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The (n)–solvable filtration of link concordance and Milnor's invariants

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We establish several new results about both the (n)–solvable filtration of the set of link concordance classes and the (n)–solvable filtration of the string link concordance group, $\mathcal{C}^m$. The set of (n)–solvable m–component string links is denoted by $\mathcal{F}_n^m$. We first establish a relationship between Milnor's invariants and links, L, with certain restrictions on the 4–manifold bounded by M_L, the zero-framed surgery of S^3 on L. Using this relationship, we can relate (n)–solvability of a link (or string link) with its Milnor's $\bar{\mu}$–invariants. Specifically, we show that if a link is (n)–solvable, then its Milnor's invariants vanish for lengths up to $2^{n+2}-1$. Previously, there were no known results about the "other half" of the filtration, namely $\mathcal{F}_{n.5}^m/\mathcal{F}_{n+1}^m$. We establish the effect of the Bing doubling operator on (n)–solvability and using this, we show that $\mathcal{F}_{n.5}^m/\mathcal{F}_{n+1}^m$ is nontrivial for links (and string links) with sufficiently many components. Moreover, we show that these quotients contain an infinite cyclic subgroup. We also show that links and string links modulo (1)–solvability is a nonabelian group. We show that we can relate other filtrations with Milnor's invariants. We show that if a link is n–positive, then its Milnor's invariants will also vanish for lengths up to $2^{n+2}-1$. Lastly, we prove that the grope filtration of the set of link concordance classes is not the same as the (n)–solvable filtration.

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1 Introduction

In order to study the structure of both the knot and (string) link concordance groups, Cochran, Orr and Teichner defined the (n)–solvable filtration [10]. Much work has been done in the quest of understanding the (n)–solvable filtration. In particular, many have studied successive quotients of this filtration and some of their contributions can be found in Cha [2], Cochran and Harvey [6], Cochran, Harvey and Leidy [9], and Harvey [14].

For example, Harvey first showed that $\mathcal{F}_n^m/\mathcal{F}_{n+1}^m$ is a nontrivial group that contains an infinitely generated subgroup [14]. She also showed that this subgroup is generated by boundary links (links with components that bound disjoint Seifert surfaces). Cochran

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and Harvey generalized this result by showing that $\mathcal{F}^m_n/\mathcal{F}^m_{n.5}$ contains an infinitely generated subgroup [6]. Again, this subgroup consists entirely of boundary links.

Up to this point, little has been known about the relationship of Milnor's $\bar{\mu}$–invariants and (n)–solvability. We first establish the following relationship, which is the main theorem of this paper.

Theorem 3.1 *Suppose L is a link whose zero-framed surgery, M_L, bounds an orientable and compact 4–manifold W such that:*

(1) *$H_1(M_L) \to H_1(W;\mathbb{Z})$ is an isomorphism induced by the inclusion map.*

(2) *$H_2(W;\mathbb{Z})$ has a basis consisting of connected compact oriented surfaces $\{L_i\}$ with $\pi_1(L_i) \subset \pi_1(W)^{(n)}$.*

Then $\bar{\mu}_L(I) = 0$ for $|I| \le 2^{n+2} - 1$, where $\bar{\mu}_L(I)$ is the length-I Milnor's invariant of L defined in Section 2.

Using this theorem we obtain the following relationship.

Corollary 3.5 *If L is an (n)–solvable link (or string link), then $\bar{\mu}_L(I) = 0$ for $|I| \le 2^{n+2} - 1$.*

In other words, if a link (or string link) is (n)–solvable, then all of its $\bar{\mu}$–invariants will vanish for lengths less than or equal to $2^{n+2} - 1$. Moreover, this theorem is sharp in the sense that we exhibit (n)–solvable links with $\bar{\mu}(I) \neq 0$ for $|I| = 2^{n+2}$.

We can also obtain a relationship between Milnor's invariants and other filtrations. Specifically, we have the following result that relates Milnor's invariants to n–positive, n–negative and n–bipolar filtrations. The definitions of these filtrations can be found in Cochran, Harvey and Horn [8].

Corollary 3.6 *If L is an n–positive, n–negative or n–bipolar link, then $\bar{\mu}_L(I) = 0$ for $|I| \le 2^{n+2} - 1$.*

We study the effects of Bing doubling on (n)–solvability. We show that solvability is not only preserved under this operator, but it increases the solvability by one.

Proposition 4.7 *If L is an (n)–solvable link, then* $\mathrm{BD}(L)$ *is $(n+1)$–solvable. Moreover, if L is an $(n.5)$–solvable link, then* $\mathrm{BD}(L)$ *is $((n+1).5)$–solvable.*

Until this point, nothing was known about the "other half" of the filtration, $\mathcal{F}^m_{n.5}/\mathcal{F}^m_{n+1}$. Using the above results, we show that the "other half" of the (n)–solvable filtration is nontrivial.

Theorem 5.1 *The "other half" of the filtration $\mathcal{F}^m_{n.5}/\mathcal{F}^m_{n+1}$ contains an infinite cyclic subgroup for $m \geq 3 \cdot 2^{n+1}$.*

The examples used come from iterated Bing doubles of links with nonvanishing $\bar{\mu}$–invariants. Thus, our examples are not concordant to boundary links, so the subgroups that they will generate will be different than those previously detected. The result of Theorem 5.1 is still unknown for knots.

Since the knot concordance group, $\mathcal{C}$, is abelian, all successive quotients of the (n)–solvable filtration are abelian. However, it is known that the m–component (string) link concordance group, $\mathcal{C}^m$, is nonabelian for $m \geq 2$; see Le Dimet [16]. We have shown that certain successive quotients are not abelian.

Theorem 5.3 *We have $\mathcal{F}^m_{-0.5}/\mathcal{F}^m_1$ is nonabelian for $m \geq 3$.*

Similar to the relationship between (n)–solvability and $\bar{\mu}$–invariants, we establish a relationship between $\bar{\mu}$–invariants and a link in which all of its components bound disjoint gropes of height n. This relationship says that if all components of a link bound disjoint gropes of a certain height, then its $\bar{\mu}$ invariants vanish for certain lengths. See Section 6 for the definition of a grope.

Corollary 6.6 *A link L with components that bound disjoint gropes of height n has $\bar{\mu}_L(I) = 0$ for $|I| \leq 2^n$.*

A result of Lin [17] states that k–cobordant links will have the same $\bar{\mu}$–invariants up to length $2k$. Using this result, the proof of this proposition relies on showing that L is 2^{n-1}–cobordant to a slice link.

These two filtrations are related by the fact that $\mathcal{G}^m_{n+2} \subseteq \mathcal{F}^m_n$ for all $n \in \mathbb{N}$ and $m \geq 1$ [10]. A natural question is whether or not these filtrations are actually the same. We show that these filtrations differ at each stage.

Corollary 6.8 *We have that $\mathcal{F}^m_n/\mathcal{G}^m_{n+2}$ is nontrivial for $m \geq 2^{n+2}$. Moreover, $\mathbb{Z} \subset \mathcal{F}^m_n/\mathcal{G}^m_{n+2}$.*

It is still unknown whether the previous result holds for knots.

2 Preliminaries

A *knot* is an embedding $S^1 \hookrightarrow S^3$. The set of knots modulo concordance forms a group under the operation of connected sum, known as the knot concordance group $\mathcal{C}$. Two knots K and J are said to be *concordant* if $K \times \{0\}$ and $J \times \{1\}$ cobound a smoothly embedded annulus in $S^3 \times [0,1]$.

An *m–component link* is an embedding $\coprod_m S^1 \hookrightarrow S^3$. The connected sum operation is not well defined for links. Therefore, in order to define a group structure on links, it is necessary to study string links.

Definition 2.1 Let D be the unit disk, I the unit interval and $\{p_1, p_2, \dots, p_k\}$ be k points in the interior of D. A *k–component (pure) string link* is a smooth proper embedding $\sigma\colon \coprod_{i=1}^k I_i \to D \times I$ such that

$$\sigma|_{I_i}(0) = \{p_i\} \times \{0\},$$
$$\sigma|_{I_i}(1) = \{p_i\} \times \{1\}.$$

The image of I_i is called the i^{th} *string* of the string link. An orientation on σ is induced by the orientation of I. Two string links σ and σ' are said to be *equivalent* if there is an orientation preserving homeomorphism $h\colon D^2 \times I \to D^2 \times I$ such that h fixes the boundary piecewise and $h(\sigma) = \sigma'$.

The operation on string links is the stacking operation seen in the braid group. If L_1 and L_2 are in $\mathcal{C}^m$, then $L_1 L_2$ is the string link obtained by stacking L_1 on top of L_2.

The notion of concordance can be generalized for string links; see Figure 1.

Definition 2.2 Two m–component string links σ_1 and σ_2 are *concordant* if there exists a smooth embedding $H\colon \coprod_m (I \times I) \to B^3 \times I$ that is transverse to the boundary and such that $H|_{\coprod_m I \times \{0\}} = \sigma_1$, $H|_{\coprod_m I \times \{1\}} = \sigma_2$, and $H|_{\coprod_m \partial I \times I} = j_0 \times \mathrm{id}_I$, where $j_0\colon \coprod_m \partial I \to S^2$.

Under the operation of stacking, the concordance classes of m–component string links form a group, denoted $\mathcal{C}^m$, and is known as the string link concordance group. The identity class of this group is the class of slice string links. The inverses are the string links obtained by reflecting the string link about $D \times \{\frac{1}{2}\}$ and reversing the orientation. When $m = 1$, $\mathcal{C}^m$ is the knot concordance group. For $m \geq 2$, it has been shown that $\mathcal{C}^m$ is not abelian [16].

