Algebraic & Geometric Topology 13 (2013) 3261–3286

Irreducibility of q–difference operators and the knot 7_4

STAVROS GAROUFALIDIS
CHRISTOPH KOUTSCHAN

Our goal is to compute the minimal-order recurrence of the colored Jones polynomial of the 7_4 knot, as well as for the first four double twist knots. As a corollary, we verify the AJ Conjecture for the simplest knot 7_4 with reducible nonabelian $\mathrm{SL}(2,\mathbb{C})$ character variety. To achieve our goal, we use symbolic summation techniques of Zeilberger's holonomic systems approach and an irreducibility criterion for q–difference operators. For the latter we use an improved version of the qHyper algorithm of Abramov–Paule–Petkovšek to show that a given q–difference operator has no linear right factors. En route, we introduce exterior power Adams operations on the ring of bivariate polynomials and on the corresponding affine curves.

57N10; 57M25, 33F10, 39A13

1 Introduction

1.1 Notation

Throughout the paper the symbol $\mathbb{K}$ denotes a field of characteristic zero; for most applications one may think of $\mathbb{K}=\mathbb{Q}$. We write $\mathbb{K}[X_1,\dots,X_n]$ for the ring of polynomials in the variables $X_1,\dots,X_n$ with coefficients in $\mathbb{K}$, and similarly $\mathbb{K}[X_1^{\pm 1},\dots,X_n^{\pm 1}]$ for the ring of Laurent polynomials, and $\mathbb{K}(X_1,\dots,X_n)$ for the field of rational functions. In a somewhat sloppy way we use angle brackets, eg $\mathbb{K}\langle X_1,\dots,X_n\rangle$, to refer to the ring of polynomials in $X_1,\dots,X_n$ with some noncommutative multiplication. This noncommutativity may occur between variables X_i and X_j, or between the coefficients in $\mathbb{K}$ and the variables X_i. It will be always clear from the context which commutation rules apply. Let $p(X,Y_1,\dots,Y_n)=\sum_{k=a}^{b} p_k(Y_1,\dots,Y_n)X^k$, $a,b\in\mathbb{Z}$, be a nonzero Laurent polynomial with $p_a\neq 0$ and $p_b\neq 0$; then we define $\deg_X(p):=b$ and $\operatorname{ldeg}_X(p):=a$. As usual, $\lfloor a\rfloor$ (resp. $\lceil a\rceil$) denotes the largest integer $\le a$ (resp. smallest integer $\ge a$).

Published: 10 October 2013 DOI: 10.2140/agt.2013.13.3261

1.2 The colored Jones polynomial of a knot and its recurrence

The *colored Jones function* $J_{K,n}(q) \in \mathbb{Z}[q^{\pm 1}]$ of a *knot* K in 3–space for $n \in \mathbb{N}$ is a powerful knot invariant which satisfies a linear recurrence (ie a linear recursion relation) with coefficients that are polynomials in q and q^n Garoufalidis and Le [15]. The *noncommutative A–polynomial* $A_K(q, M, L)$ of K is defined to be the (homogeneous and content-free) such recurrence for $J_{K,n}(q)$ that has minimal order, written in operator notation. (By "content-free" we mean that the coefficients of the recurrence, which are polynomials in q and M, do not share a common nontrivial factor.) By definition, the noncommutative A–polynomial of K is an element of the *localized q–Weyl algebra*

$$\mathbb{W} = \mathbb{K}(q, M)\langle L\rangle/(LM - qML),$$

where $\mathbb{K} = \mathbb{Q}$ and the symbols L and M denote operators which act on a sequence $f_n(q)$ by

$$(Lf)_n(q) = f_{n+1}(q), \qquad (Mf)_n(q) = q^n f_n(q)\,.$$

The noncommutative A–polynomial of a knot allows one to compute the *Kashaev invariant* of a knot in *linear time*, and to confirm numerically the *Volume Conjecture* of Kashaev, the *Generalized Volume Conjecture* of Gukov and of Garoufalidis and Le, the *Modularity Conjecture* of Zagier, the *Slope Conjecture* of Garoufalidis and the *Stability Conjecture* of Garoufalidis and Le. For a discussion of the above conjectures and for a survey of computations, see Garoufalidis [11]. This explains the importance of exact formulas for the noncommutative A–polynomial of a knot.

In Garoufalidis [12] (see also Gelca [18]) the author formulated the *AJ conjecture*, which relates the specialization $A_K(1, M, L)$ with the A–polynomial $A_K(M, L)$ of K. The latter parametrizes the affine variety of $\mathrm{SL}(2, \mathbb{C})$ representations of the knot complement, viewed from the boundary torus; see Cooper, Culler, Gillet, Long and Shalen [7].

So far, the AJ conjecture has been verified only for knots whose A–polynomial consists of a single multiplicity-free component (aside from the component of abelian representations); see Le [27] and Le and Tran [28]. For the remaining knots, and especially for the hyperbolic knots, one does not know whether the noncommutative A–polynomial detects

(a) all nongeometric components of the $\mathrm{SL}(2, \mathbb{C})$ character variety,

(b) their multiplicities.

Our goal is to compute the noncommutative A–polynomial of the simplest knot whose A–polynomial has two irreducible components of nonabelian $\mathrm{SL}(2, \mathbb{C})$ representations (see Theorem 1.2), as well as recurrences for the colored Jones polynomials of the first four double twist knots. En route, we will introduce *Adams operations* on $\mathbb{W}$ which will allow us to define Adams operations of the ring $\mathbb{Q}[M, L]$ of A–polynomials and their noncommutative counterparts.

1.3 Minimal-order recurrences

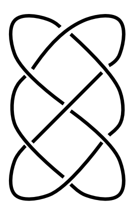
We split the problem of determining a minimal-order recurrence for a given sequence into two independent parts:

(a) Compute a recurrence: if the sequence is defined by a multidimensional sum of a proper q–hypergeometric term (as it is the case for the colored Jones polynomial), numerous algorithms can produce a linear recurrence with polynomial coefficients; see eg Petkovšek, Wilf and Zeilberger [35]. Different algorithms in general produce different recurrences, which may not be of minimal order; see Paule and Riese [32].

(b) Show that the recurrence produced in (a) has in fact minimal order: this can be achieved by proving that the corresponding operator is irreducible in $\mathbb{W}$. Criteria for certifying the irreducibility of a q–difference operator are presented in Section 4.

1.4 The noncommutative A–polynomial of the 7_4 knot

To illustrate our ideas concretely, rigorously and effectively, we focus on the simplest knot with reducible A–polynomial, namely the 7_4 knot in Rolfsen's notation [37]:



The 7_4 knot is a 2–*bridge knot* $K(11/15)$, and a *double-twist* knot obtained by $(-1/2, -1/2)$ surgery on the *Borromean rings*. Its A–polynomial can be computed with the `Mathematica` implementation by Hoste or with the `Maxima` implementation by Huynh, see also Petersen [33], and it is given by

$$\begin{aligned}A_{7_4}(M,L) = {} & (L^2M^8 - LM^8 + LM^6 + 2LM^4 + LM^2 - L + 1)^2 \\ & \times \big(L^3M^{14} - 2L^2M^{14} + LM^{14} + 6L^2M^{12} - 2LM^{12} + 2L^2M^{10} \\ & + 3LM^{10} - 7L^2M^8 + 2LM^8 + 2L^2M^6 - 7LM^6 + 3L^2M^4 \\ & + 2LM^4 - 2L^2M^2 + 6LM^2 + L^2 - 2L + 1\big).\end{aligned}$$

The first factor of $A_{7_4}(M,L)$ has multiplicity two and corresponds to a nongeometric component of the $\mathrm{SL}(2,\mathbb{C})$ character variety of 7_4. The second factor of $A_{7_4}(M,L)$ has multiplicity one and corresponds to the geometric component of the $\mathrm{SL}(2,\mathbb{C})$ character variety of 7_4. Let $A^{\mathrm{red}}_{7_4}$ denote the squarefree part of the above polynomial (ie where the second power of the first factor is replaced by the first power), called the *reduced A–polynomial*. Finally, let $A^{\mathrm{red}}_{-7_4}(M,L) = A^{\mathrm{red}}_{7_4}(M,L^{-1})L^5 \in \mathbb{Z}[M,L]$ denote the reduced A–polynomial of -7_4, the mirror of 7_4.

Definition 1.1 We say that an operator $P \in \mathbb{K}(q)\langle M^{\pm 1}, L^{\pm 1}\rangle/(LM - qML)$ is *palindromic* if and only if there exist integers $a, b \in \mathbb{Z}$ such that

$$P(q,M,L) = (-1)^a q^{bm/2} M^m L^b P(q,M^{-1},L^{-1}) L^{\ell - b}, \tag{1}$$

where $m = \deg_M(P) + \mathrm{ldeg}_M(P)$ and $\ell = \deg_L(P) + \mathrm{ldeg}_L(P)$. An operator in $\mathbb{W}$ is called palindromic if, after clearing denominators, it is palindromic in the above sense.

If $P = \sum_{i,j} p_{i,j} M^i L^j$ then Condition (1) implies that

$$p_{i,j} = (-1)^a q^{b(i-m/2)} p_{m-i,\ell-j}$$

for all $i, j \in \mathbb{Z}$. Note also that palindromic operators give rise to (skew-) symmetric solutions (if doubly-infinite sequences $(f_n)_{n\in\mathbb{Z}}$ are considered). More precisely, the equation $Pf = 0$ for palindromic P admits nontrivial symmetric (ie $f_{\lceil r+n\rceil} = f_{\lfloor r-n\rfloor}$ for all n) and skew-symmetric (ie $f_{\lceil r+n\rceil} = -f_{\lfloor r-n\rfloor}$ for all n) solutions, where $r = (\ell - b)/2$ is the reflection point.

