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Connectivity of motivic H–spaces

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In this note we prove that the $\mathbb{A}^1$–connected component sheaf $a_{\mathrm{Nis}}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))$ of an H–group $\mathcal{X}$ is $\mathbb{A}^1$–invariant.

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1 Introduction

Let Sm/k denote the category of smooth, separated k–schemes and let $\mathrm{PSh}(\mathrm{Sm}/k)$ denote the category of presheaves of sets on Sm/k. A functor $\mathcal{X}\colon \triangle^{\mathrm{op}} \to \mathrm{PSh}(\mathrm{Sm}/k)$ is called a simplicial presheaf or a space. Here $\triangle$ is the category of simplices. Let $\triangle^{\mathrm{op}}\,\mathrm{PSh}(\mathrm{Sm}/k)$ denote the category of spaces.

$\triangle^{\mathrm{op}}\,\mathrm{PSh}(\mathrm{Sm}/k)$ has a local model category structure with respect to the Nisnevich topology called the injective Nisnevich model structure. A morphism $f\colon \mathcal{X} \to \mathcal{Y}$ is a weak equivalence if the induced morphism on the Nisnevich stalks are weak equivalences of simplicial sets. Cofibrations are sectionwise injective morphisms and fibrations are defined using the right lifting property (see Jardine [6], and Morel and Voevodsky [11]). The resulting homotopy category is denoted by $\mathbf{H}_s(\mathrm{Sm}/k)$.

The Bousfield localisation of the local model structure on $\triangle^{\mathrm{op}}\,\mathrm{PSh}(\mathrm{Sm}/k)$ with respect to the class of maps $\mathcal{X}\times\mathbb{A}^1 \to \mathcal{X}$ is called the $\mathbb{A}^1$–model structure (the $\mathbb{A}^1$–model structure for simplicial sheaves on Sm/k described in [11] was extended to simplicial presheaves in Jardine [7]). The resulting homotopy category is denoted by $\mathbf{H}(k)$.

For any space $\mathcal{X}$, define $\pi_0^{\mathbb{A}^1}(\mathcal{X})$ to be the presheaf

$$U \in \mathrm{Sm}/k \mapsto \mathrm{Hom}_{\mathbf{H}(k)}(U,\mathcal{X}).$$

The presheaf $\pi_0^{\mathbb{A}^1}(\mathcal{X})$ is homotopy invariant, ie, for any $U \in \mathrm{Sm}/k$ the morphism

$$\pi_0^{\mathbb{A}^1}(\mathcal{X})(U) \to \pi_0^{\mathbb{A}^1}(\mathcal{X})(\mathbb{A}^1_U),$$

induced by the projection $\mathbb{A}^1_U \to U$, is bijective.

Let $a_{\mathrm{Nis}}\colon \mathrm{PSh}(\mathrm{Sm}/k) \to \mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}/k)$ denote the Nisnevich sheafification functor. The following conjecture of Morel states that the above property remains true after Nisnevich sheafification.

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Conjecture 1.1 For any $U \in \mathrm{Sm}/k$, the morphism

$$a_{\mathrm{Nis}}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))(U) \longrightarrow a_{\mathrm{Nis}}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))(\mathbb{A}^1_U),$$

induced by the projection $\mathbb{A}^1_U \to U$, is bijective.

In this paper, we prove the conjecture (Theorem 4.18) for H–loops with inverse (Definition 4.1) and homogeneous spaces on these (see Definitions 4.5, 4.6). For a general space, Corollary 3.2 gives a partial answer to the conjecture.

The conjecture is trivially true for $\mathbb{A}^1$–rigid smooth k–schemes and for $\mathbb{A}^1$–connected smooth k–schemes (Asok and Morel [1, Definition 2.1.4, Definition 2.1.8, Lemma 2.1.9] and Morel and Voevodsky [11, Section 3, Example 2.4]). So for smooth proper curves and smooth proper k–rational surfaces the conjecture is true (see Asok and Morel [1, Section 2.3], and Morel [10, Remark 13] for more examples). By the work of Morel [9, Theorem A.2] and Wendt [12, Section 5.2], the conjecture is true for split linear algebraic groups.

Voevodsky proved (see Mazza, Voevodsky and Weibel [8, Theorem 22.3]) that for any homotopy invariant presheaf with transfers S, the sheafification $a_{\mathrm{Nis}}(S)$ is a homotopy invariant sheaf with transfers. The proof is quite hard. It becomes harder if we consider general homotopy invariant presheaves (without transfers). For any homotopy invariant presheaf of sets S on Sm/k, one can ask to which extent the analogue of Voevodsky's result is true for S. Our results in this paper show that if S is a presheaf of groups, the canonical morphism $S \to a_{\mathrm{Nis}}(\pi_0^{\mathbb{A}^1}(S))$ is universal among all the morphisms from S to homotopy invariant sheaves of sets.

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2 Homotopy limits and colimits

In this section we prove some basic results on homotopy limits and colimits.

Let I be a small category. There is a functor $(I/-)\colon I \to \mathrm{Cat}$ such that for any $i \in I$, $(I/-)(i) = I/i$. Here Cat is the category of small categories and I/i is the over category. There is a functor $N\colon \mathrm{Cat} \to \Delta^{\mathrm{op}}\,\mathrm{Sets}$, such that for any $J \in \mathrm{Cat}$, the simplicial set $N(J)$ is the nerve of the category J. Define $N(I/-) := N \circ (I/-)$.

A set S will be considered as a simplicial set in the obvious way: in every simplicial degree it is given by S and faces and degeneracies are identities. These simplicial sets are called discrete simplicial sets. We get a functor $\iota\colon \mathrm{Sets} \to \triangle^{\mathrm{op}}\mathrm{Sets}$. For $S \in \mathrm{Sets}$ and $S'_\bullet \in \triangle^{\mathrm{op}}\mathrm{Sets}$, the set of maps $\mathrm{Hom}(S'_\bullet, \iota(S))$ is in bijection with the set of maps $f\colon S'_0 \to S$ such that $f \circ d_0 = f \circ d_1\colon S'_1 \to S$. Also note that

$$\pi_0(S'_\bullet) = \mathrm{colim}(S'_1 \underset{d_0}{\overset{d_1}{\rightrightarrows}} S'_0).$$

Therefore $\mathrm{Hom}(S'_\bullet, \iota(S)) = \mathrm{Hom}(\pi_0(S'_\bullet), S)$ and π_0 is left adjoint to ι.

For any $X\colon I \to \triangle^{\mathrm{op}}\mathrm{Sets}$, the canonical maps $I/i \to \bullet$ induce a natural map $\lim_I X \to \mathrm{holim}_I X$ (Bousfield and Kan [2, Chapter XI 3.5]).

Lemma 2.1 *Let* $X\colon I \to \triangle^{\mathrm{op}}\mathrm{Sets}$ *be a diagram of discrete simplicial sets. Then the canonical map* $\lim_I X \to \mathrm{holim}_I X$ *is a bijection of discrete simplicial sets.*

Proof By adjointness [2, Chapter XI 3.3]

$$\mathrm{Hom}(\triangle^n \times N(I/-), X) = \mathrm{Hom}(\triangle^n, \mathrm{holim}_I X).$$

The functor $\pi_0\colon (\triangle^{\mathrm{op}}\mathrm{Sets})^I \to (\mathrm{Sets})^I$ is left adjoint to the functor $\iota\colon (\mathrm{Sets})^I \to (\triangle^{\mathrm{op}}\mathrm{Sets})^I$, where ι maps a diagram of sets to the same diagram of discrete simplicial sets. Hence $\mathrm{Hom}(\triangle^n \times N(I/-), X) = \mathrm{Hom}(\bullet_I, X)$, where $\bullet_I$ is the diagram of sets given by the one element set for each $i \in I$. But $\mathrm{Hom}(\bullet_I, X) = \mathrm{Hom}(\bullet, \lim_I X)$, by adjointness. Therefore, we get our result. □

In the language of derived functors, holim_I is defined as the right derived functor of the functor $\lim_I$. Any diagram of discrete simplicial sets is already fibrant in the diagram category (projective model structure), therefore we do not need to derive anything and hence the map $\lim_I X \to \mathrm{holim}_I X$ is the identity map in this case.

