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On Kirby calculus for null-homotopic framed links in 3–manifolds

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Kirby proved that two framed links in S^3 give orientation-preserving homeomorphic results of surgery if and only if these two links are related by a sequence of two kinds of moves called stabilizations and handle-slides. Fenn and Rourke gave a necessary and sufficient condition for two framed links in a closed, oriented 3–manifold to be related by a finite sequence of these moves.

The purpose of this paper is twofold. We first give a generalization of Fenn and Rourke's result to 3–manifolds with boundary. Then we apply this result to the case of framed links whose components are null-homotopic in the 3–manifold.

57M25, 57M27

1 Introduction

In 1978, Kirby [10] proved that two framed links in S^3 have homeomorphic result of surgery if and only if they are related by a sequence of two kinds of moves called stabilizations and handle-slides. This result enables one to construct a 3–manifold invariant by constructing a link invariant which is invariant under these moves. Fenn and Rourke [5] generalized Kirby's Theorem to framed links in closed 3–manifolds, and Roberts [11] generalized it to framed links in 3–manifolds with boundary.

Fenn and Rourke [5] also considered the equivalence relation on framed links in an arbitrary closed, oriented 3–manifold generated by stabilizations and handle-slides. Here we state Fenn and Rourke's Theorem, leaving some details to the original paper [5]. Let M be a closed, oriented 3–manifold. For a framed link L in M, we will denote by W_L the 4–manifold obtained from $M \times I$ by attaching 2–handles along $L \times \{1\} \subset \partial(M \times I)$ in a way determined by the framing. Note that W_L is a cobordism between M and M_L, where M_L denotes the 3–manifold obtained from M by surgery along L. The inclusions $M_L \hookrightarrow W_L \hookleftarrow M$ induce surjective homomorphisms

$$\pi_1(M_L) \twoheadrightarrow \pi_1(W_L) \twoheadleftarrow \pi_1(M).$$

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The kernel of the homomorphism $\pi_1(M) \to \pi_1(W_L)$ is normally generated by the homotopy classes of components of L.

Theorem 1.1 (Fenn–Rourke [5]) *Let M be a closed, oriented 3–manifold, and let L and L' be two framed links in M. Then L and L' are related by a sequence of stabilizations and handle-slides if and only if there exist an orientation-preserving homeomorphism $h\colon M_L \to M_{L'}$ and an isomorphism*

$$f\colon \pi_1(W_L) \longrightarrow \pi_1(W_{L'}),$$

such that the diagram

$$\begin{tikzcd} \pi_1(M_L) \arrow[rr, "h_*"] \arrow[d] & & \pi_1(M_{L'}) \arrow[d] \\ \pi_1(W_L) \arrow[rr, "f"] & & \pi_1(W_{L'}) \\ & \pi_1(M) \arrow[ul] \arrow[ur] & \end{tikzcd} \tag{1}$$

commutes and we have $\rho_*([W]) = 0 \in H_4(\pi_1(W_L), \mathbb{Z})$. Here

- *W is the closed 4–manifold obtained from W_L and $W_{L'}$ by gluing along their boundaries using id_M and h,*
- *$[W] \in H_4(W, \mathbb{Z})$ is the fundamental class, and*
- *$\rho_*\colon H_4(W, \mathbb{Z}) \to H_4(\pi_1(W_L), \mathbb{Z})$ is induced by a map $\rho\colon W \to K(\pi_1(W_L), 1)$ obtained by gluing natural maps from W_L and $W_{L'}$ to $K(\pi_1(W_L), 1)$.*

See [5] for more details.

One of the main results of the present paper, Theorem 2.2, is a generalization of Theorem 1.1 to 3–manifolds with boundary. (A generalization of Theorem 1.1 to 3–manifolds with boundary was stated by Garoufalidis and Kricker [6], but unfortunately the statement they made therein is not correct for 3–manifolds with more than one boundary component.)

An obstruction to making Theorems 1.1 and 2.2 useful is the homological condition $\rho_*([W]) = 0$. Given framed links L, L' in M as in Theorems 1.1 and 2.2, it is not always easy to see whether we have $\rho_*([W]) = 0$ or not. However, if $H_4(\pi_1(W_L), \mathbb{Z}) = 0$, then clearly we have $\rho_*([W]) = 0$.

A large class of groups with vanishing $H_4(-, \mathbb{Z})$ is the one of 3–manifold groups. It seems to have been well known for a long time that if M is a compact, connected, oriented 3–manifold, then $H_4(\pi_1(M), \mathbb{Z}) = 0$ (see Lemma 3.3). So, if the components of the framed links L and L' in M are null-homotopic, then since $\pi_1(W_L) \cong \pi_1(M)$ is a 3–manifold group, we have $H_4(\pi_1(W_L), \mathbb{Z}) = 0$ and $\rho_*([W]) = 0$. Thus, for null-homotopic framed links, we do not need the condition $\rho_*([W]) = 0$; see Theorem 3.1.

Cochran, Gerges and Orr [3] studied surgery along null-homologous framed links with diagonal linking matrices with diagonal entries ± 1, and also surgery along more special classes of framed links. This includes null-homotopic framed links with diagonal linking matrices with diagonal entries ± 1. Let us call such a framed link π_1*–admissible*. Surgery along a π_1–admissible framed link L in a 3–manifold M gives a manifold M_L whose fundamental group is "very close" to that of M. In [3] it is proved that, for all $d \geq 1$, we have $\pi_1(M_L)/\,\Gamma_d\pi_1(M_L) \cong \pi_1(M)/\,\Gamma_d\pi_1(M)$, where for a group G, $\Gamma_d G$ denotes the d^{th} lower central series subgroup of G.

For π_1–admissible framed links in a 3–manifold, we can combine Theorem 3.1 with Proposition 4.1 proved by the first author [8] to obtain a refined version of Theorem 3.1; see Theorem 4.2. This theorem gives a necessary and sufficient condition for two π_1–admissible framed links in M to be related by a sequence of stabilizations and *band-slides* [8], which are pairs of algebraically cancelling handle-slides; see Section 4.

We apply Theorem 4.2 to surgery along null-homotopic framed links in cylinders over surfaces. Surgery along a π_1–admissible framed link in a cylinder over a surface gives a homology cylinder of a special kind.

The organization of the rest of the paper is as follows. In Section 2, we introduce some notation and preliminary facts, and then state and prove the generalization of Fenn and Rourke's Theorem to 3–manifolds with boundary. In Section 3, we focus on the case of null-homotopic framed links. In Section 4, we consider π_1–admissible framed links. In Section 5, we give an example which illustrates the conditions needed in Theorem 2.2.