The next theorem gives the noncommutative A–polynomial of 7_4 in its inhomogeneous form. Every inhomogeneous recurrence $Pf = b$ gives rise to a homogeneous recurrence $(L-1)(b^{-1}P)f = 0$.

Theorem 1.2 *The inhomogeneous noncommutative A–polynomial of 7_4 is given by the equation*

$$P_{7_4} J_{7_4,n}(q) = b_{7_4} \tag{2}$$

with $b_{7_4} \in \mathbb{Q}(q,q^n)$ and $P_{7_4} \in \mathbb{W}$ being a palindromic operator of (q,M,L)–degree $(65,24,5)$; both are given explicitly in the Appendix.

The proof of Theorem 1.2 consists of three parts:

(1) Compute the inhomogeneous recurrence (2) for the colored Jones function $J_{7_4,n}(q)$ using the iterated double sum formula for the colored Jones function (Equation (3)) and rigorous computer algebra algorithms (see Section 3).

(2) Prove that the operator P_{7_4} has no right factors of positive order (see Section 4). To this end, we discuss some natural $\mathbb{W}$–modules associated to a knot, given by the exterior algebra operations.

(3) Show that $J_{7_4,n}(q)$ does not satisfy a zero-order inhomogeneous recurrence, by using the degree of the colored Jones function (see Section 8).

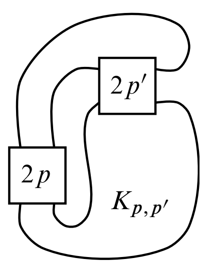
Corollary 1.3 *The AJ conjecture holds for the knot 7_4:*

$$P_{7_4}(1,M,L) = A^{\mathrm{red}}_{-7_4}(M^{1/2},L)(M-1)^5(M+1)^4(2M^4-5M^3+8M^2-5M+2).$$

Proof This follows from Theorem 1.2 by setting $q=1$. □

2 The colored Jones polynomial of double twist knots

Let $J_{K,n}(q)$ denote the colored Jones polynomial of the 0–framed knot K, colored by the n–dimensional irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$ and normalized to be 1 at the unknot; see Turaev [41; 42] and Jantzent [22]. The *double twist knot* $K_{p,p'}$ depicted below is given by $(-1/p,-1/p')$ surgery on the Borromean rings for integers p, p',



where the boxes indicate halftwists as follows:

Using the Habiro theory of the colored Jones function, (see Lauridsen [26, Section 6] following Masbaum [30] and Habiro [19]) it follows that

$$J_{K_{p,p'},n}(q) = \sum_{k=0}^{n-1} (-1)^k c_{p,k}(q) c_{p',k}(q) q^{-kn-\frac{k(k+3)}{2}} (q^{n-1};q^{-1})_k (q^{n+1};q)_k, \tag{3}$$

where $(x;q)_n$ denotes, as usual, the q–Pochhammer symbol defined as $\prod_{j=0}^{n-1}(1-xq^j)$ and

$$(4)\qquad c_{p,n}(q) = \sum_{k=0}^{n}(-1)^{k+n}q^{-\frac{k}{2}+\frac{k^2}{2}+\frac{3n}{2}+\frac{n^2}{2}+kp+k^2p}\frac{(1-q^{2k+1})(q;q)_n}{(q;q)_{n-k}(q;q)_{n+k+1}}.$$

Keep in mind that the above definition of $c_{p,n}$ differs by a power of q from the one given by Masbaum [30, Theorem 3.2]. With our definition, we have $c_{-1,n}(q)=1$ and $c_{1,n}(q)=(-1)^n q^{n(n+3)/2}$. Garoufalidis and Sun [16] have shown that for each integer p, the sequence $c_{p,n}(q)$ satisfies a monic recurrence of order $|p|$ with initial conditions $c_{p,n}(q)=0$ for $n<0$ and $c_{p,0}(q)=1$. In particular, for $p=2$ we have

$$c_{2,n+2}(q)+q^{n+3}(1+q-q^{n+2}+q^{2n+4})c_{2,n+1}(q)+q^{2n+6}(1-q^{n+1})c_{2,n}(q)=0.$$

Now, $K_{2,2}=7_4$. The first few values of the colored Jones polynomial

$$f_n(q) := J_{7_4,n}(q)$$

are listed here:

$$\begin{aligned}
f_1(q) &= 1\\
f_2(q) &= q-2q^2+3q^3-2q^4+3q^5-2q^6+q^7-q^8\\
f_3(q) &= q^2-2q^3+q^4+4q^5-6q^6+2q^7+6q^8-9q^9+3q^{10}+7q^{11}-8q^{12}\\
&\quad +q^{13}+7q^{14}-7q^{15}-q^{16}+5q^{17}-4q^{18}-q^{19}+3q^{20}-q^{21}-q^{22}+q^{23}\\
f_4(q) &= q^3-2q^4+q^5+2q^6-4q^8+q^9+6q^{10}-2q^{11}-8q^{12}+5q^{13}+9q^{14}\\
&\quad -4q^{15}-13q^{16}+7q^{17}+11q^{18}-3q^{19}-15q^{20}+6q^{21}+11q^{22}-q^{23}\\
&\quad -13q^{24}+q^{25}+10q^{26}+2q^{27}-11q^{28}-3q^{29}+9q^{30}+3q^{31}-7q^{32}-5q^{33}\\
&\quad +7q^{34}+4q^{35}-3q^{36}-5q^{37}+3q^{38}+3q^{39}-3q^{41}+q^{43}+q^{44}-q^{45}
\end{aligned}$$

The above data agrees with the `KnotAtlas` of Bar-Natan [2].

3 Computing a recurrence for the colored Jones polynomial of 7_4

We employ the definition of $J_{K_{p,p'},n}(q)$ given in (3) and (4) in terms of definite sums to compute a recurrence for the colored Jones polynomial of $7_4=K_{2,2}$. Thanks to Zeilberger's *holonomic systems approach* [45] this task can be executed in a completely automatic fashion, eg using the algorithms implemented in the Mathematica package `HolonomicFunctions` (Koutschan [25]); see Koutschan [23] for more details. The summation problem in (4) can be tackled by a q–analogue of Zeilberger's [44; 46] fast summation algorithm since the summand is a proper q–hypergeometric term; also see Wilf and Zeilberger [43] and Paule and Riese [32].

As it was mentioned above, the sequence $c_{p,n}(q)$ satisfies a recurrence of order $|p|$ and therefore the summand of (3) is not q–hypergeometric in general. Thus we apply Chyzak's generalization [5] of Zeilberger's algorithm to derive a recurrence for $J_{7_4,n}(q)$.

Both algorithms are based on the concept of *creative telescoping* [46]; see [23] for an introduction and Garoufalidis and Sun [17] for an earlier application to the computation of noncommutative A–polynomials. Let $f_{n,k}(q)$ denote the summand of (3). Our implementation of Chyzak's algorithm yields the equation

$$P_{7_4}(f_{n,k}) = c_d(q,q^n)f_{n+d,k} + \cdots + c_0(q,q^n)f_{n,k} = g_{n,k+1} - g_{n,k},$$

where $g_{n,k}$ is a $\mathbb{K}(q,q^k,q^n)$–linear combination of certain shifts of f (eg $f_{n,k}$, $f_{n+1,k}$, $f_{n,k+1}$, etc). Now creative telescoping is executed by summing this equation with respect to k. It follows that $P_{7_4}(J_{7_4,n}) = g_{n,n} - g_{n,0} = b_{7_4}$.

The summation problems (3) and (4) for $p = p' = 2$ are of moderate size: our software `HolonomicFunctions` computes the solution in less than 2 minutes. The result is given in the Appendix.

4 Irreducibility of q–difference operators

An element $P \in \mathbb{W}$ is *irreducible* if it cannot be written in the form $P = QR$ with $Q, R \in \mathbb{W}$ of positive L–degree. Since there is a (left and right) division algorithm in $\mathbb{W}$, it follows that every element P is a finite product of irreducible elements. However, it can happen that P can be factored in different ways, but any two factorizations of P into irreducible elements are related in a specific way; see Ore [31].

A factorization algorithm for elements of the *localized Weyl algebra* $\mathbb{K}(x)\langle\partial\rangle$, where $\partial x - x\partial = 1$ has been discussed by several authors that include Schwarz [38], Tsarev [40] and Bronstein [3] and also van Hoeij [20, Section 8] and van der Put and Singer [36]; the factorization of more general Ore operators (including differential, difference, and q–difference) has been investigated in Bronstein and Petkovšek [4]. Roughly, a factorization algorithm for $P \in \mathbb{K}(x)\langle\partial\rangle$ of order d (as a linear differential operator) proceeds as follows: if $P = QR$ where R is of order k, then the coefficients of R can be computed by finding the right factors of order 1 of the associated equation obtained by the k^{th} exterior power of P. The problem of finding linear right factors can be solved algorithmically.

For our purposes we do not require a full factorization algorithm, but only criteria for certifying the irreducibility of q–difference operators. Consider $P(q, M, L) \in \mathbb{W}$ and assume that the leading coefficient of P does not vanish when specialized to $q = 1$. The following is an algorithm for certifying irreducibility of P:

(1) If $P(1, M, L) \in \mathbb{K}(M)[L]$ is irreducible, then P is irreducible (see Section 4.1).