If L is a string link, the closure of L, denoted $\widehat{L}$, is the ordered, oriented link in S^3 obtained by gluing $\partial(D^2 \times I)$ to $\partial(D^2 \times I)$ of the standard trivial string link using the

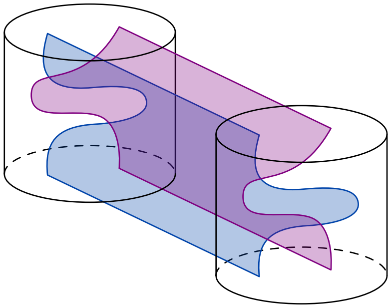
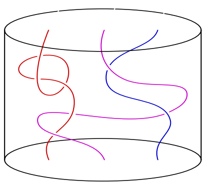
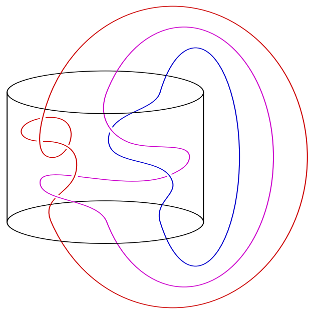


Figure 1: String link concordance



(a) An example of a three component string link



(b) The closure of a string link

Figure 2: A string link and its closure

identity map; see Figure 3(b). This gives a canonical way to obtain a link from a string link. If two string links are concordant, then their closures are concordant as links.

Every link has a string link representative. In other words, given any link, L in S^3, there exists a string link σ such that $\widehat{\sigma}$ is isotopic to L; see Habegger and Lin [13].

In order to study the structure of $\mathcal{C}^m$, Cochran, Orr and Teichner [10] defined the (n)–solvable filtration, $\{\mathcal{F}^m_n\}$, by

$$\{0\} \subset \cdots \subset \mathcal{F}^m_{n+1} \subset \mathcal{F}^m_{n.5} \subset \mathcal{F}^m_n \subset \cdots \subset \mathcal{F}^m_{0.5} \subset \mathcal{F}^m_0 \subset \mathcal{C}^m.$$

Definition 2.3 An m–component link L is (n)*–solvable* if the zero-framed surgery, denoted M_L, bounds a compact, smooth 4–manifold, W^4, such that the following conditions hold.

(i) We have $H_1(M_L;\mathbb{Z}) \cong \mathbb{Z}^m$ and $H_1(M_L) \to H_1(W;\mathbb{Z})$ is an isomorphism induced by the inclusion map.

(ii) We have $H_2(W;\mathbb{Z})$ has a basis consisting of connected, compact, oriented surfaces, $\{L_i, D_i\}_{i=1}^r$, embedded in W with trivial normal bundles, where the surfaces are pairwise disjoint except that, for each i, L_i intersects D_i transversely once with positive sign.

(iii) For all $i, \pi_1(L_i) \subset \pi_1(W)^{(n)}$ and $\pi_1(D_i) \subset \pi_1(W)^{(n)}$, where $\pi_1(W)^{(n)}$ is the n^{th} term of the derived series. The derived series of a group G, denoted $G^{(n)}$, is defined recursively by $G^{(0)} := G$ and $G^{(i)} := [G^{(i-1)}, G^{(i-1)}]$.

The manifold W is called an (n)*–solution* for L and a string link is (n)*–solvable* if its closure in S^3 is an (n)–solvable link.

A link is $(n.5)$*–solvable* if, in addition to the above, $\pi_1(L_i) \subset \pi_1(W)^{(n+1)}$ for all i. In this instance, W is called an $(n.5)$*–solution* for L.

We define the (n)–solvable filtration of the string link concordance group for $m \geq 1$ by setting $\mathcal{F}_n^m$ to be the set of (n)–solvable string links for $n \in \frac{1}{2}\mathbb{N}_0$. It is known that $\mathcal{F}_n^m$ is a normal subgroup of $\mathcal{C}^m$ for all $m \geq 1$ and $n \in \frac{1}{2}\mathbb{N}_0$. For convenience, $\mathcal{F}_{-0.5}^m$ will denote the set of string links with all pairwise linking numbers equal to zero.

In the early 1950s, John Milnor defined a family of higher-order linking numbers known as $\bar{\mu}$–invariants for oriented, ordered links in S^3 [20; 21]. These numbers are not link invariants in the typical sense since there is some indeterminacy due to the choice of meridians of a link; however, as invariants of string links they are well defined [13]. In general, Milnor's invariants determine how deep the longitudes of each component lie in the lower central series of the link group. We will show in Corollary 3.5 that these invariants give information about the solvable filtration. We will use this relationship to prove Theorem 5.1.

Suppose L is an m–component link in S^3. Let $G = \pi_1(S^3 - L)$ be the fundamental group of the complement of L in S^3. The *lower central series* of G, denoted G_i, is recursively defined by $G_1 := G$ and $G_i := [G_{i-1}, G]$. It is worthy to note that the derived series and the lower central series of a group G are related by $G^{(n)} \subset G_{2^n}$. Since $[G_r, G_s] \subseteq G_{r+s}$, it is a straightforward computation to achieve the relation.

Consider the nilpotent quotient group G/G_k. A presentation of this group, given by Milnor [21], can be written

$$G/G_k \cong \langle \alpha_1, \alpha_2, \dots, \alpha_m \mid [\alpha_1, l_1], [\alpha_2, l_2], \dots, [\alpha_m, l_m], F_k \rangle, \tag{1}$$

where $\alpha_1, \dots, \alpha_m$ are a choice of m meridians for L, F_k is the k^{th} term of the lower central series of $F = F\langle \alpha_1, \dots, \alpha_m \rangle$, the free group on m generators and l_i is the i^{th} longitude of L written as a product of the α_i.

Let $\mathbb{Z}[\![X_1,\ldots,X_m]\!]$ be the ring of power series in m noncommuting variables. The *Magnus embedding* is a map $E\colon \mathbb{Z}F \to \mathbb{Z}[\![X_1,\ldots,X_m]\!]$ defined by sending $\alpha_i \mapsto 1+X_i$ and $\alpha_i^{-1} \mapsto 1-X_i+X_i^2-X_i^3+\cdots$ for $1\le i\le m$. Let $I=i_1i_2\cdots i_{r-1}i_r$ be a string of integers amongst $\{1,\ldots,m\}$ with possible repeats and let $X_I = X_{i_1}X_{i_2}\cdots X_{i_r}$. We let $\epsilon_I(l)=\epsilon(\delta_{i_1}\cdots\delta_{i_r}(l))$; see [21] for more details. If $l\in F$, then the image of l under the Magnus embedding has the form

$$E(l) = 1+\sum_{|I|\ge 1}\epsilon_I(l)X_I.$$

For each j, let w_j denote a word in F which represents the image of l_j in G/G_k. Then, for $I=i_1i_2\cdots i_{r-1}i_r$ with $r\le k$, the coefficient $\epsilon_{I'}(w_{i_r})$, where $I'=i_1\cdots i_{r-1}$ is denoted as $\mu_L(I)$. Milnor's invariant $\overline{\mu}_L(I)$ is defined as the residue class of $\mu_L(I)$ modulo the greatest common divisor of $\mu_L(J)$, where J runs over all sequences obtained by cyclically permuting proper subsequences of I. We refer to $|I|$ as the length of the Milnor's invariant. It is useful to note that the first nonvanishing $\overline{\mu}$–invariant, $\overline{\mu}_L(I)$, will be $\mu_L(I)$ since it is well defined.

For $\overline{\mu}$–invariants of length two, the calculation measures the linking between two components, ie $\overline{\mu}_L(ij)$ is the linking number between the i^{th} and j^{th} components of L. It is also known that $\overline{\mu}$–invariants are concordance invariants; see Casson [1].

The following is a classical and well-known result of Milnor [21].

Theorem 2.4 (Milnor) *The longitudes of L lie in G_{k-1} if and only if $F/F_k\cong G/G_k$. In other words, $\overline{\mu}_L(I)=0$ for $|I|\le k-1$ if and only if $F/F_k\cong G/G_k$.*

The following corollary allows us to detect whether certain Milnor's invariants are zero using the fundamental group of M_L, the zero-framed surgery on L.

Corollary 2.5 *We have $F/F_{k+1}\cong G/G_{k+1}$ if and only if $F/F_k\cong J/J_k$, where $J=\pi_1(M_L)$.*

An outline of the proof is as follows. Let L_i be the i^{th} component of L. The group G/G_{k+1} has presentation given by

$$G/G_{k+1}\cong\langle x_1,\ldots,x_m \mid [x_i,\lambda_i],\, F_{k+1}\rangle,$$

where λ_i is the longitude of L_i and x_i is a meridian of L_i. The inclusion of S^3-L into M_L induces an epimorphism on fundamental groups that has kernel normally generated by $\lambda_1,\ldots,\lambda_m$. The fundamental group $J=\pi_1(M_L)$ is obtained from G by setting the longitudes λ_i to zero. This gives the presentation $J/J_k\cong\langle x_1,\ldots,x_m \mid \lambda_i,\, F_k\rangle$.

Suppose that the map induced from inclusion from G/G_{k+1} to F/F_{k+1} is an isomorphism. Then $[x_i, \lambda_i] \in F_{k+1}$, and thus $\lambda_i \in F_k$ since x_i is a generator of F. It is apparent that $J/J_k \cong F/F_k$.