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2 Generalization of Fenn and Rourke's Theorem

In this section we state and prove a generalization of Theorem 1.1 to 3–manifolds with nonempty boundary. We start by giving the necessary notation which is used throughout this paper. Then we introduce the conditions under which Theorem 1.1 holds for manifolds with boundary and give the statement and the proof of our generalization of Theorem 1.1. Our construction mainly follows [5] and borrows some ideas also from [6].

Let M be a compact, connected, oriented 3–manifold, possibly with nonempty boundary.

A *framed link* $L = L_1 \cup \cdots \cup L_l$ in M is a link (ie, disjoint union of finitely many embedded circles in M) such that each component L_i of L is given a framing, ie, a homotopy class of trivializations of the normal bundle. Such a framing of L_i may be given as a homotopy class of a simple closed curve γ_i in the boundary $\partial N(L_i)$ of a tubular neighborhood $N(L_i)$ of L_i in M which is homotopic to L_i in $N(L_i)$.

For a framed link $L \subset M$ as above, let M_L denote the result from surgery of M along L. This manifold is obtained from M by removing the interiors of $N(L_i)$, and gluing a solid torus $D^2 \times S^1$ to $\partial N(L_i)$ so that the curve $\partial D^2 \times \{*\}$, $* \in S^1$, is attached to $\gamma_i \subset \partial N(L_i)$ for each $i = 1, \dots, l$.

Surgery along a framed link can be defined by using 4–manifolds as well. Let L be a framed link in M. Let W_L denote the 4–manifold obtained from the cylinder $M \times I$ by attaching a 2–handle $h_i \cong D^2 \times D^2$ along $N(L_i) \times \{1\}$ using the homeomorphism

$$S^1 \times D^2 \xrightarrow{\cong} N(L_i),$$

which maps $S^1 \times \{*\}$, $* \in \partial D^2$, onto the framing γ_i. We have a natural identification

$$\partial W_L \cong M \cup_{\partial M} (\partial M \times I) \cup_{\partial M_L} M_L.$$

Thus, W_L is a cobordism between M and M_L. Note that ∂W_L is connected if $\partial M \neq \varnothing$.

We define two moves on framed links. A *handle-slide* replaces one component L_i of L with a band sum L_i' of L_i and a parallel copy of another component L_j as in Figure 1, where the blackboard framing convention is used. A *stabilization* adds to or removes from a link L an isolated ± 1–framed unknot.

2.1 Some notation

We introduce some notation which we need in the statement of our generalization of Theorem 1.1, and which will be used in later sections as well.

Let M be a compact, connected, oriented 3–manifold with *nonempty* boundary.

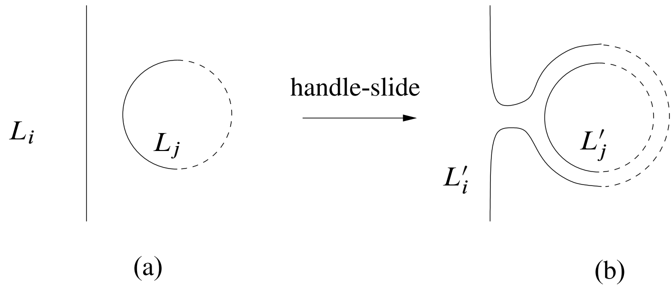

Figure 1: (a) Two components L_i and L_j of a framed link
(b) The result of a handle-slide of L_i over L_j

Let $F_1, \dots, F_n$ ($n \geq 1$) denote the components of ∂M. For each $k = 1, \dots, n$, choose a base point $p_k \in F_k$. We denote by $\pi_1(M; p_1, p_k)$ the set of homotopy classes of paths from p_1 to p_k in M. We consider p_1 as the base point of M, and write

$$\pi_1(M) = \pi_1(M; p_1) = \pi_1(M; p_1, p_1).$$

Let L be a framed link in M as before. We consider the 4–manifold W_L defined in Section 2. For $k = 1, \dots, n$, set $p_k^L = p_k \times \{1\} \in \partial M_L$ and $\gamma_k = p_k \times I \subset \partial W_L$. Note that γ_k is an arc in ∂W from $p_k \in \partial M \subset \partial W_L$ to p_k^L.

The inclusions

$$M \overset{i}{\hookrightarrow} W_L \overset{i'}{\hookleftarrow} M_L$$

induce surjective maps

$$\pi_1(M; p_1, p_k) \xrightarrow{i_k} \pi_1(W_L; p_1, p_k) \xleftarrow{i'_k} \pi_1(M_L; p_1^L, p_k^L)$$

for $k = 1, \dots, n$. Here i'_k is defined to be the composition

$$\pi_1(M_L; p_1^L, p_k^L) \xrightarrow{i'_k} \pi_1(W_L; p_1^L, p_k^L) \underset{\gamma_1, \gamma_k}{\cong} \pi_1(W_L; p_1, p_k),$$

where the second isomorphism is induced by the arcs γ_1 and γ_k.

We regard p_1^L as the base point of M_L and write $\pi_1(M_L) := \pi_1(M_L; p_1^L)$. We regard p_1 as a base point of W_L as well as of M, and we set $\pi_1(W_L) := \pi_1(W_L; p_1)$.

An Eilenberg–Mac Lane space $K(\pi_1(W_L), 1)$ can be obtained from W_L by attaching cells which kill higher homotopy groups. Thus, there is a natural inclusion

$$\rho_L\colon W_L \hookrightarrow K(\pi_1(W_L), 1).$$

2.2 Construction of a homology class

Now, consider two framed links L and L' in M, and suppose that there exists a homeomorphism $h\colon M_L \to M_{L'}$ relative to the boundary. Moreover, we assume that there exist isomorphisms $f_k\colon \pi_1(W_L; p_1, p_k) \to \pi_1(W_{L'}; p_1, p_k)$ such that the diagram

$$
\begin{tikzcd}
\pi_1(M_L; p_1^L, p_k^L) \arrow[rr, "h_k"] \arrow[d, "i'_k"'] & & \pi_1(M_{L'}; p_1^{L'}, p_k^{L'}) \arrow[d, "i'_k"] \\
\pi_1(W_L; p_1, p_k) \arrow[rr, "f_k"] & & \pi_1(W_{L'}; p_1, p_k) \\
& \pi_1(M; p_1, p_k) \arrow[ul, "i_k"] \arrow[ur, "i_k"'] &
\end{tikzcd} \tag{2}
$$

commutes for $k = 1, \dots, n$. For $k = 2, \dots, n$, that f_k *is an isomorphism* means that f_k is a bijection. (Here, if f_k is a bijection which makes the above diagram commute, then it follows that f_k is an isomorphism between the $\pi_1(W_L)$–set $\pi_1(W_L; p_1, p_k)$ and the $\pi_1(W_{L'})$–set $\pi_1(W_{L'}; p_1, p_k)$ along the group isomorphism $f_1\colon \pi_1(W_L) \to \pi_1(W_{L'})$.)