(2) If not, factor the commutative polynomial $P(1, M, L)$ into irreducible factors $P_1, \dots, P_n$.

(3) For each $k = \sum_{i\in I} \deg_L(P_i)$ such that $I \subseteq \{1, \dots, n\}$ compute the exterior power $\bigwedge^k P \in \mathbb{W}$ (see Section 4.2).

(4) Apply the algorithm qHyper (eg in its improved version described in Section 7), to show that none of the computed exterior powers has a linear right factor. Then P is irreducible.

4.1 An easy sufficient criterion for irreducibility

In this section we mention an easy irreducibility criterion in $\mathbb{W}$, which is sufficient but not necessary, as we shall see. This criterion has been used in [17] to compute the noncommutative A–polynomial of twist knots, and also in [27; 28] to verify the AJ conjecture in some cases.

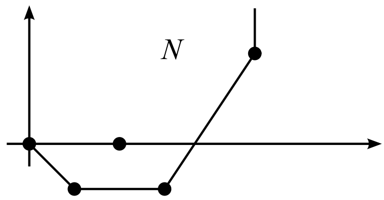
To formulate the criterion, we will use the Newton polygon at $q = 1$, in analogy with the Newton polygon at $q = 0$ studied in Garoufalidis [13]. Expanding a rational function $a(q, M) \in \mathbb{K}(q, M)$ into a formal Laurent series in $q - 1$, let $v(a(q, M)) \in \mathbb{Z} \cup \{\infty\}$ denote the lowest power of $q-1$ which has nonzero coefficient. It can be easily verified that v is a *valuation*, ie it satisfies

$$v(ab) = v(a) + v(b), \quad v(a + b) \geq \min(v(a), v(b)) .$$

Definition 4.1 For an operator $P(q, M, L) = \sum_{j=0}^{d} a_j(q, M) L^j \in \mathbb{W}$ the *Newton polygon* $N(P)$ is defined to be the lower convex hull (see Loera, Rambau and Santos [8]) of the set $\{(j, v(a_j)) \mid j = 0, \dots, d\}$. Furthermore, let $N^e(P)$ denote the union of the (nonvertical) boundary line segments of $N(P)$.

For instance, if $P = (q-1)^2 L^5 + ((q-1)(Mq-1))^{-1} L^3 + L^2 + (q-1)^{-1} L + 1$, then $N(P)$ is shown in the figure on the following page. Here, $N^e(P)$ is the path of straight line segments connecting the points $(0, 0)$, $(1, -1)$, $(3, -1)$ and $(5, 2)$.

The next lemma is elementary; it follows easily from the definitions: Recall that the *Minkowski sum* $A + B$ of two polytopes A and B is the convex hull of the set $\{a + b \mid a \in A,\ b \in B\}$ (Ziegler [47]).



Lemma 4.2 *If $Q, R \in \mathbb{W}$, then $N(QR) = N(Q) + N(R)$.*

Proposition 4.3 *Let $P(q, M, L) = \sum_{j=0}^{d} a_j(q, M)L^j \in \mathbb{W}$ with $d > 1$ and assume that $P(1, M, L) \in \mathbb{K}(M)[L]$ is well-defined and irreducible with*

$$a_0(1, M)a_d(1, M) \neq 0.$$

Then $P(q, M, L)$ is irreducible in $\mathbb{W}$.

Proof The assumptions imply that $N^e(P)$ is the horizontal line segment from the origin to $(\deg_L(P), 0)$. If $P = QR$ with $\deg_L(Q)\deg_L(R) \neq 0$, then Lemma 4.2 implies that both $N^e(Q)$ and $N^e(R)$ consist of a single horizontal segment as well. Without loss of generality, assume that the leading coefficient of Q has valuation zero; if not we can multiply Q by $(1-q)^a$ and R by $(1-q)^{-a}$ for an appropriate integer a. Then, it follows that $N^e(Q) = [0, \deg_L(Q)] \times 0$ and $N^e(R) = [0, \deg_L(R)] \times 0$. Evaluating at $q = 1$, it follows that $Q(1, M, L)$, $R(1, M, L) \in \mathbb{K}(M)[L]$ are well-defined and $P(1, M, L) = Q(1, M, L)R(1, M, L)$, where $Q(1, M, L)$ and $R(1, M, L)$ are of L–degree $\deg_L(Q)$ and $\deg_L(R)$ respectively. This contradicts the assumption that $P(1, M, L)$ is irreducible and completes the proof. □

4.2 Adams operations on $\mathbb{W}$–modules

In this section we introduce Adams (ie exterior power) operations on finitely generated $\mathbb{W}$–modules. The Adams operations were inspired by the Weyl algebra setting, and play an important role in irreducibility and factorization of elements in $\mathbb{W}$.

To begin with, a finitely generated left $\mathbb{W}$–module $\mathcal{M}$ is a direct sum of a free module of finite rank and a cyclic torsion module. The proof of this statement for $\mathbb{W}$ is identical to the proof for modules over the Weyl algebra, discussed for example in [36, Lemma 2.5 and Proposition 2.9].

Consider a torsion $\mathbb{W}$–module $\mathcal{M}$ with generator f. We will write this by $(\mathcal{M}, f)$, following the notation of [36, Section 2.3]. f is often called a *cyclic vector* for $\mathcal{M}$. It follows that $\mathcal{M} = \mathbb{W}/(\mathbb{W}P)$ where P is a generator of the left annihilator ideal $\mathrm{ann}(f) := \{Q \in \mathbb{W} \mid Qf = 0\}$ of f.

Definition 4.4 For a natural number k, we define the k^{th} exterior power of $(\mathcal{M}, f)$ by

$$\bigwedge^k(\mathcal{M}, f) = (\bigwedge^k \mathcal{M}, f \wedge Lf \wedge \cdots \wedge L^{k-1} f).$$

If $P = \operatorname{ann}(f)$, then we define $\bigwedge^k P := \operatorname{ann}(f \wedge Lf \wedge \cdots \wedge L^{k-1} f)$.

The next lemma is an effective algorithm to compute $\bigwedge^k P$. Recall the *shifted analogue of the Wronskian (also called Casoratian)* of k sequences $f_n^{(i)}$ for $i = 1, \dots, k$ given by

$$W\big(f^{(1)}, \dots, f^{(k)}\big)_n = \det_{\substack{0 \le j \le k-1 \\ 1 \le i \le k}} f_{n+j}^{(i)}. \tag{5}$$

Lemma 4.5 *Let $P \in \mathbb{W}$ and $f_n^{(1)}, \dots, f_n^{(k)}$ be k linearly independent solutions of the equation $Pf = 0$. Then $\bigwedge^k P$ is the minimal-order operator in $\mathbb{W}$ which annihilates the sequence $w_n = W\big(f^{(1)}, \dots, f^{(k)}\big)_n$. In particular, there is a unique such solution (up to left multiplication by elements from $\mathbb{K}(q, M)$).*

Proof Let $d = \deg_L(P)$ and fix $k \le d$. Let

$$\mathcal{I} = \{(i_1, \dots, i_k) \mid 1 \le i_1 < i_2 < \cdots < i_k \le d\}.$$

$\bigwedge^k \mathcal{M}$ has a basis $\{v_I \mid I \in \mathcal{I}\}$, where $v_I = L^{i_1-1} f \wedge L^{i_2-1} f \cdots \wedge L^{i_k-1} f$ for $I = (i_1, \dots, i_k) \in \mathcal{I}$. Now, using the fact that $Pf = 0$, it follows that for $v \in \mathcal{M}$ and every natural number j we have

$$L^j v = \sum_{I \in \mathcal{I}} a_{j,I} v_I,$$

where the $a_{j,I}$ are rational functions in q and q^n. It follows that the set

$$\big\{L^j v \mid j = 0, \dots, \tbinom{d}{k}\big\}$$

is linearly dependent. A minimal dependency of this set gives rise to the operator $\bigwedge^k P$ by definition of the latter.

To prove the lemma, choose a fundamental set of solutions $f^{(i)}$ for $i = 1, \dots, d$ to the recurrence equation $Pf = 0$ and consider the correspondence

$$\phi : \{v_I \mid I \in \mathcal{I}\} \longrightarrow \{w_I \mid I \in \mathcal{I}\}, \quad \phi(v_I) = w_I,$$

where $w_I = W\big(f^{(i_1)}, \dots, f^{(i_k)}\big)$ for $I = (i_1, \dots, i_k) \in \mathcal{I}$. The above correspondence is invariant with respect to the L–action since

$$LW\big(g^{(i_1)}, \dots, g^{(i_k)}\big) = W\big(Lg^{(i_1)}, \dots, Lg^{(i_k)}\big).$$

It follows that for all natural numbers j we have

$$L^j w = \sum_{I \in \mathcal{I}} a_{j,I} w_I,$$

where $w = \phi(f \wedge Lf \wedge \cdots \wedge L^{k-1} f) = W(f^{(1)}, \dots, f^{(k)})$. Since $\bigwedge^k P$ is the operator that encodes the minimal dependency among the translates of v, it follows that it is also the operator that encodes the minimal dependency among the translates of w. The result follows. □

Corollary 4.6 *Lemma 4.5 gives the following algorithm to compute $\bigwedge^k P$: The definition of w_n as a determinant together with the equations $Pf^{(i)} = 0$ for $i = 1, \dots, k$ allows to express $w_{n+\ell}$ for arbitrary $\ell \in \mathbb{N}$ as a $\mathbb{K}(q, q^n)$–linear combination of the products*

$$\prod_{i=1}^{k} f^{(i)}_{n+j_i} \quad \text{where } 0 \le j_i < \deg_L(P)$$

for $1 \le i \le k$. This allows to determine the minimal ℓ such that $w_n, \dots, w_{n+\ell}$ are $\mathbb{K}(q, q^n)$–linearly dependent. Compare also with [36, Example 2.29].