Let $X\colon I \to \triangle^{\mathrm{op}}\mathrm{Sets}$ be a diagram such that each $X(i)$ is fibrant for all $i \in I$. The canonical morphism $X(i) \to \pi_0(X(i))$ induces a morphism

$$\mathrm{holim}_I(X) \longrightarrow \mathrm{holim}_I \pi_0(X).$$

By inverting the bijection of Lemma 2.1 and applying π_0, we get the following morphism:

$$\pi_0(\mathrm{holim}_I(X)) \longrightarrow \lim_I \pi_0(X). \tag{2-1}$$

Lemma 2.2 *Suppose that I is the pullback category $1 \to 0 \leftarrow 2$ and let $D\colon I \to \triangle^{op}\,\mathrm{Sets}$ be a diagram*

$$X \xrightarrow{p} Y \xleftarrow{q} Z$$

such that X, Y, Z are fibrant. Then the map (2-1) *is surjective.*

Proof By [2, Chapter XI 4.1.(iv), 5.6], $\mathrm{holim}_I(X) \cong X' \times_Y Z$, where

$$X \longrightarrow X' \xleftarrow{p'} Y$$

is a factorisation of p into a trivial cofibration followed by a fibration p'. Since $\pi_0(X) \cong \pi_0(X')$, it is enough to show that

$$\pi_0(X' \times_Y Z) \longrightarrow \pi_0(X') \times_{\pi_0(Y)} \pi_0(Z)$$

is surjective. So we can assume that p is a fibration. Let $s \in \pi_0(X) \times_{\pi_0(Y)} \pi_0(Z)$. s can be represented (not uniquely) by (x, y, z), where $(x, z) \in X_0 \times Z_0$ and $y \in Y_1$ such that $d_0(y) = p(x)$ and $d_1(y) = q(z)$. Since p is a fibration, we can lift the path y to a path $y' \in X_1$ such that $d_0(y') = x$ and $x' := d_1(y')$ maps to $q(z)$. $\mathrm{holim}_I D \cong X \times_Y Z$. Therefore $(x', z) \in \mathrm{holim}_I D$ which maps to s. This proves the surjectivity. □

Under the condition of Lemma 2.2, the map (2-1) may not be injective. Indeed, if Y is connected, X is the universal cover of Y and $Z = \bullet$, then (2-1) is injective if and only if Y is simply connected.

Lemma 2.3 *Let $B \underset{g}{\overset{f}{\rightrightarrows}} A$ be a diagram of spaces. Then:*

$$a_{\mathrm{Nis}}(\pi_0(\mathrm{hocolim}(B \underset{g}{\overset{f}{\rightrightarrows}} A))) \cong \mathrm{colim}(a_{\mathrm{Nis}}(\pi_0(B)) \underset{g}{\overset{f}{\rightrightarrows}} a_{\mathrm{Nis}}(\pi_0(A))).$$

Proof Let $S \in \mathrm{PSh}(\mathrm{Sm}/k)$ and let $\iota(S)$ be the simplicial presheaf such that in every simplicial degree k, $\iota(S)_k = S$. The face and degeneracy morphisms are identity morphisms. This gives a functor $\iota\colon \mathrm{PSh}(\mathrm{Sm}/k) \to \triangle^{op}\,\mathrm{PSh}(\mathrm{Sm}/k)$, which is right adjoint to π_0. Hence $a_{\mathrm{Nis}}(\pi_0)$ also has a right adjoint $\iota\colon \mathrm{Sh}(\mathrm{Sm}/k) \to \triangle^{op}\,\mathrm{Sh}(\mathrm{Sm}/k)$. This implies $a_{\mathrm{Nis}}(\pi_0)$ commutes with colimits. In our case,

$$\mathrm{hocolim}(B \underset{g}{\overset{f}{\rightrightarrows}} A) = \mathrm{colim}(B \underset{g'}{\overset{f'}{\rightrightarrows}} A').$$

Here, we get A' by taking the factorisation

$$B \amalg B \xrightarrow{h} A' \xrightarrow{h'} A,$$

such that the composition is $f \amalg g$, h is a cofibration and h' is a trivial fibration. Let $e_i\colon B \to B \amalg B$ be the inclusion in the i^{th} component for $i = 1$ or 2. Then

$f' := h \circ e_1$ and $g' := h \circ e_2$. As $a_{\mathrm{Nis}}\pi_0$ commutes with colimits and $a_{\mathrm{Nis}}\pi_0(A') \cong a_{\mathrm{Nis}}\pi_0(A)$, we get our result. □

If

$$\begin{array}{ccc} B & \longrightarrow & C \\ \downarrow & & \downarrow \\ A & \longrightarrow & D \end{array}$$

is a homotopy co-Cartesian square of spaces then, after applying $a_{\mathrm{Nis}}(\pi_0)$, one gets a co-Cartesian square of sheaves. Indeed, let

$$B \xrightarrow{f} A' \xrightarrow{g} A$$

be a factorisation of $B \to A$, such that f is a cofibration and g is a trivial fibration. The homotopy colimit of the diagram $A \leftarrow B \to C$ is weakly equivalent to the colimit of $A' \leftarrow B \to C$. Similarly we get our result as $a_{\mathrm{Nis}}(\pi_0)$ commutes with colimits and $a_{\mathrm{Nis}}(\pi_0(A)) \cong a_{\mathrm{Nis}}(\pi_0(A'))$.

3 Generalities on the Nisnevich local model structure

In this section we briefly recall the Nisnevich Brown–Gersten property and give some consequences on the π_0 functor.

Recall [11, Definition 3.1.3] that a Cartesian square in Sm/k,

$$\begin{array}{ccc} W & \longrightarrow & V \\ \downarrow & & \downarrow p \\ U & \xrightarrow{i} & X, \end{array}$$

is called an elementary distinguished square (in the Nisnevich topology), if p is an étale morphism and i is an open embedding such that $p^{-1}(X-U) \to (X-U)$ is an isomorphism (endowing these closed subsets with the reduced subscheme structure). Moreover, if p is an open embedding then the above Cartesian square is an elementary distinguished square in the Zariski topology. An elementary distinguished square in the Zariski topology is an elementary distinguished square in the Nisnevich topology.

A space $\mathcal{X}$ is said to satisfy the Nisnevich (resp. Zariski) Brown–Gersten property if for any elementary distinguished square in the Nisnevich topology (resp. Zariski topology) as above, the induced square of simplicial sets

$$\begin{array}{ccc} \mathcal{X}(X) & \longrightarrow & \mathcal{X}(V) \\ \downarrow & & \downarrow \\ \mathcal{X}(U) & \longrightarrow & \mathcal{X}(W) \end{array}$$

is homotopy Cartesian (see [11, Definition 3.1.13]).

Any fibrant space for the Nisnevich local model structure satisfies the Nisnevich (and therefore Zariski) Brown–Gersten property [11, Remark 3.1.15].