In the following, we define a homology class

$$\rho_*([W]) \in H_4(\pi_1(W_L), \mathbb{Z}),$$

by constructing a closed 4–manifold W and a map $\rho\colon W \to K(\pi_1(W_L), 1)$.

As in [6], define a 4–manifold W by

$$W := W_L \cup_\partial (-W_{L'}),$$

where we glue W_L and $-W_{L'}$ (the orientation reversal of $W_{L'}$) along the boundaries using the identity map on $M \cup (\partial M \times I)$ and the homeomorphism $h\colon M_L \xrightarrow{\cong} M_{L'}$.

Consider the following diagram:

$$
\begin{tikzcd}
\partial W_L \arrow[r, "u'"] \arrow[d, "u"'] & W_{L'} \arrow[d, "j'"'] \arrow[ddr, "\widetilde{\rho}_{L'}"] & \\
W_L \arrow[r, "j"] \arrow[drr, "\rho_L"'] & W \arrow[dr, "\rho"] & \\
& & K(\pi_1(W_L), 1),
\end{tikzcd} \tag{3}
$$

where u, u', j, j' are inclusions. The map $\widetilde{\rho}_{L'}\colon W_{L'} \to K(\pi_1(W_L), 1)$ is the composite

$$W_{L'} \xrightarrow{\rho_{L'}} K(\pi_1(W_{L'}), 1) \xrightarrow[\simeq]{K(f_1^{-1},1)} K(\pi_1(W_L), 1).$$

Here $K(f_1^{-1}, 1)$ is a homotopy equivalence, unique up to homotopy. By the definition of W, the square is a pushout. Hence, to prove existence of ρ such that $\rho j = \rho_L$ and $\rho j' = \widetilde{\rho}_{L'}$, we need only to show that $\rho_L u \simeq \widetilde{\rho}_{L'} u'$, which easily follows from Lemma 2.1 below. (It is in the proof of this lemma where commutativity of Theorem 2.2(2) is necessary not only for $k = 1$ but also for $k = 2, \dots, n$.)

Lemma 2.1 *Under the above situation, the following diagram commutes:*

(4)
$$\begin{tikzcd} \pi_1(\partial W_L) \arrow[r, "u'_*"] \arrow[d, "u_*"'] & \pi_1(W_{L'}) \arrow[d, "j'_*"] \\ \pi_1(W_L) \arrow[r, "j_*"] \arrow[ur, "f_1"] & \pi_1(W) \end{tikzcd}$$

Proof Since u_* is surjective and the square is commutative, $u'_* = f_1 u_*$ implies $j_* = j'_* f_1$.

Let us prove that $u'_* = f_1 u_*$. For $k = 2, \dots, n$, choose an arc c_k in M from p_1 to p_k disjoint from L. Set

$$d_k = (c_k \times \{0, 1\}) \cup (\partial c_k \times I),$$

which is a loop in ∂W_L based at p_1. The fundamental group $\pi_1(\partial W_L)$ is then generated by the elements $d_2, \dots, d_n$ and the images of the maps $i_*\colon \pi_1(M) \to \pi_1(\partial W_L)$ and $i'_*\colon \pi_1(M_L) \to \pi_1(\partial W_L)$. Hence $u'_* = f_1 u_*$ is reduced to the following:

(a) $u'_* i_* = f_1 u_* i_*\colon \pi_1(M) \to \pi_1(W_{L'})$.

(b) $u'_* i'_* = f_1 u_* i'_*\colon \pi_1(M_L) \to \pi_1(W_{L'})$.

(c) $u'_*(d_k) = f_1 u_*(d_k)$ for $k = 2, \dots, n$.

(a) (resp. (b)) follows from commutativity of the lower (resp. upper) part of Diagram (2) for $k = 1$. (c) follows from commutativity of Diagram (2) for $k = 2, \dots, n$. □

2.3 Statement of the theorem

Now we can state our generalization of Theorem 1.1 to 3–manifolds with boundary.

Theorem 2.2 *Let M be a compact, connected, oriented 3–manifold with $n > 0$ boundary components, and let $L, L' \subset M$ be framed links. Then the following conditions are equivalent:*

(i) *L and L' are related by a sequence of stabilizations and handle-slides.*

(ii) *There exists a homeomorphism $h\colon M_L \to M_{L'}$ relative to the boundary and isomorphisms $f_k\colon \pi_1(W_L; p_1, p_k) \to \pi_1(W_{L'}; p_1, p_k)$ for $k = 1, \dots, n$ such that Diagram (2) commutes for $k = 1, \dots, n$ and $\rho_*([W]) = 0 \in H_4(\pi_1(W_L))$.*

Remark 2.3 Theorem 1.1 can be derived from the case $\partial M = S^2$ of Theorem 2.2.

Remark 2.4 In a paper in preparation [9], we will give an example in which a nonzero homology class $\rho_*([W])$ is realized.

2.4 Proof of the theorem

We need the following lemma which gives a necessary and sufficient condition for $\rho_*([W]) \in H_4(\pi_1(W_L))$ to vanish.

Lemma 2.5 [5, Lemma 9; 6, Lemma 2.1] *In the situation of Theorem 2.2, we have $\rho_*([W]) = 0$ if and only if the connected sum of W with some copies of $\pm\mathbb{C}P^2$ is the boundary of an oriented 5–manifold Ω in such a way that the diagram*

$$\begin{tikzcd} \pi_1(W_L; p_1) \arrow[rr, "f_1"] \arrow[dr, "j_*"'] & & \pi_1(W_{L'}; p_1) \arrow[dl, "j'_*"] \\ & \pi_1(\Omega; p_1) & \end{tikzcd} \tag{5}$$

commutes and j_*, j'_* are split injections induced by the inclusions $j\colon W_L \hookrightarrow \Omega$ and $j'\colon W_{L'} \hookrightarrow \Omega$.

Proof of Theorem 2.2 The proof that (i) implies (ii) is almost the same as the proof of Theorem 1.1 given in [5]. It follows from the "if" part of Lemma 2.5 and the fact that handle-slides and stabilizations on a framed link L preserve the homeomorphism class of M_L and the $\pi_1(W_L; p_1, p_k)$, $k = 1, \dots, n$.

