tau) = 1$, then $\tau \simeq \mathrm{id}_{S^{2n}}$ and

$$\begin{aligned} i \wedge f &\simeq (i \wedge \mathrm{id}_{S^n}) \circ (\mathrm{id}_{S^n} \wedge f) \simeq (i \wedge \mathrm{id}_{S^n}) \circ \tau \circ (\mathrm{id}_{S^n} \wedge f) \\ &\simeq (i \wedge \mathrm{id}_{S^n}) \circ (f \wedge \mathrm{id}_{S^n}) \circ \tau' \simeq *. \end{aligned}$$

If $\deg(\tau) = -1$, since $\mathrm{id}_{S^n} \wedge f$ is a suspension, we have

$$(\mathrm{id}_{S^n} \wedge f) \circ [-1] \simeq \tau \circ (\mathrm{id}_{S^n} \wedge f).$$

Thus we obtain that

$$\mathrm{id}_{S^n} \wedge f \simeq (\mathrm{id}_{S^n} \wedge f) \circ [-1] \circ [-1] \simeq \tau \circ (\mathrm{id}_{S^n} \wedge f) \circ [-1] \simeq (f \wedge \mathrm{id}_{S^n}) \circ \tau' \circ [-1].$$

It follows that

$$i \wedge f \simeq (i \wedge \mathrm{id}_{S^n}) \circ (\mathrm{id}_{S^n} \wedge f) \simeq (i \wedge \mathrm{id}_{S^n}) \circ (f \wedge \mathrm{id}_{S^n}) \circ \tau' \circ [-1] \simeq *.$$

Thus in either case, $i \wedge f$ is null-homotopic. Therefore

$$sk_{2m-1}(X \wedge X) \simeq (X \wedge S^n) \vee S^{m+n}.$$

Hence we can define α by the following composite of canonical inclusions:

$$S^{m+n} \hookrightarrow sk_{2m-1}(X \wedge X) \hookrightarrow X \wedge X.$$

Because $[u, v]$ is the only primitive generator of $H_{m+n}(X \wedge X; \mathbb{Z}/2\mathbb{Z})$, one gets $\alpha_*(\iota_{m+n}) = [u, v] \in H_*(X \wedge X; \mathbb{Z}/2\mathbb{Z})$. □

3 Decomposition of loop spaces and proofs of Theorems 1.1 and 1.3

Recall that for an odd integer n, the space $\widetilde{L}_n(X)$ is a homotopy retract of $\Sigma X^{(n)}$. By studying the 2–local decomposition of $\Sigma X^{(n)}$, one can investigate the spaces $\widetilde{L}_n(X)$. In Selick and Wu [4], the finest 2–primary splitting of $X^{(n)}$ is given.

Suppose that $\bar{H}_*(X;\mathbb{Z}/2\mathbb{Z})$ is generated by two elements u and v with $|u|<|v|$. Proposition 2.1 implies that, if $n=2k+1$ for some non-negative integer k, then there exists a canonical inclusion $\Sigma^{1+k(|u|+|v|)}X \hookrightarrow \Sigma X^{(n)}$.

Proposition 3.1 *Let X be a path-connected 2–local 2–cell CW–complex such that $\bar{H}_*(X;\mathbb{Z}/2\mathbb{Z})$ is of dimension 2 with generators u and v such that $|u|<|v|$. For every odd integer $n\geq 3$, let p_n be the natural projection defined in Equation* (1). *Then the following composite has a left homotopy inverse*

$$\Sigma^{1+k(|u|+|v|)}X \to \Sigma X^{(n)} \xrightarrow{p_n} \tilde{L}_n(X).$$

Proof For every odd integer $n\geq 3$, let i_n be the inclusion defined as in Equation (1). Recall that

$$(i_n\circ p_n)_* = \tilde{\beta}_n\colon H_*(\Sigma X^{(n)}\mathbb{Z}/2\mathbb{Z}) \to H_*(\Sigma X^{(n)}\mathbb{Z}/2\mathbb{Z}).$$

Let α be the spherical class given in Proposition 2.1 and let ϕ_1 be the composite

$$\Sigma^{1+|u|+|v|}X \simeq \Sigma X\wedge S^{|u|+|v|} \xrightarrow{\mathrm{id}\wedge\alpha} \Sigma X\wedge X\wedge X \xrightarrow{i_3\circ p_3} \Sigma X\wedge X\wedge X.$$

Take the mod 2 reduced homology:

$$\begin{aligned}\phi_{1*}(\iota_1\otimes u\otimes\iota_{|u|+|v|}) &= (i_{3*}\circ p_{3*})(\iota_1\otimes(uuv+uvu))\\ &= \tilde{\beta}_{3*}(\iota_1\otimes(uuv+uvu)) = \iota_1\otimes[[u,v],u].\end{aligned}$$