A space is $\mathbb{A}^1$–fibrant if and only if it is fibrant in the local model structure and $\mathbb{A}^1$–local [11, Proposition 2.3.19].

There exist endofunctors Ex (resp. $\mathrm{Ex}_{\mathbb{A}^1}$) of $\triangle^{\mathrm{op}}\,\mathrm{PSh}(\mathrm{Sm}/k)$ such that for any space $\mathcal{X}$, the object $\mathrm{Ex}(\mathcal{X})$ is fibrant (resp. $\mathrm{Ex}_{\mathbb{A}^1}\,\mathcal{X}$ is $\mathbb{A}^1$–fibrant). Moreover, there exists a natural morphism $\mathcal{X} \to \mathrm{Ex}(\mathcal{X})$ (resp. $\mathcal{X} \to Ex_{\mathbb{A}^1}(\mathcal{X})$), which is a local weak equivalence (resp. $\mathbb{A}^1$–weak equivalence) [11, Remark 3.2.5, Lemma 3.2.6, Theorem 2.1.66].

For the injective local model structure all spaces are cofibrant. Hence for any space $\mathcal{X}$ and for any $U \in \mathrm{Sm}/k$,

$$\mathrm{Hom}_{\mathbf{H}_s(\mathrm{Sm}/k)}(U,\mathcal{X}) = \pi_0(\mathrm{Ex}(\mathcal{X})(U)). \tag{3-1}$$

Since $\mathrm{Ex}_{\mathbb{A}^1}(\mathcal{X})$ is $\mathbb{A}^1$–local,

$$\mathrm{Hom}_{\mathbf{H}(k)}(U,\mathcal{X}) = \mathrm{Hom}_{\mathbf{H}_s(\mathrm{Sm}/k)}(U,\mathrm{Ex}_{\mathbb{A}^1}(\mathcal{X})).$$

Moreover $\mathrm{Ex}_{\mathbb{A}^1}(\mathcal{X})$ is fibrant. Hence,

$$\mathrm{Hom}_{\mathbf{H}(k)}(U,\mathcal{X}) = \pi_0(\mathrm{Ex}_{\mathbb{A}^1}(\mathcal{X})(U)). \tag{3-2}$$

For any space $\mathcal{X}$, let $\pi_0(\mathcal{X})$ be the presheaf defined by

$$U \in \mathrm{Sm}/k \mapsto \mathrm{Hom}_{\mathbf{H}_s(\mathrm{Sm}/k)}(U,\mathcal{X}).$$

Theorem 3.1 *Let $\mathcal{X}$ be a space. For any $X \in \mathrm{Sm}/k$, such that $\dim(X) \le 1$, the canonical morphism*

$$\pi_0(\mathcal{X})(X) \longrightarrow a_{\mathrm{Nis}}(\pi_0(\mathcal{X}))(X)$$

is surjective.

Before giving the proof we note the following consequence.

Corollary 3.2 *For any space $\mathcal{X}$, the canonical morphism*

$$\pi_0^{\mathbb{A}^1}(\mathcal{X})(\mathbb{A}^1_F) \longrightarrow a_{\mathrm{Nis}}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))(\mathbb{A}^1_F)$$

is bijective for all finitely generated separable field extensions F/k.

Proof For any $X \in \mathrm{Sm}/k$,

$$\pi_0^{\mathbb{A}^1}(\mathcal{X})(X) = \pi_0(Ex_{\mathbb{A}^1}\mathcal{X})(X).$$

The canonical morphism

$$\pi_0^{\mathbb{A}^1}(\mathcal{X})(\mathbb{A}^1_F) \longrightarrow a_{\mathrm{Nis}}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))(\mathbb{A}^1_F)$$

is surjective (applying Theorem 3.1 for the space $\mathrm{Ex}_{\mathbb{A}^1}(\mathcal{X})$). On the other hand, consider the following commutative diagram

$$\begin{array}{ccc}
\pi_0^{\mathbb{A}^1}(\mathcal{X})(\mathbb{A}^1_F) & \longrightarrow & \pi_0^{\mathbb{A}^1}(\mathcal{X})(F) \\
\downarrow & & \downarrow \wr \\
a_{\mathrm{Nis}}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))(\mathbb{A}^1_F) & \longrightarrow & a_{\mathrm{Nis}}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))(F),
\end{array}$$

where the horizontal morphisms are induced by the zero section $s_0\colon F \to \mathbb{A}^1_F$. The top horizontal morphism is bijective by construction of $\pi_0^{\mathbb{A}^1}$ and the right vertical morphism is bijective because finitely generated separable field extensions of k are stalks in the Nisnevich topology. Hence the left vertical surjective morphism is injective. □

The proof of Theorem 3.1 depends on the relation between homotopy pullback of spaces and pullback of the presheaves of connected components of those spaces.

A Noetherian k–scheme X, which is the inverse limit of a left filtering system $(X_\alpha)_\alpha$ with each transition morphism $X_\beta \to X_\alpha$ being an étale affine morphism between smooth k–schemes, is called an essentially smooth k–scheme. For any $X \in \mathrm{Sm}/k$ and any $x \in X$, the local schemes $\mathrm{Spec}(O_{X,x})$ and $\mathrm{Spec}(O^h_{X,x})$ are essentially smooth k–schemes.

Lemma 3.3 *Let $\mathcal{X}$ be a space. For any essentially smooth discrete valuation ring R, the canonical morphism*

$$\pi_0(\mathcal{X})(R) \longrightarrow a_{\mathrm{Nis}}(\pi_0(\mathcal{X}))(R)$$

is surjective.

Proof By Equation (3-1) we can assume that $\mathcal{X}$ is fibrant.

Let $F = \mathrm{Frac}(R)$ and let R^h be the Henselisation of R at its maximal ideal. Suppose $s \in a_{\mathrm{Nis}}(\pi_0(\mathcal{X}))(R)$. Then for the image of s in $a_{\mathrm{Nis}}(\pi_0(\mathcal{X}))(R^h)$, there exists a Nisnevich neighbourhood of the closed point p: $W \to \mathrm{Spec}(R)$ and $s' \in \pi_0(\mathcal{X})(W)$, such that s' gets mapped to $s|_W \in a_{\mathrm{Nis}}(\pi_0(\mathcal{X}))(W)$. Let $L = \mathrm{Frac}(W)$. For any finitely generated separable field extension F'/k, the map $\pi_0(\mathcal{X})(F') \to a_{\mathrm{Nis}}(\pi_0(\mathcal{X}))(F')$ is bijective. Hence, $s'|_L$ is same as $s|_L$. We get two sections $s' \in \pi_0(\mathcal{X})(W)$ and $s|_F \in \pi_0(\mathcal{X})(F)$, such that $s'|_L = s|_L$. By Lemma 2.2 and the fact that $\mathcal{X}$ satisfies the Nisnevich Brown–Gersten property, we find an element $s_v \in \pi_0(\mathcal{X})(R)$ that gets mapped to s. Therefore, $\pi_0(\mathcal{X})(R) \to a_{\mathrm{Nis}}(\pi_0(\mathcal{X}))(R)$ is surjective. □

Proof of Theorem 3.1 Let $X \in \mathrm{Sm}/k$ and $\dim(X) = 1$. Let α be an element of $a_{\mathrm{Nis}}(\pi_0(\mathcal{X}))(X)$. This α gives $\alpha_p \in a_{\mathrm{Nis}}(\pi_0(\mathcal{X}))(O_{X,p})$ for every codimension-1 point $p \in X$, such that $\alpha_p|_{K(X)} = \alpha_q|_{K(X)}$, for all $p, q \in X^{(1)}$. By Lemma 3.3 the map

$$\pi_0(\mathcal{X})(O_{X,p}) \longrightarrow a_{\mathrm{Nis}}(\pi_0(\mathcal{X}))(O_{X,p})$$

is surjective. Also, the map

$$\pi_0(\mathcal{X})(K(X)) \longrightarrow a_{\mathrm{Nis}}(\pi_0(\mathcal{X}))(K(X))$$

is bijective, because finitely generated separable field extensions are stalks in the Nisnevich topology. Therefore, we get elements $\alpha'_p \in \pi_0(\mathcal{X})(O_{X,p})$ mapping to α_p, such that $\alpha'_p|_{K(X)} = \alpha'_q|_{K(X)}$ for $p, q \in X^{(1)}$.