Now we prove that (ii) implies (i). Assume that all the algebraic conditions are satisfied. By Lemma 2.5, we may assume, after some stabilizations, that $W = \partial\Omega$, where Ω is a 5–manifold such that Diagram (5) commutes and j_* and j'_* are split injections. Now we alter Ω, as in the original proof in [5], by doing surgery on Ω until we have $\pi_1(\Omega) \cong \pi_1(W_L)$. Then we modify L and L' to $\widetilde{L}$ and $\widetilde{L}'$ by some specific stabilizations and handle-slides until we obtain a trivial cobordism Ω' joining $W_{\widetilde{L}}$ and $W_{\widetilde{L}'}$. Thus $W_{\widetilde{L}}$ and $W_{\widetilde{L}'}$ are two different relative handle decompositions of the same manifold.

By a famous theorem of Jean Cerf [2], any two relative handle decomposition of the same manifold are connected by a sequence of handle slides, creating/annihilating canceling handle pairs and isotopies; see Gompf and Stipsicz [7, Theorem 4.2.12]. Note that Cerf's Theorem applies in the case when W_L has two boundary components, as well as in the case where the boundary of the 4–manifold is connected. Fenn and Rourke have shown in [5] that these handle slides (1–handle slides and 2–handle slides) and creating or annihilating canceling handle pairs can be achieved by modifying the links using stabilization and handle-slides. Hence the proof is complete. □

3 Null-homotopic framed links

In this section we apply Theorem 2.2 to null-homotopic framed links.

Let M be a compact, connected, oriented 3–manifold with $n > 0$ boundary components as before. We use the notation given in Section 2.

A framed link L in M is said to be *null-homotopic* if each component of L is null-homotopic in M. In this case, the map

$$i_k\colon \pi_1(M; p_1, p_k) \longrightarrow \pi_1(W_L; p_1, p_k)$$

is bijective for $k = 1, \dots, n$. Define

$$e_k\colon \pi_1(M_L; p_1^L, p_k^L) \longrightarrow \pi_1(M; p_1, p_k)$$

to be the composition

$$e_k\colon \pi_1(M_L; p_1^L, p_k^L) \xrightarrow{i'_k} \pi_1(W_L; p_1, p_k) \xrightarrow[\cong]{i_k^{-1}} \pi_1(M; p_1, p_k),$$

which is surjective.

Theorem 3.1 *Let M be a compact, connected, oriented 3–manifold with $n > 0$ boundary components, and let $L, L' \subset M$ be null-homotopic framed links. Then the following conditions are equivalent:*

(i) *L and L' are related by a sequence of stabilizations and handle-slides.*

(ii) *There exists a homeomorphism $h\colon M_L \to M_{L'}$ relative to the boundary such that the following diagram commutes for $k = 1, \dots, n$:*

$$\begin{tikzcd} \pi_1(M_L; p_1^L, p_k^L) \arrow[rr, "h_k"] \arrow[dr, "e_k"'] & & \pi_1(M_{L'}; p_1^{L'}, p_k^{L'}) \arrow[dl, "e'_k"] \\ & \pi_1(M; p_1, p_k) & \end{tikzcd} \tag{6}$$

Remark 3.2 For a closed 3–manifold M, the variant of Theorem 3.1 is implicitly obtained in [5]. Two null-homotopic framed links L and L' in a closed, connected, oriented 3–manifold M are related by a sequence of stabilizations and handle-slides if and only if there is a homeomorphism $h\colon M_L \to M_{L'}$ such that the diagram

$$\begin{tikzcd} \pi_1(M_L) \arrow[rr, "h_*"] \arrow[dr, "e"'] & & \pi_1(M_{L'}) \arrow[dl, "e'"] \\ & \pi_1(M) & \end{tikzcd} \tag{7}$$

commutes. Here e and e' are defined similarly as before.

Theorem 3.1 follows easily from Theorem 2.2 and the following lemma, which seems to be well known. In fact, it seems implicit in Fenn and Rourke [5, page 8, lines 8–9], where it reads, "For many other groups, $\eta(\Delta)$ vanishes, eg the fundamental group of any 3–manifold." We give a sketch of proof of this fact since we have not been able to find a suitable reference for it.

Lemma 3.3 *If M is a compact, connected, oriented 3–manifold, then we have $H_4(\pi_1 M, \mathbb{Z}) = 0$.*

Proof Consider a connected sum decomposition $M \cong M_1 \sharp \cdots \sharp M_k$, $k \geq 0$, where each M_i is prime. Since $\pi_1 M \cong \pi_1 M_1 * \cdots * \pi_1 M_k$, we have

$$H_4(\pi_1 M, \mathbb{Z}) \cong H_4(\pi_1 M_1, \mathbb{Z}) \oplus \cdots \oplus H_4(\pi_1 M_k, \mathbb{Z}).$$

Thus, we may assume without loss of generality that M is prime. If $M = S^2 \times S^1$, then we have $H_4(\pi_1 M, \mathbb{Z}) = H_4(\mathbb{Z}, \mathbb{Z}) = 0$. Hence we may assume that M is irreducible.

If $\pi_1 M$ is infinite, then M is a $K(\pi_1 M, 1)$ space. Hence

$$H_4(\pi_1 M, \mathbb{Z}) \cong H_4(M, \mathbb{Z}) = 0.$$

Suppose that $\pi_1 M$ is finite. If $\partial M \neq \varnothing$, then we have $M \cong B^3$ and clearly $H_4(\pi_1 M, \mathbb{Z}) = 0$. Thus we may assume that M is closed. Then the universal cover of M is a homotopy 3–sphere, which is S^3 by the Poincaré conjecture established by Perelman. From Adem and Milgram [1, Lemma 6.2], we have

$$H^5(\pi_1 M, \mathbb{Z}) \cong H^1(\pi_1 M, \mathbb{Z}). \tag{8}$$

Recall that, for any finite group G, $H_n(G,\mathbb{Z})$ is finite for all $n \geq 1$. This fact and the universal coefficient theorem imply

$$H^1(\pi_1 M,\mathbb{Z}) \cong \mathrm{Hom}(H_1(\pi_1 M,\mathbb{Z}),\mathbb{Z}) = 0, \tag{9}$$

$$\begin{aligned} H^5(\pi_1 M,\mathbb{Z}) &\cong \mathrm{Hom}(H_5(\pi_1 M,\mathbb{Z}),\mathbb{Z}) \oplus \mathrm{Ext}(H_4(\pi_1 M,\mathbb{Z}),\mathbb{Z}) \\ &\cong H_4(\pi_1 M,\mathbb{Z}), \end{aligned} \tag{10}$$

where the last $\cong$ follows since $H_4(\pi_1 M,\mathbb{Z})$ is finite. Now, (8), (9) and (10) imply that $H_4(\pi_1 M,\mathbb{Z}) = 0$. □

4 π_1–admissible framed links

In this section we consider π_1–admissible framed links and give a refinement of Theorem 3.1. We also consider π_1–admissible framed links in cylinders over surfaces.