Conversely, if $J/J_k \cong F/F_k$ then the relations show $\lambda_i \in F_k$ and thus $[x_i, \lambda_i] \in F_{k+1}$. This gives that $G/G_{k+1} \cong F/F_{k+1}$. It follows that $\bar{\mu}_L(I) = 0$ for $|I| \le k$ if and only if $F/F_k \cong J/J_k$.

3 Main theorem

Before now, little has been known about the relationship between Milnor's invariants and (n)–solvability. The following theorem demonstrates a relationship between Milnor's invariants and links, L, with certain restrictions on the 4–manifold bounded by M_L. This theorem will be used to find an obstruction that detects an infinite subgroup of $\mathcal{F}^m_{n.5}/\mathcal{F}^m_{n+1}$ as well as to show that $\mathcal{F}^m_{-0.5}/\mathcal{F}^m_1$ is a nonabelian group.

Theorem 3.1 *Suppose L is a link whose zero-framed surgery, M_L, bounds an orientable and compact 4–manifold W such that:*

(1) *$H_1(M_L) \to H_1(W;\mathbb{Z})$ is an isomorphism induced by the inclusion map.*

(2) *$H_2(W;\mathbb{Z})$ has a basis consisting of connected compact oriented surfaces $\{L_i\}$ with $\pi_1(L_i) \subset \pi_1(W)^{(n)}$.*

Then $\bar{\mu}_L(I) = 0$, where $|I| \le 2^{n+2} - 1$.

Proof As mentioned above in Theorem 2.4, $\bar{\mu}_L(I)$=0 for all $|I| \le k$ for any m–component link L in S^3 if and only if $F/F_{k+1} \cong G/G_{k+1}$, where $F = F\langle x_1, \dots, x_m\rangle$ and $G = \pi_1(S^3 - L)$. Using Corollary 2.5, this is equivalent to F/F_k being isomorphic to J/J_k, where $J = \pi_1(M_L)$.

Consider the meridians about the link components of L. By connecting these meridians by arcs in $S^3 - L$, we can view this as a wedge of circles. Let F be the fundamental group of this wedge of circles, and consider the following sequence of maps on π_1 induced by inclusion

$$F \xrightarrow{\phi_1} G \xrightarrow{\phi_2} J \xrightarrow{\phi_3} E = \pi_1(W).$$

The map ϕ_2 is the surjection induced by the inclusion of $S^3 - L$ into M_L and has kernel normally generated by the longitudes. The quotients of all of these groups by the k^{th} terms of their lower central series gives another sequence of maps

$$F/F_k \xrightarrow{\bar{\phi}_1} G/G_k \xrightarrow{\bar{\phi}_2} J/J_k \xrightarrow{\bar{\phi}_3} E/E_k.$$

Since ϕ_2 is surjective, the map $\overline{\phi}_2\colon G/G_k \to J/J_k$ is a surjection for all values of k.

Dwyer's theorem [12] is of particular importance and is stated here for convenience.

Theorem 3.2 (Dwyer's integral theorem) *Let $\phi\colon A \to B$ be a homomorphism that induces an isomorphism on $H_1(-;\mathbb{Z})$. Then for any positive integer k, the following are equivalent:*

(i) *ϕ induces an isomorphism $A/A_{k+1} \cong B/B_{k+1}$.*

(ii) *ϕ induces an epimorphism $H_2(A;\mathbb{Z})/\Phi_k(A) \to H_2(B;\mathbb{Z})/\Phi_k(B)$.*

Here $\Phi_k(A) = \ker(H_2(A) \to H_2(A/A_k))$ for $k \geq 1$.

Consider the map induced by $\phi_3 \circ \phi_2 \circ \phi_1$,

(2) $$H_2(F;\mathbb{Z})/\Phi_k(F) \to H_2(E;\mathbb{Z})/\Phi_k(E),$$

where $E = \pi_1(W)$. By Theorem 3.2, showing (2) is a surjection is equivalent to showing $\phi := \overline{\phi}_3 \circ \overline{\phi}_2 \circ \overline{\phi}_1\colon F/F_{k+1} \to E/E_{k+1}$ is an isomorphism.

Since F is the free group on m generators, $H_2(F;\mathbb{Z}) = 0$. The map of (2) is a surjection precisely when $\Phi_k(E) = H_2(E,\mathbb{Z})$. We need to determine for which k we have $\Phi_k(E) = H_2(E,\mathbb{Z})$.

Consider the diagram

$$\begin{tikzcd} H_2(W_k) \arrow[r, "p_*"] \arrow[d] & H_2(W) \arrow[d] & \\ H_2(E_k) \arrow[r, "i_*"] & H_2(E) \arrow[r, "\pi_*"] & H_2(E/E_{2k-1}), \end{tikzcd}$$

where W_k is the covering space of W that corresponds with the k^{th} term of the lower central series of $\pi_1(W)$. The vertical maps are surjections obtained from the exact sequence induced by the Hurewicz map

(3) $$\pi_2(X) \to H_2(X) \to H_2(\pi_1(X)) \to 1.$$

The maps p_*, i_* and π_* are the maps induced by the covering map p, inclusion and projection respectively.

By assumption, there is a basis of $H_2(W)$ consisting of surfaces, denoted $\{L_i\}$. The group $H_2(E)$ is generated by the images of the L_i since $H_2(W) \to H_2(E)$ is a surjection.

We claim that the map i_* is a surjection. This can be seen by viewing $H_2(E_k)$ as the second homology group for the covering space, $K(E_k,1)$ of the Eilenberg–Mac Lane space $K(E,1)$. Note that $K(E_k,1)$ is the covering space of $K(E,1)$ corresponding to the subgroup E_k of E. When $k=2^n$, $\pi_1(L_i)\subset E_k$, hence the images of $\{L_i\}$ in $H_2(E)$ will lift to $H_2(E_k)$ and $i_*\colon H_2(E_k)\to H_2(E)$ is surjective.

Cochran and Harvey [7, Lemma 5.4] showed that the composition of the maps

$$H_2(E_k)\xrightarrow{i_*}H_2(E)\xrightarrow{\pi_*}H_2(E/E_{2k-1})$$

is the zero map for all k. Since i_* is surjective, this implies that π_* is the zero map. Hence $\Phi_{2k-1}(E)=H_2(E)$ and Dwyer's theorem gives an isomorphism $H_2(F)/\Phi_{2k-1}(F)\to H_2(E)/\Phi_{2k-1}(E)$ induced by $\phi_3\circ\phi_2\circ\phi_1$ for $k=2^n$. In turn, this gives an isomorphism $F/F_{2k}\cong E/E_{2k}$ when $k=2^n$. Thus we have that $\widehat{\phi}:=\overline{\phi}_2\circ\overline{\phi}_1\colon F/F_{2^{n+1}}\to J/J_{2^{n+1}}$ is a monomorphism. Since $\overline{\phi}_1$ is a map $F/F_k\to F/\langle\text{relations}, F_k\rangle$ and ϕ_2 is a surjection, by Milnor's presentation (1), $\widehat{\phi}$ is a surjection and thus an isomorphism. It is also of note that the map $\overline{\phi}_3$ is an isomorphism.

By Theorem 2.4 and Corollary 2.5, the $\overline{\mu}$–invariants of length less than or equal to 2^{n+1} vanish for (n)–solvable links.

This result can be improved slightly. Let $g=(\widehat{\phi})^{-1}$ be a specified isomorphism. Let f be the composite of the maps

$$J\xrightarrow{\pi_J}J/J_{2^{n+1}}\xrightarrow{g}F/F_{2^{n+1}},$$

where π_J is the canonical quotient map. Consider the commutative diagram

$$\begin{tikzcd}
E/E_{2^{n+1}} & J/J_{2^{n+1}} \arrow[l, "\overline{\phi}_3"', "\cong"] & F/F_{2^{n+1}} \arrow[l, "g^{-1}"', "\cong"] \\
E \arrow[u, "\pi_E"'] & J \arrow[l] \arrow[u, "\pi_J"] \arrow[ur, "g\circ\pi_J=f"']
\end{tikzcd}$$

where ϕ is the isomorphism between $J/J_{2^{n+1}}$ and $E/E_{2^{n+1}}$ established earlier in the proof and π_E is the canonical quotient map. Thus we have an extension of f to E, namely $\overline{f}=g\circ\phi^{-1}\circ\pi_E\colon E\to F/F_{2^{n+1}}$. This gives the following commutative diagram:

$$\begin{tikzcd}
\pi_1(M_L) \arrow[r, "f"] \arrow[d, "i_*"'] & F/F_{2^{n+1}} \\
\pi_1(W) \arrow[ur, "\overline{f}"']
\end{tikzcd}$$

The commutative diagram below on homology is achieved by the induced maps obtained from the above maps:

$$\begin{tikzcd} H_3(M_L) \arrow[r, "f"] \arrow[d, "i_*"'] & H_3(F/F_{2^{n+1}}) \\ H_3(W) \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

Since $M_L = \partial W$, we have that $i_*\colon H_3(M_L) \to H_3(W)$ is the zero map. Also, the map $f\colon H_3(M_L) \to H_3(F/F_{2^{n+1}})$ is the zero map since the diagram commutes. Consider the sequence of maps

$$H_3(F/F_{2k-1}) \xrightarrow{h_{k-1}} H_3(F/F_{2k-2}) \xrightarrow{h_{k-2}} \cdots \xrightarrow{h_2} H_3(F/F_{k+1}) \xrightarrow{h_1} H_3(F/F_k),$$

where $h_i\colon H_3(F/F_{k+i}) \to H_3(F/F_{k+i-1})$ and $k = 2^{n+1}$. The image of the fundamental class under the map $H_3(M_L) \to H_3(F/F_m)$ will be denoted by $\theta_m(M_L, f)$.