Similarly $\phi_{1*}(\iota_1\otimes v\otimes\iota_{|u|+|v|})=\iota_1\otimes[[u,v],v]$. Since $\bar{H}_*(\tilde{L}_3(X))$ is of dimension 2, the composite $p_3\circ\phi_1$ induces an isomorphism in mod 2 homology. Because the spaces involved are CW–complexes of finite type, hence we have the homotopy equivalence

$$p_3\circ\phi_1\colon \Sigma^{1+|u|+|v|}X \to \tilde{L}_3(X).$$

Define $\phi_k\colon \Sigma^{1+k(|u|+|v|)}X \to \Sigma X^{(2k+1)}$ inductively as the composite

$$\begin{aligned}\Sigma^{1+k(|u|+|v|)}X &\simeq \Sigma^{1+(k-1)(|u|+|v|)}X\wedge S^{|u|+|v|}\\ &\xrightarrow{\phi_{k-1}\wedge\alpha} \Sigma X^{(2k-1)}\wedge X^{(2)} \xrightarrow{i_{2k+1}\circ p_{2k+1}} \Sigma X^{(2k+1)}.\end{aligned}$$

Set $\varphi_1=p_3$. Define $\varphi_k\colon \Sigma X^{(2k+1)} \to \Sigma^{1+k(|u|+|v|)}X$ inductively as the composite

$$\Sigma X^{(2k+1)} \xrightarrow{\varphi_{k-1}\wedge\mathrm{id}} \Sigma^{1+(k-1)(|u|+|v|)}X\wedge X\wedge X \xrightarrow{\Sigma^{(k-1)(|u|+|v|)}p_3} \Sigma^{1+k(|u|+|v|)}X.$$

Since $i_{2k+1}\circ p_{2k+1}$ factors through $\tilde{L}_{2k+1}(X)$, the composite $\varphi_k\circ\phi_k$ factors through $\tilde{L}_{2k+1}(X)$. Let $\mathrm{ad}(x)(y)=[y,x]$ and $\mathrm{ad}^{i+1}(x)(y)=[\mathrm{ad}^i(x)(y),x]$ for $i\geq 1$. The coefficients are taken mod 2, hence $\mathrm{ad}([u,v])(u)=[[u,v],u]=[u,[u,v]]$.

Thus it is sufficient to show that the following composite is a homotopy equivalence for all $k \geq 1$:

$$\Sigma^{1+k(|u|+|v|)} X \xrightarrow{\phi_k} \Sigma X^{(2k+1)} \xrightarrow{\varphi_k} \Sigma^{1+k(|u|+|v|)} X.$$

Since the spaces involved are CW–complexes of finite type, it is enough to show that $\varphi_k \circ \phi_k$ induces an isomorphism on mod 2 homology. Explicitly, we will prove the following statements by induction on k:

$$\begin{aligned} &\varphi_{k*} \circ \phi_{k*}(\iota_{1+k(|u|+|v|)} \otimes u) = \varphi_{k*}(\iota_1 \otimes \mathrm{ad}^k([u,v])(u)) = \iota_{1+k(|u|+|v|)} \otimes u, \\ &\varphi_{k*} \circ \phi_{k*}(\iota_{1+k(|u|+|v|)} \otimes v) = \varphi_{k*}(\iota_1 \otimes \mathrm{ad}^k([u,v])(v)) = \iota_{1+k(|u|+|v|)} \otimes v. \end{aligned} \tag{2}$$

The case of $k=1$ has already been shown. Suppose that the statement is true for all $k' < k$. We have

$$\widetilde{\beta}_{3*}(\iota_1 \otimes \mathrm{ad}([u,v])(u)) = \widetilde{\beta}_{3*}(\iota_1 \otimes [[u,v],u]) = \iota_1 \otimes [[u,v],u] = \iota_1 \otimes \mathrm{ad}([u,v])(u).$$

Also

$$\begin{aligned} &\widetilde{\beta}_{2k+1*}(\iota_1 \otimes [u,v] \otimes (\mathrm{ad}^{k-1}([u,v])(u)) \\ &\qquad = \widetilde{\beta}_{2k+1*}(\iota_1 \otimes uv \otimes (\mathrm{ad}^{k-1}([u,v])(u)) + \widetilde{\beta}_{2k+1*}(\iota_1 \otimes vu \otimes (\mathrm{ad}^{k}([u,v])(u)) \\ &\qquad = 2\widetilde{\beta}_{2k+1*}(\iota_1 \otimes uv \otimes (\mathrm{ad}^{k-1}([u,v])(u)) \\ &\qquad = 0. \end{aligned}$$