4.1 π_1–admissible framed links in 3–manifolds

Let M be a compact, connected, oriented 3–manifold. Let us call a framed link L in M *π_1–admissible* if

- L is null-homotopic, and
- the linking matrix of L is diagonal with diagonal entries ± 1, or, in other words, L is algebraically split and ± 1–framed.

Surgery along π_1–admissible framed links has been studied by Cochran, Gerges and Orr [3]. (They considered mainly more general framed links.) They proved that for all $d \geq 1$, $\pi_1(M_L)/\Gamma_d\pi_1(M_L) \cong \pi_1(M)/\Gamma_d\pi_1(M)$, where for a group G, $\Gamma_d G$ denotes the d^{th} lower central series subgroup of G defined by $\Gamma_1 G = G$ and $\Gamma_d G = [G, \Gamma_{d-1}G]$ for $d \geq 2$. In this sense, surgery along a π_1–admissible framed link L in a 3–manifold M gives a 3–manifold M_L whose fundamental group is very close to that of M.

Surgery along π_1–admissible framed links was also studied by the first author [8]. To state the result from [8] that we use in this section, we introduce "band-slides" and "Hoste moves", which are two special kinds of moves on π_1–admissible framed links.

A *band-slide* is a pair of algebraically cancelling pair of handle-slides of one component over another; see Figure 2. A band-slide on a π_1–admissible framed link produces a π_1–admissible framed link.

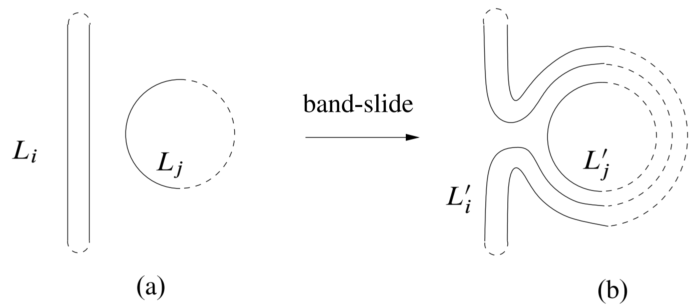

Figure 2: (a) Two components L_i and L_j of a framed link
(b) The result of a band-slide of L_i over L_j

A *Hoste move* is depicted in Figure 3. Let $L = L_1 \cup \cdots \cup L_l$ be a π_1–admissible framed link in M, with an unknotted component L_i with framing $\epsilon = \pm 1$. Since L is π_1–admissible, the linking number of L_i and each component of $L' := L \setminus L_i$ is zero. Let L'_{L_i} denote the framed link obtained from L' by surgery along L_i, which is regarded as a framed link in $M \cong M_{L_i}$. The link L'_{L_i} is again π_1–admissible. Then the framed links L and L'_{L_i} are said to be related by a Hoste move.

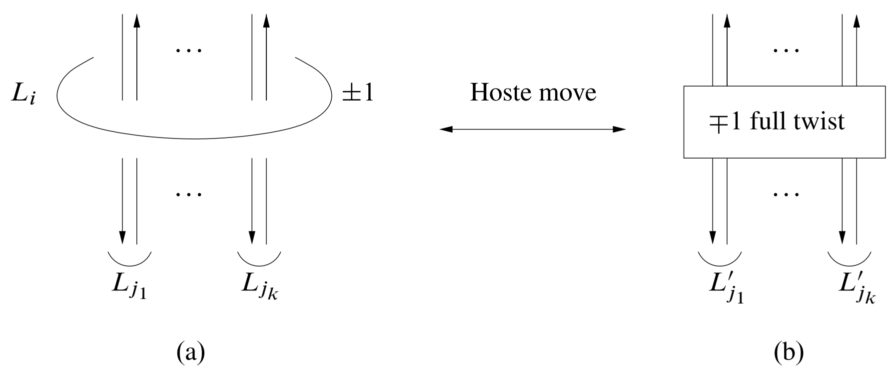

Figure 3: (a) The component L_i of L is unknotted and of framing ± 1
(b) The result L'_{L_i} of a Hoste move on L_i

Proposition 4.1 [8, Proposition 6.1] *For two π_1–admissible framed links L and L' in a connected, oriented 3–manifold M, the following conditions are equivalent:*

(i) *L and L' are related by a sequence of stabilizations and handle-slides.*

(ii) *L and L' are related by a sequence of stabilizations and band-slides.*

(iii) *L and L' are related by a sequence of Hoste moves.*

Theorem 3.1 and Proposition 4.1 immediately imply the following result.

Theorem 4.2 *Let M be a compact, connected, oriented 3–manifold with $n > 0$ boundary components, and let $L, L' \subset M$ be π_1–admissible, framed links. Then the following conditions are equivalent:*

(i) *L and L' are related by a sequence of stabilizations and band-slides.*

(ii) *L and L' are related by a sequence of Hoste moves.*

(iii) *There exists a homeomorphism $h\colon M_L \to M_{L'}$ relative to the boundary such that the following diagram commutes for $k = 1, \dots, n$:*

(11)
$$\begin{tikzcd} \pi_1(M_L; p_1^L, p_k^L) \arrow[rr, "h_k"] \arrow[dr, "e_k"'] & & \pi_1(M_{L'}; p_1^{L'}, p_k^{L'}) \arrow[dl, "e'_k"] \\ & \pi_1(M; p_1, p_k) & \end{tikzcd}$$

4.2 π_1–admissible framed links in cylinders over surfaces

In this subsection, we consider the special cases of Theorem 4.2 where $M = \Sigma_{g,n} \times I$ is the cylinder over a surface $\Sigma_{g,n}$ of genus $g \geq 0$ with $n \geq 0$ boundary components. In this case, Condition (3) in Theorem 4.2 can be weakened.

Let L be a π_1–admissible framed link in the cylinder $M = \Sigma_{g,n} \times I$. By [3, Theorem 6.1], there are natural isomorphisms between nilpotent quotients,

(12)
$$\pi_1 M_L / \Gamma_d \pi_1 M_L \cong \pi_1 M / \Gamma_d \pi_1 M \cong \pi_1 \Sigma_{g,n} / \Gamma_d \pi_1 \Sigma_{g,n},$$

for all $d \geq 1$.

4.2.1 Surfaces with nonempty boundary Consider the case $n \geq 1$. Note that $\partial M = \partial(\Sigma_{g,n} \times I)$ is connected.