Lemma 4.7 *Let $P \in \mathbb{W}$ be of the form $P = L^d + \sum_{j=0}^{d-1} a_j L^j$ with $a_0 \neq 0$, and let $\{f^{(1)}_n, \dots, f^{(d)}_n\}$ be a fundamental solution set of the equation $Pf = 0$. Then $w_{n+1} - (-1)^d a_0 w_n = 0$, where $w = W(f^{(1)}, \dots, f^{(d)})$.*

Proof The proof is done by an elementary calculation

$$w_{n+1} = \det \begin{pmatrix} f^{(1)}_{n+1} & \cdots & f^{(d)}_{n+1} \\ \vdots & & \vdots \\ f^{(1)}_{n+d} & \cdots & f^{(d)}_{n+d} \end{pmatrix} = \det \begin{pmatrix} f^{(1)}_{n+1} & \cdots & f^{(d)}_{n+1} \\ \vdots & & \vdots \\ f^{(1)}_{n+d-1} & \cdots & f^{(d)}_{n+d-1} \\ -a_0 f^{(1)}_n & \cdots & -a_0 f^{(d)}_n \end{pmatrix} = (-)^d a_0 w_n,$$

where in the second step the identities

$$f^{(i)}_{n+d} = -\sum_{j=0}^{d-1} a_j f^{(i)}_{n+j}$$

and some row operations have been employed. □

Theorem 4.8 *Let $P, Q, R \in \mathbb{W}$ such that $P = QR$ is a factorization of P, and let k denote the order of R, ie $k = \deg_L(R)$. Then $\bigwedge^k P$ has a linear right factor of the form $L - a$ for some $a \in \mathbb{K}(q, M)$.*

Proof Let $F = \{f^{(1)}, \dots, f^{(k)}\}$ be a fundamental solution set of R. By Lemma 4.7 it follows that $w = W(f^{(1)}, \dots, f^{(k)})$ satisfies a recurrence of order 1, say $w_{n+1} = a w_n, a \in \mathbb{K}(q, M)$. But F is also a set of linearly independent solutions of $Pf = 0$, and therefore w is contained in the solution space of $\bigwedge^k P$. It follows that $\bigwedge^k P$ has the right factor $L - a$. $\square$

5 Plethysm

In this section we define Adams operations on the ring $\mathbb{Q}(M)[L]$, and in particular on the set of affine curves in $\mathbb{C}^* \times \mathbb{C}^*$.

Let $\mathbb{Q}(M)_+[L]$ denote the subring of $\mathbb{Q}(M)[L]$ which consists of $p(M, L) \in \mathbb{Q}(M)[L]$ with $p(M, 0) = 1$. If $p(M, L) \in \mathbb{Q}(M)_+[L]$ has degree $d = \deg_L(p)$, then we can write

$$p(M, L) = \prod_{i=1}^{d} (1 + L_i(M) L^i)$$

in an appropriate algebraic closure of $\mathbb{Q}(M)[L]$.

Definition 5.1 For $k \in \mathbb{N}$ we define ψ: $\mathbb{Q}(M)_+[L] \longrightarrow \mathbb{Q}(M)_+[L]$ by

$$\psi_k(p)(M, L) = \prod_{1 \le i_i < i_2 < \dots < i_k \le d} (1 + L_{i_1}(M) \dots L_{i_k}(M) L).$$

The next lemma expresses the coefficients of $\psi_k(p)$ in terms of those of p using the plethysm operations on the basis e_i of the ring of symmetric functions Λ. For a definition of the latter, see Macdonald [29, Section I–8].

Lemma 5.2 *If* $p = \prod_{i=1}^{\infty} (1 + x_i L) = \sum_{i=0}^{\infty} e_i L^i$ *then*

$$\psi_k(p) = \sum_{i=0}^{\infty} (e_i \circ e_k) L^i.$$

Corollary 5.3 *In particular for* $d = 5$ *and* $k = 2, 3$ *(as is the case of interest for the knot* 7_4*) the* SF *package (Stembridge [39]) gives:*

$$p = 1 + e_1 L + e_2 L^2 + e_3 L^3 + e_4 L^4 + e_5 L^5$$

$$
\begin{aligned}
\psi_2(p) = 1 + e_2 L + (e_1e_3 - e_4)L^2 + (-2e_2e_4 + e_3^2 + e_1^2e_4 - e_1e_5)L^3 \\
+ (e_1^3e_5 + e_3e_5 - e_4^2 - 3e_1e_2e_5 + e_1e_3e_4)L^4 \\
+ (e_1^2e_3e_5 - 2e_1e_4e_5 - 2e_2e_3e_5 + 2e_5^2 + e_2e_4^2)L^5 \\
+ (e_1e_2e_4e_5 - e_1^2e_5^2 + e_2e_5^2 - 3e_3e_4e_5 + e_4^3)L^6 \\
+ (-e_4e_5^2 + e_1e_4^2e_5 - 2e_1e_3e_5^2 + e_2^2e_5^2)L^7 \\
+ (e_2e_4e_5^2 - e_1e_5^3)L^8 + e_3e_5^3L^9 + e_5^4L^{10}
\end{aligned}
$$

$$
\begin{aligned}
\psi_3(p) = 1 + e_3 L + (e_2e_4 - e_1e_5)L^2 + (-2e_1e_3e_5 - e_4e_5 + e_1e_4^2 + e_2^2e_5)L^3 \\
+ (e_1e_2e_4e_5 - e_1^2e_5^2 + e_2e_5^2 - 3e_3e_4e_5 + e_4^3)L^4 \\
+ (-2e_2e_3e_5^2 + 2e_5^3 + e_1^2e_3e_5^2 + e_2e_4^2e_5 - 2e_1e_4e_5^2)L^5 \\
+ (-e_4^2e_5^2 + e_3e_5^3 + e_1^3e_5^3 + e_1e_3e_4e_5^2 - 3e_1e_2e_5^3)L^6 \\
+ (e_3^2e_5^3 - e_1e_5^4 + e_1^2e_4e_5^3 - 2e_2e_4e_5^3)L^7 \\
+ (-e_4e_5^4 + e_1e_3e_5^4)L^8 + e_2e_5^5L^9 + e_5^6L^{10}
\end{aligned}
$$

6 Factorization of q–difference operators after Bronstein–Petkovšek

This section is not needed for the results of our paper, but may be of independent interest. Bronstein and Petkovšek [4] developed a factorization algorithm for q–difference operators, and more generally, for Ore operators. A key component of their algorithm, which predated and motivated the work of Abramov, Paule and Petkovšek [1], is to reduce the problem of factorization into computing all linear right factors of a finite list of so-called associated operators. Since this factorization algorithm is not widely known, we will describe it in this section, following [4]. All results in this section are due to [4].

Definition 6.1 Let $P \in \mathbb{W}$ be of the form $P = L^d + \sum_{j=0}^{d-1} a_j L^j$ with $a_0 \neq 0$, and let $\{f_n^{(1)}, \dots, f_n^{(d)}\}$ be a fundamental solution set of the equation $Pf = 0$. Let

$$
\sum_{l=0}^{d} w^{(d-l)} L^l f = \det \begin{pmatrix} f & f^{(1)} & \cdots & f^{(d)} \\ \vdots & \vdots & & \vdots \\ L^d f & L^d f^{(1)} & \cdots & L^d f^{(d)} \end{pmatrix}. \tag{6}
$$

Lemma 6.2 *With the notation of Definition 6.1 we have:*

(a) $w^{(d-j)}/w^{(0)} = a_j, \quad j = 0, \dots, d.$

(b) $w^{(0)} = W(f^{(1)}, \dots, f^{(d)})$ *satisfies* $w^{(0)}_{n+1} + (-1)^d a_0 w^{(0)}_n = 0$.

(c) *For* $j = 0, \dots, d-1$ *and* $n \in \mathbb{N}$ *we have*

$$a_j(q, q^n) w^{(d-j)}_{n+1} + (-1)^d a_j(q, q^{n+1}) a_0(q, q^n) w^{(d-j)}_n = 0. \tag{7}$$

Proof Since $\sum_{l=0}^{d} w^{(d-l)} L^l f^{(i)} = 0$ for $i = 1, \dots, d$ and $\{f^{(1)}, \dots, f^{(d)}\}$ is a fundamental solution of the equation $Pf = 0$, it follows that $P = \sum_{l=0}^{d} w^{(d-l)} L^l f$. This proves (a). The definition of $w^{(0)}$ implies that $w^{(0)} = W(f^{(1)}, \dots, f^{(d)})$ and likewise

$$w^{(d)} = W(Lf^{(1)}, \dots, Lf^{(d)}) = LW(Lf^{(1)}, \dots, Lf^{(d)}).$$

Using $w^{(d)} = a_0 w^{(0)}$ (by Part (a)) and the above, we obtain (b). Now, (a) gives $w^{(d-j)} = a_j w^{(0)}$, hence

$$Lw^{(d-j)} = (La_j) L w^{(0)} = (-1)^{d-1} (La_j) a_0 w^{(0)}.$$

Eliminating $w^{(0)}$, (c) follows. □

Lemma 6.2 gives the following algorithm that produce a finite set of all possible right factors $R = L^k + \sum_{j=0}^{k-1} a_j L^j$ of an element $P \in \mathbb{W}$.