Fix a $p \in X^{(1)}$. There exists an open set W containing p and $\beta \in \pi_0(\mathcal{X})(W)$, such that $\beta|_{O_{X,p}} = \alpha'_p$. Let $\beta' \in a_{\mathrm{Nis}}(\pi_0(\mathcal{X})(W))$ be the image of β. Suppose that $\beta' \neq \alpha|_W$, but $\beta'|_{O_{X,p}} = \alpha_p$. Hence there exists $U \subset W$, such that $\beta'|_U = \alpha|_U$.

So we can assume that there exists an open set $U \subset X$ and $\alpha' \in \pi_0(\mathcal{X})(U)$, such that α' gets mapped to $\alpha|_U$. If $U \neq X$, then there exists a codimension one point $q \in X \setminus U$. We can get an open neighborhood U_q and an element $\alpha'' \in \pi_0(\mathcal{X})(U_q)$, such that α'' gets mapped to $\alpha|_{U_q}$. But by construction of these α'', α' we know that $\alpha''|_{K(X)} = \alpha'|_{K(X)}$. Hence there exists an open set $U' \subset U_q \cap U$, such that $\alpha''|_{U'} = \alpha'|_{U'}$. Let $Z = U_q \cap U \setminus U'$. Since $\dim(X) = 1$, the set Z is finite collection of closed points. Therefore, Z is closed in U. Let $U'' = U \setminus Z$ be the open subset of U. Note that $U'' \cap U_q = U'$. Denote $U'' \cup U_q = U \cup U_q$ by V.

Let $\alpha'|_{U''} \in \pi_0(\mathcal{X})(U'')$ be the restriction of α' to U''. Hence, $\alpha'|_{U''}$ gets mapped to $\alpha|_{U''}$ and $\alpha'|_{U''}$ restricted to U' is same as α'' restricted to U'. As $\mathcal{X}$ is Nisnevich fibrant, it satisfies the Zariski Brown–Gersten property. By Lemma 2.2, we get a section $s_V \in \pi_0(\mathcal{X})(V)$ that gets mapped to $\alpha|_V$. If $V = X$, then we are done.

Otherwise, we use the same procedure and the fact that X is Noetherian to get an element $s_X \in \pi_0(\mathcal{X})(X)$ that maps to α. This finishes the proof of the theorem. □

4 H–groups and homogeneous spaces

In this section we prove $\mathbb{A}^1$–invariance of $a_{\mathrm{Nis}}(\pi_0^{\mathbb{A}^1})$ for H–groups and homogeneous spaces for H–groups.

Definition 4.1 Let $\mathcal{X}$ be a pointed space, ie, $\mathcal{X}$ is a space endowed with a morphism $x\colon \mathrm{Spec}(k) \to \mathcal{X}$. It is called an H–space if there exists a base point preserving morphism $\mu\colon (\mathcal{X} \times \mathcal{X}) \to \mathcal{X}$, such that $\mu \circ (x \times \mathrm{id}_{\mathcal{X}})$ and $\mu \circ (\mathrm{id}_{\mathcal{X}} \times x)$ are equal to $\mathrm{id}_{\mathcal{X}}$ in $\mathbf{H}(k)$. Here $\mathcal{X} \times \mathcal{X}$ is pointed by (x, x).

It is called an H–loop with inverse if:

(1) (inverse) There exists a morphism $(\,.\,)^*\colon \mathcal{X} \to \mathcal{X}$, such that $\mu \circ (\mathrm{id}_{\mathcal{X}}, (\,.\,)^*)$ and $\mu \circ ((\,.\,)^*, \mathrm{id}_{\mathcal{X}})$ are equal to the constant map $c\colon \mathcal{X} \to \mathcal{X}$ in $\mathbf{H}(k)$. Here the image of the constant map c is x.

(2) (weak associative) The composition $\mu \circ (\mathrm{id} \times \mu) \circ ((((\,.\,)^* \times \mathrm{id}) \circ d) \times \mathrm{id})$ is equal to pr_2 and $\mu \circ (\mu \times \mathrm{id}) \circ (\mathrm{id} \times ((\mathrm{id} \times (\,.\,)^*) \circ d))$ is equal to pr_1 in $\mathbf{H}(k)$. Here $d\colon \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is the diagonal map.

An H–space is called an H–group if:

(1) (inverse) It satisfies the inverse property of H–loop with inverse.

(2) (associative) $\mu \circ (\mu \times \mathrm{id}_{\mathcal{X}})$ is equal to $\mu \circ (\mathrm{id}_{\mathcal{X}} \times \mu)$ in $\mathbf{H}(k)$ modulo the canonical isomorphism $\alpha\colon \mathcal{X} \times (\mathcal{X} \times \mathcal{X}) \to (\mathcal{X} \times \mathcal{X}) \times \mathcal{X}$.

Remark 4.2 An H–group is an H–loop with inverse. There are H–loops with inverse that are not H–groups. For example, let $\mathbf{O}$ be a split octonion algebra over k. The affine quadric defined by the norm 1 elements of $\mathbf{O}$ is a smooth affine quadric that is an H–loop with inverse. However, it is not an H–group as the complex realisation [11, section 3.3.2] of this space is S^7 with the Cayley multiplication. But the Cayley multiplication on S^7 is not homotopy associative by James [5, Theorem 1.4].

Recall from [11, Section 3.2.1] that

$$\mathrm{Ex}_{\mathbb{A}^1} = \mathrm{Ex}^{\mathcal{G}} \circ (\mathrm{Ex}^{\mathcal{G}} \circ \mathrm{Sing}_*^{\mathbb{A}^1})^{\mathbb{N}} \circ \mathrm{Ex}^{\mathcal{G}}.$$

The functors $\mathrm{Ex}^{\mathcal{G}}$ and $\mathrm{Sing}_*^{\mathbb{A}^1}$ commute with finite limits by [11, Section 2.3.2, Theorem 2.1.66]. Also filtered colimit commutes with finite products in the category of spaces. Therefore, $\mathrm{Ex}_{\mathbb{A}^1}$ commutes with finite products.

Definition 4.3 A set S with a composition law μ and an unit $\mathrm{id}_S \in S$ is a loop with right and left inverse property if for every element $s \in S$, there exists a $s^{-1} \in S$ such that for all $s' \in S$, $\mu(\mu(s',s),s^{-1}) = \mu(s^{-1},\mu(s,s')) = s'$.

For a loop S with left and right inverse property and for any element $s \in S$, the element s^{-1} is the unique left and right inverse of s.