Proposition 4.3 *Let L and L' be two π_1–admissible, framed links in $M = \Sigma_{g,n} \times I$ with $n > 0$. Then the following conditions are equivalent:*

(i) *L and L' are related by a sequence of stabilizations and band-slides.*

(ii) *There exists a homeomorphism $h\colon M_L \to M_{L'}$ relative to the boundary.*

Proof That (i) implies (ii) immediately follows from Theorem 4.2.

To prove that (ii) implies (i), one has to show that Diagram (11) commutes for $k = 1$, ie that

$$\begin{tikzcd} \pi_1(M_L; p_1^L) \arrow[rr, "h_1"] \arrow[dr, "e_1"'] & & \pi_1(M_{L'}; p_1^{L'}) \arrow[dl, "e_1'"] \\ & \pi_1(M; p_1) & \end{tikzcd} \tag{13}$$

commutes. This can be checked by using the isomorphism (12). Let $x \in \pi_1(M_L; p_1^L)$. For $d \geq 1$, take the nilpotent quotient of Diagram (13):

$$\begin{tikzcd} \pi_1(M_L; p_1^L)/\,\Gamma_d \arrow[rr, "h_1", "\cong"'] \arrow[dr, "\cong", "e_1"'] & & \pi_1(M_{L'}; p_1^{L'})/\,\Gamma_d \arrow[dl, "\cong"', "e_1'"] \\ & \pi_1(M; p_1)/\,\Gamma_d & \end{tikzcd} \tag{14}$$

where all arrows are isomorphisms. Since the homeomorphism $h\colon M_L \xrightarrow{\cong} M_{L'}$ respects the boundary, Diagram (14) commutes. Hence, for $x \in \pi_1(M_L; p_1^L)$ we have

$$e_1(x) \equiv e_1' h_1(x) \pmod{\Gamma_d \pi_1(M; p_1)}. \tag{15}$$

Since (15) holds for all $d \geq 1$, and since we have $\bigcap_{d\geq 1} \Gamma_d \pi_1(M; p_1) = \{1\}$, it follows that $e_1(x) = e_1' h_1(x)$. Hence Diagram (13) commutes. □

4.2.2 Closed surfaces Now, we consider the case $n = 0$. In this case, the manifold $M = \Sigma_{g,0} \times I$ has two boundary components. Set $F_1 = \Sigma_{g,0} \times \{0\}$ and $F_2 = \Sigma_{g,0} \times \{1\}$. Choose a base point p of $\Sigma_{g,0}$ and set $p_1 = (p, 0) \in F_1$ and $p_2 = (p, 1) \in F_2$.

Proposition 4.4 *Let L and L' be two π_1–admissible, framed links in $M = \Sigma_{g,0} \times I$. Then the following conditions are equivalent:*

(i) *L and L' are related by a sequence of stabilizations and band-slides.*

(ii) *There exists a homeomorphism $h\colon M_L \to M_{L'}$ relative to the boundary such that the following diagram commutes:*

$$\begin{tikzcd} \pi_1(M_L; p_1^L, p_2^L) \arrow[rr, "h_2"] \arrow[dr, "e_2"'] & & \pi_1(M_{L'}; p_1^{L'}, p_2^{L'}) \arrow[dl, "e_2'"] \\ & \pi_1(M; p_1, p_2) & \end{tikzcd} \tag{16}$$

Proof The proof is similar to the proof that (ii) implies (i) for Proposition 4.3; one has to prove that Diagram (11) commutes for $k=1$. This can be done similarly using the fact that

$$\bigcap_{d\geq 1}\Gamma_d\pi_1(M;p_1)=\bigcap_{d\geq 1}\Gamma_d\pi_1(\Sigma_{g,0};p_1)=\{1\}. \qquad \square$$

For the cylinder over the torus $T^2=\Sigma_{1,0}$, we do not need commutativity of (16) in Proposition 4.4.

Proposition 4.5 *Let L and L' be two π_1–admissible, framed links in the cylinder $M=T^2\times I$. Then the following conditions are equivalent:*

(i) *L and L' are related by a sequence of stabilizations and band-slides.*

(ii) *There exists a homeomorphism $h\colon M_L\to M_{L'}$ relative to the boundary.*

Proof By Proposition 4.4 we just have to show that if there exists a homeomorphism $h\colon M_L\to M_{L'}$ relative to the boundary, then there exists a homeomorphism $h'\colon M_L\to M_{L'}$ such that Diagram (16), with h_2 replaced by h_2', commutes.

Consider the cylinder $T^2\times I$. Fix one boundary component while twisting the other once along the meridian (resp. the longitude) of T^2. This defines a self-homeomorphism τ_m (resp. τ_l) on $T^2\times I$ relative to the boundary which maps $\{*\}\times I$, $*\in T^2$, to a line with the same endpoints but which travels once along the meridian (resp. the longitude). A sequence of τ_m and τ_l defines a self-homeomorphism s on $T^2\times I$ by using the composition of maps. Any bijective map $b\colon \pi_1(T^2\times I;p_1,p_2)\to\pi_1(T^2\times I;p_1,p_2)$ of $\pi_1(T^2\times I)$–sets can be induced by such a self-homeomorphism. Let

$$M'_{L'}=M_{L'}\cup_{T^2}(T^2\times I)$$

be a homeomorphic copy of $M_{L'}$ obtained by gluing together $M_{L'}$ and $T^2\times I$ along $F_2\cong T^2\subset M_{L'}$ and $T^2\times\{0\}$ using the identity map. Any self-homeomorphism s on $T^2\times I$ as defined above, extends to a self-homeomorphism $\widetilde{s}$ on $M'_{L'}$. Thus, we can find a self-homeomorphism s on $T^2\times I$ such that the composition $h'=\widetilde{s}\circ h$ defines the commutative Diagram (16). $\square$

Remark 4.6 If $g>1$, then the above proof can not be extended to the closed surface $\Sigma_{g,0}$. In this case, every self-homeomorphism of $\Sigma_{g,0}$ is homotopic to the identity. This can be seen as follows. Every diffeomorphism $g\in\mathrm{Diff}(\Sigma_{g,0}\times I)$ relative to the boundary is homotopic to a diffeomorphism $g'(x,t):=(g_t(x),t)$ with $g_t(x)\in\mathrm{Diff}(\Sigma_{g,0})$. Since g is the identity on the boundaries we have $g_0(x)=g_1(x)=\mathrm{id}_{\Sigma_{g,0}}(x)$. Hence, g_t defines a loop in $\mathrm{Diff}(\Sigma_{g,0})$ and every g_t is homotopic to $\mathrm{id}_{\Sigma_{g,0}}$. Thus, g_t is a loop in the group $\mathrm{Diff}_0(\Sigma_{g,0})$ of diffeomorphisms of $\Sigma_{g,0}$ homotopic to the identity. By a theorem of Earle and Eells [4] the group $\mathrm{Diff}_0(\Sigma_{g,0})$ is contractible when $g>1$. Hence, the loop formed by g_t is homotopic to $\mathrm{id}_{\Sigma_{g,0}}$ and therefore g is homotopic to $\mathrm{id}_{\Sigma_{g,0}\times I}$.