Corollary 6.3 *Using the definition of* $w^{(k-j)}$ *for* $j = 0, \dots, k$ *together with the equations* $Pf^{(i)} = 0$ *for* $i = 1, \dots, k$ *allows to express* $w^{(k-j)}_{n+\ell}$ *for arbitrary* $\ell \in \mathbb{N}$ *as a* $\mathbb{K}(q, q^n)$*–linear combination of the products* $\prod_{i=1}^{k} f^{(i)}_{n+j_i}$ *where* $0 \le j_i < \deg_L(P)$ *for* $1 \le i \le k$*. This allows to determine the minimal* ℓ *such that*

$$w^{(k-j)}_n, \dots, w^{(k-j)}_{n+\ell}$$

are $\mathbb{K}(q, q^n)$*–linearly dependent. Let* $\bigwedge_j^k P$ *denote the corresponding monic minimal-order operators. Using qHyper, list all right factors of* $\bigwedge_j^k P$*. If* $a_j = 0$*, include it in the list of possible values of* a_j*. Otherwise, use the computed finite list and Equation* (7) *to list all possible values of* a_j*. The result follows.*

7 Irreducibility of the computed recurrence for 7_4

The irreducibility of a monic operator $P \in \mathbb{W}$ of order d can be established by Theorem 4.8, ie by showing that none of the exterior powers $\bigwedge^k P$ for $1 \le k < d$ has a linear right factor. In the case of the fifth-order operator P_{7_4} we observed that its $q = 1$ specialization factors into two irreducible factors of L–degrees 2 and 3, respectively (and hence Proposition 4.3 is not applicable). We conclude that P_{7_4} cannot have right factors of L–degrees 1 or 4. Thus it suffices to inspect its second and third exterior powers only.

The computation of an exterior power $\bigwedge^k P$ is immediate from its definition. We start with an ansatz for a linear recurrence for the Wronskian:

$$c_\ell(q,M)w_{n+\ell}+\cdots+c_1(q,M)w_{n+1}+c_0(q,M)w_n=0. \tag{8}$$

In the next step, all occurrences of w_{n+j} in (8) are replaced by the expansion of the determinant (5), eg for $k=2$ we have

$$w_{n+j}=f^{(1)}_{n+j}f^{(2)}_{n+j+1}-f^{(1)}_{n+j+1}f^{(2)}_{n+j}.$$

As before let d denote the L–degree of P. Now each $f^{(i)}_{n+j}$ with $j\geq d$ is rewritten as a $\mathbb{Q}(q,M)$–linear combination of

$$f^{(i)}_n,\ldots,f^{(i)}_{n+d-1},$$

using the equation $Pf^{(i)}=0$. Finally, coefficient comparison with respect to $f^{(i)}_{n+j}$, $1\leq i\leq k$, $0\leq j<d$ yields a linear system for the unknown coefficients $c_0,\ldots,c_\ell$. The minimal-order recurrence for w_n can be found by trying $\ell=0$, $\ell=1$, …, until a solution is found. This methodology was employed to compute $\bigwedge^2 P_{7_4}$ and $\bigwedge^3 P_{7_4}$ (see Table 1 for their sizes).

	L–degree	M–degree	q–degree	`ByteCount`
P_{7_4}	5	24	65	463544
$\bigwedge^2 P_{7_4}$	10	134	749	37293800
$\bigwedge^3 P_{7_4}$	10	183	1108	62150408

Table 1: Some statistics concerning P_{7_4} and its exterior powers

Having the exterior powers of P_{7_4} at hand, we can now apply Theorem 4.8 to it: for establishing the irreducibility of P_{7_4} we have to show that its exterior powers do not have right factors of order one. Note that for our application we would not necessarily need the minimal-order recurrences for the Wronskian; as long as they have no linear right factors, the irreducibility follows as a consequence. Note also that one could try to use Proposition 4.3 for this task; unfortunately this is not going to work, since from the discussion in Section 5 it is clear that, after the substitution $q=1$, the exterior powers in question do have a linear factor.

It is well known that a linear right factor of a q–difference equation corresponds to a q–hypergeometric solution, ie a solution $f_n(q)$ such that f_{n+1}/f_n is a rational function in q and q^n. The problem of computing all such solutions has been solved in [1] and the corresponding algorithm has been implemented by Petkovšek in the Mathematica package `qHyper`.

Let $P(q,M,L)=\sum_{i=0}^{d} p_i(q,M)L^i$ be an operator such that all p_i are polynomials in q and M. The qHyper algorithm described in [1] attempts to find a right factor $L-r(q,M)$ of P where the rational function r is assumed to be written in the normal form

$$r(q,M)=z(q)\frac{a(q,M)}{b(q,M)}\frac{c(q,qM)}{c(q,M)}$$

subject to the conditions

$$\gcd\big(a(q,M),b(q,q^nM)\big)=1 \quad \text{for all } n\in\mathbb{N} \tag{9}$$

and

$$\gcd\big(a(q,M),c(q,M)\big)=1,\quad \gcd\big(b(q,M),c(q,qM)\big)=1,\quad c(0)\neq 0$$

(see [1] for the existence proof). It is not difficult to show that under these assumptions $a(q,M)\mid p_0(q,M)$ and $b(q,M)\mid p_d(q,q^{1-d}M)$. Therefore the algorithm qHyper proceeds by testing all admissible choices of a and b. Each such choice yields a q–difference equation for $c(q,M)$ which also involves the unknown algebraic expression $z(q)$. The techniques for solving this kind of equations (or for showing that no solution exists) are described in detail in [1].

Now let's apply qHyper to $P^{(2)}(q,M,L):=\bigwedge^2 P_{7_4}$ whose trailing and leading coefficients are given by

$$\begin{aligned}
p_0(q,M)&=q^{162}M^{44}(M-1)\Big(\prod_{i=6}^{9}(q^iM-1)\Big)\\
&\qquad\times\Big(\prod_{i=6}^{10}(q^iM+1)(q^{2i+1}M^2-1)\Big)F_1(q,M),\\
p_{10}(q,q^{-9}M)&=q^{-397}(q^2M-1)\Big(\prod_{i=4}^{7}(M-q^i)\Big)\\
&\qquad\times\Big(\prod_{i=4}^{8}(M+q^i)(M^2-q^{2i+1})\Big)F_2(q,M),
\end{aligned}$$

where F_1 and F_2 are large irreducible polynomials, related by $q^{280}F_1(q,M)=F_2(q,q^{10}M)$. A blind application of qHyper would result in

$$45\cdot 2^{16}\cdot 2^{16}=193\,273\,528\,320$$

possible choices for a and b; far too many to be tested in reasonable time. Cluzeau and van Hoeij [6] and Horn [21] have presented improvements to qHyper which are based on local types and exclude a large number of possible choices; however, the simple criteria described below seem to be more efficient.

In order to confine the number of qHyper's test cases we exploit two facts. The first is the fact that $P^{(2)}(1, M, L) = R_1(M) \cdot (L - M^4) \cdot Q_1(M, L) \cdot Q_2(M, L)$, where Q_1 and Q_2 are irreducible of L–degree 3 and 6, respectively. In other words, we need only to test pairs (a, b) which satisfy the condition

$$a(1, M) = M^4 b(1, M). \tag{10}$$

The second fact is that a and b must fulfill Condition (9); Petkovšek [34, Remark 4.1] has already suggested this improvement, formulated in the setting of difference equations. In our example we are lucky because the two criteria exclude most of the possible choices for a and b; the process of figuring out which cases remain to be tested is now presented in detail.

(1) Equation (10) implies that either both F_1 and F_2 must be present or none of them; Condition (9) then excludes them entirely.

(2) Clearly the factor M^4 in Equation (10) can only come from M^{44} in p_0; thus all other (linear and quadratic) factors in $a(1, M)/b(1, M)$ must cancel completely.

(3) The most simple admissible choice is $a(q, M) = M^4$ and $b(q, M) = 1$.

(4) Because of Equation (9) a cancellation can almost never take place among factors which are equivalent under the substitution $q = 1$. This is reflected by the fact that the entries in the first column of Table 2 are (row-wise) larger than those in the second column, eg $(q^6 M + 1) \mid a(q, M)$ and $(q^{-4} M + 1) \mid b(q, M)$ violates Equation (9).

(5) The only exception is that $(M-1) \mid a(q, M)$ cancels with $(q^2 M-1) \mid b(q, M)$ in $a(1, M)/b(1, M)$. In that case, Equation (9) excludes further factors of the form $q^i M - 1$, and together with Equation (10) we see that no other factors at all can occur. This gives the choice $a(q, M) = M^4(M-1)$ and $b(q, M) = q^2 M - 1$.

(6) We may assume that $a(q, M)$ contains some of the quadratic factors $q^i M^2 - 1$. For $q = 1$ they factor as $(M-1)(M+1)$ and therefore can be canceled with corresponding pairs of linear factors in $b(q, M)$. Condition (9) forces $a(q, M)$ to be free of linear factors and $b(q, M)$ to be free of quadratic factors. Thus we obtain $\sum_{m=1}^{5} \binom{5}{m}^3 = 2251$ possible choices.

(7) Analogously $a(q, M)$ can have some linear factors which for $q = 1$ must cancel with quadratic factors in $b(q, M)$; this gives 2251 further choices.