Hence we obtain

$$\begin{aligned} \widetilde{\beta}_{2k+1*}(\iota_1 \otimes \mathrm{ad}^k([u,v])(u)) &= \widetilde{\beta}_{2k+1*}(\iota_1 \otimes [\mathrm{ad}^{k-1}([u,v])(u),[u,v]]) \\ &= \widetilde{\beta}_{2k+1*}(\iota_1 \otimes (\mathrm{ad}^{k-1}([u,v])(u)) \otimes [u,v]) \\ &\qquad + \widetilde{\beta}_{2k+1*}(\iota_1 \otimes [u,v] \otimes (\mathrm{ad}^{k-1}([u,v])(u))) \\ &= \widetilde{\beta}_{2k+1*}(\iota_1 \otimes (\mathrm{ad}^{k-1}([u,v])(u)) \otimes [u,v]). \end{aligned}$$

Furthermore

$$\begin{aligned} &\widetilde{\beta}_{2k+1*}(\iota_1 \otimes (\mathrm{ad}^{k-1}([u,v])(u)) \otimes [u,v]) \\ &\quad = \widetilde{\beta}_{2k+1*}(\iota_1 \otimes (\mathrm{ad}^{k-1}([u,v])(u)) \otimes uv) + \widetilde{\beta}_{2k+1*}(\iota_1 \otimes (\mathrm{ad}^{k-1}([u,v])(u)) \otimes vu) \\ &\quad = [[[\widetilde{\beta}_{2k-1*}(\iota_1 \otimes (\mathrm{ad}^{k-1}([u,v])(u)))],u],v] \\ &\qquad\qquad + [[[\widetilde{\beta}_{2k-1*}(\iota_1 \otimes (\mathrm{ad}^{k-1}([u,v])(u)))],v],u] \\ &\quad = [\widetilde{\beta}_{2k-1*}(\iota_1 \otimes \mathrm{ad}^{k-1}([u,v])(u)),[u,v]] \\ &\quad = \mathrm{ad}([u,v])(\widetilde{\beta}_{2k-1*}(\iota_1 \otimes \mathrm{ad}^{k-1}([u,v])(u))), \end{aligned}$$

where the third equality is due to Jacobi identity. Thus we have

$$\begin{aligned}\tilde{\beta}_{2k+1*}(\iota_1\otimes \mathrm{ad}^k([u,v])(u)) &= \mathrm{ad}([u,v])(\tilde{\beta}_{2k-1*}(\iota_1\otimes \mathrm{ad}^{k-1}([u,v])(u)))\\ &= \mathrm{ad}^2([u,v])(\tilde{\beta}_{2k-3*}(\iota_1\otimes \mathrm{ad}^{k-2}([u,v])(u))).\end{aligned}$$

Hence we obtain that

$$\tilde{\beta}_{2k+1*}(\iota_1\otimes \mathrm{ad}^k([u,v])(u)) = \iota_1\otimes \mathrm{ad}^k([u,v])(u).$$

By induction hypothesis, we have

$$\phi_{k-1*}(\iota_{1+(k-1)(|u|+|v|)}\otimes u) = \iota_1\otimes \mathrm{ad}^{k-1}([u,v])(u).$$

It follows that

$$\begin{aligned}\phi_{k*}(\iota_{1+k(|u|+|v|)}\otimes u) &= (i_{2k+1}\circ p_{2k+1})_*(\phi_{k-1*}(\iota_{1+(k-1)(|u|+|v|)}\otimes u)\otimes [u,v])\\ &= \tilde{\beta}_{2k+1*}(\iota_1\otimes \mathrm{ad}^{k-1}([u,v])(u)\otimes [u,v])\\ &= \tilde{\beta}_{2k+1*}(\iota_1\otimes \mathrm{ad}^{k}([u,v])(u))\\ &= \iota_1\otimes \mathrm{ad}^{k}([u,v])(u).\end{aligned}$$

Similarly

$$\phi_{k*}(\iota_{1+k(|u|+|v|)}\otimes v) = \iota_1\otimes \mathrm{ad}^k([u,v])(v).$$

Also we have

$$\begin{aligned}&(\varphi_{k*}\circ\tilde{\beta}_{2k+1*})(\iota_1\otimes \mathrm{ad}^k([u,v])(u))\\ &\qquad= \varphi_{k*}(\iota_1\otimes \mathrm{ad}^{k-1}([u,v])(u)\otimes [u,v]))\\ &\qquad= \left(\Sigma^{(k-1)(|u|+|v|)}p_3\right)_*\circ((\varphi_{k-1*}\otimes \mathrm{id})(\iota_1\otimes \mathrm{ad}^{k-1}([u,v])(u)\otimes [u,v]))\\ &\qquad= (\Sigma^{(k-1)(|u|+|v|)}p_3)_*\circ(\varphi_{k-1*}((\iota_1\otimes \mathrm{ad}^{k-1}([u,v])(u))\otimes [u,v])\\ &\qquad= (\Sigma^{(k-1)(|u|+|v|)}p_3)_*(\iota_{1+(k-1)(|u|+|v|)}\otimes u\otimes [u,v])\\ &\qquad= \iota_{1+k(|u|+|v|)}\otimes u.\end{aligned}$$