5 Example

5.1 An example

Let us call the equivalence relation on framed links generated by stabilizations and handle-slides the *δ–equivalence*.

The following example shows that commutativity of Diagram (2) for $k=2,\dots,n$ is necessary as well as that for $k=1$.

Let V_1 and V_2 be handlebodies of genus 2 and 1, respectively, embedded in S^3 in a trivial way, and set $M=S^3\setminus\mathrm{int}(V_1\cup V_2)$, $F_k=\partial V_k$ $(k=1,2)$; see Figure 4(a).

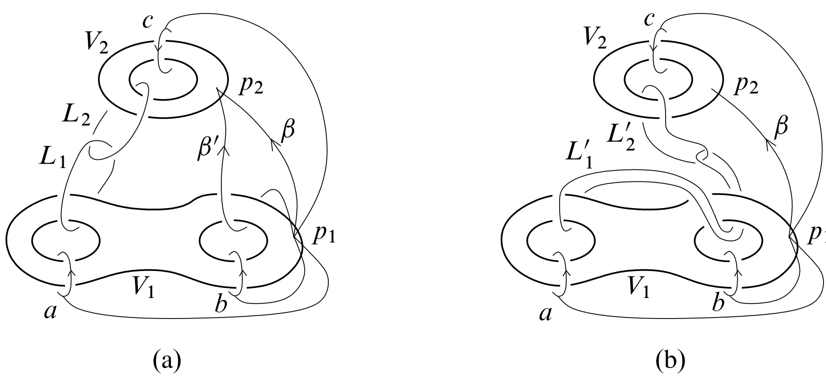

Figure 4

Let $\beta,\beta'\subset M$ be two arcs from $p_1\in F_1$ to $p_2\in F_2$, and let a,b and c be loops based at p_1, as depicted. The fundamental group $\pi_1 M$ is freely generated by $a,b,c\in\pi_1 M$.

Let $L=L_1\cup L_2$ be the framed link in M as depicted in Figure 4(a), where L_1 and L_2 are of framing 0. The result M_L of surgery along L is obtained from M by letting the two handles in V_1 and V_2 clasp each other. $\pi_1 M_L$ has a presentation $\langle a,b,c \mid aca^{-1}c^{-1}=1\rangle$.

Let $f\colon M\xrightarrow{\cong}M$ be a homeomorphism relative to the boundary such that $f(\beta')=\beta$. The image $f(L)=L'=L'_1\cup L'_2$ looks as depicted in Figure 4(b). Let $h\colon M_L\xrightarrow{\cong}M_{L'}$ be the homeomorphism induced by f. Note that $\pi_1 W_L\cong\langle b\rangle\cong\mathbb{Z}$ and $\pi_1 W_{L'}\cong\langle b\rangle\cong\mathbb{Z}$.

Observe that Diagram (2) is commutative for $k = 1$ but not for $k = 2$. Hence Theorem 2.2 can not be used here to deduce that L and L' are δ–equivalent.

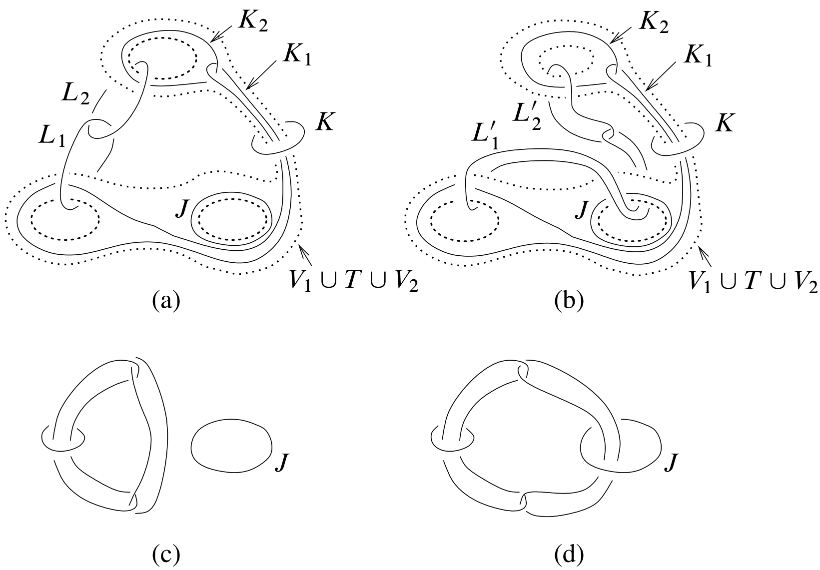

Figure 5

In fact, L and L' are *not* δ–equivalent. We can verify this as follows. Let T be a tubular neighborhood of β in M. Let K be a small 0–framed unknot meridional to T. Let J be a knot in $\operatorname{int} V_1$, to which the loop b is meridional, as depicted in Figure 5(a), (b), and let $N(J)$ denote a small tubular neighborhood of J in V_1. Set $M' = S^3 \setminus \operatorname{int} N(J)$, which is homeomorphic to a solid torus. Let K_1 and K_2 be framed knots as depicted. It suffices to prove that the framed links $\widetilde{L} = L \cup K \cup K_1 \cup K_2$ and $\widetilde{L}' = L' \cup K \cup K_1 \cup K_2$ in M' are not δ–equivalent. Observe that $\widetilde{L}$ (resp. $\widetilde{L}'$) is δ–equivalent to the 3–component link in Figure 5(c) (resp. (d)). (These links are the Borromean rings in S^3 with 0–framings.) The invariant B of framed links defined in Section 5.2 shows that these two links are not δ–equivalent. For the framed links L_c and L_d of Figure 5(c) and (d), respectively, we have $B(L_c) = \{0\}$ and $B(L_d) = \mathbb{Z}$.

5.2 An invariant of O_n–π_1–admissible framed links in the exterior of an unknot in S^3

For $n \geq 0$, let O_n and I_n denote the zero matrix and the identity matrix, respectively, of size n. For $p, q \geq 0$, set $I_{p,q} = I_p \oplus (-I_q)$, where $\oplus$ denotes block sum.

Let J be an unknot in S^3 and set $E = S^3 \setminus \operatorname{int} N(J) \cong S^1 \times D^2$, where $N(J)$ is a tubular neighborhood of J.