Summing up, we have to test 4504 cases which can be done in relatively short time on a computer. None of these cases delivered a solution for $c(q,M)$ and $z(q)$ which proves that $P^{(2)}$ does not have a linear right factor.

	$\bigwedge^2 P_{7_4}$		$\bigwedge^3 P_{7_4}$	
	$p_0(q,M)$	$p_{10}(q,q^{-9}M)$	$p_0(q,M)$	$p_{10}(q,q^{-9}M)$
q^iM-1	0, 6, 7, 8, 9	$-7, -6, -5, -4, 2$	0, 7, 8, 9	$-6, -5, -4, 3$
q^iM+1	6, 7, 8, 9, 10	$-8, -7, -6, -5, -4$	7, 8, 9, 10, 11	$-8, -7, -6, -5, -4$
q^iM^2-1	13, 15, 17, 19, 21	$-17, -15, -13, -11, -9$	$5, 7, 9, 11, 13^2, 15^2, 17^2, 19^2, 21^2, 23$	$-17, -15^2, -13^2, -11^2, -9^2, -7^2, -5, -3, -1, 1$

Table 2: Factors of the leading and trailing coefficients of the exterior powers of P_{7_4}; each cell contains the values of i of the corresponding factors. Superscripts indicate that factors occur with multiplicities.

The situation for $P^{(3)}(q,M,L) := \bigwedge^3 P_{7_4}$ is very similar. Now the trailing and leading coefficients turn out to be

$$p_0(q,M) = q^{297}M^{66}(M-1)(q^7M-1)\cdots(q^{23}M^2-1)F_3(q,M),$$
$$p_{10}(q,q^{-9}M) = q^{-456}(q^3M-1)(M-q^4)\cdots(M^2-q^{17})F_4(q,M),$$

where the linear and quadratic factors can be extracted from Table 2. Also not explicitly displayed are the large irreducible factors F_3 and F_4 which satisfy $q^{275}F_3(q,M) = F_4(q,q^{10}M)$. For $q=1$ we obtain the factorization

$$P^{(3)}(1,M,L) = R_2(M)\cdot(L+M^7)\cdot Q_3(M,L)\cdot Q_4(M,L),$$

where Q_3 and Q_4 are irreducible of L–degree 3 and 6, respectively. As before we get two special cases, the first with $a(q,M) = M^7$ and $b(q,M) = 1$, and the second with $a(q,M) = M^7(M-1)$ and $b(q,M) = q^3M-1$. For the choices where we cancel quadratic against linear factors, we obtain

$$2\sum_{m=1}^{4}\binom{4}{m}\binom{5}{m}\sum_{j=0}^{\lfloor m/2\rfloor}\binom{5}{j}\binom{10-j}{m-2j} = 23600$$

possibilities. Again, none of these cases yields a solution for $c(q,M)$ and therefore we have shown that $P^{(3)}$ does not have a linear right factor.

Theorem 4.8 now implies that P_{7_4} cannot have a right factor of order 2 or 3. We conclude that the operator P_{7_4} is irreducible.

8 No recurrence of order zero

In this section we give an elementary criterion to deduce that a q–holonomic sequence does not satisfy an inhomogeneous recurrence of order zero, and apply it in the case of the 7_4 knot to conclude the proof of Theorem 1.2. The next lemma is obvious.

Lemma 8.1 *If $\deg_q(f_n(q))$ is not a linear function of n, then $f_n(q)$ does not satisfy $af = b$ for $a, b \in \mathbb{K}(q, q^n)$.*

The degree $\deg_q(J_{K,n}(q))$ of an alternating knot is well-known and given by a quadratic polynomial in n; see eg [14; 27]. In the case of the alternating knot 7_4, we have

$$\deg_q(J_{7_4,n}(q)) = \tfrac{7}{2}n^2 - \tfrac{5}{2}n - 1 .$$

It follows that $J_{7_4,n}$ does not satisfy an inhomogeneous recurrence of order zero.

9 Proof of Theorem 1.2

In this section we will finish the proof of Theorem 1.2. It follows from the following lemma, of independent interest.

Lemma 9.1 *Suppose f is a q–holonomic sequence such that*

(1) *f satisfies the inhomogeneous recurrence $Pf = b$,*

(2) *$P \in \mathbb{W}$ is irreducible, $\deg_L(P) > 1$ and $b \in \mathbb{K}(q, q^n) \neq 0$,*

(3) *f does not satisfy a recurrence of the form $af = c$ for $a, c \in \mathbb{K}(q, q^n)$, $a \neq 0$.*

Then the minimal-order homogeneous recurrence relation that f satisfies is given by $(L-1)(b^{-1}P)f = 0$.

Proof Let $d = \deg_L(P)$ and $P' = (L-1)(b^{-1}P)$. P' is the product of two irreducible elements of $\mathbb{W}$ (namely, $L-1$ and $b^{-1}P$) of L–degrees 1 and d respectively. Recall that $\mathbb{W}$ is a Euclidean domain. Although the factorization of an element in $\mathbb{W}$ into irreducible factors is not unique in general, [31, Theorem 1] proves that the number of irreducible factors of a fixed order is independent of the factorization. It follows that any factorization of P' into a product of irreducible factors has exactly two factors, one of L–degree 1 and another of L–degree d.

Suppose $P'' f = 0$ where P'' has minimal L–degree strictly less than $d+1$. Since $\mathbb{W}$ is a Euclidean domain, it follows that P'' is a right factor of P', and P'' is a product of irreducible factors. The above discussion implies that P'' is irreducible of L–degree 1 or d. Since $\mathbb{W}$ is a Euclidean domain, we can write $P = QP'' + R$ where $R \neq 0$ and $\deg_L(R) < \deg_L(P'')$. It follows that $Rf = b$, thus $(L-1)(b^{-1}R)f = 0$. By the choice of P'', it follows that P'' is a right factor of $(L-1)(b^{-1}R)$.

Case 1 $(\deg_L(P'')=1)$ Then $\deg_L(R) = 0$ and f satisfies $Rf = b$ contrary to the hypothesis.

Case 2 $(\deg_L(P'')=d)$ Then P'' is irreducible and is a right factor of $(L-1)(b^{-1}R)$ where $\deg_L(b^{-1}R) < d$. It follows that any factorization of $b^{-1}R$, extended to a factorization of $(L-1)(b^{-1}R)$, will contain an irreducible factor of L–degree d. This is impossible since $\deg_L(b^{-1}R) < d$. □

10 Extension to double twist knots

10.1 The A–polynomial of double twist knots

The $\mathrm{SL}(2,\mathbb{C})$ character variety of nonabelian representations of $K_{p,p}$ for $p > 1$ consists of two components, the geometric one, and the nongeometric one; see Petersen [33]. It follows that the A–polynomial of $K_{p,p}$ is the product of two factors, with multiplicities. The values of $A_{K_{p,p}}(M,L)$ for $p = 2,\dots,8$, as well as the recurrences presented in Section 10.2, are available from [9]. For $p = 2,\dots,8$ we have that

$$A_{K_{p,p}}(M,L) = A^{\mathrm{geom}}_{K_{p,p}}(M,L)\, A^{\mathrm{ngeom}}_{K_{p,p}}(M,L)^2$$

is the product of two irreducible factors: the geometric component has (M,L)–degree $(2p-1, 8p-2)$ and multiplicity one, and the nongeometric one has (M,L)–degree $(p^2-p, 4p^2-4)$ and multiplicity two. The Newton polygons of $A^{\mathrm{geom}}_{K_{p,p}}$ and $A^{\mathrm{ngeom}}_{K_{p,p}}$ are parallelograms given by the convex hull of

$$\bigl\{(2p-1, 8p-2), (1, 8p-2), (0,0), (2p-2, 0)\bigr\}$$

and

$$\bigl\{(p^2-p, 4p^2-4p), (p-1, 4p^2-4p), (0,0), (p^2-2p+1, 0)\bigr\}$$

respectively in (M,L)–coordinates. The area of the above Newton polygons is $4(4p-1)(p-1)$ and $4p(p-1)^3$, respectively. The behavior of the Newton polygon of $A_{p,p}(M,L)$ as a function of p is in agreement with a theorem of [10].

10.2 The noncommutative A–polynomial of double twist knots

We have rigorously computed an inhomogeneous recurrence for the double twist knot $K_{3,3}$, using the creative telescoping algorithm proposed in Koutschan [24]; see also Section 3. It has order 11 and its (q, q^n)–degree is $(458, 74)$. Moreover, it verifies the AJ conjecture using the reduced A–polynomial. The corresponding operator factors for $q = 1$ into two irreducible factors of L–degrees 5 and 6. In order to show the irreducibility of the operator itself (to prove that the computed recurrence is of minimal order), we would have to investigate its fifth and sixth exterior powers, a challenge that currently seems hopeless.

For $K_{4,4}$ and $K_{5,5}$ we were able to obtain recurrences, using an ansatz with undetermined coefficients ("guessing"). Although they were derived in a nonrigorous way, they both confirm the AJ conjecture using the reduced A–polynomial. Again, both recurrences are inhomogeneous; the one for $K_{4,4}$ has order 19 and (q, q^n)–degree $(2045, 184)$, the one for $K_{5,5}$ is a truly gigantic one: it is of order 29, has (q, q^n)–degree $(6922, 396)$, and its total size is nearly 8GB (according to Mathematica's `ByteCount`). These data qualify it as a good candidate for the largest q–difference equation that has ever been computed explicitly. A rigorous derivation of these two recurrences using creative telescoping, or even the application of the irreducibility criterion using exterior powers, is far beyond our current computing abilities.