Lemma 4.4 *If $\mathcal{X}$ is an H–group as described in Definition 4.1, then $\pi_0(Ex_{\mathbb{A}^1}(\mathcal{X}))$ is a presheaf of groups. If $\mathcal{X}$ is an H–loop with inverse then $\pi_0(Ex_{\mathbb{A}^1}(\mathcal{X}))$ is a presheaf of loops with right and left inverse property.*

Proof Suppose $\mathcal{X}$ is an H–group. The morphisms $\mathrm{Ex}_{\mathbb{A}^1}(x)$, $\mathrm{Ex}_{\mathbb{A}^1}(\mu)$ and $\mathrm{Ex}_{\mathbb{A}^1}((\,.\,)^*)$ satisfy the conditions of the Definition 4.1. Hence, $\mathrm{Ex}_{\mathbb{A}^1}(\mathcal{X})$ is also an H–group. For an H–group $\mathcal{X}$, suppose that $a,b,c \in \pi_0(Ex_{\mathbb{A}^1}(\mathcal{X}))(U)$ for some $U \in \mathrm{Sm}/k$. Let $f,g\colon \mathcal{Y} \to \mathcal{Z}$ be morphisms between $\mathbb{A}^1$–fibrant spaces such that f is equal to g in $\mathbf{H}(k)$. Then f and g are simplicially homotopic. Using this, we get $\mu(a,x) = a = \mu(x,a)$, $\mu(a,a^*) = \mu(a^*,a) = x$ and $\mu(a,\mu(b,c)) = \mu(\mu(a,b),c)$. Hence, $\pi_0(Ex_{\mathbb{A}^1}(\mathcal{X}))$ is a presheaf of groups. Now, suppose $\mathcal{X}$ is an H–loop with inverse. For every $U \in \mathrm{Sm}/k$, we have $\mu(a^*,\mu(a,b)) = b$ and $\mu(\mu(a,b),b^*) = a$, for all $a,b \in \pi_0(Ex_{\mathbb{A}^1}(\mathcal{X}))(U)$. Therefore, $\pi_0(Ex_{\mathbb{A}^1}(\mathcal{X}))$ is a presheaf of loops with right and left inverse property. □

Let S and S' be loops with right and left inverse property, and let 1_S (resp. $1_{S'}$) be the unit of S (resp. S'). Let $f\colon S \to S'$ be a map of pointed sets (S and S' are pointed by 1_S and $1_{S'}$ respectively), such that f preserves the composition laws. If f is injective as a map of pointed sets then f is injective as a map of sets. Indeed, if $f(s_1) = f(s_2)$, then $f(s_1.s_2^{-1}) = 1_{S'}$. This implies $s_1.s_2^{-1} = 1_S$. Therefore, $s_1 = s_2$.

Definition 4.5 Let $\mathcal{X}$ be an H–loop with inverse. Let $\mathcal{Y}$ be a space. The space $\mathcal{Y}$ is called an $\mathcal{X}$–space if there exists a morphism $a\colon \mathcal{X} \times \mathcal{Y} \to \mathcal{Y}$, such that the diagram

$$\begin{array}{ccc} \mathcal{X} \times (\mathcal{X} \times \mathcal{Y}) & \xrightarrow{\mathrm{id}_{\mathcal{X}} \times a} & \mathcal{X} \times \mathcal{Y} \\ \downarrow{\scriptstyle a_{\mathcal{X}} \times \mathrm{id}_{\mathcal{Y}}} & & \downarrow{\scriptstyle a} \\ \mathcal{X} \times \mathcal{Y} & \xrightarrow{\quad a \quad} & \mathcal{Y} \end{array}$$

commutes in $\mathbf{H}(k)$ and the canonical map $a \circ (x \times \mathrm{id}_{\mathcal{Y}})$ is equal to $\mathrm{id}_{\mathcal{Y}}$ in $\mathbf{H}(k)$.

Definition 4.6 Let $\mathcal{X}$ be an H–loop with inverse and let $\mathcal{Y}$ be an $\mathcal{X}$–space. $\mathcal{Y}$ is called a homogeneous $\mathcal{X}$–space if for any essentially smooth Henselian R, the loop with right and left inverse property $\pi_0^{\mathbb{A}^1}(\mathcal{X})(R)$ acts transitively on $\pi_0^{\mathbb{A}^1}(\mathcal{Y})(R)$.

We give the following easy characterisations of pointed homogeneous $\mathcal{X}$–spaces.

Lemma 4.7 *$\mathcal{Y}$ is a pointed homogeneous $\mathcal{X}$–space if and only if the Nisnevich sheaf associated to the colimit of the diagram*

$$\pi_0^{\mathbb{A}^1}(\mathcal{X}) \times \pi_0^{\mathbb{A}^1}(\mathcal{Y}) \underset{\mathrm{pr}}{\overset{a}{\rightrightarrows}} \pi_0^{\mathbb{A}^1}(\mathcal{Y})$$

is the trivial sheaf.

Proof Let S be the Nisnevich sheaf associated to

$$\operatorname{colim}(\pi_0^{\mathbb{A}^1}(\mathcal{X}) \times \pi_0^{\mathbb{A}^1}(\mathcal{Y}) \underset{\mathrm{pr}}{\overset{a}{\rightrightarrows}} \pi_0^{\mathbb{A}^1}(\mathcal{Y})).$$

Then for any essentially smooth Henselian R, the set $S(R)$ is the orbit space of the action of $\pi_0^{\mathbb{A}^1}(\mathcal{X})(R)$ on $\pi_0^{\mathbb{A}^1}(\mathcal{Y})(R)$. The orbit space is trivial if and only if the action is transitive. □

Corollary 4.8 *Let $\mathcal{Y}$ be a pointed $\mathcal{X}$–space. $\mathcal{Y}$ is a homogeneous $\mathcal{X}$–space if and only if the homotopy colimit of*

$$\mathrm{Ex}_{\mathbb{A}^1}(\mathcal{X}) \times \mathrm{Ex}_{\mathbb{A}^1}(\mathcal{Y}) \underset{\mathrm{pr}}{\overset{a}{\rightrightarrows}} \mathrm{Ex}_{\mathbb{A}^1}(\mathcal{Y})$$

is simplicially connected (ie, $a_{\mathrm{Nis}}(\pi_0)$ of the homotopy colimit is trivial).

Proof The proof follows from Lemma 2.3 and Lemma 4.7. □

Lemma 4.9 *Let $\mathcal{Y}$ be a pointed $\mathcal{X}$–space. $\mathcal{Y}$ is a homogeneous $\mathcal{X}$–space if the homotopy colimit of*

$$\mathcal{X} \times \mathcal{Y} \underset{\mathrm{pr}}{\overset{a}{\rightrightarrows}} \mathcal{Y}$$

is simplicially connected.

Proof By [11, Corollary 2.3.22], the canonical morphism

$$a_{\mathrm{Nis}}(\pi_0(\mathcal{X})) \longrightarrow a_{\mathrm{Nis}}(\pi_0^{\mathbb{A}^1}(\mathcal{X})) \quad (\text{resp. } a_{\mathrm{Nis}}(\pi_0(\mathcal{Y})) \longrightarrow a_{\mathrm{Nis}}(\pi_0^{\mathbb{A}^1}(\mathcal{Y}))$$

is surjective as a morphism of Nisnevich sheaves. Using Lemma 2.3 and the previous fact we deduce that there is a surjective map from the connected component sheaf

$$S := a_{\mathrm{Nis}}(\pi_0(\operatorname{hocolim}(\mathcal{X} \times \mathcal{Y} \underset{\mathrm{pr}}{\overset{a}{\rightrightarrows}} \mathcal{Y})))$$

to the connected component sheaf

$$S' := a_{\mathrm{Nis}}(\pi_0(\operatorname{hocolim}(Ex_{\mathbb{A}^1}(\mathcal{X}) \times Ex_{\mathbb{A}^1}(\mathcal{Y}) \underset{\mathrm{pr}}{\overset{a}{\rightrightarrows}} Ex_{\mathbb{A}^1}(\mathcal{Y})))).$$

By assumption S is trivial. Thus, S' is also trivial. By Corollary 4.8, $\mathcal{Y}$ is a homogeneous $\mathcal{X}$–space. □

Remark 4.10 Note that the definition of homogeneous spaces does not include quotients of algebraic groups, for instance, if G is an algebraic group and Γ a finite subgroup scheme of G. Then the algebraic quotient G/Γ is not a G–homogeneous space for us, but the Nisnevich quotient $G/_{\mathrm{Nis}}\Gamma$ is a G–homogeneous space.