Since $\tilde{\beta}_{2k+1*} = i_{2k+1*}\circ p_{2k+1*}$ and $\tilde{\beta}_{2k+1*} = \tilde{\beta}_{2k+1*}\circ\tilde{\beta}_{2k+1*}$, we have

$$\varphi_{k*}\circ\phi_{k*} = \varphi_{k*}\circ\tilde{\beta}_{2k+1*}\circ\phi_{k*}.$$

Hence

$$\begin{aligned}&(\varphi_{k*}\circ\phi_{k*})(\iota_{1+k(|u|+|v|)}\otimes u)\\ &\qquad= (\varphi_{k*}\circ\tilde{\beta}_{2k+1*}\circ\phi_{k*})(\iota_{1+(k-1)(|u|+|v|)}\otimes u\otimes \iota_{|u|+|v|}) = \iota_{1+k(|u|+|v|)}\otimes u.\end{aligned}$$

Similarly we obtain

$$\begin{aligned}(\varphi_{k*}\circ\phi_{k*})(\iota_{1+k(|u|+|v|)}\otimes v)\\ &= (\varphi_{k*}\circ\widetilde{\beta}_{2k+1*}\circ\phi_{k*})(\iota_{1+(k-1)(|u|+|v|)}\otimes v\otimes\iota_{|u|+|v|}) = \iota_{1+k(|u|+|v|)}\otimes v.\end{aligned}$$

This completes the induction step. Thus (2) holds. As noted earlier, this implies the composite $\varphi_k\circ\phi_k$ is a homotopy equivalence, which completes the proof. □

We can obtain a weaker result when the space studied is a finite wedge product of 2–cell CW–complexes.

Proposition 3.2 *For every $1\le k\le n$, let X_k be a path-connected 2–local 2–cell CW–complexes such that $\bar{H}_*(X_k;\mathbb{Z}/2\mathbb{Z})$ is of dimension 2 with generators u_k and v_k such that $|u_k|<|v_k|$. Then the following map has a left homotopy inverse:*

$$\Sigma\wedge\bigwedge_{k=1}^n\Sigma^{|u_k|+|v_k|}X_k\to\widetilde{L}_3\big(\bigwedge_{k=1}^n X_k\big).$$

Proof As shown in the proof of Proposition 3.1, for each space X_k we have a canonical projection

$$p_3^k\colon \Sigma X_k^{(3)}\to\Sigma^{1+|u_k|+|v_k|}X_k\simeq\widetilde{L}_3(X_k)$$

and a canonical inclusion

$$i_3^k\colon \Sigma^{1+|u_k|+|v_k|}X_k\simeq\widetilde{L}_3(X_k)\to\Sigma X_k^{(3)}.$$

The map $p_3^1,\dots,p_3^n$ induce the projection

$$\bar{p}\colon \Sigma\wedge\bigwedge_{k=1}^n {X_k}^{(3)}\to\Sigma\wedge\bigwedge_{k=1}^n\Sigma^{|u_k|+|v_k|}X_k.$$

The map $i_3^1,\dots,i_3^n$ induce the inclusion

$$\bar{i}\colon \Sigma\wedge\bigwedge_{k=1}^n\Sigma^{|u_k|+|v_k|}X_k\to\Sigma\wedge\bigwedge_{k=1}^n {X_k}^{(3)}.$$

Let θ be the composite

$$\begin{aligned}\Sigma\wedge\bigwedge_{k=1}^n {X_k}^{(3)}\xrightarrow{\Sigma\tau}\Sigma\big(\bigwedge_{k=1}^n X_k\big)^{(3)}\xrightarrow{p}\widetilde{L}_3\big(\bigwedge_{k=1}^n X_k\big)\\ &\xrightarrow{i}\Sigma\big(\bigwedge_{k=1}^n X_k\big)^{(3)}\xrightarrow{\Sigma\tau'}\Sigma\wedge\bigwedge_{k=1}^n {X_k}^{(3)},\end{aligned}$$

where the maps τ and τ' switch positions. Explicitly, for $x_k,y_k,z_k\in H_*(X_k;\mathbb{Z}/2\mathbb{Z})$, the maps τ and τ' induce the following maps on homology:

$$\begin{aligned}\tau_*\big(\bigotimes_{k=1}^n x_ky_kz_k\big)&=\big(\prod_{k=1}^n x_k\big)\otimes\big(\prod_{k=1}^n y_k\big)\otimes\big(\prod_{k=1}^n z_k\big),\\ \tau'_*\big(\big(\prod_{k=1}^n x_k\big)\otimes\big(\prod_{k=1}^n y_k\big)\big)\otimes\big(\prod_{k=1}^n z_k\big)\big)&=\bigotimes_{k=1}^n x_ky_kz_k.\end{aligned}$$