We will use the following two results of Cochran, Gerges and Orr [5]. These results rely heavily on deep work of Igusa and Orr [15].

Lemma 3.3 (Cochran–Gerges–Orr) *We have*

$$\theta_m(M_L, f) \in \mathrm{Image}(\pi_*\colon H_3(F/F_{m+1}) \to H_3(F/F_m))$$

if and only if there is some isomorphism $\tilde{f}\colon J/J_{m+1} \to F/F_{m+1}$ extending f such that $\pi_*(\theta_{m+1}(M_L, \tilde{f})) = \theta_m(M_L, f)$.

Corollary 3.4 (Cochran–Gerges–Orr) *The map $H_3(F/F_{2m-1}) \to H_3(F/F_m)$ is the zero map. Any element in the kernel of $H_3(F/F_{m+j}) \to H_3(F/F_m)$, $j \le m-1$, lies in the image of $H_3(F/F_{2m-1}) \to H_3(F/F_{m+j})$.*

Since $\theta_{2^{n+1}}(M_L, f) = [0]$ and $[0]$ is always in the image of a homomorphism, there is an extension of f to an isomorphism $\tilde{f}\colon J/J_{k+1} \to F/F_{k+1}$ satisfying $h_1(\theta_{k+1}(M_L, \tilde{f})) = \theta_k(M_L, f) = 0$ by Lemma 3.3 and $\theta_{k+1}(M_L, \tilde{f})$ is in the kernel of h_1. Then $\theta_{k+1}(M_L, \tilde{f})$ lies in the image of $H_3(F/F_{2k-1}) \to H_3(F/F_{k+1})$ by Corollary 3.4. In other words, it lies in the image of the map $h_2 \circ h_3 \circ \cdots \circ h_{k-1}$ and in turn lies in the image of h_2. By Lemma 3.3, there is an extension of $\tilde{f}$ that is an isomorphism between J/J_{k+2} and F/F_{k+2}. By continuing this process, an isomorphism between J/J_{2k-1} and F/F_{2k-1} with $k = 2^{n+1}$ is obtained. Thus we have that the $\bar{\mu}$–invariants of lengths less than or equal to $2^{n+2}-1$ of our link vanish. This concludes the proof of Theorem 3.1. □

We end this section with several remarks involving the limitations and generalizations of Theorem 3.1. We first notice that the proof of Theorem 3.1 did not rely on the intersection form seen in the definition of (n)–solvability. As a consequence, we obtain the following corollary.

Corollary 3.5 *If L is an (n)–solvable link (or string link), then $\bar{\mu}_L(I) = 0$ for $|I| \leq 2^{n+2} - 1$.*

The converse of Corollary 3.5 is false. Consider the Whitehead link $= W$ in Figure 3. The first nonvanishing $\bar{\mu}$–invariant occurs at length four. One of these invariants is $\bar{\mu}_W(1122) = \pm 1$, depending on orientation. The figure eight knot, 4_1, may be obtained as the result of band summing the two components of W. It is known that this knot is not (0)–solvable since its Arf invariant is nonzero [10]. It is known that if a link is (n)–solvable, then the result of bandsumming any two components is (n)–solvable; see Cochran, Orr and Teichner [25] for a complete proof. This implies that the Whitehead link is not (0)–solvable.

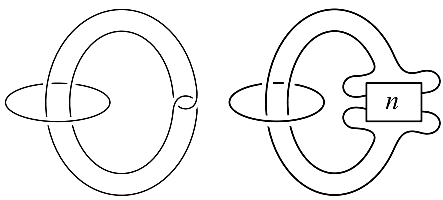


Figure 3: The Whitehead link and the n–twisted Whitehead link

We also remark that the result of Corollary 3.5 is sharp in the sense the length of vanishing $\bar{\mu}$–invariants cannot be extended. Consider the n–twisted Whitehead link in Figure 3. The number n represents the number of full twists. When n is even, this link is band pass equivalent to the trivial link. We will see in Section 4 that this implies the link is (0)–solvable. However, $\bar{\mu}(1122) = -n$ and $\bar{\mu}(1212) = 2n$ are the first nonvanishing $\bar{\mu}$–invariants. There are examples of (n)–solvable links L with $\bar{\mu}_L(I) = \pm 1$ for some $|I| = 2^{n+1}$; see Section 5 for examples using Bing doubling.

Recently, Cochran, Harvey and Horn defined several new filtrations of $\mathcal{C}^m$ [8], namely the n–positive, n–negative, and n–bipolar filtrations. The proof of Theorem 3.1 also can be applied to all these filtations as well.

Corollary 3.6 *If L is an n–positive, n–negative or n–bipolar link, then $\bar{\mu}_L(I) = 0$ for $|I| \leq 2^{n+2} - 1$.*

4 Bing doubling and solvability

The goal of this section is to understand the effect that Bing doubling has on solvability. Bing doubling is a doubling operator performed on knots and links. If K is a knot, then the two-component link in Figure 4 is the Bing double of K, denoted $\mathrm{BD}(K)$. If L is an m–component link, $\mathrm{BD}(L)$ denotes be the $2m$–component link obtained by Bing doubling every component of L.

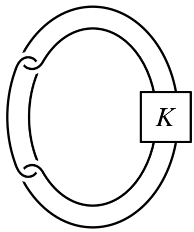
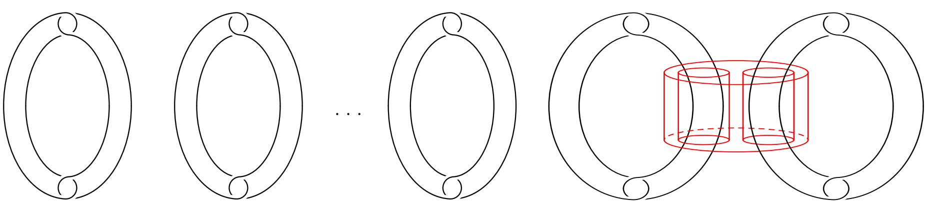


Figure 4: The Bing double of a knot K, $\mathrm{BD}(K)$

Bing doubling can also be viewed as multi-infection by a string link. Let $L = L_1 \cup L_2 \cup \cdots \cup L_m$ in S^3 be an m component link in S^3. Let L_{BD} be the $2m$–component link pictured in Figure 5 that is isotopic to the $2m$–component trivial link. Then there is a handlebody H in $S^3 - L_{\mathrm{BD}}$ which is the exterior of a trivial string link with m components; see Figure 5 for an example. The η_i are curves in $S^3 - L_{\mathrm{BD}}$ and are the canonical meridians of the trivial string link.

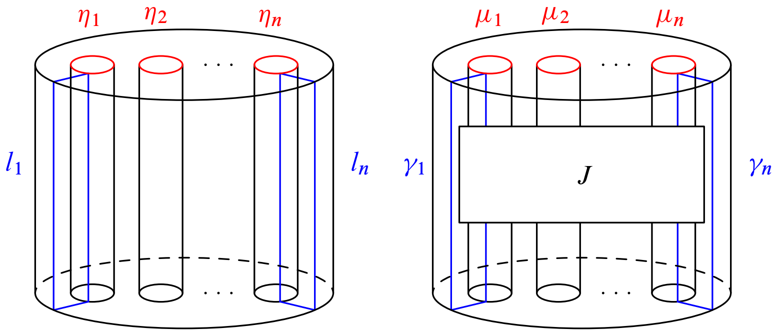


(a) The trivial link L_{BD} (b) A handlebody in $S^3 - L_{\mathrm{BD}}$

Figure 5

Take a string link J, such that $\widehat{J}$ is isotopic to L. There are an infinite number of string links that meet this criterion. Then $S^3 = ((S^3 - H) \cup_\phi ((D^2 \times I) - J)$ and $\mathrm{BD}(L)$ is the image of L_{BD} in this new S^3. The map ϕ maps $l_i \mapsto \gamma_i$ and $\mu_i \mapsto \eta_i^{-1}$, where the l_i, γ_i, μ_i and η_i are depicted in Figure 6.

We will consider geometric moves that can be performed on knots and links and determine their effects on solvability. The first move we consider is the band pass move, illustrated in Figure 7.



(a) Exterior of the trivial string link, the handlebody H

(b) Exterior of the string link J

Figure 6: The longitudes l_i, γ_i and meridians μ_i, η_i

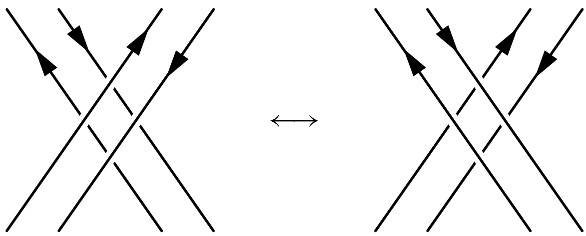


Figure 7: Band pass move

Remark 4.1 Lemmas 4.2 and 4.5 are results of Taylor Martin and complete proofs can be found in Martin [18].