Lemma 4.11 *Let G, G' be loops with right and left inverse property acting on pointed sets S, S' by action maps a, a' respectively. Suppose that $f\colon G \to G'$ is a loop homomorphism (ie, f preserves the composition laws and $f(1_G) = 1_{G'}$) and let $s\colon S \to S'$ be a morphism of pointed sets with trivial kernel such that $s \circ a = a' \circ (f \times s)$. If G acts transitively on S, then s is injective.*

Proof Let b_S (resp. $b_{S'}$) be the base point of S (resp. S') and let $a, b \in S$. Since G acts transitively on S, there exist $g, g' \in G$ such that $a(g, b_S) = a$ and $a(g', b_S) = b$. If $s(a) = s(b)$, then $a'(f(g), b_{S'}) = a'(f(g'), b_{S'})$. Hence $a'(f(g^{-1}.g'), b_{S'}) = b_{S'}$. So $s(a(g^{-1}.g', b_S)) = b_{S'}$. But s is a morphism of pointed sets with trivial kernel, therefore $a(g^{-1}.g', b_S) = b_S$. Since G is a loop with right and left inverse property, we have $a = a(g, b_S) = a(g', b_S) = b$. □

Let $\widetilde{\mathrm{Sm}/k}$ be the category whose objects are same as objects of Sm/k, but the morphisms are smooth morphisms. The following argument is taken from [10, Corollary 5.9]:

Lemma 4.12 *Let S be a Nisnevich sheaf on Sm/k. Suppose that for all essentially smooth Henselian X, the map $S(X) \to S(K(X))$ is injective. Then $S(Y) \to S(K(Y))$ is injective, for all connected $Y \in \mathrm{Sm}/k$.*

Proof Let S' be the presheaf on $\widetilde{\mathrm{Sm}/k}$, given by

$$X \in \widetilde{\mathrm{Sm}/k} \mapsto \prod_i S(K(X_i)),$$

where the X_i are the connected components of X. Then S' is a Nisnevich sheaf on $\widetilde{\mathrm{Sm}/k}$ (as every Nisnevich covering of some $X \in \widetilde{\mathrm{Sm}/k}$ splits over some open dense $U \subset X$). The canonical morphism $S \to S'$ is injective on Nisnevich stalks. Hence $S \to S'$ is sectionwise injective. □

Corollary 4.13 *Let S be a Nisnevich sheaf on Sm/k. Suppose that for all essentially smooth Henselian X, the map $S(X) \to S(K(X))$ is injective. Then $S(Y) \to S(U)$ is injective for any $Y \in \mathrm{Sm}/k$ and any open dense $U \subset Y$.*

Proof We can assume that Y is connected. By Lemma 4.12, the morphism $S(Y) \to S(K(Y))$ is injective and $S(U) \to S(K(Y))$ is injective, hence $S(Y) \to S(U)$ is injective. □

Lemma 4.14 *Let S be a Zariski sheaf on Sm/k, such that $S(X) \to S(U)$ is injective for any $X \in \mathrm{Sm}/k$ and for any open dense $U \subset X$. Then S is $\mathbb{A}^1$–invariant if and only if $S(F) \to S(\mathbb{A}^1_F)$ is bijective for every finitely generated separable field extension F/k.*

Proof The only if part is clear. We need to show that for any connected $X \in \mathrm{Sm}/k$, the morphism $S(\mathbb{A}^1_X) \to S(X)$ (induced by the zero section), is bijective. Let $F := K(X)$. In the commutative diagram

$$\begin{array}{ccc} S(\mathbb{A}^1_X) & \longrightarrow & S(X) \\ \downarrow & & \downarrow \\ S(\mathbb{A}^1_F) & \longrightarrow & S(F) \end{array}$$

the left vertical, the right vertical and the bottom horizontal morphisms are injective, thus the top horizontal surjective morphism is injective. □

We recall the following from Colliot-Thélène, Hoobler and Kahn [3] and Morel [10, Corollary 5.7]:

Theorem 4.15 *Let X be a smooth (or essentially smooth) k–scheme, $s \in X$ be a point and $Z \subset X$ be a closed subscheme of codimension $d > 0$. Then there exists an open subscheme $\Omega \subset X$ containing s and a closed subscheme $Z' \subset \Omega$, of codimension $d-1$, containing $Z_\Omega := Z \cap \Omega$ and such that for any $n \in \mathbb{N}$ and for any $\mathbb{A}^1$–fibrant space $\mathcal{X}$, the map*

$$\pi_n(\mathcal{X}(\Omega/(\Omega - Z_\Omega))) \longrightarrow \pi_n(\mathcal{X}(\Omega/(\Omega - Z')))$$

is the trivial map. In particular, if Z has codimension 1 and X is irreducible, Z' must be Ω. Thus for any $n \in \mathbb{N}$ the map

$$\pi_n(\mathcal{X}(\Omega/(\Omega - Z_\Omega))) \longrightarrow \pi_n(\mathcal{X}(\Omega))$$

is the trivial map.

Remark 4.16 The proof of the previous theorem relies on a version of Gabber presentation lemma that is stated in [10, Lemma 15] without proof. The form available in the literature assumes the base field k to be an infinite field.

Corollary 4.17 [10] *Let X be an essentially smooth local scheme and let $\mathcal{X}$ be a $\mathbb{A}^1$–fibrant space with a base point x. Then $\pi_0(\mathcal{X})(X)$ and $\pi_0(\mathcal{X})(K(X))$ are pointed and the restriction $\pi_0(\mathcal{X})(X) \to \pi_0(\mathcal{X})(K(X))$ is a morphism of pointed sets with trivial kernel. In particular if X is Henselian, then the morphism of pointed sets*

$$a_{\mathrm{Nis}}(\pi_0(\mathcal{X}))(X) \longrightarrow a_{\mathrm{Nis}}(\pi_0(\mathcal{X}))(K(X))$$

has trivial kernel.

Proof Let z be the closed point. Let $U \subset X$ be an open set. Adding disjoint base points to U and X and making X/U pointed by the image of U, we get a cofiber sequence $U_+ \to X_+ \to X/U \to \sum_s^1 U_+ \to \cdots$.

For the $\mathbb{A}^1$–fibrant space $\mathcal{X}$ with a base point x, we have the following exact sequence of pointed sets and groups:

$$\cdots \longrightarrow \pi_1(\mathcal{X}, x)(X) \longrightarrow \pi_1(\mathcal{X}, x)(U) \\ \longrightarrow \pi_0(\mathcal{X})(X/U) \longrightarrow [X_+, \mathcal{X}]_{H_{\bullet,s}(\mathrm{Sm}/k)} \longrightarrow [U_+, \mathcal{X}]_{H_{\bullet,s}(\mathrm{Sm}/k)}.$$

The last two terms are pointed by the map that sends everything to the base point of $\mathcal{X}$. Forgetting the base point, $[X_+, \mathcal{X}]$ (resp. $[U_+, \mathcal{X}]$) is $\pi_0(\mathcal{X})(X)$ (resp. $\pi_0(\mathcal{X})(U)$).