Recall that for the canonical inclusion i and canonical projection p, we have $\widetilde{\beta}_{3*} = i_* \circ p_*$. Then $\theta_*\big(\iota_1 \otimes \bigotimes_{k=1}^n x_k y_k z_k\big)$ is given by the following:

$$\begin{aligned}
\iota_1 \otimes \textstyle\bigotimes_{k=1}^n x_k y_k z_k &\xrightarrow{(\Sigma\tau)_*} \iota_1 \otimes \big(\prod_{k=1}^n x_k\big) \otimes \big(\prod_{k=1}^n y_k\big) \otimes \big(\prod_{k=1}^n z_k\big)\\
&\xrightarrow{(\widetilde{\beta}_3)_*} \iota_1 \otimes \big[\big[\big(\textstyle\prod_{k=1}^n x_k\big), \big(\prod_{k=1}^n y_k\big)\big], \big(\prod_{k=1}^n z_k\big)\big]\\
&\xrightarrow{(\Sigma\tau)_*} \iota_1 \otimes \big(\textstyle\bigotimes_{k=1}^n x_k y_k z_k + \bigotimes_{k=1}^n y_k x_k z_k\\
&\qquad\qquad + \textstyle\bigotimes_{k=1}^n z_k x_k y_k + \bigotimes_{k=1}^n z_k y_k x_k\big)
\end{aligned}$$

Let γ_1, γ_2, γ_3, γ_4 be the mapping defined by:

$$\begin{aligned}
\gamma_1\big(\iota_1 \otimes \textstyle\bigotimes_{k=1}^n x_k y_k z_k\big) &= \iota_1 \otimes \big(\textstyle\bigotimes_{k=1}^n x_k y_k z_k\big),\\
\gamma_2\big(\iota_1 \otimes \textstyle\bigotimes_{k=1}^n x_k y_k z_k\big) &= \iota_1 \otimes \big(\textstyle\bigotimes_{k=1}^n y_k x_k z_k\big),\\
\gamma_3\big(\iota_1 \otimes \textstyle\bigotimes_{k=1}^n x_k y_k z_k\big) &= \iota_1 \otimes \big(\textstyle\bigotimes_{k=1}^n z_k x_k y_k\big),\\
\gamma_4\big(\iota_1 \otimes \textstyle\bigotimes_{k=1}^n x_k y_k z_k\big) &= \iota_1 \otimes \big(\textstyle\bigotimes_{k=1}^n z_k y_k x_k\big).
\end{aligned}$$

Therefore $\theta_* = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$ and $(\overline{p} \circ \theta)_* = \overline{p}_* \circ \theta_* = \overline{p}_* \circ \gamma_1 + \overline{p}_* \circ \gamma_2 + \overline{p}_* \circ \gamma_3 + \overline{p}_* \circ \gamma_4$.

Since $0 = \widetilde{\beta}_{3*}(\iota_1 \otimes u_k u_k v_k) = i^k_{3*} \circ p^k_{3*}(\iota_1 \otimes u_k u_k v_k)$ and i^k_{3*} is a monomorphism, one gets $p^k_{3*}(\iota_1 \otimes u_k u_k v_k) = 0$. Recall that $\overline{p}$ is induced by $p^1_3, \dots, p^n_3$. If $x_l y_l z_l = u_l u_l v_l$ for some $1 \le l \le n$, then $\overline{p}_*(\iota_1 \otimes \bigotimes_{k=1}^n x_k y_k z_k) = 0$. Since tensor product is bilinear,

$$\textstyle\bigotimes_{k=1}^n (u_k u_k v_k + v_k u_k u_k) = \sum_{j=1}^{2^n} \bigotimes_{k=1}^n x^j_k y^j_k z^j_k,$$

where $x^j_k y^j_k z^j_k = u_k u_k v_k$ or $v_k u_k u_k$ for $1 \le j \le 2^n$.

Hence if $x^j_l y^j_l z^j_l = u_l u_l v_l$ for some $1 \le l \le n$, we have

$$\begin{aligned}
\overline{p}_* \circ \gamma_1\big(\iota_1 \otimes \textstyle\bigotimes_{k=1}^n x^j_k y^j_k z^j_k\big) &= \overline{p}_*\big(\iota_1 \otimes \textstyle\bigotimes_{k=1}^n x^j_k y^j_k z^j_k\big)\\
&= \overline{p}_*(\iota_1 \otimes \cdots \otimes u_l u_l v_l \otimes \cdots) = 0.
\end{aligned}$$

It follows that $\overline{p}_* \circ \gamma_1\big(\iota_1 \otimes \bigotimes_{k=1}^n x^j_k y^j_k z^j_k\big)$ is non-zero if and only if

$$\textstyle\bigotimes_{k=1}^n x^j_k y^j_k z^j_k = \bigotimes_{k=1}^n v_k u_k u_k.$$