Let $L = L^z_1 \cup \cdots \cup L^z_n \cup L^a_1 \cup \cdots \cup L^a_{p+q}$, $n, p, q \geq 0$, be an oriented, ordered, null-homotopic framed link in E whose linking matrix is of the form $O_n \oplus I_{p,q}$. Let us call such a framed link O_n*–*π_1*–admissible*. Let us call $L^z_1, \dots, L^z_n$ the z*–components* of L, and $L^a_1, \dots, L^a_{p+q}$ the a*–components* of L.

Since $L^z_1 \cup \cdots \cup L^z_n \cup J$ is algebraically split, for $1 \leq i < j \leq n$ the triple Milnor invariant $\overline{\mu}(L^z_i, L^z_j, J) \in \mathbb{Z}$ is well defined. Set

$$B(L) = \mathrm{Span}_{\mathbb{Z}}\{\overline{\mu}(L^z_i, L^z_j, J) \mid 1 \leq i < j \leq n\},$$

which is a subgroup of $\mathbb{Z}$. Note that $B(L)$ does not depend on the a–components of L. Note also that $B(L)$ does not depend on the ordering and orientations of the z–components of L.

Lemma 5.1 *$B(L)$ is invariant under handle-slide of a z–component over another z–component.*

Proof It suffices to consider a handle-slide of L^z_1 over L^z_2. The link obtained from L by this handle-slide is

$$L' = (L')^z_1 \cup (L')^z_2 \cup \cdots \cup (L')^z_n \cup (L')^a_1 \cup \cdots \cup (L')^a_{p+q},$$

where $(L')^z_1 = L^z_1 \sharp_b \widetilde{L}^z_2$ is a band sum of L^z_1 and a parallel copy $\widetilde{L}^z_2$ of L^z_2 along a band b, and $(L')^z_i = L^z_i$ for $i = 2, \dots, n$. We have

$$\begin{aligned}
&\overline{\mu}((L')^z_1, (L')^z_2, J) = \overline{\mu}(L^z_1, L^z_2, J), \\
&\overline{\mu}((L')^z_1, (L')^z_i, J) = \overline{\mu}(L^z_1, L^z_i, J) + \overline{\mu}(L^z_2, L^z_i, J) \quad (2 \leq i \leq n), \\
&\overline{\mu}((L')^z_i, (L')^z_j, J) = \overline{\mu}(L^z_i, L^z_j, J) \quad (2 \leq i < j \leq n).
\end{aligned}$$

Hence we have $B(L') = B(L)$. ◻

Lemma 5.2 *$B(L)$ is invariant under band-slides.*

Proof Clearly, a band-slide of an a–component over another (z– or a–) component preserves B. Lemma 5.1 implies that a band-slide of a z–component over another z–component preserves B.

Consider a band-slide of a z–component L^z_1 of L over an a–component L^a_1 of L. Let L' be the resulting link. Let L'' denote the result from L by the same band-slide as before, but we use here the 0–framing of L^a_1 for the band-slide. By the previous case, it follows that $B(L'') = B(L)$. The z–part $(L')^z (= (L')^z_1 \cup \cdots \cup (L')^z_n)$ of L' differs from the z–part $(L'')^z$ of L'' by self-crossing change of the component $(L')^z_1$. Since the triple Milnor invariant is invariant under link homotopy, it follows that $B(L') = B(L'')$. Hence $B(L) = B(L')$. ◻

Proposition 5.3 *If two O_n–π_1–admissible framed links L and L' are δ–equivalent, then we have $B(L) = B(L')$.*

Proof We give a sketch proof assuming familiarity with techniques on framed links developed in [8].

If L and L' are δ–equivalent, then after adding to L and L' some unknotted ± 1–framed components by stabilizations, L and L' become related by a sequence of handle-slides. Clearly, stabilization on an O_n–π_1–admissible framed link preserves B. So, we may assume that L and L' are related by a sequence of handle-slides. It follows that L and L' have the same linking matrix $O_n \oplus I_{p,q}$, $n, p, q \geq 0$.

Recall that for each sequence S of handle-slides between oriented, ordered framed links there is an associated invertible matrix $\varphi(S)$ with coefficients in $\mathbb{Z}$; see eg [8]. In our case, a sequence from L to L' gives a matrix $P \in \mathrm{GL}(n+p+q;\mathbb{Z})$ such that

$$P(O_n \oplus I_{p,q})P^t = (O_n \oplus I_{p,q}). \tag{17}$$

(Here P^t denotes the transpose of P.) Let $H_{n,p,q} < \mathrm{GL}(n+p+q;\mathbb{Z})$ denote the subgroup consisting of matrices satisfying (17). It is easy to see that $H_{n,p,q}$ is generated by the following elements:

(a) $Q \oplus I_{p+q}$, where $Q \in \mathrm{GL}(n;\mathbb{Z})$.

(b) $\begin{pmatrix} I_n & 0 \\ X & I_{p+q} \end{pmatrix}$, where $X \in \mathrm{Mat}_{\mathbb{Z}}(p+q, n)$.

(c) $I_n \oplus R$, where $R \in O(p,q;\mathbb{Z}) = \{T \in \mathrm{GL}(p+q;\mathbb{Z}) \mid T I_{p,q} T^t = I_{p,q}\}$.

Hence $\varphi(S)$ can be expressed as

$$\varphi(S) = w_1^{\epsilon_1} \cdots w_k^{\epsilon_k},$$

where $k \geq 0$, $\epsilon_1, \dots, \epsilon_k \in \{\pm 1\}$, and $w_1, \dots, w_k \in H_{n,p,q}$ are generators of the above form.

By an argument similar to that in [8], we can show that there are framed links $L^{(0)} = L$, $L^{(1)}, \dots, L^{(k)} = L''$ such that:

(i) For $i = 1, \dots, k$, $L^{(k-1)}$ and $L^{(k)}$ are related by a sequence S_i of handle-slides, orientation changes and permutations with associated matrix $\varphi(S_i) = w_i^{\epsilon_i}$.

(ii) There is a sequence of band-slides from L'' and L'.

Here the framed links $L^{(0)}, \dots, L^{(k)}$ are O_n–π_1–admissible.

Let $i=1,\dots,k$. If w_i is a generator of type (b) or (c), then $B(L^{(i-1)})=B(L^{(i)})$ since S_i is a sequence of handle-slides of a–components over other (z– or a–) components. If w_i is a generator of type (a), then S_i is a sequence of orientation changes of z–components, permutations of z–components, and handle-slides of z–components over z–components. Clearly, orientation changes and permutations preserve B. Handle-slides of z–components over z–components also preserve B by Lemma 5.1.

By Lemma 5.2, we have $B(L'')=B(L')$. Hence we have $B(L)=B(L')$. □

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