Applying Theorem 4.15 to X and its closed point z, we see that $\Omega = X$ and the morphisms

$$\pi_n(\mathcal{X})(X/U) \longrightarrow \pi_n(\mathcal{X})(X)$$

are trivial. Hence the morphism of pointed sets

$$\pi_0(\mathcal{X})(X) \longrightarrow \pi_0(\mathcal{X})(U)$$

has trivial kernel. Taking colimit over open sets, this gives the morphism of pointed sets

$$\pi_0(\mathcal{X})(X) \longrightarrow \pi_0(\mathcal{X})(K(X))$$

which has trivial kernel. In particular if X is Henselian, then the morphism of pointed sets

$$a_{\mathrm{Nis}}(\pi_0(\mathcal{X}))(X) \longrightarrow a_{\mathrm{Nis}}(\pi_0(\mathcal{X}))(K(X))$$

has trivial kernel. □

Theorem 4.18 *Let $\mathcal{X}$ be an H–loop with inverse. Then $a_{\mathrm{Nis}}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))$ is $\mathbb{A}^1$–invariant. If $\mathcal{Y}$ is a pointed homogeneous $\mathcal{X}$–space, then $a_{\mathrm{Nis}}(\pi_0^{\mathbb{A}^1}(\mathcal{Y}))$ is $\mathbb{A}^1$–invariant.*

Proof For any connected $X \in \mathrm{Sm}/k$ and any $x \in X$, the morphisms of pointed sets

$$a_{\mathrm{Nis}}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))(O_{X,x}^h) \longrightarrow a_{\mathrm{Nis}}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))(K(O_{X,x}^h)),$$
$$a_{\mathrm{Nis}}(\pi_0^{\mathbb{A}^1}(\mathcal{Y}))(O_{X,x}^h) \longrightarrow a_{\mathrm{Nis}}(\pi_0^{\mathbb{A}^1}(\mathcal{Y}))(K(O_{X,x}^h))$$

have trivial kernel by Corollary 4.17. By Lemma 4.11 and that $a_{\mathrm{Nis}}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))(O_{X,x}^h)$ is a loop with right and left inverse property, the morphisms mentioned above are injective morphisms of sets. By Lemma 4.12, for every $X \in \mathrm{Sm}/k$, the morphisms

$$a_{\mathrm{Nis}}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))(X) \longrightarrow a_{\mathrm{Nis}}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))(K(X)),$$
$$a_{\mathrm{Nis}}(\pi_0^{\mathbb{A}^1}(\mathcal{Y}))(X) \longrightarrow a_{\mathrm{Nis}}(\pi_0^{\mathbb{A}^1}(\mathcal{Y}))(K(X))$$

are injective. Hence for any $X \in \mathrm{Sm}/k$ and any open dense subscheme $U \subset X$, the morphisms

$$a_{\mathrm{Nis}}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))(X) \longrightarrow a_{\mathrm{Nis}}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))(U),$$
$$a_{\mathrm{Nis}}(\pi_0^{\mathbb{A}^1}(\mathcal{Y}))(X) \longrightarrow a_{\mathrm{Nis}}(\pi_0^{\mathbb{A}^1}(\mathcal{Y}))(U)$$

are injective by Corollary 4.13. Now applying Corollary 3.2 and Lemma 4.14, we get our result. □

If $\mathcal{X}$ is an H–loop with inverse, then

$$\pi_0^{\mathbb{A}^1}(\mathcal{X})(R) \longrightarrow a_{\mathrm{Nis}}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))(R)$$

is bijective for any essentially smooth discrete valuation ring R. Indeed, using Corollary 4.17 one can easily show that for any essentially smooth discrete valuation ring R, the loop homomorphism

$$\pi_0^{\mathbb{A}^1}(\mathcal{X})(R) \longrightarrow \pi_0^{\mathbb{A}^1}(\mathcal{X}))(K(R))$$

is injective. On the other hand, consider the following commutative diagram:

$$\begin{array}{ccc} \pi_0^{\mathbb{A}^1}(\mathcal{X})(R) & \longrightarrow & \pi_0^{\mathbb{A}^1}(\mathcal{X}))(K(R)) \\ \downarrow & & \downarrow \wr \\ a_{\mathrm{Nis}}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))(R) & \longrightarrow & a_{\mathrm{Nis}}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))(K(R)), \end{array}$$

where the bottom horizontal morphism is injective by Theorem 4.18. The left vertical injective morphism is surjective by Lemma 3.3. Hence it is bijective.

5 Application and comments

We recall from [10, Definition 7] the following definition.

Definition 5.1 A sheaf of groups G on Sm/k is called strongly $\mathbb{A}^1$–invariant if for any $X \in \mathrm{Sm}/k$, the map

$$H^i_{\mathrm{Nis}}(X, G) \longrightarrow H^i_{\mathrm{Nis}}(\mathbb{A}^1_X, G)$$

induced by the projection $\mathbb{A}^1_X \to X$ is bijective for $i \in \{0, 1\}$.

By gathering known facts from Wendt [12], Morel [9, Theorem A.2], and Gille [4, Corollary 5.10], one can show that for any connected linear algebraic group G, such that the almost simple factors of the universal covering (in algebraic group theory sense) of the semisimple part of G is isotropic and retract k–rational [4, Definition 2.2], the sheaf $a_{\mathrm{Nis}}(\pi_0^{\mathbb{A}^1}(G))$ is $\mathbb{A}^1$–invariant (in fact strongly $\mathbb{A}^1$–invariant). By Theorem 4.18, we have the following generalisation.

Corollary 5.2 *Let G be any sheaf of groups on Sm/k and B be any subsheaf of groups. Then $a_{\mathrm{Nis}}(\pi_0^{\mathbb{A}^1}(G))$ is $\mathbb{A}^1$–invariant and $a_{\mathrm{Nis}}(\pi_0^{\mathbb{A}^1}(G/B))$ is $\mathbb{A}^1$–invariant. Here G/B is the quotient sheaf in Nisnevich topology.*

Let $\mathcal{X}$ be a pointed space. By [10, Theorem 9], for any pointed simplicial presheaf $\mathcal{X}$, the sheaf of groups $a_{\mathrm{Nis}}(\pi_0(\Omega(Ex_{\mathbb{A}^1}(\mathcal{X})))) = \pi_1^{\mathbb{A}^1}(\mathcal{X}, x)$ is strongly $\mathbb{A}^1$–invariant. Here x is the base point of $\mathcal{X}$ and $\Omega(\mathrm{Ex}_{\mathbb{A}^1}(\mathcal{X}))$ is the simplicial loop space of $\mathrm{Ex}_{\mathbb{A}^1}(\mathcal{X})$. So for any space $\mathcal{X}$ that is the loop space of some $\mathbb{A}^1$–local space $\mathcal{Y}$, [10, Theorem 9] gives the $\mathbb{A}^1$–invariance property for $a_{\mathrm{Nis}}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))$. We end this section by showing that there exists an $\mathbb{A}^1$–local H–group that is not a loop space of some $\mathbb{A}^1$–local space. This will imply that the statement of the Theorem 4.18 for H–groups is not a direct consequence of [10, Theorem 9]. It is enough to show that there exists sheaf of groups G that is $\mathbb{A}^1$–invariant, but not strongly $\mathbb{A}^1$–invariant.