We must have

$$
\begin{aligned}
&\bar{p}_* \circ \gamma_1\big(\iota_1 \otimes \textstyle\bigotimes_{k=1}^n (u_k u_k v_k + v_k u_k u_k)\big)\\
&\qquad = \bar{p}_* \circ \gamma_1\big(\textstyle\sum_{j=1}^{2^n} \iota_1 \otimes \bigotimes_{k=1}^n x_k^j y_k^j z_k^j\big) = \textstyle\sum_{j=1}^{2^n} \bar{p}_* \circ \gamma_1\big(\iota_1 \otimes \bigotimes_{k=1}^n x_k^j y_k^j z_k^j\big)\\
&\qquad = \bar{p}_* \circ \gamma_1\big(\iota_1 \otimes \textstyle\bigotimes_{k=1}^n v_k u_k u_k\big) = \bar{p}_*\big(\iota_1 \otimes \bigotimes_{k=1}^n v_k u_k u_k\big)\\
&\qquad = \iota_1 \otimes \textstyle\bigotimes_{k=1}^n (\iota_{|u_k|+|v_k|} \otimes u_k).
\end{aligned}
$$

Similarly:

$$
\begin{aligned}
&\bar{p}_* \circ \gamma_2\big(\iota_1 \otimes \textstyle\bigotimes_{k=1}^n (u_k u_k v_k + v_k u_k u_k)\big)\\
&\qquad = \bar{p}_* \circ \gamma_2\big(\textstyle\sum_{j=1}^{2^n} \iota_1 \otimes \bigotimes_{k=1}^n x_k^j y_k^j z_k^j\big) = \textstyle\sum_{j=1}^{2^n} \bar{p}_* \circ \gamma_2\big(\iota_1 \otimes \bigotimes_{k=1}^n x_k^j y_k^j z_k^j\big)\\
&\qquad = \bar{p}_* \circ \gamma_2\big(\iota_1 \otimes \textstyle\bigotimes_{k=1}^n v_k u_k u_k\big) = \bar{p}_*\big(\iota_1 \otimes \bigotimes_{k=1}^n u_k v_k u_k\big)\\
&\qquad = \iota_1 \otimes \textstyle\bigotimes_{k=1}^n (\iota_{|u_k|+|v_k|} \otimes u_k),\\
&\bar{p}_* \circ \gamma_4\big(\iota_1 \otimes \textstyle\bigotimes_{k=1}^n (u_k u_k v_k + v_k u_k u_k)\big)\\
&\qquad = \bar{p}_* \circ \gamma_4\big(\textstyle\sum_{j=1}^{2^n} \iota_1 \otimes \bigotimes_{k=1}^n x_k^j y_k^j z_k^j\big) = \textstyle\sum_{j=1}^{2^n} \bar{p}_* \circ \gamma_4\big(\iota_1 \otimes \bigotimes_{k=1}^n x_k^j y_k^j z_k^j\big)\\
&\qquad = \bar{p}_* \circ \gamma_4\big(\iota_1 \otimes \textstyle\bigotimes_{k=1}^n u_k u_k v_k\big) = \bar{p}_*\big(\iota_1 \otimes \bigotimes_{k=1}^n v_k u_k u_k\big)\\
&\qquad = \iota_1 \otimes \textstyle\bigotimes_{k=1}^n (\iota_{|u_k|+|v_k|} \otimes u_k).
\end{aligned}
$$

Also notice that $\bar{p}_* \circ \gamma_3\big(\bigotimes_{k=1}^n x_k^j y_k^j z_k^j\big)$ is non-zero for $1 \le j \le 2^n$. We get

$$
\begin{aligned}
&\bar{p}_* \circ \gamma_3\big(\iota_1 \otimes \textstyle\bigotimes_{k=1}^n (u_k u_k v_k + v_k u_k u_k)\big)\\
&\qquad = \bar{p}_* \circ \gamma_3\big(\textstyle\sum_{j=1}^{2^n} \iota_1 \otimes \bigotimes_{k=1}^n x_k^j y_k^j z_k^j\big) = \textstyle\sum_{j=1}^{2^n} \bar{p}_* \circ \gamma_3\big(\iota_1 \otimes \bigotimes_{k=1}^n x_k^j y_k^j z_k^j\big)\\
&\qquad = 2^n \iota_1 \otimes \textstyle\bigotimes_{k=1}^n (\iota_{|u_k|+|v_k|} \otimes u_k) = 0.
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
(\bar{p} \circ \theta)_*(\iota_1 \otimes \textstyle\bigotimes_{k=1}^n (u_k u_k v_k + v_k u_k u_k))&\\
= \bar{p}_* \circ \gamma_1(\iota_1 \otimes \textstyle\bigotimes_{k=1}^n (u_k u_k v_k + v_k u_k u_k))&\\
+ \bar{p}_* \circ \gamma_2(\iota_1 \otimes \textstyle\bigotimes_{k=1}^n (u_k u_k v_k + v_k u_k u_k))&\\
+ \bar{p}_* \circ \gamma_3(\iota_1 \otimes \textstyle\bigotimes_{k=1}^n (u_k u_k v_k + v_k u_k u_k))&\\
+ \bar{p}_* \circ \gamma_4(\iota_1 \otimes \textstyle\bigotimes_{k=1}^n (u_k u_k v_k + v_k u_k u_k))&\\
= \iota_1 \otimes \textstyle\bigotimes_{k=1}^n (\iota_{|u_k|+|v_k|} \otimes u_k).&
\end{aligned}
$$