Lemma 4.2 (Martin) *A band pass move preserves* (0)*–solvability.*

Proposition 4.3 *If* L *is any link of* m *components, then* $\mathrm{BD}(L)$ *is* (0)*–solvable.*

Proof Let L be an m–component link in S^3. The Bing double, $\mathrm{BD}(L)$ is band pass equivalent to the trivial link of $2m$ components, arising from the fact that any link can be transformed into the trivial link by a finite number of crossing changes. Since the trivial link is 0–solvable and band pass moves preserve 0–solvability, $\mathrm{BD}(L)$ is 0–solvable. □

We will also consider several other geometric moves, illustrated in Figures 8 and 9.

Lemma 4.4 *The delta move can be realized as a half-clasp move. Moreover, the double delta move can be realized by a double half-clasp move.*

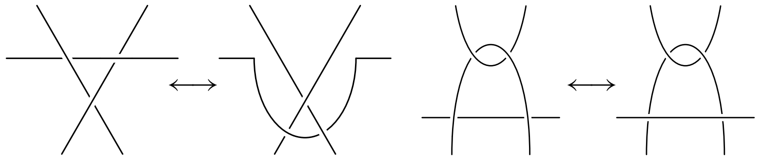


Figure 8: The delta move and the half-clasp move

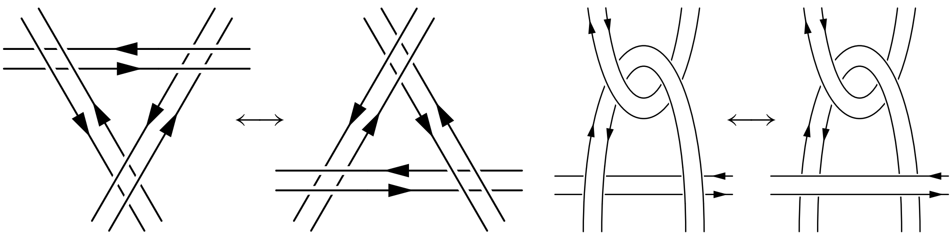


Figure 9: The double delta move and the double half-clasp move

Proof The images in Figure 10 illustrate how to use isotopy and a half-clasp move to achieve the delta move. This result is easily adaptable for the double of the moves.

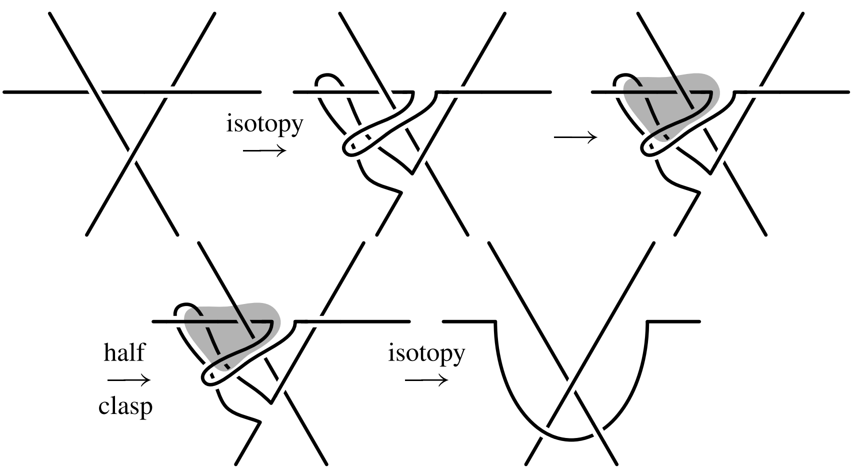


Figure 10: Sequence of isotopy and half-clasp moves to achieve the delta move

This completes the proof. □

Martin established a relationship between (0.5)–solvability and the double half-clasp move which is given in the following lemma.

Lemma 4.5 (Martin) *The double half-clasp move preserves* (0.5)*–solvability.*

Proposition 4.6 *If $L \in \mathcal{F}^m_{-0.5}$, then* $\mathrm{BD}(L)$ *is* (0.5)*–solvable.*

Proof Suppose L has all pairwise linking numbers equal to zero. It was shown (see Murakami and Nakanishi [22] and Matveev [19]) that two links are equivalent by delta moves if and only if they have the same pairwise linking numbers. This result was generalized for string links; see Naik and Stanford [23]. Recall that in our construction of Bing doubling of a link, we chose a string link J such that $\widehat{J}$ is isotopic to L. Since $\widehat{J}$ has all pairwise linking numbers equal to zero by assumption, J can be chosen to have all pairwise linking numbers equal to zero as a string link.

In the construction of Bing doubling we can see that the handlebody H was replaced with the exterior of J. As a result of this replacement, we have a new string link $\widetilde{J}$. Using double delta moves, we are able to get the trivial link (delta moves on J will be double delta moves on $\widetilde{J}$). Since the double half-clasp move preserves (0.5)–solvability by Lemma 4.5, the double delta move will also preserve (0.5)–solvability. Thus $\mathrm{BD}(L)$ is (0.5)–solvable. □

Proposition 4.7 *If L is an (n)–solvable link, then* $\mathrm{BD}(L)$ *is $(n+1)$–solvable. Moreover, if L is an $(n.5)$–solvable link, then* $\mathrm{BD}(L)$ *is $((n+1).5)$–solvable.*

Proof Suppose L is an (n)–solvable link with m components. We will construct an $(n+1)$–solution for $\mathrm{BD}(L)$ by first constructing a cobordism between M_L and $M_{\mathrm{BD}(L)}$. Suppose that J is a string link such that $\widehat{J}$ is isotopic to L. Then $M_L = (D^2 \times I - J) \cup (D^2 \times I - \text{trivial string link})$; see Cochran, Friedl and Teichner [4, page 623] for more details. Consider $M_L \times [0,1]$ and $M_{L_{\mathrm{BD}}} \times [0,1]$. Recall that L_{BD} was isotopic to the $2m$–component trivial link. Let V be the handlebody $D^2 \times I$–trivial string link. Glue $M_L \times \{1\}$ to $M_{L_{\mathrm{BD}}} \times \{1\}$ by identifying $V \subset M_L \times \{1\}$ with $V \subset M_{L_{\mathrm{BD}}} \times \{1\}$. Call the resulting space X; see Figure 11. Then $\partial X = M_L \sqcup M_{L_{\mathrm{BD}}} \sqcup -M_{\mathrm{BD}(L)}$.

To proceed, we need the following lemma.

Lemma 4.8 *With X as above, the inclusion maps induce*

(i) *isomorphisms* $H_1(M_{L_{\mathrm{BD}}}) \to H_1(X)$ *and* $H_1(M_{\mathrm{BD}(L)}) \to H_1(X)$*,*

(ii) *an isomorphism* $H_2(X) \cong H_2(M_{L_{\mathrm{BD}}}) \oplus H_2(M_L)$*.*

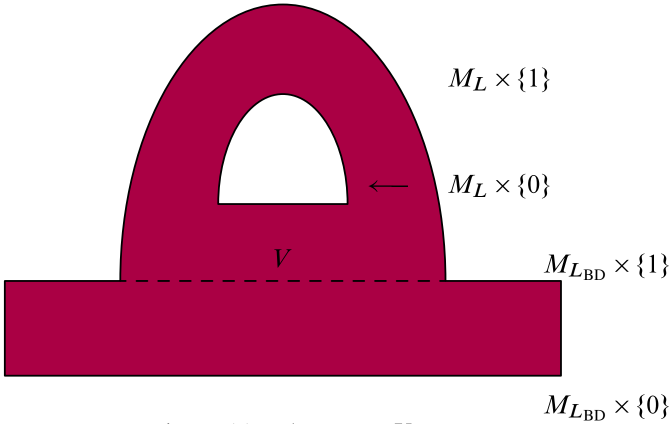


Figure 11: The space X

Proof Consider the following diagram of inclusion maps:

$$\begin{tikzcd} & V \arrow[dl, "i_1"'] \arrow[dr, "i_2"] & \\ M_{L_{\mathrm{BD}}} \times [0,1] \arrow[dr, "j_2"'] & & M_L \times [0,1] \arrow[dl, "j_1"] \\ & X & \end{tikzcd}$$

Using Mayer–Vietoris, the maps above induce the following long exact sequence (in reduced homology), where $I_* = (i_{1*}, i_{2*})$ and $J_* = j_{1*} - j_{2*}$ (the homology groups are with $\mathbb{Z}$ coefficients):

$$\cdots \xrightarrow{\partial_*} H_2(V) \xrightarrow{I_*} H_2(M_{L_{\mathrm{BD}}}) \oplus H_2(M_L) \xrightarrow{J_*} H_2(X)$$
$$\xrightarrow{\partial_*} H_1(V) \xrightarrow{I_*} H_1(M_{L_{\mathrm{BD}}}) \oplus H_1(M_L) \xrightarrow{J_*} H_1(X) \xrightarrow{\partial_*} 0$$

The homology group $H_1(V) \cong \mathbb{Z}^m$ is generated by the meridians, μ_i of the trivial string link. Recall that the η_i s were defined in the construction of Bing doubling. Now $i_{1*}(\mu_i) = 0$ in $S^3 - L_{\mathrm{BD}} \subset M_{L_{\mathrm{BD}}}$ since $\mu_i \sim \eta_i$ and η_i is in a commutator subgroup. Also, $i_{2*}(\mu_i)$ is of infinite order in $H_1(M_L)$ since μ_i is identified with a meridian of L. Hence I_* is a monomorphism. Thus the map $\partial_*\colon H_2(X) \to H_1(V)$ is the zero map. By the properties of a long exact sequence, $H_2(M_{L_{\mathrm{BD}}}) \oplus H_2(M_L) \cong H_2(X)$.