Acknowledgments

The authors wish to thank Marko Petkovšek for sharing his expertise on the qHyper algorithm and the factorization of linear operators. This work was supported in part by NSF grant number DMS-0805078.

Appendix: Formula for the noncommutative A–polynomial of 7_4

In the following, Equation (2) from Theorem 1.2 is given explicitly; note that the operator $P_{7_4}(q, M, L) = \sum_{j=0}^{5} a_j(q, M) L^j$ is palindromic since

$$a_j(q, M) = -q^{60} M^{24} a_{5-j}(q, (q^5 M)^{-1})$$

(and therefore only a_5, a_4, and a_3 are displayed).

$$\begin{aligned}
a_5 = {} & (qM-1)(qM+1)(qM^2-1)(q^2M-1)(q^2M+1)(q^3M^2-1)(q^5M-1) \\
& \times \Big(q^8(q+1)M^4 - q^5(q^3+2q^2+q+1)M^3 \\
& \qquad + q^2(2q^4+q^3+2q^2+2q+1)M^2 \\
& \qquad - q(q^3+2q^2+q+1)M + (q+1)\Big)
\end{aligned}$$

$$\begin{aligned}a_4 = {} & q(qM-1)(qM+1)(qM^2-1)(q^3M^2-1)(q^4M-1)^2(q^4M+1)\\ & \times\Big(q^{33}(q+1)M^{11}-q^{29}(q+2)(q^3+q+1)M^{10}+q^{24}(q+1)\big(2q^6-2q^5+5q^4+q^3\\ & \quad +4q^2+3q-1\big)M^9-q^{20}\big(4q^7+2q^6+9q^5+10q^4+6q^3+6q^2-q-2\big)M^8\\ & \quad -q^{16}\big(2q^{11}+q^9-2q^8-4q^7-12q^5-10q^4-3q^3+6q+3\big)M^7+q^{12}\big(q^{13}+2q^{12}\\ & \quad +5q^{11}+q^{10}+4q^9-2q^7-8q^5+q^4+7q^3+7q^2+7q+2\big)M^6-q^9\big(q^{13}+3q^{12}+8q^{11}\\ & \quad +8q^{10}+q^9+4q^8+q^7+3q^6+q^5-4q^4+7q^3+10q^2+7q+3\big)M^5+q^6\big(4q^{12}+7q^{11}\\ & \quad +9q^{10}+4q^9-2q^8+q^7-4q^6-3q^5-3q^4-q^3+5q^2+4q+2\big)M^4-q^5\big(q^{10}+5q^9\\ & \quad +6q^8+3q^7-7q^6-10q^5-7q^4-9q^3-9q^2-9q-3\big)M^3+q^2(q^2+q+1)\big(q^7+2q^6\\ & \quad -5q^5-5q^4-3q^3-2q^2-3q-2\big)M^2+q\big(q^5+6q^4+9q^3+8q^2+3q+2\big)M\\ & \quad -(q+1)(q+2)\Big)\end{aligned}$$

$$\begin{aligned}a_3 = {} & -q^2(qM-1)(qM+1)(qM^2-1)(q^3M-1)^2(q^3M+1)(q^9M^2-1)\times\Big(q^{41}(q+1)M^{15}\\ & -q^{37}\big(q^4+2q^3+3q^2+4q+1\big)M^{14}+q^{34}\big(q^5+q^4+7q^3+9q^2+8q+3\big)M^{13}\\ & +q^{29}\big(q^9+2q^8-2q^7-2q^6-10q^5-17q^4-12q^3-3q^2+2q+1\big)M^{12}\\ & -q^{25}(2q^{11}+4q^{10}+5q^9+4q^8-3q^7-11q^6-17q^5-11q^4+2q^3+8q^2+5q+1)M^{11}\\ & +q^{22}\big(6q^{11}+12q^{10}+8q^9+8q^8-14q^6-19q^5-6q^4+11q^3+16q^2+9q+2\big)M^{10}\\ & +q^{18}\big(2q^{14}-2q^{13}-9q^{12}-17q^{11}-11q^{10}+10q^8+20q^7+24q^6+7q^5-15q^4-20q^3\\ & -10q^2+1\big)M^9-q^{15}\big(q^{15}+6q^{14}-3q^{13}-14q^{12}-14q^{11}-4q^{10}+11q^9+25q^8+36q^7\\ & +35q^6+16q^5-9q^4-13q^3-6q^2+3q+3\big)M^8+q^{12}\big(4q^{15}+6q^{14}-3q^{13}-18q^{12}\\ & -16q^{11}+4q^{10}+23q^9+30q^8+39q^7+31q^6+12q^5-14q^4-14q^3-q^2+3q+3\big)M^7\\ & -q^9\big(5q^{15}+3q^{14}-11q^{13}-23q^{12}-18q^{11}+2q^{10}+19q^9+20q^8+21q^7+8q^6-7q^5\\ & -20q^4-22q^3-5q^2+q+1\big)M^6+q^8(q+1)\big(2q^{12}-4q^{11}-13q^{10}-17q^9-q^8+2q^7\\ & +11q^6-2q^5+5q^4-9q^3-13q^2-12q-6\big)M^5+q^5\big(5q^{12}+16q^{11}+25q^{10}+11q^9\\ & -8q^8-19q^7-16q^6-4q^5-2q^4+6q^3+11q^2+5q+1\big)M^4-q^4\big(2q^{10}+10q^9+9q^8\\ & -3q^7-22q^6-23q^5-20q^4-13q^3-6q^2-3q+1\big)M^3+q^2(q+1)\big(2q^7-4q^6-6q^5-17q^4\\ & -6q^3-6q^2-2q-1\big)M^2+q\big(2q^5+8q^4+11q^3+10q^2+3q+1\big)M-(q+1)(2q+1)\Big)\end{aligned}$$

$$\begin{aligned}b_{7_4} = {} & -q^{10}M^3(qM+1)(q^2M+1)(q^3M+1)(q^4M+1)(qM^2-1)(q^3M^2-1)(q^5M^2-1)(q^7M^2-1)\\ & \times(q^9M^2-1)\Big(q^{10}(q^3+q^2-q+1)M^4-q^6(2q^5+2q^3+q^2-q+1)M^3+q^2(q+1)\\ & \times(q^7-2q^6+4q^5-q^4+q^3+q^2-q+1)M^2-q(2q^5+2q^3+q^2-q+1)M+(q^3+q^2-q+1)\Big)\end{aligned}$$

When q is set to 1, the above expressions simplify drastically. For a concise presentation we introduce the following notation for some frequently appearing irreducible factors:

$$\begin{aligned}v_1 &= M^4-M^3-2M^2-M+1\\ v_2 &= M^4-2M^3+6M^2-2M+1\\ v_3 &= 2M^4-5M^3+8M^2-5M+2\end{aligned}$$

$$
\begin{aligned}
v_4 &= M^7 - 2M^6 + 3M^5 + 2M^4 - 7M^3 + 2M^2 + 6M - 2\\
v_5 &= M^8 - 2M^7 + 6M^6 + 2M^5 - 10M^4 + 2M^3 + 6M^2 - 2M + 1\\
v_6 &= M^{12} - 6M^{11} + 16M^{10} - 24M^9 + 15M^8 + 14M^7 - 36M^6 + 14M^5 + 15M^4 - 24M^3\\
&\quad + 16M^2 - 6M + 1\\
v_7 &= 2M^{14} - 10M^{13} + 16M^{12} - 4M^{11} - 46M^{10} + 67M^9 + 28M^8 - 116M^7 + 28M^6 + 67M^5\\
&\quad - 46M^4 - 4M^3 + 16M^2 - 10M + 2\\
v_8 &= M^{18} - 4M^{17} + 10M^{16} - 10M^{15} - 3M^{14} + 40M^{13} - 67M^{12} - 34M^{11} + 157M^{10} - 14M^9\\
&\quad - 140M^8 + 40M^7 + 66M^6 - 18M^5 - 14M^4 + 4M^3 + 4M^2 - 4M + 1\\
v_9 &= M^{26} - 8M^{25} + 42M^{24} - 142M^{23} + 345M^{22} - 554M^{21} + 521M^{20} + 51M^{19} - 729M^{18}\\
&\quad + 827M^{17} + 234M^{16} - 843M^{15} + 707M^{14} - 45M^{13} + 707M^{12} - 843M^{11} + 234M^{10}\\
&\quad + 827M^9 - 729M^8 + 51M^7 + 521M^6 - 554M^5 + 345M^4 - 142M^3 + 42M^2 - 8M + 1
\end{aligned}
$$

Now the inhomogeneous part b_{7_4} and the operator P_{7_4}, together with its second and third exterior power, evaluated at $q=1$, can be written in a few lines. A bar is used to denote the mirror of a polynomial, ie $\overline{v} = M^{\deg(v)} v(1/M)$.