Similarly we have

$$(\bar{p} \circ \theta)_*\big(\iota_1 \otimes \textstyle\bigotimes_{k=1}^n (v_k v_k u_k + u_k v_k v_k)\big) = \iota_1 \otimes \bigotimes_{k=1}^n (\iota_{|u_k|+|v_k|} \otimes v_k).$$

Thus

$$\Sigma \wedge \bigwedge_{k=1}^{n} \Sigma^{|u_k|+|v_k|} X_k \xrightarrow{\bar{i}} \Sigma \wedge \bigwedge_{k=1}^{n} {X_k}^{(3)} \xrightarrow{\theta} \Sigma \wedge \bigwedge_{k=1}^{n} {X_k}^{(3)} \xrightarrow{\bar{p}} \Sigma \wedge \bigwedge_{k=1}^{n} \Sigma^{|u_k|+|v_k|} X_k$$

induces an isomorphism on mod 2 homology. Since the spaces considered are CW–complexes of finite type, the composite $\bar{p} \circ \theta \circ \bar{i}$ is a homotopy equivalence. Because θ factors through $\widetilde{L}_3\big(\bigwedge_{k=1}^{n} X_k\big)$, the statement follows. □

Proposition 3.3 is due to Paul Selick and Jie Wu. The case when $X = \Sigma X'$ is a suspension is shown in [6]; if X is not a suspension, one can modify an idea of Paul Selick and Jie Wu in [5] to prove this proposition.

Proposition 3.3 (Paul Selick and Jie Wu) *Let X be a path-connected 2–local CW–complex of finite type. Let $2 < k_1 < k_2 < \cdots$ be all the odd prime numbers in increasing order. Then there exists a topological space A such that*

$$\Omega\Sigma X \simeq \prod_j \Omega(\widetilde{L}_{k_j}(X)) \times A$$

localized at 2. □

Theorems 1.1 and 1.3 are consequences of Propositions 3.1, 3.2 and 3.3.

Proof of Theorem 1.1 By Proposition 3.2 and 3.3, the following map has a left homotopy inverse:

$$\Omega\Sigma \wedge \bigwedge_{i=1}^{n} \Sigma^{|u_i|+|v_i|} X_i \to \Omega\Sigma \wedge \bigwedge_{i=1}^{n} X_i.$$

For each i, the space $\Sigma^{|u_i|+|v_i|} X_i$ is again a 2–cell complex. By induction, we can conclude that

$$\Omega\Sigma \wedge \bigwedge_{i=1}^{n} \Sigma^{(3^k-1)/2(|u_i|+|v_i|)} X_i$$

retracts off $\Omega\Sigma \wedge \bigwedge_{i=1}^{n} X_i$. □

Proof of Theorem 1.3 Recall from in Proposition 3.1 that the following map has a left homotopy inverse:

$$\Sigma^{1+k(|u|+|v|)} X \to \widetilde{L}_{2k+1}(X).$$

Therefore $\Omega\Sigma^{1+k(|u|+|v|)} X$ retracts off $\Omega\widetilde{L}_{2k+1}(X)$. The statement follows from Proposition 3.3. □

4 Proofs of Corollaries 1.2 and 1.4 and some remarks

First we give proofs of Corollaries 1.2 and 1.4.

Proof of Corollary 1.2 Theorem 1.1 implies that

$$\pi_m\big(\Sigma \wedge \textstyle\bigwedge_{i=1}^n \Sigma^{(3^k-1)/2(|u_i|+|v_i|)} X_i\big)$$

is a summand of $\pi_m(\Sigma \wedge \bigwedge_{i=1}^n X_i)$ for $m \geq 1$. For $k \geq 0$, let

$$b_k = \sum_{i=1}^n \Big(\frac{3^k-1}{2}(|u_i| + |v_i|)\Big).$$

When k is large enough such that $j \leq b_k + 2m$, the Freudenthal Suspension Theorem implies:

$$\pi_{j+b_k}\big(\Sigma \wedge \textstyle\bigwedge_{i=1}^n \Sigma^{(3^k-1)/2(|u_i|+|v_i|)} X_i\big) \cong \pi^s_{j+b_k}\big(\Sigma \wedge \textstyle\bigwedge_{i=1}^n \Sigma^{(3^k-1)/2(|u_i|+|v_i|)} X_i\big)$$