For the other part of the lemma, consider the first isomorphism theorem. This gives

$$H_1(X) \cong \frac{H_1(M_{L_{\mathrm{BD}}}) \oplus H_1(M_L)}{\mathrm{Image}(I_*\colon H_1(V) \to H_1(M_{L_{\mathrm{BD}}}) \oplus H_1(M_L))}.$$

The image of I_* is precisely $H_1(M_L)$. Thus $H_1(X) \cong H_1(M_{L_{\mathrm{BD}}}) \cong \mathbb{Z}^{2m}$. Now $H_1(M_{\mathrm{BD}(L)})$ is generated by the meridians of $\mathrm{BD}(L)$ which are isotopic (in X) to the meridians of L_{BD}. This means that $H_1(X) \cong H_1(M_{\mathrm{BD}(L)})$ which is the desired result. $\square$

We now continue with the proof of the proposition. Let $S = B^4 - \mathbb{D}$ be a slice disk complement, where $\mathbb{D} \subset B^4$ is a collection of disjoint and smoothly embedded disks with boundary L_{BD}. Let W be an (n)–solution for L and let E be the space obtained by attaching W and S to X along $M_{L\times\{0\}}$ and $M_{L_{\mathrm{BD}}\times\{0\}}$ respectively. Thus E is a 4–manifold with boundary $M_{\mathrm{BD}(L)}$.

We claim that E is an $(n+1)$–solution for $\mathrm{BD}(L)$. We begin by showing E is an (n)–solution. Let $\bar{E} = X \cup W$. Consider the following long exact sequence (with $\mathbb{Z}$–coefficients in reduced homology) obtained by Mayer–Vietoris:

$$\cdots \xrightarrow{\partial_*} H_2(M_L) \xrightarrow{I_1} H_2(X) \oplus H_2(W) \xrightarrow{I_2} H_2(\bar{E})$$
$$\xrightarrow{\partial_*} H_1(M_L) \xrightarrow{I_1} H_1(X) \oplus H_1(W) \xrightarrow{I_2} H_1(\bar{E}) \xrightarrow{\partial_*} 0$$

We have that inclusion induces an isomorphism $H_1(M_L) \cong H_1(W)$. This together with the facts that I_2 on H_1 is surjective and $H_1(M_L) \to H_1(X)$ is the zero map, gives that $H_1(\bar{E}) \cong H_1(X)$. From Lemma 4.8, the inclusion maps induce an isomorphism $H_2(X) \cong H_2(M_{L_{\mathrm{BD}}}) \oplus H_2(M_L)$. Thus, by the first isomorphism theorem, definition of exact sequence and the fact that $H_2(M_L) \to H_2(W)$ is the zero map, we obtain

$$H_2(\bar{E}) \cong \frac{H_2(X) \oplus H_2(W)}{\ker(I_2\colon H_2(X) \oplus H_2(W) \to H_2(\bar{E}))} \cong H_2(M_{L_{\mathrm{BD}}}) \oplus H_2(W).$$

Notice that $E = \bar{E} \cup S$. Consider the following long exact sequence on homology given by Mayer–Vietoris:

$$\cdots \xrightarrow{\partial_*} H_2(M_{L_{\mathrm{BD}}}) \xrightarrow{\rho_1} H_2(\bar{E}) \oplus H_2(S) \xrightarrow{\rho_2} H_2(E)$$
$$\xrightarrow{\partial_*} H_1(M_{L_{\mathrm{BD}}}) \xrightarrow{\rho_1} H_1(\bar{E}) \oplus H_1(S) \xrightarrow{\rho_2} H_1(E) \xrightarrow{\partial_*} 0$$

Using these facts, $H_1(M_{L_{\mathrm{BD}}}) \cong H_1(X)$ induced by inclusion (Lemma 4.8), $H_2(S) = 0$, and $H_1(X) \cong H_1(\bar{E})$, we can again use the first isomorphism theorem to attain the following:

$$H_2(E) \cong \frac{H_2(\bar{E})}{\ker(\rho_2)} \cong H_2(W)$$

This shows that the second condition of (n)–solvability of Definition 2.3 is satisfied for the 4–manifold E.

For the third condition, the inclusion map $i: W \hookrightarrow E$ gives $i_*(\pi_1(W)^{(n)}) \subseteq \pi_1(E)^{(n)}$. Since no elements were added to the basis of $H_2(E)$, it has the same basis as $H_2(W)$. Thus $\pi_1(L_i) \subset \pi_1(W)^{(n)} \subseteq \pi_1(E)^{(n)}$ and similarly for $\pi_1(D_i)$, where $\{L_i, D_i\}$ is a basis for $H_2(W)$.

To check the first condition of (n)–solvability, consider again the previous long exact sequence. The first isomorphism theorem tells us that

$$\frac{H_1(M_{L_{\mathrm{BD}}})}{\ker(\rho_1\colon H_1(M_{L_{\mathrm{BD}}}) \to H_1(\bar{E}) \oplus H_1(S))} \cong \mathrm{Image}(\rho_1).$$

Now, S is an (n)–solution for $M_{L_{\mathrm{BD}}}$, and thus $H_1(M_{L_{\mathrm{BD}}}) \cong H_1(S)$ induced by inclusion. We also have that $H_1(\bar{E}) \cong H_1(S)$. Using the first isomorphism a final time and the definition of the maps ρ_1 and ρ_2, gives that $H_1(E) \cong H_1(\bar{E})$.

By Lemma 4.8 and the above results, the first condition to being (n)–solvable is met and E is an (n)–solution for $\mathrm{BD}(L)$.

We claim further that E is actually an $(n+1)$–solution. Showing $\pi_1(W) \subset \pi_1(E)^{(1)}$ (or more precisely, $i_*(\pi_1(W)) \subset \pi_1(E)^{(1)}$) is enough to imply $\pi_1(W)^{(n)} \subset \pi_1(E)^{(n+1)}$ and then L_i and D_i lift to $\pi_1(E)^{(n+1)}$.

Consider the following commutative diagram of maps where i_* is induced by inclusion and both p_{1_*} and p_{2*} are canonical quotient maps:

$$\begin{tikzcd}
\pi_1(W) \arrow[rr, "i_*"] \arrow[d, "p_{1*}"'] \arrow[drr, "h"] & & \pi_1(E) \arrow[d, "p_{2*}"] \\
H_1(W) = \frac{\pi_1(W)}{\pi_1(W)^{(1)}} \arrow[rr, "i_*"'] & & \frac{\pi_1(E)}{\pi_1(E)^{(1)}} = H_1(E)
\end{tikzcd}$$

Showing that $h \equiv 0$ is equivalent to showing that $\pi_1(W) \subset \pi_1(E)^{(1)}$. Examining this further shows that $h \equiv 0$ if and only if $i_*\colon H_1(W) \to H_1(E)$ is the zero map, since our diagram commutes. Consider the following commutative diagram:

$$\begin{tikzcd}
H_1(M_L) \arrow[r, "\cong"] \arrow[dr, "p"'] & H_1(W) \arrow[d, "i_*"] \\
& H_1(E)
\end{tikzcd}$$

To show that $i_* \equiv 0$ is equivalent to showing that the map $p\colon H_1(M_L) \to H_1(E)$ is the zero map. Consider $[\mu_i] \in H_1(M_L)$, where the μ_is generate $H_1(M_L)$. Under the map p, $[\mu_i] = [\eta_i] \in H_1(M_{L_{\mathrm{BD}}}) \subset H_1(E)$ ($\eta_i \subset S^3 - L_{\mathrm{BD}} \subset M_{L_{\mathrm{BD}}}$). But recall that $[\eta_i]$ lie in a commutator subgroup and thus $[\eta_i]=0$ in homology, and thus p is the zero map. This shows that E is an $(n+1)$–solution and the desired result is achieved.

The case when L is $(n.5)$–solvable is similar. □

5 Applications to $\{\mathcal{F}_n^m\}$

In studying the (n)–solvable filtration, we often look at successive quotients of the filtration. Recently, progress has been made towards understanding the structure of its quotients; see [2; 6; 9; 14]. Harvey first showed that $\mathcal{F}_n^m/\mathcal{F}_{n+1}^m$ is a nontrivial group that contains an infinitely generated subgroup [14]. She showed that this subgroup is generated by boundary links (links with components that bound disjoint Seifert surfaces). Cochran and Harvey improved this result by showing that $\mathcal{F}_n^m/\mathcal{F}_{n.5}^m$ contains an infinitely generated subgroup [6]. Again, this subgroup consists entirely of boundary links. Boundary links have vanishing $\bar{\mu}$–invariants at all lengths.

Using the relationship between Milnor's $\bar{\mu}$–invariants and (n)–solvability, given in Corollary 3.5, we are able to establish new results that are disjoint from previous work. Until now, nothing has been known about the "other half" of the (n)–solvable filtration, namely $\mathcal{F}_{n.5}^m/\mathcal{F}_{n+1}^m$.