Let $\mathbb{Z}[\mathbb{G}_m]$ be the free presheaf of Abelian groups generated by $\mathbb{G}_m$.

Lemma 5.3 *For any $X \in \mathrm{Sm}/k$ and a dominant morphism $U \to X$, the canonical morphism $\mathbb{Z}[\mathbb{G}_m](X) \to \mathbb{Z}[\mathbb{G}_m](U)$ is injective.*

Proof Any nonzero $a \in \mathbb{Z}[\mathbb{G}_m](X)$ can be written as $a = \sum_{i=1}^n a_i.g_i$, where $g_i \in \mathbb{G}_m(X)$ and $a_i \in \mathbb{Z} \setminus \{0\}$ such that $g_i \neq g_{i'}$ for $i \neq i'$. Suppose $a|_U = 0$, ie, $\sum_{i=1}^n a_i.g_i|_U = 0$. Since $\mathbb{G}_m(X) \to \mathbb{G}_m(U)$ is injective, $g_i|_U \neq g_{i'}|_U$ for $i \neq i'$. This implies $a_i = 0$ for all i. Hence $a = 0$. □

The presheaf $\mathbb{Z}[\mathbb{G}_m]$ is not a Nisnevich sheaf. But it is not far from being a Nisnevich sheaf.

Lemma 5.4 *The Nisnevich sheafification $a_{\mathrm{Nis}}(\mathbb{Z}[\mathbb{G}_m])$ is the presheaf that associates, to every smooth k–scheme $X = \coprod_i X_i$, the Abelian group $\prod_i \mathbb{Z}[\mathbb{G}_m](X_i)$, where the X_i are the connected components of X.*

Proof Let $\mathcal{F}$ be the presheaf that associates, to every smooth k–scheme $X = \coprod_i X_i$, the Abelian group $\prod_i \mathbb{Z}[\mathbb{G}_m](X_i)$, where the X_i are the connected components of X. It is enough to prove that $\mathcal{F}$ is a Nisnevich sheaf. We need to show that for any elementary distinguished square in Sm/k,

$$\begin{array}{ccc} W & \longrightarrow & V \\ \downarrow & & \downarrow p \\ U & \xrightarrow{i} & X, \end{array}$$

the induced commutative square

$$\begin{array}{ccc} \mathcal{F}(X) & \longrightarrow & \mathcal{F}(V) \\ \downarrow & & \downarrow \\ \mathcal{F}(U) & \longrightarrow & \mathcal{F}(W) \end{array}$$

is Cartesian. By the construction of $\mathcal{F}$ we can assume that X, W, V, U are connected. So, it is enough to prove that

$$\begin{array}{ccc} \mathbb{Z}[\mathbb{G}_m](X) & \longrightarrow & \mathbb{Z}[\mathbb{G}_m](V) \\ \downarrow & & \downarrow \\ \mathbb{Z}[\mathbb{G}_m](U) & \longrightarrow & \mathbb{Z}[\mathbb{G}_m](W) \end{array}$$

is Cartesian. Let $a \in \mathbb{Z}[\mathbb{G}_m](U)$ and let $b \in \mathbb{Z}[\mathbb{G}_m](V)$ such that $a|_W = b|_W$. We can write $a = \sum_{i=1}^n a_i.f_i$ and $b = \sum_{j=1}^m b_j.g_j$, where $a_i, b_j \in \mathbb{Z} \setminus \{0\}$ and $(f_i, g_j) \in \mathbb{G}_m(U) \times \mathbb{G}_m(V)$ such that $f_i \neq f_{i'}$ and $g_j \neq g_{j'}$ for all $i \neq i'$ and $j \neq j'$. Since all the morphisms are dominant, $g_j|_W \neq g_{j'}|_W$ and $f_i|_W \neq f_{i'}|_W$ for all $i \neq i'$ and $j \neq j'$. Hence, for every i there exists at most one j such that $f_i|_W = g_j|_W$. Suppose for some $f_{i'}$, $f_{i'}|_W \neq g_j|_W$ for all j. Then we can write

$$\left(\sum_{i=1}^n a_i.f_i|_W\right) - \left(\sum_{j=1}^m b_j.g_j|_W\right) = a_{i'} f_{i'} + \sum_{k=1}^l c_k.h_k = 0,$$

where $h_k \neq h_{k'}$ for all $k \neq k'$ and $f_{i'} \neq h_k$ for all k. This implies $a_{i'} = 0$, which gives a contradiction. Hence, for every i there exists exactly one j such that $f_i|_W = g_j|_W$. Therefore, $m = n$. Also we can write $a = \sum_{i=1}^{n} a'_i . f'_i$, such that $a'_i = b_i$ and $f'_i|_W = g_i|_W$. Since $\mathbb{G}_m$ is a Nisnevich sheaf, we get $g'_i \in \mathbb{G}_m(X)$, which restricts to f'_i and g_i. This gives a section $c = \sum_{i=1}^{n} b_i . g'_i \in \mathbb{Z}[\mathbb{G}_m](X)$, which restricts to a and b. The uniqueness of c follows from the Lemma 5.3. □

As $\mathbb{G}_m$ is pointed by 1, $a_{\mathrm{Nis}}(\mathbb{Z}[\mathbb{G}_m]) \cong \mathbb{Z} \oplus \mathbb{Z}(\mathbb{G}_m)$. Here $\mathbb{Z}$ is the sheaf generated by the point 1. Let A be a sheaf of Abelian groups on Sm/k. To give a morphism $\mathbb{G}_m \to A$, such that 1 gets mapped to $0 \in A$, is equivalent to give a morphism $\mathbb{Z}(\mathbb{G}_m) \to A$ of Abelian sheaves. Since $\mathbb{G}_m$ is $\mathbb{A}^1$–invariant, $a_{\mathrm{Nis}}(\mathbb{Z}[\mathbb{G}_m])$ is $\mathbb{A}^1$–invariant. This implies $\mathbb{Z}(\mathbb{G}_m)$ is $\mathbb{A}^1$–invariant.

Let $\sigma_1\colon \mathbb{G}_m \to \underline{K}_1^{MW}$ be the canonical pointed morphism (see [10, page 86]). For any finitely generated separable field extension F/k, the morphism maps $u \in F^*$ to the corresponding symbol $[u] \in K_1^{MW}(F)$.

Lemma 5.5 *The induced morphism $\mathbb{Z}(\mathbb{G}_m) \to \underline{K}_1^{MW}$ is not injective.*

Proof We can choose $u \in F^* \setminus 1$ such that $u(u-1)$ is not 1. The element

$$[u(u-1)] - [u] - [u-1]$$

is zero in $K_1^{MW}(F)$, but it is nonzero in $\mathbb{Z}(\mathbb{G}_m)(F)$. □

Lemma 5.6 *The $\mathbb{A}^1$–invariant sheaf of Abelian groups $\mathbb{Z}(\mathbb{G}_m)$ is not strongly $\mathbb{A}^1$–invariant.*

Proof Suppose $\mathbb{Z}(\mathbb{G}_m)$ is strongly $\mathbb{A}^1$–invariant. Then by [10, Theorem 2.37], the morphism id: $\mathbb{Z}(\mathbb{G}_m) \to \mathbb{Z}(\mathbb{G}_m)$ can be written as $\phi \circ \sigma_1$ for some unique ϕ. This implies σ_1 is injective, which contradicts Lemma 5.5. □

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