Thus

$$\pi^s_{j+b_k}\big(\Sigma \wedge \textstyle\bigwedge_{i=1}^n \Sigma^{(3^k-1)/2(|u_i|+|v_i|)} X_i\big) \cong \pi^s_j\big(\Sigma \wedge \textstyle\bigwedge_{i=1}^n X_i\big).$$

The statement follows. □

Proof of Corollary 1.4 As an immediate consequence of Theorem 1.3, for $k \geq 1$ and $n \geq 2$, the following map has a left homotopy inverse

$$\Omega P^{n+1+k(2n-1)}(2) \hookrightarrow \Omega P^{n+1}(2).$$

Further, for $0 \leq m \leq 3$, the following map has a left homotopy inverse:

$$\Omega P^{(4+8k)n+(2k+1)m-3k}(2) \hookrightarrow \Omega P^{4n+m}(2).$$

Cohen and Wu [2] showed that if $n \geq 4$ and $n \equiv 1 \pmod 2$, then $\pi_{4n-2}(P^{2n}(2))$ has a $\mathbb{Z}/8\mathbb{Z}$–summand. Therefore, if $(4+8k)n + (2k+1)m - 3k \equiv 2 \pmod 4$, then $\pi_*(\Omega P^{(4+8k)n+(2k+1)m-3k}(2))$ has a $\mathbb{Z}/8\mathbb{Z}$–summand, The statement follows when we set $m = 0, 1, 2, 3$. □

Next we remark that Beben and Wu's result [1, Proposition 5.2] can be combined with Corollary 1.2 to give a uniform formula. First recall Beben and Wu's result.

Proposition 4.1 [1, Proposition 5.2] *Let X be the p–localization of a suspended CW–complex. Set $V = \bar{H}_*(X;\mathbb{Z}/p\mathbb{Z})$. Let M denote the sum of the degrees of the generators of V. Define the sequence of integers b_i recursively, with $b_0 = 0$ and*

$$b_i = (1+\dim V)b_{i-1} + M.$$

Let $V = \bar{H}_*(X;\mathbb{Z}/p\mathbb{Z})$, $1 < \dim V < p-1$ and $V_{\mathrm{odd}} = 0$ or $V_{\mathrm{even}} = 0$. Assume X is $(m-1)$–connected for some $m \geq 1$. Then for each j, the stable homotopy group $\pi_j^s(\Sigma X)$ is a homotopy retract of $\pi_{j+b_i}(\Sigma X)$ for i large enough such that $j \leq b_i + 2m$. □

Notice that when we set $X = \bigwedge_{i=1}^n X_i$, all the b_i in Corollary 1.2 are the same as the b_i in Proposition 4.1. Combine Proposition 4.1 and Corollary 1.2 to obtain the following.

Theorem 4.2 *Let X be the p–localization of a suspended CW–complex. Set $V = \bar{H}_*(X;\mathbb{Z}/p\mathbb{Z})$. Let M denote the sum of the degrees of the generators of V. Define the sequence of integers b_i recursively by setting $b_0 = 0$ and*

$$b_i = (1+\dim V)b_{i-1} + M.$$

Assume X is $(m-1)$–connected for some $m \geq 1$ and let $V = \bar{H}_*(X)$. If either one of the following is satisfied:

- $1 < \dim V < p-1$, *and* $V_{\mathrm{odd}} = 0$ *or* $V_{\mathrm{even}} = 0$.
- $2 = p = \dim(W_i)$, *and* $X = \bigwedge_{i=1}^n X_i$ *with* $W_i = \bar{H}_*(X_i)$ *for* $1 \leq i \leq n$.

Then for each j, the stable homotopy group $\pi_j^s(\Sigma X)$ is a homotopy retract of $\pi_{j+b_i}(\Sigma X)$ for i large enough such that $j \leq b_i + 2m$. □

References

[1] **P Beben**, **J Wu**, *Modular representations and the homotopy of low rank p–local CW–complexes*, Math. Z. 273 (2013) 735–751

[2] **F R Cohen**, **J Wu**, *A remark on the homotopy groups of $\Sigma^n\mathbf{R}P^2$*, from: "The Čech centennial", (M Cenkl, H Miller, editors), Contemp. Math. 181, Amer. Math. Soc. (1995) 65–81 MR1320988

[3] **H Freudenthal**, *Über die Klassen der Sphärenabbildungen, I: Große Dimensionen*, Compositio Math. 5 (1938) 299–314

[4] **P Selick**, **J Wu**, *On functorial decompositions of self-smash products*, Manuscripta Math. 111 (2003) 435–457 MR2002820

[5] **P Selick**, **J Wu**, *The functor $A^{\min}$ on p–local spaces*, Math. Z. 253 (2006) 435–451 MR2221079

[6] **J Wu**, *On combinatorial calculations for the James–Hopf maps*, Topology 37 (1998) 1011–1023 MR1650426

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