$$
\begin{aligned}
b_{7_4}(1,M) &= -M^3(M-1)^5(M+1)^9 v_3\\
P_{7_4}(1,M,L) &= (M-1)^5(M+1)^4 v_3 (L^2 - v_1 L + M^4)(L^3 + v_4 L^2 + \overline{v_4} L + M^7)\\
P^{(2)}_{7_4}(1,M,L) &= (M-1)^{10}(M+1)^{10}(M^2+1)^2 v_2 v_3^4 v_5 v_6 v_9 (L - M^4)(L^3 - \overline{v_4} L^2 + M^7 v_4 L - M^{14})\\
&\quad \times \left(L^6 + v_1 v_4 L^5 + v_8 L^4 - M^4 v_1 v_7 L^3 + M^8 \overline{v_8} L^2 + M^{15} v_1 \overline{v_4} L + M^{26}\right)\\
P^{(3)}_{7_4}(1,M,L) &= (M-1)^{19}(M+1)^{20}(M^2+1)^2 v_2 v_3^6 v_5 v_6 v_9 (L + M^7)\\
&\quad \times (L^3 + M^4 v_4 L^2 + M^8 \overline{v_4} L + M^{19})\\
&\quad \times \left(L^6 - v_1 \overline{v_4} L^5 + M^4 \overline{v_8} L^4 + M^{11} v_1 v_7 L^3 + M^{18} v_8 L^2 - M^{29} v_1 v_4 L + M^{40}\right)
\end{aligned}
$$

References

[1] **S A Abramov**, **P Paule**, **M Petkovšek**, *q–hypergeometric solutions of q–difference equations*, from: "Proceedings of the 7th Conference on Formal Power Series and Algebraic Combinatorics", Discrete Math. 180 (1998) 3–22 MR1603685

[2] **D Bar-Natan**, *The Knot Atlas* Available at `http://www.katlas.org`

[3] **M Bronstein**, *On the factorisation of linear ordinary differential operators*, from: "Symbolic computation, new trends and developments", Math. Comput. Simulation 42 (1996) 387–389 MR1430826

[4] **M Bronstein**, **M Petkovšek**, *An introduction to pseudo-linear algebra*, from: "Algorithmic complexity of algebraic and geometric models", Theoret. Comput. Sci. 157 (1996) 3–33 MR1383396

[5] **F Chyzak**, *An extension of Zeilberger's fast algorithm to general holonomic functions*, from: "Formal power series and algebraic combinatorics", Discrete Math. 217 (2000) 115–134 MR1766263

[6] **T Cluzeau**, **M van Hoeij**, *Computing hypergeometric solutions of linear recurrence equations*, Appl. Algebra Engrg. Comm. Comput. 17 (2006) 83–115 MR2233774

[7] **D Cooper**, **M Culler**, **H Gillet**, **D D Long**, **P B Shalen**, *Plane curves associated to character varieties of* 3*–manifolds*, Invent. Math. 118 (1994) 47–84 MR1288467

[8] **J A De Loera**, **J Rambau**, **F Santos**, *Triangulations*, Algorithms and Computation in Mathematics 25, Springer, Berlin (2010) MR2743368

[9] **S Garoufalidis**, *The classical and quantum A–polynomial of double twist knots* Available at the author's website: `http://www.math.gatech.edu/~stavros/publications/double.twist.data`

[10] **S Garoufalidis**, *The Newton polygon of a recurrence sequence of polynomials and its role in TQFT* `arXiv:1102.1346`

[11] **S Garoufalidis**, *Quantum knot invariants* `arXiv:1201.3314`

[12] **S Garoufalidis**, *On the characteristic and deformation varieties of a knot*, from: "Proceedings of the Casson Fest", Geom. Topol. Monogr. 7 (2004) 291–309 MR2172488

[13] **S Garoufalidis**, *The degree of a q–holonomic sequence is a quadratic quasi-polynomial*, Electron. J. Combin. 18 (2011) 23 MR2795781

[14] **S Garoufalidis**, *The Jones slopes of a knot*, Quantum Topol. 2 (2011) 43–69 MR2763086

[15] **S Garoufalidis**, **T T Q Lê**, *The colored Jones function is q–holonomic*, Geom. Topol. 9 (2005) 1253–1293 MR2174266

[16] **S Garoufalidis**, **X Sun**, *The C–polynomial of a knot*, Algebr. Geom. Topol. 6 (2006) 1623–1653 MR2253459

[17] **S Garoufalidis**, **X Sun**, *The non-commutative A–polynomial of twist knots*, J. Knot Theory Ramifications 19 (2010) 1571–1595 MR2755491

[18] **R Gelca**, *On the relation between the A–polynomial and the Jones polynomial*, Proc. Amer. Math. Soc. 130 (2002) 1235–1241 MR1873802

[19] **K Habiro**, *A unified Witten–Reshetikhin–Turaev invariant for integral homology spheres*, Invent. Math. 171 (2008) 1–81 MR2358055

[20] **M van Hoeij**, *Factorization of differential operators with rational functions coefficients*, J. Symbolic Comput. 24 (1997) 537–561 MR1484068

[21] **P Horn**, *Faktorisierung in Schief-Polynomringen*, PhD thesis, Universität Kassel (2008)

[22] **J C Jantzen**, *Lectures on quantum groups*, Graduate Studies in Mathematics 6, Amer. Math. Soc. (1996) MR1359532

[23] **C Koutschan**, *Advanced applications of the holonomic systems approach*, PhD thesis, Johannes Kepler University (2009)

[24] **C Koutschan**, *A fast approach to creative telescoping*, Math. Comput. Sci. 4 (2010) 259–266 MR2775992

[25] **C Koutschan**, *HolonomicFunctions (user's guide)*, RISC report 10-01, Johannes Kepler University, Linz, Austria (2010)

[26] **M R Lauridsen**, *Aspects of quantum mathematics, Hitchin connections and AJ conjectures*, PhD thesis, Aarhus University (2010)

[27] **T T Q Lê**, *The colored Jones polynomial and the A–polynomial of knots*, Adv. Math. 207 (2006) 782–804 MR2271986

[28] **T T Q Lê**, **A T Tran**, *On the AJ conjecture for knots* `arXiv:1111.5258`

[29] **I G Macdonald**, *Symmetric functions and Hall polynomials*, 2nd edition, Oxford University Press (1995) MR1354144

[30] **G Masbaum**, *Skein-theoretical derivation of some formulas of Habiro*, Algebr. Geom. Topol. 3 (2003) 537–556 MR1997328

[31] **O Ore**, *Theory of non-commutative polynomials*, Ann. of Math. 34 (1933) 480–508 MR1503119

[32] **P Paule**, **A Riese**, *A Mathematica q–analogue of Zeilberger's algorithm based on an algebraically motivated approach to q–hypergeometric telescoping*, from: "Special functions, q–series and related topics", (M E H Ismail, D R Masson, M Rahman, editors), Fields Inst. Commun. 14, Amer. Math. Soc. (1997) 179–210 MR1448687

[33] **K L Petersen**, *A–polynomials of a family of 2–bridge knots*, preprint (2011) Available at `http://www.math.fsu.edu/~petersen/apoly.pdf`

[34] **M Petkovšek**, *Hypergeometric solutions of linear recurrences with polynomial coefficients*, J. Symbolic Comput. 14 (1992) 243–264 MR1187234

[35] **M Petkovšek**, **H S Wilf**, **D Zeilberger**, $A = B$, A K Peters Ltd, Wellesley, MA (1996) MR1379802

[36] **M van der Put**, **M F Singer**, *Galois theory of linear differential equations*, Grundl. Math. Wissen. 328, Springer, Berlin (2003) MR1960772

[37] **D Rolfsen**, *Knots and links*, Mathematics Lecture Series 7, Publish or Perish Inc, Houston, TX (1990) MR1277811

[38] **F Schwarz**, *A factorization algorithm for linear ordinary differential equations*, from: "ISSAC 1989 — Proceedings of the 1989 International Symposium on Symbolic and Algebraic Computation", (Y N Lakshman, editor), ACM Press, New York (1989) 17–25

[39] **J R Stembridge**, *The SF package* (2005) `http://www.math.lsa.umich.edu/~jrs/maple.html#SF`

[40] **S P Tsarev**, *An algorithm for complete enumeration of all factorizations of a linear ordinary differential operator*, from: "ISSAC 1996 — Proceedings of the 1996 International Symposium on Symbolic and Algebraic Computation", (Y N Lakshman, editor), ACM Press, New York (1996) 226–231

[41] **V G Turaev**, *The Yang–Baxter equation and invariants of links*, Invent. Math. 92 (1988) 527–553 MR939474

[42] **V G Turaev**, *Quantum invariants of knots and 3–manifolds*, de Gruyter Studies in Mathematics 18, Walter de Gruyter & Co., Berlin (1994) MR1292673

[43] **H S Wilf**, **D Zeilberger**, *An algorithmic proof theory for hypergeometric (ordinary and "q") multisum/integral identities*, Invent. Math. 108 (1992) 575–633 MR1163239

[44] **D Zeilberger**, *A fast algorithm for proving terminating hypergeometric identities*, Discrete Math. 80 (1990) 207–211 MR1048463

[45] **D Zeilberger**, *A holonomic systems approach to special functions identities*, J. Comput. Appl. Math. 32 (1990) 321–368 MR1090884

[46] **D Zeilberger**, *The method of creative telescoping*, J. Symbolic Comput. 11 (1991) 195–204 MR1103727

[47] **G M Ziegler**, *Lectures on polytopes*, Graduate Texts in Mathematics 152, Springer, New York (1995) MR1311028

School of Mathematics, Georgia Institute of Technology
686 Cherry Street, Atlanta, GA 30332-0160, USA

Johann Radon Institute for Computational and Applied Mathematics (RICAM)
Austrian Academy of Sciences, Altenberger Straße 69, A-4040 Linz, Austria

stavros@math.gatech.edu, christoph.koutschan@ricam.oeaw.ac.at

http://www.math.gatech.edu/~stavros, http://www.koutschan.de

Received: 26 November 2012