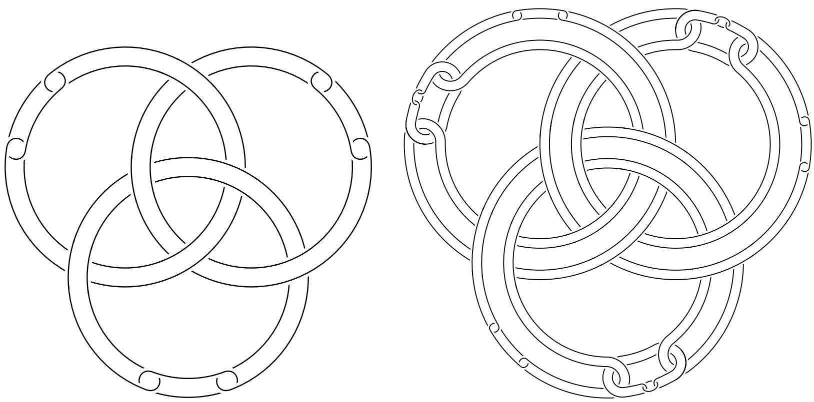
Theorem 5.1 *The "other half" of the filtration $\mathcal{F}_{n.5}^m/\mathcal{F}_{n+1}^m$ contains an infinite cyclic subgroup for $m\geq 3\cdot 2^{n+1}$.*

Proof Let $\mathrm{BR}=$ the Borromean rings. It is clear that $\mathrm{BR}\in\mathcal{F}_{-0.5}^3$. A direct calculation from the definition of Milnor's invariants will show that $\bar{\mu}_{\mathrm{BR}}(123)=\pm 1$ depending on orientation. Using Corollary 3.5, BR is a nontrivial link in $\mathcal{F}_{-0.5}^3/\mathcal{F}_0^3$. When we apply the Bing double operator on BR, denoted BD(BR) (see Figure 13(a)), this new link is in $\mathcal{F}_{0.5}^6$ by Proposition 4.7. However, the first nonvanishing $\bar{\mu}$–invariant is $\bar{\mu}_{\mathrm{BD(BR)}}(I)=\pm 1$ for a certain I with $|I|=6$; see [3, Chapter 8] for the specific details. Hence BD(BR) is not (1)–solvable by Corollary 3.5. Then BD(BR) is nontrivial in $\mathcal{F}_{0.5}^6/\mathcal{F}_1^6$ since it has a nonvanishing $\bar{\mu}$–invariant.

We can perform the Bing doubling operation on this new link to form BD(BD(BR)), or more simply, $\mathrm{BD}_2(\mathrm{BR})$; see Figure 13(b). Using Proposition 4.7, $\mathrm{BD}_2(\mathrm{BR})$ is nontrivial in $\mathcal{F}_{1.5}^{12}$. Looking at its $\bar{\mu}$–invariants, we will have that $\bar{\mu}_{\mathrm{BD}_2(\mathrm{BR})}(I)=\pm 1$ for a certain I of length 12 and our link cannot be (2)–solvable, again by Corollary 3.5. Therefore $\mathrm{BD}_2(\mathrm{BR})$ is nontrivial in $\mathcal{F}_{1.5}^{12}/\mathcal{F}_2^{12}$. We can continue this process to have $\mathrm{BD}_{n+1}(\mathrm{BR})$ nontrivial in $\mathcal{F}_{n.5}^m/\mathcal{F}_{n+1}^m$ for $m\geq 3\cdot 2^{n+1}$.

We claim that $\mathrm{BD}_{n+1}(\mathrm{BR})$ will have infinite order in $\mathcal{F}^m_{n.5}/\mathcal{F}^m_{n+1}$. Orr showed the first nonvanishing $\overline{\mu}$–invariant is additive [24]. Consider an arbitrary string link L with the following properties instead of the specific link $\mathrm{BD}_{n+1}(\mathrm{BR})$ for the moment. Suppose that $\overline{\mu}_L(I) = 0$ and that $\overline{\mu}_L(J) \neq 0$ for $|J| = |I| + 1$. Then $\overline{\mu}_{\widehat{LL}}(J) = \overline{\mu}_{\widehat{L}}(J) + \overline{\mu}_{\widehat{L}}(J) = 2\overline{\mu}_{\widehat{L}}(J)$. If we were to take the closure of the stack of n copies of L, denoted $\widehat{nL}$, we would obtain $\overline{\mu}_{\widehat{nL}}(J) = n\overline{\mu}_{\widehat{L}}$.

This gives that L generates an infinite cyclic subgroup $\mathbb{Z}$ of the string link concordance group. In our case, since $\mathrm{BD}_{n+1}(\mathrm{BR})$ has a nonzero $\overline{\mu}$–invariant of length $3 \cdot 2^{n+1}$, the same reasoning can be used to show that it generates an infinite cyclic subgroup of $\mathcal{F}^m_{n.5}/\mathcal{F}^m_{n+1}$. □



(a) BD(BR) (b) $\mathrm{BD}(\mathrm{BD}(\mathrm{BR})) = \mathrm{BD}_2(\mathrm{BR})$

Figure 12: Examples of iterated Bing doubles of the Borromean rings

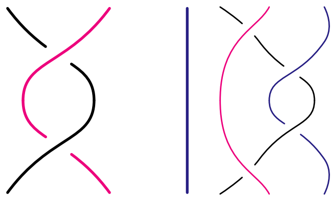
The example exhibited in the above proof came from iterated Bing doubles of a link with certain nonvanishing $\overline{\mu}$–invariants. These iterated Bing doubles will always have a nonzero $\overline{\mu}$–invariant, see Cochran [3], and the above example is also not concordant to a boundary link. Hence our results are not concordant to those previously known.

Since the knot concordance group $\mathcal{C}$ is abelian, all successive quotients of the (n)–solvable filtration are abelian. It is known, however, that $\mathcal{C}^m$ is a nonabelian group for $m \geq 2$ [16]. We briefly recall some facts known about certain quotient groups of $\{\mathcal{F}^m_n\}$.

The quotient $\mathcal{F}^m_{-0.5}/\mathcal{F}^m_0$ has been classified by Martin and is known to be abelian [18].

We also know that the quotient $\mathcal{C}^m/\mathcal{F}^m_0$ is a nonabelian group for $m \geq 3$.

Example 5.2 Consider the pure braids in Figures 14(a) and 14(b). We build the commutator $ABA^{-1}B^{-1}$ seen in Figure 14(c). The link $\widehat{ABA^{-1}B^{-1}}$ is isotopic to the Borromean rings which are not 0–solvable. We conclude that $\mathcal{C}^m/\mathcal{F}^m_0$ is not abelian.



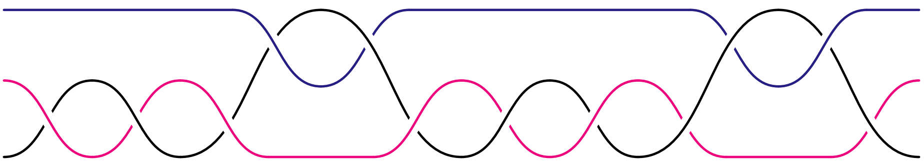


Figure 13: Example of a commutator of pure braids that is not 0–solvable

We continue on with our investigation of quotients of $\{\mathcal{F}^m_n\}$. Again using Corollary 3.5, we will show that $\mathcal{F}^m_{-0.5}/\mathcal{F}^m_1$ is a nonabelian group.

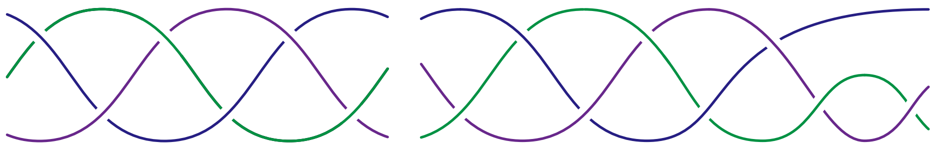
Theorem 5.3 *We have $\mathcal{F}^m_{-0.5}/\mathcal{F}^m_1$ is a nonabelian group for $m \geq 3$.*

In order to prove this theorem, we need to demonstrate that there exists two string links with pairwise linking numbers equal to zero such that when we construct the commutator we get a string link that is not (1)–solvable.

Proof The Borromean rings, BR, can be written as a pure braid, specifically, $\mathrm{BR} = \sigma_2\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2\sigma_1^{-1}$; see Figure 15(a). Consider the pure braid $\sigma_1\mathrm{BR}\sigma_1^{-1}$, the Borromean rings conjugated by σ_1; see Figure 15(b). We look at the commutator $L = (\mathrm{BR})(\sigma_1\mathrm{BR}\sigma_1^{-1})(\mathrm{BR})^{-1}(\sigma_1\mathrm{BR}\sigma_1^{-1})^{-1}$. Notice that L is also a pure braid.

For braids, the canonical meridians, m_i, will freely generate the fundamental group and any other meridian of L_i (the i^{th} string of L) in π_1 will be a conjugate of m_i. This allows us to write l_i of $\widehat{L}$ as a product of the m_i using an algorithmic procedure.

Using this idea, Davis designed a computer program to compute this invariants for braids [11]. We found that $\bar{\mu}_L(313323) = -1$. By Corollary 3.5, L is not (1)–solvable. Therefore $\mathcal{F}^m_{-0.5}/\mathcal{F}^m_1$ is a nonabelian group. □



(a) BR as a pure braid (b) $\sigma_1 \mathrm{BR} \sigma_1^{-1}$

Figure 14: Borromean rings as a pure braid and a conjugate of them

6 The grope filtration and the (n)–solvable filtration

In addition to defining the (n)–solvable filtration, Cochran, Orr and Teichner [10] also defined the grope filtration, $\{\mathcal{G}_n^m\}$ of the (string) link concordance group,

$$\{0\} \subset \cdots \subset \mathcal{G}_{n+1}^m \subset \mathcal{G}_{n.5}^m \subset \mathcal{G}_n^m \subset \cdots \subset \mathcal{G}_{0.5}^m \subset \mathcal{G}_0^m \subset \mathcal{C}^m.$$

Definition 6.1 A *grope* is a special pair (2–complex, base circle) which has a height $n \in \frac{1}{2}\mathbb{N}$ assigned to it. A grope of height 1 is precisely a compact, oriented surface Σ with a single boundary component, which is the base circle; see Figure 15.

A grope of height $n+1$ can be defined recursively by the following construction. Let $\{\alpha_i, \beta_i : i = 1, \ldots, 2g\}$, where g is the genus of Σ be a symplectic basis of curves for $H_1(\Sigma)$, where Σ is a height-one grope. The surface Σ is also known as the first stage grope. Then a grope of height $n+1$ is formed by attaching gropes of height n to each α_i and β_i along the base circles; see Figure 15. A grope of height 1.5 is a surface with surfaces attached to 'half' of the basis curves. A grope of height $n+1.5$ is obtained by gluing gropes of height n to the α_i and gropes of height $n+1$ to the β_i.

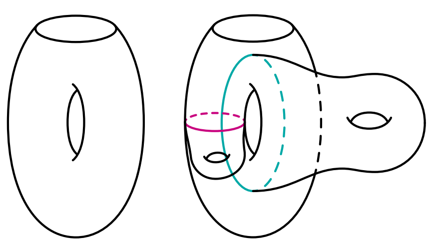


Figure 15: A height-1 and height-2 grope

Given a 4–manifold, W, with boundary S^3 and a framed circle $\gamma \subset S^3$, we say that γ bounds a *grope* in W if γ extends to a smooth embedding of a grope with its untwisting framing (parallel pushoffs of gropes can be taken in W).

We denote $\mathcal{G}_n^m$ to be the subset of $\mathcal{C}^m$ defined by the following. A string link L is in $\mathcal{G}_n^m$ if the components of $\widehat{L}$ bound disjoint gropes of height n in D^4. It can be shown that these subsets are actually normal subgroups of $\mathcal{C}^m$. Harvey showed that this filtration is nontrivial by looking at the filtration of boundary string links [6].

There is also a notion of grope concordance. To define this, the following definition is needed.

Definition 6.2 An *annular grope* of height n is a grope of height n that has an extra boundary component on its first stage.

The two boundary components of an annular grope are said to cobound an annular grope. Two links, L_0 and L_1, are *height n grope concordant* if their components cobound disjoint height n annular gropes, G_i, in $S^3 \times [0,1]$ such that $G_i \cap (S^3 \times \{j\})$ is equal to the i^{th} component of L_j, where $j = 0, 1$.

Thus far, two filtrations of the string link concordance group $\mathcal{C}^m$ have been defined. The (n)–solvable filtration is an algebraic approximation while the grope filtration is a geometric approximation to a link being slice. It is a natural question to ask whether these two filtrations are related. In order to answer this question, we need to analyze the relationship between a link bounding disjoint gropes and the $\overline{\mu}$–invariants of links.

Definition 6.3 Let $L = L_1 \cup L_2 \cup \cdots \cup L_m$ and $L' = L'_1 \cup L'_2 \cup \cdots \cup L'_m$ be ordered, oriented links in S^3. We say that L is *k–cobordant* to L', where $k \in \mathbb{Z}^+$, if there are disjoint, smoothly embedded compact, connected, oriented surfaces $V_1, V_2, \dots, V_m$ in $S^3 \times [0,1]$ with $\partial V_i = \partial_0 V_i \cup \partial_1 V_i$ such that for all $i = 1, \dots, m$, we have:

(i) $V_i \cap (S^3 \times \{0\}) = \partial_0 V_i = L_i$ and $V_i \cap (S^3 \times \{1\}) = \partial_1 V_i = L'_i$.

(ii) There is a tubular neighborhood $V_i \times D^2$ of V_i in $S^3 \times [0,1]$ which extends the "longitudinal" ones of $\partial V_i = L_i \cup L'_i$ in $S^3 \times \{0\}$ and $S^3 \times \{1\}$ respectively such that the image of the homomorphism

$$\pi_1(V_i) \to \pi_1(V_i \times \partial D^2) \to \pi_1(S^3 \times [0,1] - V) = G$$

lies in the k^{th} term of the lower central series of G, denoted G_k, and the image of $\pi_1(\partial V_i)$.

A link that is k–cobordant to a slice link is called *null k–cobordant*.

The concept of k–cobordism is related to the grope filtration as seen in the following proposition.

Proposition 6.4 *If $L \in \mathcal{G}_n^m$, then it is 2^{n-1}–cobordant to a slice link.*

Proof Suppose $L \in \mathcal{G}_n^m$. Then the components of L, say ℓ_i, bound disjoint gropes of height n in $D^4 \cong S^3 \times [0,1]$. Moreover, the ℓ_i extend to smooth embeddings of gropes with their untwisting framing. Also, L is height n grope concordant to a slice link L'. Let V_i be the first stage grope bounded by ℓ_i and ℓ_i' (ie the annular grope in the concordance). Let $V = \coprod_{i=1}^m V_i$.

Now consider the homomorphism

$$\pi_1(V_i) \to \pi_1(V_i \times \partial D^2) \to \pi_1(S^3 \times [0,1] - V) = G$$

that is induced by pushing V_i off itself in the normal direction. Let $\{\alpha_i, \beta_i\}$ be a sympletic basis for $H_1(V_i/\partial V_i)$; see Figure 16. The parallel pushoffs of gropes can be taken in $S^3 \times [0,1]$ and thus are now in $S^3 \times [0,1] - V$. By the construction of the gropes, each of the α_i and β_i bound gropes of height $n-1$ in the exterior of V. Thus

$$[\alpha_i], [\beta_i] \in G^{(n-1)} \subset G_{2^{n-1}}$$

by the fact that if a curve ℓ bounds a (map of a) grope of height n in a space X, then $[\ell] \in \pi_1(X)^{(n)}$. This concludes the proof. □

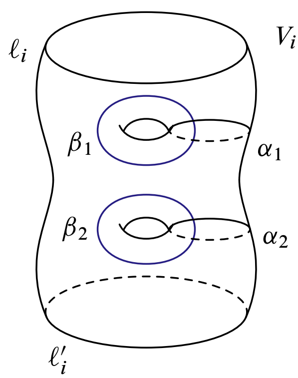


Figure 16: The first stage grope, V_i, with symplectic basis $\{\alpha_i, \beta_i\}_{i=1,2}$

The following corollary of Lin [17] relates Milnor's invariants with k–cobordant links.

Corollary 6.5 (Lin) *If L and L' are k–cobordant, then Milnor's $\overline{\mu}$–invariants of L and L' with lengths less than or equal to $2k$ are the same. In particular, if L is null k–cobordant, then $\overline{\mu}_L(I) = 0$ for $|I| \le 2k$.*

Corollary 6.6 *A link L with components that bound disjoint gropes of height n has $\overline{\mu}_L(I) = 0$ for $|I| \le 2^n$.*

Proof The proof of this is immediate from the previous two results. □

Cochran, Orr and Teichner [10] showed that these two filtrations are related.

Theorem 6.7 (Cochran–Orr–Teichner) *If a link L bounds a grope of height $n+2$ in D^4, then L is (n)–solvable, ie $\mathcal{G}^m_{n+2} \subseteq \mathcal{F}^m_n$ for all m and n.*

The natural question is whether or not the inclusion goes in the other direction. Recall from Corollary 3.5 that an (n)–solvable string link has vanishing $\bar{\mu}$–invariants for lengths less than or equal to $2^{n+2}-1$, whereas in Corollary 6.6 a string link in $\mathcal{G}^m_{n+2}$ has vanishing $\bar{\mu}$–invariants for lengths less than or equal to 2^{n+2}. This difference of one gives motivation to try to find a nontrivial element in $\mathcal{F}^m_n/\mathcal{G}^m_{n+2}$.

Corollary 6.8 *We have that $\mathcal{F}^m_n/\mathcal{G}^m_{n+2}$ is nontrivial for $m \geq 2^{n+2}$. Moreover, $\mathbb{Z} \subset \mathcal{F}^m_n/\mathcal{G}^m_{n+2}$ in this case.*

Proof Let H be the Hopf link. By Proposition 4.7, $\mathrm{BD}(H) \in \mathcal{F}_0$, where $\mathrm{BD}(H)$ is the Bing double H. The invariant $\bar{\mu}_H(12) = \pm 1$ depending on orientation, as it is just the linking number between the two components. Again, by work of Cochran given in Chapter 8 of [3], $\bar{\mu}_{\mathrm{BD}(H)}(I) = \pm 1$ for some I of length 4. Using iterated Bing doubling we achieve $\mathrm{BD}_{n+1}(H)$ in $\mathcal{F}_n$ by Proposition 4.7, and $\bar{\mu}_{\mathrm{BD}_{n+1}(H)}(I) = \pm 1$ for some I of length 2^{n+2}. $\mathrm{BD}_{n+1}(H)$ is (n)–solvable, but since some $\bar{\mu}_{\mathrm{BD}_{n+1}(H)}$ does not vanish for a length of 2^{n+2} it cannot bound a grope of height $n+2$.

To show that there is an infinite cyclic subgroup contained within this quotient, we look at string link representatives of H and $\mathrm{BD}_{n+1}(H)$. The proof of this result is completely analogous to the proof of Theorem 5.1. □

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